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INEFFICIENCY OF NASH EQUILIBRIA: I

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by

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1. Introduction

The main theme of this paper and its sequel ([5]) is that Nash equilibria (N.E.) are generally inefficient (in the Pareto sense). Suppose that a game with n players is given by n maps

$$\phi_i : S_1 \times \ldots \times S_n \to Y_i$$

and n maps

$$u^i : Y_i \to \mathbb{R} ,$$

where S_i is the strategy set, Y_i the outcome space, and u^i the utility function of the i^{th} player. Thus the i^{th} player gets the payoff $u^i(\phi_i(\overline{s}))$, where $\overline{s} = (s_1, \ldots, s_n)$ and $s_j \in S_j$ denotes the strategy chosen by the j^{th} player. The S_i and Y_i are taken to be "S-manifolds" (as defined in Section 2), for instance simplices or manifolds,[2] and the maps ϕ_i and u^i are C^2. The general question is: "to what extent are the Nash equilibria of these games Pareto efficient or inefficient?"

The focus of the present paper is the case where all the Y_i

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[2] Throughout this paper, all manifolds are assumed to be C^∞-manifolds.

coincide with a fixed Y and the ϕ_i with a fixed ϕ ; this may be thought of as corresponding to "pure public outcomes." We begin its analysis in Section 2, assuming that $Y = S_1 \times \dots \times S_n$ and that ϕ is the identity map. Our main result (Theorem 2.4) states that if the space of utilities satisfies a certain condition "T.C." given in Section 2, then generically (for an open dense set of utilities): (a) the set of N.E. is finite and varies continuously, (b) if an N.E. is efficient, then at least one player is on a "vertex" of his strategy set, (c) if an N.E. is strong, then at most one player is not on a vertex of his strategy set. Note that (b) implies generic inefficiency of N.E. if the strategy sets are vertex-free (e.g. manifolds) or if vertices can a priori be ruled out of N.E. in the given case. The result applies to the multi-matrix games of Nash. (Section 3). Here a vertex corresponds to a pure-strategy and, given the special structure of payoff functions, (c) can be strengthened to: if an N.E. is strong, every player is using a pure strategy. These results were obtained when S_j is a simplex in [2] and the present treatment is based on the same ideas in a more general framework. It is also shown that the set of efficient strategies is contained in a union of submanifolds of Y of codimension at least $1 + (N - n)$ where N is the dimension of Y.

In Section 4, we present a simple example of the results of Section 2. It illustrates all of the general phenomena and the reader who wishes to understand Theorem 2.4 without reading the proof is encouraged to read the definitions of Section 2 and then turn directly to Section 4.

In the final section, we discuss what happens for arbitrary ϕ . When the strategy sets and the outcome space open submanifolds of Euclidean space, it turns out that a certain inequality relating the number of players, the dimension of the strategy sets, and the dimension of the outcome space guarantees that the conclusions of Theorem 2.4 hold for generic ϕ .

In Part II ([5]) we consider the "pure private outcomes" case, in which the ϕ_i are distinct and $Y_1 \times \ldots \times Y_n$ is the set of all reallocations of privately-owned commodities. The ϕ_i are subject to certain constraints in the spirit of [4] (which also includes a survey of recent articles on such "strategic market games"). The precise statement of inefficiency will be made in its place. Here again there is a precursor ([3]), in which a particular example is worked out. But our approach is significantly different from that of [3]. We show that for <u>fixