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INFORMATION ABOUT MOVES IN EXTENSIVE GAMES: I

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by

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1. INTRODUCTION

In this paper we explore the relation between information patterns and Nash Equilibria in extensive games. By information we mean what players know about each other's moves.[1] Also we confine ourselves throughout to pure strategies. Our main result is that in games in which the level of information is intrinsically "low," the Nash outcomes are invariant of the information.

The extensive game model is of fundamental importance and captures the interplay between information and decision-making. Yet we find that its definition, as set forth by Kuhn in [6], is insufficient from certain points-of-view. It is unable to incorporate games with a continuum of players. Also it often makes for an unnaturally complex representation. For instance, a game in which n players move simultaneously can be described in the Kuhn framework. But first we would have to order the players artificially and then have them move in sequence with suitably enlarged information sets. If we try to carry this out when n is not finite but a continuum, the difficulty of the procedure becomes clear. Therefore

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[1] Chance moves are in essence absent though they can be inserted in certain special circumstances--see Remark (6).

we are led to develop a variant model which has the feature that several players can move simultaneously at any position in the game. Games of the type in [6] are, of course, included as a special case of our set-up.

In Section 2 we develop our model and illustrate it with examples. In Section 3 we show that among all possible information patterns in a game there is a unique minimal one. This is done under the assumption of perfect recall. Otherwise the conclusion is false. (See the example in Section 3.)

In the rest of the paper, we focus on the effect on Nash Equilibria (N.E.) that is caused solely by changes in the information pattern of an extensive game. In Section 4.2 we show that if information is refined then the N.E.'s of the coarse game do not disappear. But the converse is not true: in general there is a rapid proliferation of new N.E.'s. In the next two sections, Section 4.3 and Section 4.4, we explore conditions under which this proliferation is arrested. The notion of "no informational influence" is introduced. It says that if a single player unilaterally changes his strategy, then the resultant new path passes through the same information sets of the others as the old one did. This is a purely set-theoretic condition and can hold not only in non-atomic, but also in finite, games--see the example in Section 4.3. We prove that if it holds then a Nash outcome of the refined game is also that of its coarse form, i.e., is not a "new" N.E. brought about by the increased strategic (threat) possibilities. When we turn to non-atomic games, no informational influence holds in full force and we get: Nash outcomes are invariant of the information pattern. (See Section 5.2.) This leads to the "Anti-folk Theorem" (Section 5.4). It also has some relevance to strategic market games (Section 5.4).

No informational influence arises naturally in games that involve a "large number of small players," and constitutes an essential criterion for the onset of "perfect competition." The very notion of a Nash Equilibrium becomes more viable in its presence.[1] (If his unilateral change of strategy can be observed by others, why should a player imagine that they will stay put when he changes?) Our result reveals that then the N.E.'s are "robust" to changes in information. It also has another upshot: the fine distinctions raised by the notion of "perfect" equilibrium ([8]) become irrelevant in this setting.

2. EXTENSIVE GAMES IN SIMULTANEOUS MOVE FORM

To motivate the definition it will help to consider an example. Let the player-set N be the $[0,1]$ closed interval equipped with the Lebesgue measure. Consider the subsets $A = [0, 1/4)$, $B = [1/4, 1/2)$, $C = [1/2, 3/4)$ and $D = [3/4, 1]$. The moves available to these players are as follows. A player in A (B,C,D) can select any real numbers in the interval $[0,10]$ $([0,5], [0,1], [0,1])$. A and B move simultaneously at the start, after which C and D move. Only measurable choices by $A \cup B$ and by $C \cup D$ are considered. The players in C can observe the integral of the choice made by $A \cup B$. On the other hand, those in D can observe the entire measurable function of $A \cup B$'s choice, upto null sets. The payoffs depend on the "play" induced by everyone's choice.

We wish to consider games of this type, as well as repetitions of them (possibly infinite in number). They arise naturally in several contexts. It is clear that the Kuhn formulation cannot easily handle them. This motivates our new model. After describing it, we shall return to this example and see how it is expressed in terms of our model.

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[1] See Remark (5).

2.1. The Basic Definition

An extensive game Γ in simultaneous move form is a seven-tuple:

(2.1) $$\Gamma = (N, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}, \{I_i\}_{i \in N}) .$$

Let us explain our symbols. (Unless otherwise stated, all sets are assumed to be non-empty.)

(i) N is the set of all players.

(ii) X is the set of all positions in the game, one of which, x_0, is distinguished and represents the start of the game.

(iii) π maps X to subsets of N. For $x \in X$, $\pi(x)$ is the set of players who move simultaneously at the position x.

(iv) For each $x \in X$, S^x is a set of functions from $\pi(x)$ to some set Y^x. We assume that $\pi(x) = \phi \Longleftrightarrow S^x = \phi$. If $\pi(x) = \phi$, then x is called an ending position of the game Γ. Given $s^x \in S^x$, $t \in Y^x$ and $i \in \pi(x)$, denote by (s^x_{-i}, t) the function from $\pi(x)$ to Y^x which assigns t to i, and agrees with s^x elsewhere. Also let s^x_i stand for $s^x(i)$. Our assumption on S^x is:

(2.2) $$\text{if } s^x, r^x \in S^x, \text{ then } (s^x_{-i}, r^x_i) \in S^x \text{ for all } i \in \pi(x) .$$

Define $S^x_i = \{s^x_i : s^x \in S^x\}$ for $i \in \pi(x)$. Note that, by (2.2),

(2.3) $$\text{if } t \in S^x_i \text{ and } s^x \in S^x, \text{ then } (s^x_{-i}, t) \in S^x$$

and,[1] by (2.3),

(2.4) $$\text{if } \pi(x) \text{ is finite, } S^x = \prod_{i \in \pi(x)} S^x_i .$$

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[1] Π denotes a Cartesian product.

S_i^x is _the set of moves available to player_ i _at the position_ x and S^x is the set of move selections at x by the players in $\pi(x)$ that are feasible in the game.

(v) Φ links positions to moves. Put $X^* = X \setminus \{x_0\}$. Let F be the collection of all finite sequences $(s^{x_0}, s^{x_1}, \ldots, s^{x_m})$ with $s^{x_k} \in S^{x_k}$ for $k = 0, 1, \ldots, m$. Then Φ is an injection,

$$\Phi : X^* \to F ,$$

such that:

(a) if $s^{x_0} \in S^{x_0}$, then $(s^{x_0}) \in \Phi(X^*)$;

(b) if $(s^{x_0}, s^{x_1}, \ldots, s^{x_{m-1}}, s^{x_m}) \in \Phi(X^*)$, then $(s^{x_0}, s^{x_1}, \ldots, s^{x_{m-1}}) = \Phi(x_m)$;

(c) if $\Phi(x) = (s^{x_0}, \ldots, s^{x_m})$ and $s^x \in S^x$, then $(s^{x_0}, \ldots, s^{x_m}, s^x) \in \Phi(X^*)$.

Since Φ is an injection, we will sometimes _identify_ x _with_ $\Phi(x)$, and say that $x = (s^{x_0}, \ldots, s^{x_m})$ for $x \in X^*$. This should cause no confusion. Also, if $\Phi(x) = (s^{x_0}, \ldots, s^{x_m})$ we will write (x, s^x) for $(s^{x_0}, \ldots, s^{x_m}, s^x)$.

(vi) An infinite sequence $(s^{x_0}, \ldots, s^{x_m}, \ldots)$ or a finite sequence $(r^{y_0}, r^{y_1}, \ldots, r^{y_k})$, where $y_0 \equiv x_0$, is called a _play_[1] of the game Γ if

(a) $(s^{x_0}, \ldots, s^{x_\ell}) \in \Phi(X^*)$ for all $\ell = 0, 1, \ldots$

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[1]This also is an "outcome" in our context.

or

(b) $(r^{y_0}, \ldots, r^{y_\ell}) \in \Phi(X^*)$ for $\ell = 0, \ldots, k$ and $\pi(z) = \phi$,

where $z = (r^{y_0}, \ldots, r^{y_k})$.

Denote the set of all plays of Γ by $P(\Gamma)$. Each h_i is simply a real-valued function on $P(\Gamma)$ and gives the <u>payoff</u> to player i for any play of the game.

(vii) I_i is a partition of $X_i = \{x \in X : i \in \pi(x)\}$ and is called the <u>information partition of player</u> i. If x and y are two positions in the same set of player i's partition $(x, y \in u \in I_i)$, then this means that i cannot distinguish between x and y. It is natural to impose some constraints on the $\{I_i\}_{i \in N}$ in view of this interpretation. First we require

(2.5) $$\text{if } x, y \in u \in I_i, \text{ then } S_i^x = S_i^y .$$

If this were not so, then i could distinguish intrinsically between x and y. Given (2.5) we will, without confusion, talk of the set of moves S_i^u which is available to i at (each position in) his information set u.

To describe the other property of $\{I_i\}_{i \in N}$ we need to develop some more terminology. If $x = (s^{x_0}, \ldots, s^{x_m})$ and $y \in \{x_0, \ldots, x_m\}$, we will say that x <u>follows from</u> y and write this as $y \prec x$. (Then $\prec$ is a partial order on X with x_0 as its unique minimal element.) Furthermore, if $t = s_i^y$, then x <u>follows from</u> y <u>via the move</u> t <u>of player</u> i (in symbols: $y \underset{t}{\prec} x$). And if, in addition $y \in u \in I_i$, write $u \underset{t}{\prec} x$, as well as $u \prec x$.

The other condition on $\{I_i\}_{i \in N}$ is given by:

(2.6) If $x, y \in u \in I_i$ and $v \underset{t}{\prec} x$ (for some $v \in I_i$ and $t \in S_i^v$) , then $v \underset{t}{\prec} y$.

It goes by the name of "perfect recall". It says that, at any position, each player can fully recall the entire history of his previous moves.

This completes the definition of the game Γ .

Sometimes we will need to talk only about the <u>five</u> tuple: $(N, X, \pi, \{S^x\}_{x \in X}, \Phi)$, and will call it the <u>extensive form</u> of the game Γ , and denote it by Γ_- . The set of plays $\mathcal{P}(\Gamma)$ is in fact now better written $\mathcal{P}(\Gamma_-)$ because it does not depend on the payoffs or the information pattern.

2.2. <u>Examples</u>

Let us formulate the example given at the start in our model. Let $X^{**} = \{t : t(\cdot)$ is a measurable function on $A \cup B$ such that $t(i) \in [0,10]$ for all $i \in A$ and $t(i) \in [0,5]$ for all $i \in B\}$ and $F = \{t : t(\cdot)$ is a measurable function on $C \cup D$ such that $t(i) \in [0,1]$ for all $i \in C \cup D\}$. Then $X = \{x_0\} \cup X^{**} \cup (X^{**} \times F)$. The function $\pi(\cdot)$ is given by $\pi(x_0) = A \cup B$, $\pi(x) = C \cup D$ for all $x \in X^{**}$ and $\pi(x) = \phi$ for all $x \in X^{**} \times F$. Each Y^x is the reals. $\{S^x\}_{x \in X}$ is given by $S^{x_0} = X^{**}$, $S^x = F$ for all $x \in X^{**}$ and $S^x = \phi$ for all $x \in X^{**} \times F$. Note that Φ is the identity map from $X - \{x_0\}$ to $X - \{x_0\}$. The information pattern $\{I_i\}_{i \in N}$ is as follows:

$$I_i = \{x_0\} \text{ for all } i \in A \cup B ;$$

$$I_i = \{\{s^{x_0} \in X^{**}(=S^{x_0}) : \int_{A\cup B} s_i^{x_0} d\mu = \int_{A\cup B} \bar{s}_i^{x_0} d\mu\} : \bar{s}^{x_0} \in X^{**}\}$$

for all $i \in C$; and

$$I_i = \{\{s^{x_0} \in X^{**} : \mu(\{i \in A \cup B : s_i^{x_0} \neq \bar{s}_i^{x_0}\}) = 0\} : \bar{s}^{x_0} \in X^{**}\}$$

for all $i \in D$.

The payoff functions h_i are just real-valued functions on the set of plays (s^{x_0}, s^x) $(x = s^{x_0})$.

One could associate a tree to Γ by letting the branches be given by the binary relation $\overset{*}{\succ}$ on X :

$$x \overset{*}{\succ} y \text{ if } x \succ y \text{ but not } x \succ z \succ y \text{ for any } z \in X .$$

The tree for the above example will then be:

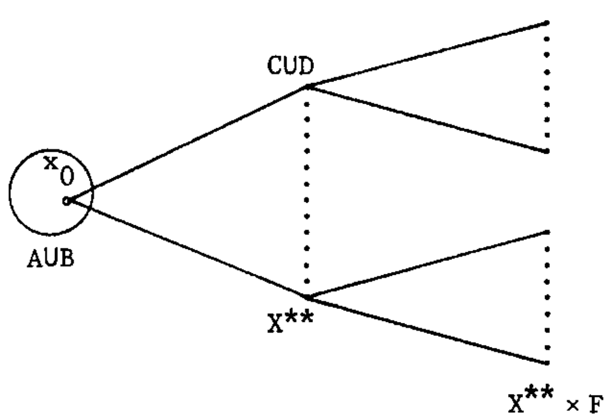

FIGURE 1

Because of its infinite structure the example can not be fully illustrated. But if all the constituents of Γ are finite, then we can draw a tree which will completely describe its extensive form at a glance. Consider:

$$N = \{1,2,3,4\} \ ; \quad X = \{x_0, x_1, \ldots, x_{20}\} \ ;$$

$$\pi(x_0) = \{1,2\} \ , \quad \pi(x_t) = \{3,4\} \quad \text{for} \quad t = 1, 2, 3, 4 \ , \quad \text{and}$$

$$\pi(x_t) = \phi \quad \text{for} \quad t = 5, \ldots, 20 \ ;$$

$$S^{x_0} = \{(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_2, \beta_1), (\alpha_2, \beta_2)\} \ , \quad \text{i.e.,}$$

$$S_1^{x_0} = \{\alpha_1, \alpha_2\} \quad \text{and} \quad S_2^{x_0} = \{\beta_1, \beta_2\} \ ;$$

$$S^{x_t} = \{(\gamma_1, \delta_1), (\gamma_1, \delta_2), (\gamma_2, \delta_1), (\gamma_2, \delta_2)\} \ , \quad \text{i.e.,}$$

$$S_3^{x_t} = \{\gamma_1, \gamma_2\} \quad \text{and} \quad S_4^{x_t} = \{\delta_1, \delta_2\} \quad \text{for} \quad t = 1, 2, 3, 4 \ ;$$

$$S^{x_t} = \phi \quad \text{for} \quad t = 5, \ldots, 20 \ ;$$

$$\Phi(x_1) = (\alpha_1, \beta_1) \ , \quad \Phi(x_2) = (\alpha_1, \beta_2) \ , \quad \Phi(x_3) = (\alpha_2, \beta_1)$$

$$\Phi(x_4) = (\alpha_2, \beta_2) \ , \quad \text{etc.;}$$

$$I_1 = I_2 = \{\{x_0\}\} \ , \quad I_3 = \{\{x_1, x_2\}, \{x_3, x_4\}\} \quad \text{and}$$

$$I_4 = \{\{x_1, x_2, x_3\}, \{x_4\}\} \ .$$

The tree of this game is:

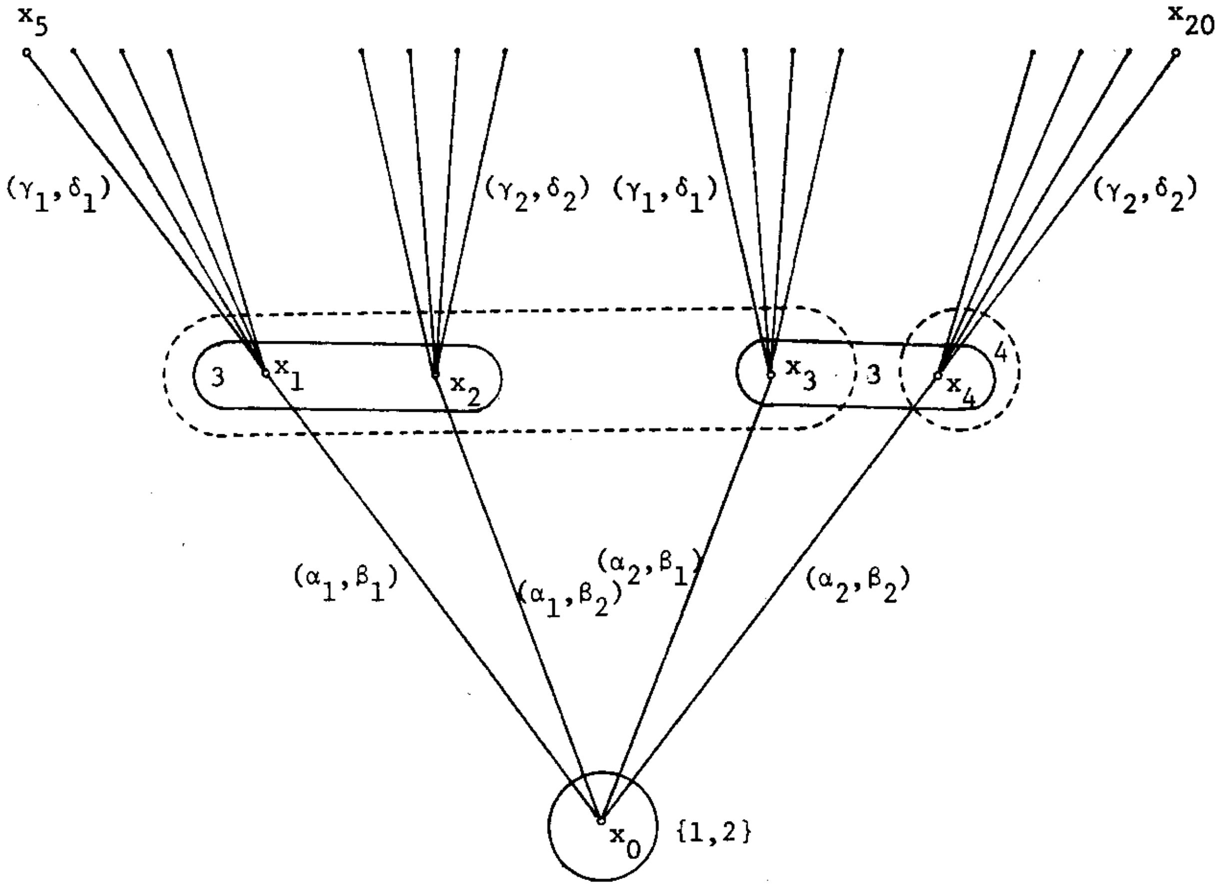

FIGURE 2

2.3. The Problem of Reduction

If the game Γ satisfies:

(i) $|N|$ is finite,[1]

(ii) $|\pi(x)| \leqq 1$ for all x,

(iii) $|S^x|$ is finite for all x,

(iv) there is no infinite play,

then it corresponds to a "Kuhn-type" game as formulated in [6].

However every finite game can also be cast in the Kuhn format. For instance, the game of Figure 2 may be given an alternative description as in Figure 3 without changing any of its game-theoretical features. But this is at the expense of enlarging the informational structure.

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[1] $|S|$ denotes the cardinality of the set S.

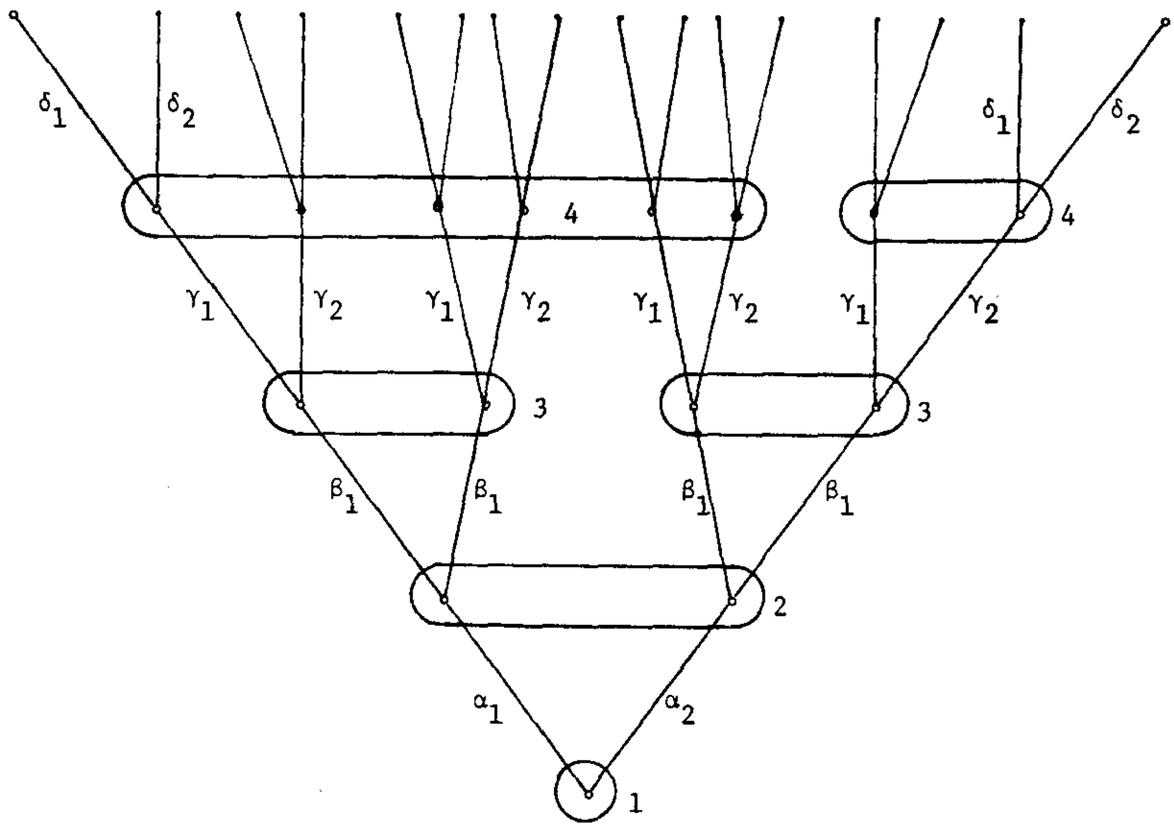

FIGURE 3

In contrast, because of the possibility of simultaneous moves in our model, the representation of this structure was **more** concise in Figure 2. We can "reduce" Figure 3 to Figure 2. This immediately raises the question: is there, for any game, a unique completely reduced form? We have not yet quite formulated the question in precise terms, but we suspect that its answer is "Yes." We plan to look into this in a future paper.

2.4. Elimination of Ending Positions

For simplicity of notation we will assume that Γ has no ending positions. This is done without loss of generality. For each ending position x of Γ add an infinite sequence $(x_1, x_2, \ldots)$ to x, and extend the game as follows:

(i) For all $\ell = 1, 2, \ldots$

$$\pi(x) = \pi(x_\ell) = N ,$$

$$|S^x| = |S^{x_\ell}| = 1 ,$$

$$\Phi(x^\ell) = (\Phi(x), s^x, s^{x_1}, \ldots, s^{x_{\ell-1}}) ;$$

(ii) The payoff at x is now attached to the play $(\Phi(x), s^x, s^{x_1}, \ldots, s^{x_\ell}, \ldots)$.

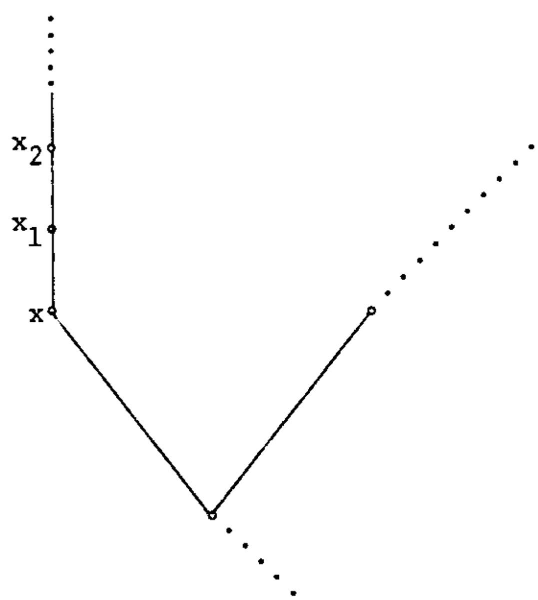

FIGURE 4

2.5. Remarks

(1) A somewhat weaker requirement than (2.6) is:

(2.7) No position follows another within an information set, i.e., if $x, y \in u \in I_i$ then it is not the case that $x \prec y$.

Clearly (2.6) implies (2.7) but not vice-versa. We postulate (2.6) as a fundamental defining property of a game for reasons given in the next remark. However all our Propositions, with the exception of Proposition 1, continue to hold by the same proofs if (2.6) is replaced by (2.7).

(2) In [6], the "story" told to support (2.7) is that the players have distinct "agents" on each of their information sets. Therefore the violation of (2.7) would imply lapse of memory by an agent, and this is not to be tolerated.

We prefer to extend this logic to the player himself and to think of him as a game-theoretic individual who has no such shortcomings. This inexorably forces us to postulate the condition (2.6) of perfect recall. Thus, in contrast with [6], we would rather think of bridge as a 4-person game with perfect recall, in which the two players of any team have identical payoff functions.[1]

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[1]For a further discussion, see [8, Section 1].

3. THE MINIMAL AND MAXIMAL INFORMATION PATTERNS

Let us consider the six-tuple[1]

$$T = (N, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N})$$

and denote by $\Delta(T)$ the set of all games obtained from T by adding all possible information patterns $\{I_i\}_{i \in N}$, subject to conditions (2.5) and (2.6).

Consider $\Gamma, \Gamma^* \in \Delta(T)$ where--in careless notation-- $\Gamma = \{I_i\}_{i \in N}$ and $\Gamma^* = \{I_i^*\}_{i \in N}$. We will say that Γ^* is a refinement of Γ (or Γ is a coarsening of Γ^*) if each I_i^* is obtained by a refinement of I_i, i.e., for any $v \in I_i$, $v = \bigcup_{\substack{v^* \in I_i^* \\ v^* \cap v \neq \phi}} v^*$. In this case we will write $\Gamma \prec \Gamma^*$. It is clear that

(a) $\prec$ is a partial order[2] on $\Delta(T)$,

(b) $\prec$ has a unique maximal element, namely $\{I_i\}_{i \in N}$, with

$$I_i = \{\{x\} : x \in X_i\} .$$

Our main result in this section is that $\prec$ also has a unique minimal element in $\Delta(T)$. To pave the way for this we need two lemmas.

Lemma 1. Let $\{\Gamma_\lambda\}_{\lambda \in \Lambda}$ be a totally ordered subset[2] of $\prec$ in $\Delta(T)$. Then there exists a Γ in $\Delta(T)$ such that $\Gamma \prec \Gamma_\lambda$ for every $\lambda \in \Lambda$.

Proof. Let $\Gamma_\lambda = \{I_i\}_{i \in N}$ for $\lambda \in \Lambda$. For any $i \in N$, $x \in X_i$ and $\lambda \in \Lambda$, put

$$I_i^\lambda(x) = \text{the information set in } I_i^\lambda \text{ which contains } x .$$

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[1] Actually the minimal and maximal information patterns are independent of the payoffs $\{h_i\}_{i \in N}$ and pertain only to the extensive form.

[2] See the Appendix.

Let $I_i(x) = \bigcup_{\lambda \in \Lambda} I_i^{\lambda}(x)$. We submit that the $I_i(x)$ yield an information partition for i on T . Since $\bigcup_{x \in X_i} I_i^{\lambda}(x) = X_i$ for any λ , we get

(c) $\bigcup_{x \in X_i} I_i(x) = X_i$.

Suppose $z \in I_i(x)$ and $I_i(z) \neq I_i(x)$. Since it is clear that $I_i(z) \not\subset I_i(x)$, we can take $y \in I_i(z) \setminus I_i(x)$. Then $y \in I_i^{\lambda_1}(z)$ and $z \in I_i^{\lambda_2}(x)$ for some λ_1 and λ_2 in Λ . Pick $\lambda^* \in \{\lambda_1, \lambda_2\}$ such that $\Gamma^{\lambda^*} \to \Gamma^{\lambda_i}$ for $i = 1, 2$. (Clearly this can be done.) But then $z \in I_i^{\lambda_2}(x) \subset I_i^{\lambda^*}(x)$ and $y \in I_i^{\lambda_1}(z) \subset I_i^{\lambda^*}(z)$. Since $I_i^{\lambda^*}$ is an information partition, $z \in I_i^{\lambda^*}(x)$ implies $I_i^{\lambda^*}(z) = I_i^{\lambda^*}(x)$. But then $y \in I_i^{\lambda^*}(x)$, *a fortoiri* $y \in I_i(x)$, a contradiction. We have proved

(d) The sets $\{I_i(x) : x \in X_i\}$ are either equal or disjoint.

From (c) and (d) it follows that if we define $I_i = \{I_i(x) : x \in X_i\}$ then we get a partition of X_i . We will now verify that I_i satisfies conditions (2.5) and (2.6). If $y, z \in I_i(x)$, then $y \in I_i^{\lambda_1}(x)$ and $z \in I_i^{\lambda_2}(x)$ for some λ_1 and λ_2 in Λ . But then $S_i^y = S_i^x$ and $S_i^z = S_i^x$, which implies $S_i^y = S_i^z$. This verifies (2.5) for I_i .

Suppose $y, z \in I_i(x)$ and $v \underset{t}{<} y$ for some $v \in I_i$ and $t \in S_i^v$, i.e., $y' \underset{t}{<} y$ for some $y' \in v$. Then there are λ_1 and λ_2 in Λ such that $y \; I_i^{\lambda_1}(x)$ and $z \in I_i^{\lambda_2}(x)$. Pick $\lambda^* \in \{\lambda_1, \lambda_2\}$ such that $\Gamma^{\lambda^*} \to \Gamma^{\lambda_k}$ for $k = 1, 2$. Since $I_i^{\lambda^*}(x) \supset I_i^{\lambda_k}(x)$ for $k = 1, 2$, y and z are in $I_i^{\lambda^*}(x)$. By (2.6) there is a $z' \in I_i^{\lambda^*}(y')$ such that $z' \underset{t}{<} z$. Of course, $I_i^{\lambda^*}(y') \subset v$, which implies $v \underset{t}{<} z$.

Take the game Γ obtained by adding the information partition $\{I_i\}_{i \in N}$ to T. We just checked that $\Gamma \in \Delta(T)$. It is obvious from the construction of Γ that $\Gamma \dashv \Gamma_\lambda$ for any $\lambda \in \Lambda$. Q.E.D.

Lemma 2. For any Γ' and Γ'' in $\Delta(T)$, there exists a Γ^* in $\Delta(T)$ such that $\Gamma^* \dashv \Gamma'$ and $\Gamma^* \dashv \Gamma''$.

Proof. Put $\Gamma' = \{I'_i\}_{i \in N}$ and $\Gamma'' = \{I''_i\}_{i \in N}$. Define the binary relation $\beta_i \subset X_i \times X_i$ by:

$$(x,y) \in \beta_i \Longleftrightarrow \left\{ \begin{array}{c} x,y \in v' \in I'_i \\ \text{or} \\ x,y \in v'' \in I''_i \end{array} \right\}$$

Note that β_i is symmetric and reflexive.[1] Let B_i be the set of all equivalence relations β'_i on X_i such that $\beta_i \subset \beta'_i$. Put

$$\beta^*_i = \bigcap_{\beta'_i \in B_i} \beta'_i .$$

Observe that the intersection is nonempty because $X_i \times X_i \in B_i$, and--by definition-- $\beta_i \subset \beta^*_i$. It is easily checked that an intersection of equivalence relations is itself an equivalence relation. Thus β^*_i is <u>the</u> minimal equivalence relation on X_i that contains β_i.

Let I^*_i be the set of equivalence classes induced by β^*_i. I^*_i is clearly a partition of X_i. We will show that it satisfies conditions (2.5) and (2.6).

Suppose there is a $v \in I^*_i$ such that for some $x, y \in v$ we have

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[1] See the Appendix.

$S_i^x \neq S_i^y$. Then partition v into the two nonempty sets

$$u = \{z \in v : S_i^z = S_i^x\}$$

$$\tilde{u} = \{z \in v : S_i^z \neq S_i^x\} .$$

If $(z,\tilde{z}) \in \beta_i$ for some $z \in u$ and $\tilde{z} \in \tilde{u}$, this means that either $\{z,\tilde{z}\} \subset v' \in I_i'$ or $\{z,\tilde{z}\} \subset v'' \in I_i''$ for some v' and v'' . In either case $S_i^z = S_i^{\tilde{z}}$ which is a contradiction. Thus $(u \times \tilde{u}) \cap \beta_i$ is empty. Now let $\tilde{I}_i$ be given by:

$$\tilde{I}_i = (I_i^* \setminus \{v\}) \cup \{u,\tilde{u}\} .$$

Consider the equivalence relation $\tilde{\beta}_i$ induced by $\tilde{I}_i$, i.e. $\tilde{\beta}_i = \{(x,y) \in X_i \times X_i : x, y \in v$ for some $v \in \tilde{I}_i\}$. Then $\tilde{\beta}_i \in B_i$ (because $(u \times \tilde{u}) \cap \beta_i$ is empty; and hence from $\beta_i \subset \beta_i^*$, and the construction of $\tilde{\beta}_i$, we have $\beta_i \subset \tilde{\beta}_i$). On the other hand $\tilde{\beta}_i$ is strictly contained in β_i^* . This contradicts the minimality of β_i^* . We have proved that I_i^* satisfies condition (2.5).

To check (2.6) suppose that there are $x, y \in v \in I_i^*$ such that for some $u \in I_i^*$ we have:

$$u \prec_t x \quad \text{for some} \quad t \in S_i^u .$$

We must then show that $u \prec_t y$ also. Suppose not. Then define w and w' by:

$$w = \{z \in v : u \prec_t z\}$$

$$w' = v \setminus w .$$

Both w and w' are nonempty by supposition. Moreover if $\{z, z'\} \in \beta_i$ for some $(z, z') \in w \times w'$, then either $\{z, z'\} \subset v' \subset I_i'$ or $\{z, z'\} \subset v'' \subset I_i''$. This contradicts respectively the fact that I_i' and I_i'' each satisfy condition (2.6). Thus $(w \times w') \cap \beta_i$ is empty. We now proceed exactly as in the proof that (2.5) holds. Define $\hat{I}_i$ by $\hat{I}_i = (I_i^* \setminus \{v\}) \cup \{w, w'\}$, and consider the equivalence relation $\hat{\beta}_i$ induced by $\hat{I}_i$. Again it can be checked that $\beta_i \subset \hat{\beta}_i \in B_i$, and $\hat{\beta}_i$ is strictly contained in β_i^* , contradicting the minimality of β_i^* . Therefore we conclude that I_i^* satisfies (2.6).

So if we put $\Gamma^* = \{I_i^*\}_{i \in N}$ then $\Gamma^* \in \Delta(T)$. It is clear from our construction that $\Gamma^* \dashv \Gamma'$ and $\Gamma^* \dashv \Gamma''$. Q.E.D.

Putting the two lemmas together we easily get

Proposition 1. There is a unique minimal element of $\dashv$ in $\Delta(T)$.

Proof. Every totally ordered set in $\Delta(T)$ has a lower bound[1] in $\Delta(T)$ by Lemma 1. By Zorn's lemma[1] there is a minimal element Γ^* of $\dashv$ in $\Delta(T)$. Suppose $\tilde{\Gamma}$ is another minimal element in $\Delta(T)$ with $\Gamma^* \not\dashv \tilde{\Gamma}$. Then by Lemma 2 we can find a $\hat{\Gamma}$ in $\Delta(T)$ such that $\hat{\Gamma} \dashv \tilde{\Gamma}$ and $\hat{\Gamma} \dashv \Gamma^*$. Clearly $\hat{\Gamma} \neq \Gamma^*$ which contradicts the fact that Γ^* is a minimal element of $\dashv$. This proves that Γ^* is the unique minimal element of $\dashv$ in $\Delta(T)$. Q.E.D.

If condition (2.7) is used in place of (2.6) then Proposition 1 breaks down. Consider the following 1-person extensive form in which the

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[1] See the Appendix.

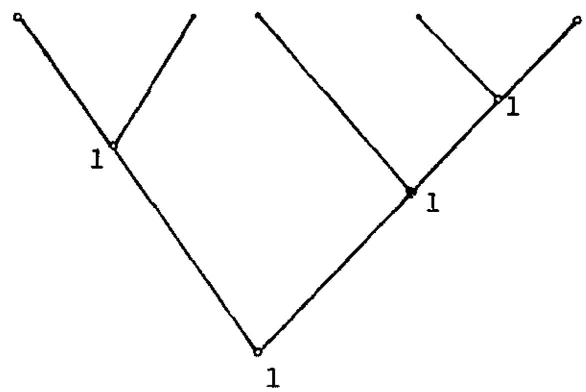

FIGURE 5

moves at all the positions are identical. Then, assuming only (2.6) there are two minimal information patterns

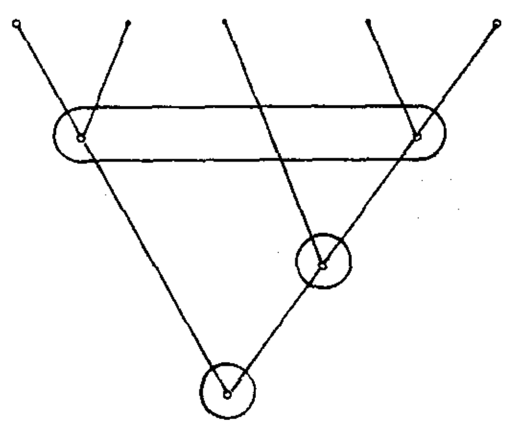

FIGURE 6

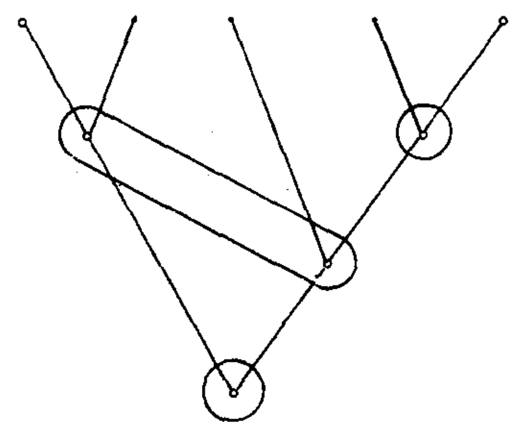

FIGURE 7

Remark

(3) The counterexample reinforces the view expressed in Remark (2). The non-uniqueness of minimal information is unnatural and arises from the (incomplete) model of players who do not have "perfect recall."

4. NASH EQUILIBRIA AND INFORMATION PATTERNS

4.1. The Definition of a Nash Equilibrium

From now on, we will focus on the effect on Nash Equilibria that is caused solely by changes in the information pattern of an extensive game. First let us recall the notion of a Nash Equilibrium. Fix a game

$$\Gamma = (N, X, \pi, \{S^x\}_{x\in X}, \Phi, \{h_i\}_{i\in N}, \{I_i\}_{i\in N})$$

as in Section 2.1. Let $Z(\Gamma\_)$ be the set of all $\{s^x\}_{x\in X}$ which satisfy $s^x \in S^x$ for all x . Any $\{s^x\}_{x\in X} = z \in Z(\Gamma\_)$ gives rise to a play $p(z) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ by the rule:

$$x_1 = \Phi^{-1}(s^{x_0})$$
$$\vdots$$
$$x_{k+1} = \Phi^{-1}(s^{x_0}, \ldots, s^{x_k}) , \quad \text{for} \quad k \geq 1 .$$

The <u>strategy-set</u> of a player i is made up of all possible choices of moves available to him in X_i , under the proviso that he must make the same choice at positions that are indistinguishable in his information partition. It is the set S^i consisting of all maps s_i from X_i to $\bigcup_{x\in X_i} S_i^x$ which satisfy:

(i) $s_i(x) \in S^x$,

(ii) $s_i(x) = s_i(y)$ if $x, y \in u \in I_i$.

The choice $z = \{s^x\}_{x\in X}$ in $Z(\Gamma)$ is said to be <u>feasible in</u> Γ if the induced maps τ_i belong to S^i for each i in N , where $\tau_i(x) = s_i^x$ for $x \in X_i$. Let $Z_F(\Gamma\_)$ denote the set of all z in $Z(\Gamma\_)$ that are feasible in Γ . <u>We assume throughout that</u> $Z_F(\Gamma)$ <u>is not empty</u>.

For any $z = \{s^x\}_{x\in X}$ in $Z(\Gamma\_)$ and $t_i \in S^i$, $(z|t_i)$ is the element of $Z(\Gamma_{-})$ obtained from z by replacing player i's choices in accordance with t_i, i.e., $(z|t_i) = \{q^x\}_{x\in X}$ where

$$q_j^x = \begin{cases} s_j^x & \text{if } j \in N\backslash\{i\} \text{ and } j \in \pi(x) \\ t_i(x) & \text{if } j = i \text{ and } j \in \pi(x) \end{cases}$$

Clearly $(z|t_i)$ is in $Z(\Gamma_{-})$; by (2.2), it is also in $Z_F(\Gamma)$ if z is.

We will call $z \in Z(\Gamma_{-})$ a <u>Nash Equilibrium</u> (N.E.) of Γ if

(4.1) z is feasible in Γ (i.e., $z \in Z_F(\Gamma)$);

(4.2) for each $i \in N$, $h_i(p(z|t_i)) \leqq h_i(p(z))$ for all $t_i \in S^i$.

The play $p(z)$ produced by an N.E. z of Γ will be called a <u>Nash play</u> of Γ.

<u>Remarks</u>

(4) It is clear that $\mathcal{P}(\Gamma_{-}) = \{p(z) : z \in Z(\Gamma_{-})\}$. Later on we will need to talk about the <u>set</u> $\mathcal{F}(\Gamma)$ <u>of feasible plays</u>, $\mathcal{F}(\Gamma) = \{p(z) : z \in Z_F(\Gamma)\}$.

(5) It must be stressed that we do not think of an N.E. of Γ as a possible solution if Γ is played once. Instead, we interpret an N.E. as a stationary state that may accrue if Γ is repeated several times. This repetition of Γ is also an extensive game Γ^*. To consider an N.E. of Γ^* we are now obliged to think of a repetition of Γ^* itself. This goes on without end and there is no largest game to which an N.E. can be applied. For a fuller discussion of this interpretation see [5], [6].

4.2. The Nestedness of Nash Equilibria under Refinement

For much of the remainder of this paper we shall be concerned with a pair of games Γ, $\tilde{\Gamma}$ in[1] $\Delta(T)$ with $\Gamma \dashv \tilde{\Gamma}$. In this case we will use the symbols S_i, $\tilde{S}_i$ and I_i, $\tilde{I}_i$ to denote the strategy sets and information partitions of player i in Γ, $\tilde{\Gamma}$ respectively.

If $\Gamma \dashv \tilde{\Gamma}$, there is a natural sense in which $S^i \subset \tilde{S}^i$: simply identify $s_i \in S^i$ with $\tilde{s}_i \in \tilde{S}^i$ where $\tilde{s}_i(x) = s_i(x)$ for any $x \in X_i$. We first show that not only the strategy-sets but also the N.E. are nested under refinement of information.

Proposition 2. If $\Gamma \dashv \tilde{\Gamma}$ (for Γ, $\tilde{\Gamma}$ in $\Delta(T)$), then any N.E. of Γ is also an N.E. of $\tilde{\Gamma}$.

Proof. Let $z = \{s^x\}_{x \in X} \in Z$ be an N.E. of Γ. Define $z = \{\tilde{s}^x\}_{x \in X}$ by $\tilde{s}^x = s^x$ for all x. Then $\tilde{z}$ is feasible in $\tilde{\Gamma}$. Also it is easily checked that $p(\tilde{z}) = p(z)$. If $\tilde{z}$ is not an N.E. of $\tilde{\Gamma}$, then for some player j there is a $\tilde{q}_j \in \tilde{S}^j$ such that

(i) $h_j(p(\tilde{z}|\tilde{q}_j)) > h_j(p(\tilde{z}))$.

Let $\{u^\ell\}_{\ell=1}^k$ --where, possibly, $k = \infty$ --be the information sets of player j in $\tilde{\Gamma}$ through which the play $p(\tilde{z}|\tilde{q}_j)$ "passes." (If $(r^{x_0}, \ldots, r^{x_m}, \ldots)$ is a play and, for some ℓ, x_ℓ is in an information set $\tilde{u}$ of j, then the play is said to "pass through" $\tilde{u}$.) Clearly $\{u^\ell\}_{\ell=1}^k$ is not empty, otherwise $p(\tilde{z}|\tilde{q}_j) = p(\tilde{z})$, contradicting (i).

Let I_j, $\tilde{I}_j$ be the information partitions of j in Γ, $\tilde{\Gamma}$. For each ℓ, there is a unique $u^\ell \in I_j$ such that $\tilde{u}^\ell \subset u^\ell$. Moreover, by (2.6), the sets $\{u^\ell\}_{\ell=1}^k$ are disjoint. Define $q_j \in S^j$ by:

---

[1] Where T and $\Delta(T)$ are as in Section 3.

$$q_j(x) = \tilde{q}_j(y) \quad \text{if} \quad y \in \tilde{u}^\ell \quad \text{and} \quad x \in u^\ell \ ;$$

and, for other $x \in X_i$, $q_j(x)$ is arbitrary subject to the condition (2.5). (There is no problem in doing this.) Then $p(z|q_j) = p(\tilde{z}|\tilde{q}_j)$, therefore:

$$\begin{aligned} h_j(p(z|q_j)) &= h_j(p(\tilde{z}|\tilde{q}_j)) \\ &> h_j(p(\tilde{z})) \\ &= h_j(p(z)) \end{aligned}$$

contradicting that z is an N.E. of Γ . Q.E.D.

This proposition was established in a somewhat less general context in [3], though by essentially the same proof. It shows that, if we refine information, the N.E.'s of the coarse game are not lost. But there is no dearth of examples to convince one that, more often than not, there is a rapid proliferation of new N.E.'s. Consider:[1]

[1] For a more interesting example, drawn from an economic context, see [1].

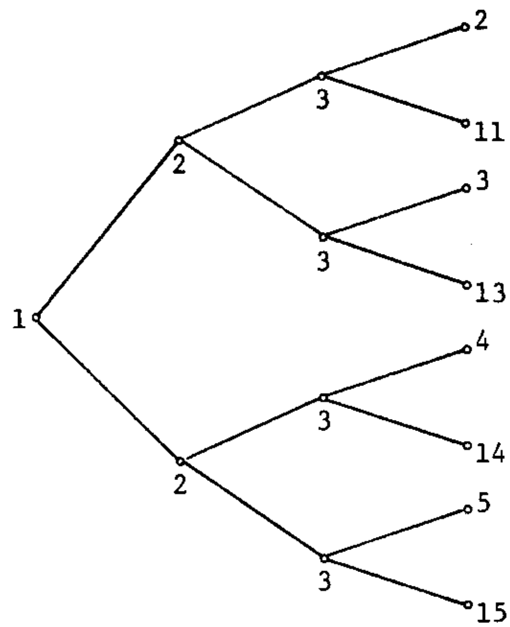

FIGURE 8

The numbers at the ending positions give the common payoff to each of the three players. Consider the three games Γ_1 , Γ_2 , Γ_3 with the information patterns given below.

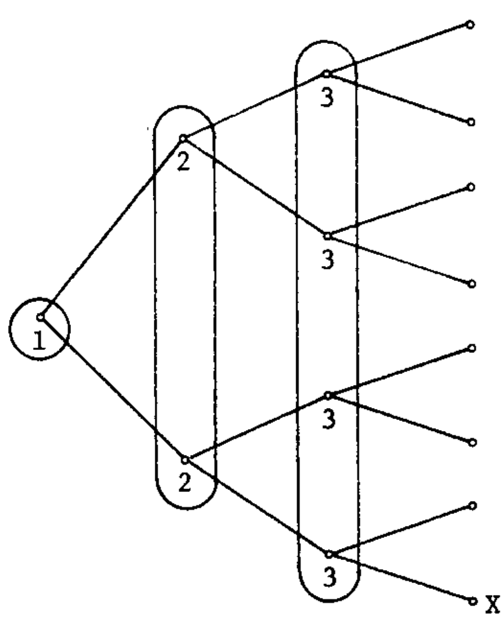

FIGURE 8. The Game Γ_1

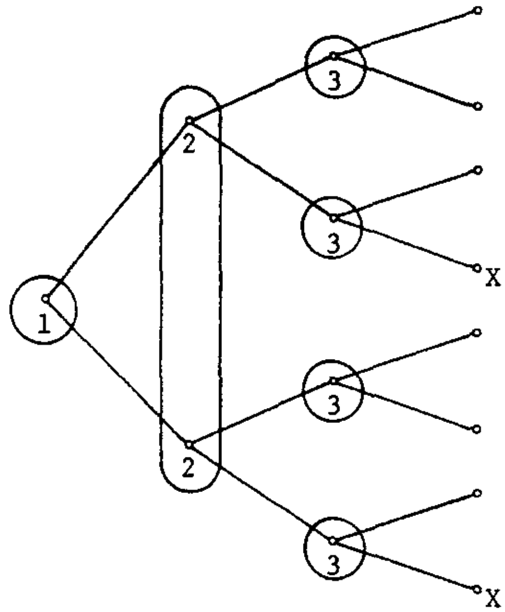

FIGURE 9. The Game Γ_2

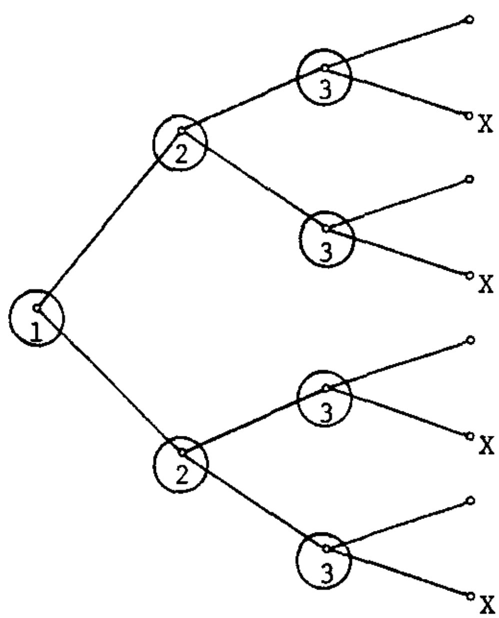

FIGURE 10. The Game Γ_3

The Nash plays in each case are marked by X . Those of Γ_ℓ are preserved in $\Gamma_{\ell+1}$ $(\ell = 1, 2)$ in accordance with Proposition 2.

4.3. No Informational Influence

We are interested in investigating conditions under which this proliferation of Nash plays is arrested. The next two propositions make an advance in that direction, and constitute partial converses to Proposition 2.

Γ is fixed as in Section 4.2. Take any $z = \{s^x\}_{x \in X} \in Z(\Gamma_-)$. Let i and j be two distinct players. We say that i <u>has no informational influence on j at z in Γ</u> (and denote this by " $i \not\to j \bmod z, \Gamma$ ") if:

(4.3) for any $s_i \in S^i$, the path $p(z|s_i)$ passes through exactly the same information sets of player j .

In other words if $p(z) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ and $p(z|s_i) = (r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots)$ (where $y_0 \equiv x_0$), then we require that

$$x_\ell \in u_j^\ell \in I_j \Leftrightarrow y_\ell \in u_j^\ell \in I_j \ .$$

Also define i <u>has no informational influence at z in Γ</u> if:

$$i \not\to j \bmod z, \Gamma \quad \text{for all} \quad j \in N \setminus \{i\} \ ;$$

and i <u>is not informationally influenced at z in Γ</u> if:

$$j \not\to i \bmod z, \Gamma \quad \text{for all} \quad j \in N \setminus \{i\} \ .$$

If N is finite then, by (2.4), $\mathcal{F}(\Gamma) = \mathcal{P}(\Gamma_-)$. The same conclusion can be derived (in the appropriate setting with $\Delta(T)$ replaced by $\nabla(L)$ --see Lemma 3 in Section 5) if N is non-atomic. Thus the stipulation (d) of the forthcoming Proposition 3 is a bit of a red herring.

But, given the very general setting in which we are at the moment, there does not seem to be a way to avoid it.

<u>Proposition 3</u>. For Γ, $\tilde{\Gamma}$ in $\Delta(T)$ suppose: (a) $\Gamma \dashv \tilde{\Gamma}$, (b) $\tilde{z}$ is an N.E. of $\tilde{\Gamma}$, (c) no player has informational influence at $\tilde{z}$ in $\tilde{\Gamma}$, (d) $p(\tilde{z}) \in F(\Gamma)$. Then there exists an N.E. z of Γ such that $p(z) = p(\tilde{z})$.

<u>Proof</u>. Let $\{\tilde{u}_i^\ell\}_{\ell=1}^{k(i)}$ be the information sets of player i in $\tilde{\Gamma}$ through which the play $p(z)$ passes. (We allow $\{\tilde{u}_i^\ell\}_{\ell=1}^{k(i)}$ to be empty or infinite.) Denote by u_i^ℓ the information set of i in $\tilde{\Gamma}$ for which $\tilde{u}_i^\ell \subset u_i^\ell$. By (d) there is a $z \in Z_F(\Gamma)$ such that $p(z) = p(\tilde{z})$. Then it must be that

$$s_i^x = \tilde{s}_i^y \quad \text{if} \quad x \in u_i^\ell \quad \text{and} \quad y \in \tilde{u}_i^\ell .$$

If z is an N.E. of Γ, we are done. If not, there is some player j and $q_j \in S^j$ such that

$$h_j(p(z|q_j)) > h_j(p(z)) .$$

Construct $\tilde{q}_j$ as follows:

$$\tilde{q}_j(x) = q_j(x) \quad \text{for all} \quad x \in X_i .$$

Clearly $\tilde{q}_j \in \tilde{S}^j$. We assert that $p(\tilde{z}|\tilde{q}_j) = p(z|q_j)$. Put $p(\tilde{z}|\tilde{q}_j) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$, $p(z|q_j) = (r^{x_0}, r^{y_1}, \ldots, y^{y_m}, \ldots)$, and $p(\tilde{z}) = (t^{x_0}, t^{z_1}, \ldots, t^{z_m}, \ldots)$. Also denote by $\tilde{I}_i(x)$ the information set of i in $\tilde{\Gamma}$ that contains $x \in X$ (where $\tilde{I}_i(x)$ could be empty). No informational influence at $\tilde{z}$ implies

(*) $\tilde{\Gamma}_i(x_\ell) = \tilde{\Gamma}_i(z_\ell)$ for all $i \in N \setminus \{j\}$, and all ℓ.

It is clear that $s^{x_0} = r^{x_0}$ and thus $x_1 = y_1$. Assume inductively that $x_\ell = y_\ell$ for $\ell = 1, \ldots, k$. Then by (*) and our construction of z, we get:

$$s_i^{x_k} = r_i^{y_k} \text{ for all } i \in \pi(x_k) \setminus \{j\}.$$

If $j \in \pi(x_k)$, then $\tilde{q}_j(x_k) = q_j(x_k)$ by construction. Thus $s^{x_k} = r^{y_k}$ and, therefore, $x_{k+1} = y_{k+1}$. This proves that $p(\tilde{z}|\tilde{q}_j) = p(z|q_j)$. But then $h_j(p(\tilde{z}|\tilde{q}_j)) = h_j(p(z|q_j)) > h_j(p(z)) = h_j(p(\tilde{z}))$, contradicting that z is an N.E. of $\tilde{\Gamma}$. Q.E.D.

The condition (c) of Proposition 3 is undoubtedly severe, though it is a natural one in the context of a "large number of small players," not necessarily non-atomic. To stress this last point let us give an example of a finite-player game in which (c) holds. Suppose $N = \{1, \ldots, 1000\}$. Let $S = \{1, \ldots, 500\}$ and $T = \{501, \ldots, 1000\}$. The game Γ is as follows. First all players in S move simultaneously, and each $i \in S$ selects a real number r_i in the closed interval $[0,1]$. The players in S can observe $\sum_{i \in S} r_i$. But there is a grid on their scale which does not permit very fine measurements. They can tell only that $\sum_{i \in S} r_i$ lies in one of the intervals

$$[0,10), [10,20), \ldots, [490,500).$$

After S has moved, then the players in T move simultaneously, and again each of them can select a real number in $[0,1]$. Suppose there is a Nash equilibrium in which $\sum_{i \in S} r_i = 145$. (One can easily concoct payoffs to make this so.) Then no player will have any informational influence at

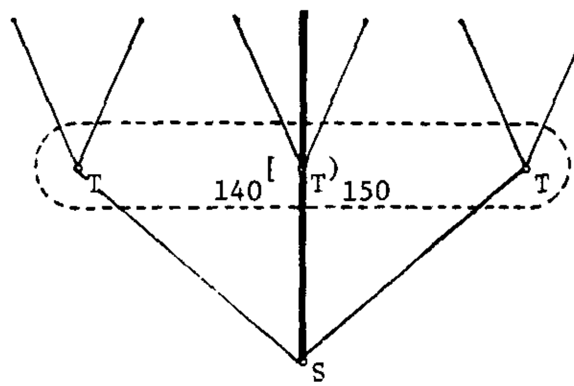

FIGURE 11. The Games Γ , $\tilde{\Gamma}$

this N.E. The resulting N.E. play is marked in Figure 11. If any one player in S changes his strategy, this will change the play but no one in T can observe it because the new play continues to pass through [140,150) . If we call the above game $\tilde{\Gamma}$ and let Γ be its coarsening in which players in T observe nothing (i.e. have the information set marked by dotted lines in Figure 11) then all conditions of Proposition 3 are met.

It is worth noting that if the Nash play were to pass instead through one of the "boundary points" 10, 20, ..., 490--or even sufficiently close to them--then (c) of Proposition 3 would break down. At these points information is "discontinuous" as a function of moves. This is a little disconcerting. One often wishes to model each player with an intrinsic bound on his capacity of observation, so that he cannot detect very small changes in others' behavior. But the points of discontinuity reveal that the extensive game model, taken literally, is not capable of expressing this and needs to be modified. This will be the topic of [1].

4.4. No Informational Loss

The conclusion of Proposition 3 can be obtained under a considerably weaker hypothesis than "no informational influence" (at least in the case when N is finite). Let $\Gamma \dashv \tilde{\Gamma}$ be as before. Consider a $\tilde{z} = \{\tilde{s}^x\}_{x \in X}$ in $Z_F(\tilde{\Gamma})$. For any $\tilde{u} \in \tilde{I}_i$, let $I_i(\tilde{u})$ be the set in I_i for which $\tilde{u} \subset I_i(\tilde{u})$. Let $\tilde{J}_i(\tilde{z}) \subset \tilde{I}_i$ be defined by:

$$\tilde{J}_i(\tilde{z}) = \{\tilde{v} \in \tilde{I}_i : p(\tilde{z}|\tilde{s}_j) \text{ passes through } \tilde{v} \text{ for some } \tilde{s}_j \in \tilde{S}^j, \text{ and } j \in N \setminus \{i\}\} .$$

We say that <u>player</u> i <u>incurs no informational loss in going from</u> $\tilde{\Gamma}$ <u>to</u> Γ <u>at</u> $\tilde{z}$ if

$$\text{(4.4)} \qquad \tilde{v}, \tilde{u} \in J_i(\tilde{z}) \text{ and } \tilde{v} \neq \tilde{u} \Rightarrow I_i(\tilde{v}) \neq I_i(\tilde{u}) .$$

Intuitively we now allow i to detect changes of a single player's strategy, but require that he remain capable in Γ of making all the distinctions that he could in $\tilde{\Gamma}$.

Observe

i is not informationally influenced in $\tilde{\Gamma}$ at $\tilde{z} \Rightarrow i$ incurs no informational loss in going from $\tilde{\Gamma}$ to Γ at $\tilde{z}$.

But, as examples later will show, the converse implication is not true.

<u>Proposition 4</u>. Suppose (a) $\Gamma \dashv \tilde{\Gamma}$ for $\Gamma, \tilde{\Gamma} \in \Delta(T)$, (b) $\tilde{z}$ is an N.E. of $\tilde{\Gamma}$, (c) no player incurs informational loss in going from Γ to $\tilde{\Gamma}$ at $\tilde{z}$, (d) the player-set N is finite. Then there exists an N.E. of z of Γ such that $p(z) = p(\tilde{z})$.

Proof. For each $i \in N$, let

$$A_i = I_i(\tilde{v}) : \tilde{v} \in \tilde{J}_i(\tilde{z})$$

$$B_i = I_i \setminus A_i .$$

Then define $z = \{s^x\}_{x \in X}$ by:

$$s_i^x = \begin{cases} \tilde{s}_i^y \text{ if } x \in u \in A_i , \quad y \in \tilde{v} , \quad u = I_i(\tilde{v}) ; \\ \text{arbitrary if } x \in B_i , \text{ subject to the condition} \\ \quad s_i^x = s_i^y \text{ if } x, y \in u \in B_i . \end{cases}$$

Given (c) the definition makes sense. Moreover since N is finite, we have $S^x = \prod_{i \in \pi(x)} S_i^x$ by (2.2), hence z is feasible. The rest of the proof proceeds exactly as the proof of Proposition 3. We first check (inductively, starting at x_0) that $p(z) = p(\tilde{z})$. If z is not an N.E. of Γ , then we get a contradiction as before. Q.E.D.

Consider the games Γ , $\tilde{\Gamma}$ of Figure 11. Construct a new pair Γ' , $\tilde{\Gamma}'$ from them as follows. Leave all other data fixed, but let some subset of the players in T have perfect information in both Γ' and $\tilde{\Gamma}'$, i.e., their information sets are singletons. Clearly Proposition 4 applies to Γ' , $\tilde{\Gamma}'$.

5. NON-ATOMIC GAMES

5.1. The Definition

We need to specialize the set-theoretic structure of Γ to treat non-atomic games. The player-set N is now equipped with a non-atomic measure. Precisely, we have a measure space $\{N,\mathcal{B},\mu\}$. $\mathcal{B}$ is an σ-field of subsets of N which includes the singleton sets $\{i\}$, $i \in N$; μ is a non-atomic probability measure on $\{N,\mathcal{B}\}$. Each Y^x (for $x \in X$) is also assumed to be a measurable space and C^x denotes its σ-field of subsets. We now add the following conditions on the constituents of Γ , over and above those in Section 1, (i)-(vi).

(vii) For any $x \in X$, $\pi(x)$ is a non-null[1] set in $\mathcal{B}$.

(viii) For any $x \in X$, there is a measurable correspondence f^x from $\pi(x)$ to Y^x , and[2] S^x consists of all measurable selections from f^x , i.e., of all functions $g : \pi(x) \to Y^x$ which satisfy:

(a) $g(i) \in f^x(i)$

(b) g is measurable.

(ix) For any $x, y \in X$, the set $\{i \in N : y \in I_i(x)\}$ is measurable.

These conditions are fairly innocuous. The sine qua non of the nonatomic assumption is in the next, and final, condition. It says that null sets of players and their moves cannot be observed by any of the others.

(x) If $x = (s^{x_0}, s^{x_1}, \ldots, s^{x_m})$, $y = (r^{y_0}, r^{y_1}, \ldots, r^{y_m})$, and $i \in N$ satisfy (where $y_0 \equiv x_0$):

(a) $x_\ell \in v \in I_i \Longleftrightarrow y_\ell \in v \in I_i$,

(b) If $x_\ell, y_\ell \in v \in I_i$, then $s_i^{x_\ell} = r_i^{r_\ell}$,

---

[1] $S \in \mathcal{B}$ is called null if $\mu(S) = 0$; non-null if it is not null.

[2] S^x is assumed to be always non-empty.

(c) $\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$

for $\ell = 0, 1, \ldots, m$, then

$$x \in v \in I_i \leftrightarrow y \in v \in I_i .$$

This completes our definition of a non-atomic game. Note that (viii) easily implies

(5.1) $S_i^x = f^x(i)$

(5.2) If $\pi(x)$ is a disjoint union of $\pi_1(x)$ and $\pi_2(x)$, and $g_1 : \pi_1(x) \to Y^x$, $g_2 : \pi_2(x) \to Y^x$ are measurable functions which satisfy $g_1(i) \in f^x(i)$ for $i \in \pi_1(x)$, $g_2(i) \in f^x(i)$ for $i \in \pi_2(x)$, then the function $g : \pi(x) \to Y^x$, obtained by putting together g_1 and g_2, will belong to S^x.

It can be checked that (vii)-(x) are consistent with the earlier assumptions in (i)-(vi), i.e., there are models of games that satisfy (i)-(x). See the example in Section 5.4.

5.2. Invariance of Nash Plays on Information Patterns

We will establish that if (i)-(x) hold for a game, then the Nash plays are invariant of the information pattern that the game is endowed with.

We prepare for this with

Lemma 3. Let Γ satisfy (i)-(ix). Then $P(\Gamma_-) = F(\Gamma)$.

Proof. Let $p = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots) \in \mathcal{P}(\Gamma)$. Put $U_\ell = \bigcup_{i \in \pi(x_\ell)} I_i(x_\ell)$ and $U = \bigcup_{\ell=0}^{\infty} U_\ell$. For $x \in U_\ell$, let

$$\gamma^\ell(x) = \{i \in \pi(x) : x \in I_i(x_\ell)\} .$$

By (2.6), if $\ell \neq \ell'$, then $\gamma^\ell(x) \cap \gamma^{\ell'}(x) = \phi$. By (ix), each $\gamma^\ell(x)$ is measurable. Therefore, by (vii), so is

$$\alpha(x) = \pi(x) - \bigcup_{\ell=0}^{\infty} \gamma^\ell(x) .$$

Let $\{q^x\}_{x \in X}$ be any element of $Z_F(\Gamma)$, and now define $\hat{z} = \{r^x\}_{x \in X}$ by:

$$r_i^x = \begin{cases} s_i^{x_\ell} & \text{if } x \in U_\ell \text{ and } i \in \gamma^\ell(x) \text{ for some } \ell \geqq 0 , \\ q_i^x & \text{if } i \in \pi(x) \text{ but } i \notin \bigcup_{\ell=0}^{\infty} \gamma^\ell(x) . \end{cases}$$

Since $\{\gamma^\ell(x)\}_{\ell=0}^{\infty}$ are disjoint, this $\hat{z}$ is well-defined. It can be checked (inductively, starting at x_0) that $p(\hat{z}) = p$. It remains to verify that $\hat{z} \in Z_F(\Gamma)$. It is clear that if $x, y \in u$ for some $u \in I_i$, then $r_i^x = r_i^y$. Therefore it is sufficient to show $r^x \in S^x$ for all $x \in X$. If $x \in X \setminus U$, then $r^x = q^x$ and $q^x \in S^x$ by assumption. If $x \in U_\ell$ for some $\ell \geqq 0$, then $\pi(x)$ is the disjoint union of $\{\gamma^\ell(x)\}_{\ell=0}^{\infty}$ and $\alpha(x)$. By (2.5) and (5.1), $f^x(i) = f^{x_\ell}(i)$ for $i \in \gamma^\ell(x)$. Also, clearly $\bigcup_{\ell=0}^{\infty} \gamma^\ell(x) \subset \pi(x_\ell)$. But then by construction, the map r^x coincides with s^{x_ℓ} on $\gamma^\ell(x)$ for all $\ell \geqq 0$. Hence r^x is a measurable selection from f^x on $\bigcup_{\ell=0}^{\infty} \gamma^\ell(x)$. On the other hand, r^x coincides with q^x on

$\alpha(x)$ and is, *a fortiori*, a measurable selection from f^x on $\alpha(x)$.
Therefore by (5.2), $r^x \in S^x$. Q.E.D.

Lemma 4. Suppose Γ satisfies (i)-(x), and $z \in Z(\Gamma\_)$. Then no player has informational influence at z in Γ.

Proof. Let $z = \{s^x\}_{x \in X}$. Consider $t_j \in S^j$. Put

$p(z) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m}, \ldots)$ and $p(z|t_j) = (r^{y_0}, r^{y_1}, \ldots, r^{y_m}, \ldots)$,

where $y_0 \equiv x_0$. It will suffice to show that for any ℓ and any $i \in N \setminus \{j\}$, if $x = (s^{x_0}, \ldots, s^{x_\ell})$ and $y = (r^{y_0}, \ldots, r^{y_\ell})$ then (a), (b), (c) of (x) are satisfied. Make the inductive hypothesis that we have shown this for $\ell = 0, 1, \ldots, k$ and consider the case[1] $\ell = k+1$. Now

$x_{k+1} = (s^{x_0}, \ldots, s^{x_k})$ and $y_{k+1} = (r^{y_0}, \ldots, r^{y_k})$. Then, by (x),

(d) $x_{k+1} \in v \in I_i \Longleftrightarrow y_{k+1} \in v \in I_i$ for $j \in N \setminus \{j\}$.

Hence

(e) $\pi(x_{k+1}) \setminus \{j\} = \pi(y_{k+1}) \setminus \{j\} \; (\equiv A_{k+1})$

(f) $s_i^{x_{k+1}} = r_i^{y_{k+1}}$ for $i \in A_{k+1}$.

From (e) and (f):

$$A_{k+1} = \{i \in \pi(x_{k+1}) \cap \pi(y_{k+1}) : s_i^{x_{k+1}} = r_i^{y_{k+1}}\}$$

hence, since $\mu(\{j\}) = 0$

(g) $\mu(A_{k+1}) = \mu(\pi(x_{k+1})) = \mu(\pi(y_{k+1}))$.

This verifies the hypothesis for $\ell = k+1$. Q.E.D.

Fix a *six*-tuple $L = \{N, X, \pi, \{S^x\}_{x \in X}, \Phi, \{h_i\}_{i \in N}\}$ for which all the assumptions in (i)-(vi), *as well as* (vii), (viii) hold. Denote by

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[1]For $\ell = 0$ the hypothesis obviously holds.

$\nabla(L)$ the set of all games obtained by adding information patterns to L subject to (2.5) and (2.6), as well as (ix) and (x). For any $\Gamma \in \nabla(L)$ let $\eta(\Gamma)$ be the set of all its Nash plays.

Proposition 5. $\eta(\Gamma) = \eta(\tilde{\Gamma})$ for any Γ, $\tilde{\Gamma}$ in $\nabla(L)$.

Proof. Denote by $\{I_i\}_{i\in N}$, $\{\tilde{I}_i\}_{i\in N}$ the information patterns in Γ, $\tilde{\Gamma}$. For each $i \in N$, let I_i^* be the common refinement of I_i and $\tilde{I}_i$, i.e.,

$$I_i^* = \{v^* \subset X_i : v^* \neq \phi \ , \ v^* = v \cap \tilde{v} \ \text{ for some } \ v \in I_i \ \text{ and } \ v \in \tilde{I}_i\} \ .$$

Consider the game Γ^* obtained by adding I_i^* to L. We will show that $\Gamma^* \in \nabla(L)$. Clearly I_i^* is a partition of X_i. For any $x, y \in X$: $\{i \in N : y \in I_i^*(x)\} = \{i \in N : y \in I_i(x) \cap \tilde{I}_i(x)\} = \{i \in N : y \in I_i(x)\} \cap \{i \in N : y \in \tilde{I}_i(x)\}$. Since each of the last two sets is measurable, so is the first, and thus I_i^* satisfies (ix). We omit the straightforward check that I_i^* satisfies (2.6). Finally take $x = (s^{x_0}, s^{x_1}, \ldots, s^{x_m})$, $y = (r^{y_0}, r^{y_1}, \ldots, r^{y_m})$, and $i \in N$ (where $x_0 \equiv y_0$) such that:

(a*) $x_\ell \in v^* \in I_i^* \Longleftrightarrow y_\ell \in v^* \in I_i^*$;

(b*) If $x_\ell, y_\ell \in v^* \in I_i^*$, then $s_i^{x_\ell} = s_i^{y_\ell}$;

(c*) Condition (c) of (x) holds.

In (a*) let $v^* = v \cap \tilde{v}$ for $v \in I_i$, $\tilde{v} \in \tilde{I}_i$. Then $x_\ell \in v^* \Rightarrow x_\ell \in v$, and $y_\ell \in v^* \Rightarrow y_\ell \in v$. From this it follows that (a*) implies:

$$x_\ell \in v \in I_i \Longleftrightarrow y_\ell \in v \in I_i \ ,$$

i.e., (a) of (x) holds for I_i. In the same manner (a) of (x) holds for $\tilde{I}_i$, and (b) of (x) holds for both I_i, $\tilde{I}_i$. (c) is independent of the information pattern and depends only on x and y. To sum up, (a), (b), (c) of (x) are satisfied for x, y, and i in both Γ, $\tilde{\Gamma}$. Then by (x),

$$(d^*) \quad x \in v \in I_i \Longleftrightarrow y \in v \in I_i ;$$

$$(e^*) \quad x \in \tilde{v} \in \tilde{I}_i \Longleftrightarrow y \in v \in \tilde{I}_i .$$

Let $x \in w^* \in I_i^*$, $w^* = w \cap \tilde{w}$ for $w \in I_i$, $\tilde{w} \in \tilde{I}_i$. Then $x \in w$ and from (d^*), $y \in w$. Similarly, $y \in \tilde{w}$. Hence $y \in w^*$. In the same way, $y \in w^* \Rightarrow x \in w^*$. This proves that condition (x) is also satisfied by Γ^*. Consequently $\Gamma^* \in \nabla(L)$.

By construction, $\Gamma \dashv \Gamma^*$ and $\tilde{\Gamma} \dashv \Gamma^*$. By Proposition 2, $\eta(\Gamma) \subset \eta(\Gamma^*)$ and $\eta(\tilde{\Gamma}) \subset \eta(\Gamma^*)$. Let z^* be any N.E. of Γ^*. In the wake of Lemmas 3 and 4, we can apply Proposition 3. This tells us that there are N.E.'s z in Γ and $\tilde{z}$ in $\tilde{\Gamma}$, such that $p(z) = p(z^*) = p(\tilde{z})$. Since z^* was arbitrary, $\eta(\Gamma^*) \subset \eta(\tilde{\Gamma})$ and $\eta(\Gamma^*) \subset \eta(\tilde{\Gamma})$, therefore $\eta(\Gamma) = \eta(\Gamma^*) = \eta(\tilde{\Gamma})$. Q.E.D.

5.3. A Variation on the Theme

The condition (x) is fairly stringent. Each player has no informational influence on others, not even on a null set. On the other hand absolutely no assumption was made on the payoff functions in proving Proposition 5. We now relax (x) to (x)* but at the expense of having to add conditions (xi) and (xii) below. Then Proposition 5 can still be retrieved, as Proposition 5*.

Condition (xi) says, roughly, that if two positions differ only on account of null sets not containing a particular player i, then i cannot *intrinsically* tell them apart.

(xi) If $\Phi(x) = (s^{x_0}, s^{x_1}, \ldots, s^{x_m})$ and $\Phi(y) = (r^{y_0}, r^{y_1}, \ldots, r^{y_m})$ satisfy, for $i \in \pi(x)$ (with $y_0 \equiv x_0$),

(f) $\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$ for $\ell = 0, 1, \ldots, m$;

(g) for all $\ell = 0, 1, \ldots, m$, $i \in \pi(x_\ell) \Longleftrightarrow i \in \pi(y_\ell)$;

(h) for all $\ell = 0, 1, \ldots, m$, $i \in \pi(x_\ell) \Rightarrow s_i^{x_\ell} = r_i^{y_\ell}$;

then $i \in \pi(y)$ and $S_i^x = S_i^y$.

The next condition (xii) is on payoffs. It says that they depend on plays "modulo" null sets.

(xii) If two plays $p = (s^{x_0}, s^{x_1}, \ldots)$ and $p' = (r^{y_0}, r^{y_1}, \ldots)$ satisfy

(i) $\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$ for all $\ell \geqq 0$;

(j) $i \in \pi(x_\ell) \Longleftrightarrow i \in \pi(y_\ell)$, for all $\ell \geqq 0$;

(k) $i \in \pi(x_\ell) \Rightarrow s_i^{x_\ell} = r_i^{y_\ell}$, for all $\ell \geqq 0$;

then $h_i(p) = h_i(p')$.

In the light of (xi) and (xii) we weaken (x) to:

(x)* No positive informational influence. Each player i has no informational influence on almost all other players (i.e. all except a null set).

Let L^* be a six-tuple as before, but assume this time that the assumptions (i)-(vi), (vii)-(ix), as well as (xi), (xii) hold. Define $\nabla^*(L^*)$ exactly as $\nabla(L)$ but with (x) replaced by the weaker (x)*.

Proposition 5*. $\eta(\Gamma) = \eta(\tilde{\Gamma})$ for any Γ , $\tilde{\Gamma}$ in $\nabla^*(L^*)$.

Proof. It is sufficient to show that for any N.E. z of Γ, there is a N.E. $\tilde{z}$ of $\tilde{\Gamma}$ such that $p(z) = p(\tilde{z})$.

Let $p(z) = (s^{x_0}, s^{x_1}, \dots)$. Select a $\tilde{z}$ in $F(\tilde{\Gamma})$ such that $p(\tilde{z}) = p(z)$. This is possible by Lemma 3.

Suppose $\tilde{z}$ is not a N.E. of $\tilde{\Gamma}$. Then there is an $s_i' \in \tilde{S}^i$ for some $i \in N$ such that $h_i(p(\tilde{z}|s_i')) > h_i(p(\tilde{z}))$. Let $p(z|s_i') = (r^{y_0}, r^{y_1}, \dots)$ $(y_0 \equiv x_0)$. Then condition (x)* implies

$$\mu(\{j \in \pi(x_\ell) \cap \pi(y_\ell) : s_j^{x_\ell} = r_j^{y_\ell}\}) = \mu(\pi(x_\ell)) = \mu(\pi(y_\ell))$$
$$\text{for all } \ell \geqq 0 .$$

Choose an $s_i \in S^i$ such that if $\Phi(x) = (t^{w_0}, t^{w_1}, \dots, t^{w_m})$ satisfies

(f*) $\mu(\{j \in \pi(y_\ell) \cap \pi(w_\ell) : r_j^{y_\ell} = t_j^{w_\ell}\}) = \mu(\pi(y_\ell)) = \mu(\pi(w_\ell))$ for all $\ell = 0, 1, \dots, m$;

(g*) for $\ell = 0, 1, \dots, m$, $i \in \pi(y_\ell) \leftrightarrow i \in \pi(w_\ell)$;

(h*) for $\ell = 0, 1, \dots, m$, if $i \in \pi(w_\ell)$ then $r_i^{y_\ell} = t_i^{w_\ell}$;

then $s_i(x) = s_i'(y_{m+1})$. Assumption (xi) ensures that this choice of s_i is possible. Let $p(z|s_i) = (q^{a_0}, q^{a_1}, \dots)$, with $a_0 \equiv x_0$. From the construction, it is clear that

(i*) for all $\ell \geqq 0$, $\mu(\{\ell \in \pi(a_\ell) \cap \pi(y_\ell) : q_j^{a_\ell} = r_j^{y_\ell}\}) = \mu(\pi(a_\ell)) = \mu(\pi(y_\ell))$;

(j*) for all $\ell \geqq 0$, $i \in \pi(a_\ell) \leftrightarrow i \in \pi(y_\ell)$;

(k*) for all $\ell \geqq 0$, if $i \in \pi(a_\ell)$, then $q_i^{a_\ell} = r_i^{y_\ell}$.

Therefore, by (xii), we have $h_i(p(z|s_i)) = h_i(p(\tilde{z}|s_i'))$. That is, $h_i(p(z|s_i)) = h_i(p(\tilde{z}|s_i')) > h_i(p(\tilde{z})) = h_i(p(z))$. This is a contradiction.

Q.E.D.

If (x), (x)* are violated then Propositions 5, 5* break down. Non-trivial counterexamples can easily be obtained by modifying the "dilemma game with rumour" in [4].

The careful reader must have noticed that we have defined a Nash Equilibrium by requiring that all--as opposed to "almost all"--players must be optimal in accordance with (4.2).[1] This is because, in our opinion, the very basis of an N.E. is individual optimization, and ignoring even a single player would go against the grain of this notion.

5.4. Two Examples

5.4.1. The Anti-Folk Theorem[2]

Let Γ be a non-atomic game in strategic form, i.e., $\pi(x_0) = N$ and every $s^{x_0} \in S^{x_0}$ constitutes an ending position. Further assume that the condition (xii) holds. In this context that simply says: if $\mu(\{j \in N : s_j^{x_0} \neq r_j^{x_0}\}) = 0$ and $s_i^{x_0} = r_i^{x_0}$ then $h_i(s^{x_0}) = h_i(r^{x_0})$, i.e., the payoff to any player depends on his strategy and the measurable function of others strategies modulo null sets.

Consider an infinite repetition Γ^∞ of Γ, in which each player can observe at each stage the entire past history of (a) his own moves and payoffs, (b) the measurable functions of others' moves, modulo null sets. The payoffs to plays in Γ^∞ are assigned by some rule (e.g., lim inf, discounted sum)...it doesn't much matter. Then Γ^∞ satisfies (x)* (and, also, of course (i)-(ix), (xi), (xii)). Consider the game Γ_C^∞ obtained by coarsening Γ^∞ as shown in the figure below, i.e., each player observes nothing at the end of any stage in Γ_C^∞.

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[1] Indeed this is why the "almost all" variations of assumptions (x), (x)*, (xi), (xii) would not suffice for our Propositions 5, 5*.

[2] For a further discussion of this topic see [5].

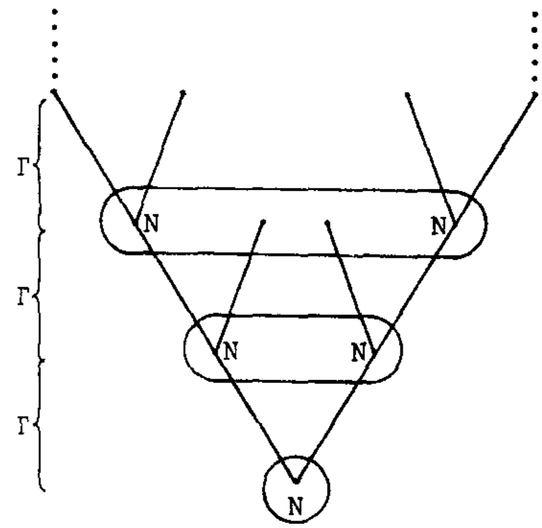

FIGURE 12

Now while Γ_C^∞ does not satisfy the perfect recall assumption (2.6), it does satisfy (2.7). Thus, by virtue of Remark (1), Proposition 5*holds. This says that the Nash plays of Γ_C^∞ are identical with the Nash plays of Γ^∞. If we denote the strategy set of i in Γ by S^i, then clearly his strategy set in Γ_C^∞ is $(S^i)^\infty$, i.e. a strategy for him is to simply pick an infinite sequence each of whose elements is in S^i. It is a short step from this to verify that the Nash plays of Γ_C^∞ (hence of Γ^∞) are typically "small." Indeed if we assign the payoff to a play of Γ^∞ by the discounted sum[1] of payoffs in each stage, then it is obvious that

$$(s^1, s^2, \ldots, s^n, \ldots) \text{ is an N.E. of } \Gamma_C^\infty \Longleftrightarrow \text{each } s_\ell \text{ is an N.E. of } \Gamma \text{ for } \ell = 1, 2, \ldots .$$

This is in sharp contrast with the "folk theorem" ([5], [7]). There players have enormous informational influence, and a stupendous proliferation of Nash plays is obtained in Γ^∞.

[1]Assuming this will always exist, e.g. by requiring that the payoffs are uniformly bounded in Γ.

5.4.2. Strategic Market Games

The fact that it was the <u>same</u> game Γ being repeated is not actually relevant to the above argument. Indeed let Γ^{∞} refer to a sequence of different non-atomic strategic games $\Gamma_1, \Gamma_2, \ldots, \Gamma_{\ell}, \ldots$. Assume again that payoffs are defined in Γ^{∞} by discounted sums. Our argument tells us that

$$(s^1, \ldots, s^{\ell}, \ldots) \text{ is an N.E. of } \Gamma_C^{\infty} \Longleftrightarrow \text{each } s_{\ell} \text{ in an N.E. of } \Gamma_{\ell} \text{ for } \ell = 1, 2, \ldots$$

and

$$\text{Nash plays of } \Gamma_C^{\infty} = \text{Nash plays of } \Gamma^{\infty} .$$

This may be of some interest in the analysis of market games,[1] where typically each round redefines the initial conditions of the next one. Our Propositions suggest that it may be possible to study questions of "growth" in this setting without being devastated out-of-hand by the super-abundance of Nash Equilibria.

On another note, our result shows that if in a one-period model we were to let traders go in a discrete sequence to the market, the N.E. would remain unperturbed by the sequencing. This perhaps gives a somewhat more realistic flavor to the simultaneous move models that have mostly been considered so far.

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[1]For a survey, see [2].

6. CONCLUDING REMARKS

(6) If there are chance moves in the game, and information patterns are varied with the proviso that what a player knows about the outcome of chance moves is constant, then the propositions of Sections 4 and 5 continue to hold. Thus we allow the variation depicted in Figure 13 but not the one in Figure 14. (In these figures the refined information is either the same as the coarse one or else indicated with broken lines. Also C denotes a position for chance moves.)

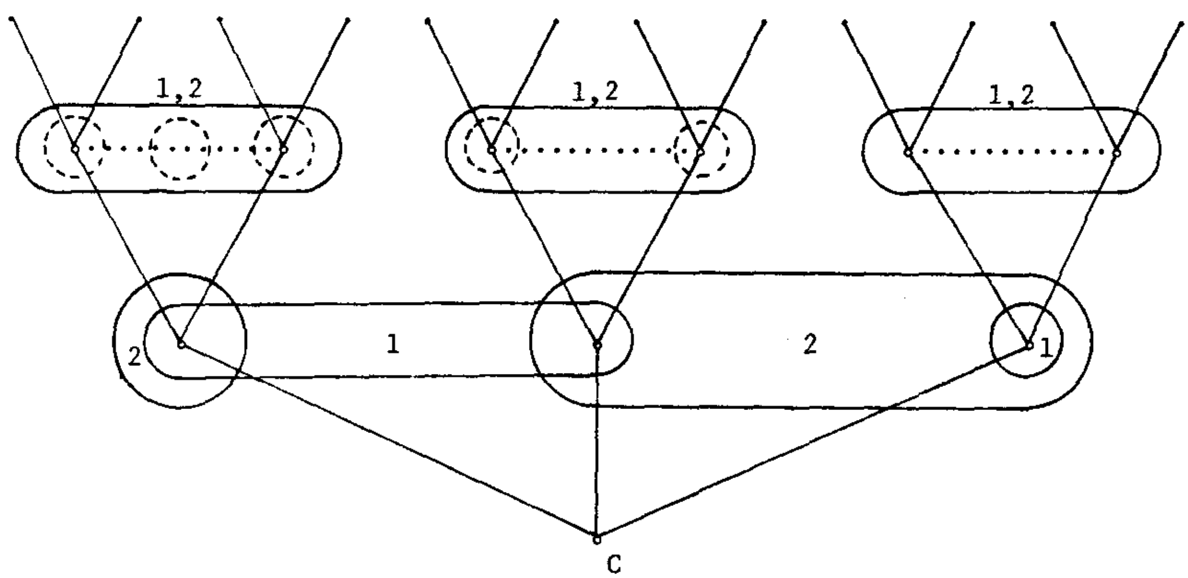

FIGURE 13

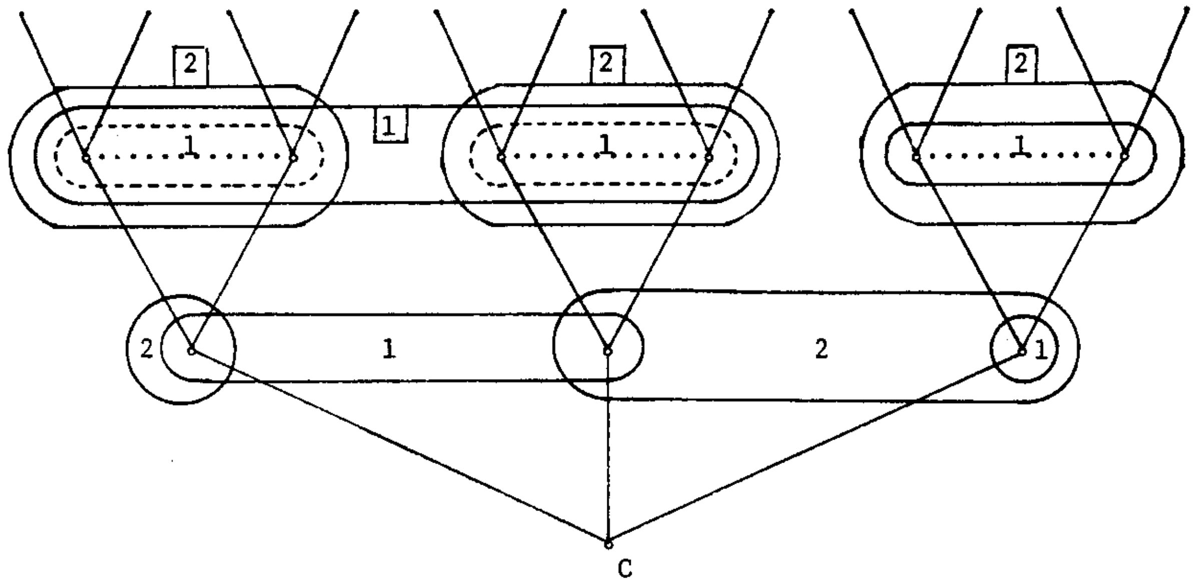

FIGURE 14

(7) The "invariance" result of Propositions 5, 5* in the non-atomic setting immediately leads to the question: what do they imply, asymptotically, for finite games? For the reasons that we have already pointed out in Section 4.3, we cannot give an asymptotic version within the model of this paper. Indeed there are examples of finite games that "converge" to a non-atomic one, while the corresponding set of Nash plays diverge (see [1]). To remedy matters, and to interpret the non-atomic results asymptotically, we will introduce the concept of "bounded capacity of observation." When that is added to the current model, an asymptotic interpretation becomes possible.

(8a) The assumption that $|N|$ is finite was needed in Proposition 4 only to ensure that the z constructed in the proof was feasible. Probably this is also true if N is non-atomic provided we make suitable additional assumptions on the game.

(b) Our proof (in Proposition 1) of the existence of a unique minimal information pattern does not carry over if N is non-atomic, i.e., in the context of conditions (i)-(ix). This is because the application of Zorn's Lemma may produce non-measurable sets. Again we feel that with further assumptions on the game this problem can be overcome and Proposition 1 retrieved. We have not worked out the details.

APPENDIX

For convenience we state Zorn's Lemma. A binary relation R on a set A is a subset of $A \times A$. If $(x,y) \in R$, we write xRy. R is called:

(1) <u>reflexive</u> if aRa for all $a \in A$;

(2) <u>symmetric</u> if for all $a, b \in A$, $aRb \Rightarrow bRa$;

(3) <u>anti-symmetric</u> if for all $a, b \in A$, aRb & $bRa \Rightarrow a = b$;

(4) <u>transitive</u> if aRb & $bRc \Rightarrow aRc$;

(5) <u>total</u> if for all $a, b \in A$, aRb or bRa.

A binary relation on R is called an <u>equivalence relation</u> if (1), (2) and (4) hold, and a <u>partial ordering</u> (a <u>total ordering</u>) if (1), (3), (4) (& (5)) hold.

<u>Zorn's Lemma</u>. Let R be a partial ordering on A. Assume that every totally ordered subset B of A (i.e., aRb or bRa for all $a, b \in B$) has a <u>lower bound</u> $\underline{b}$ in A (i.e., $bR\underline{b}$ for all $b \in B$). Then there exists a <u>minimal</u> element $\underline{a}$ in A, i.e., $\underline{a}Ra \Rightarrow \underline{a} = a$.

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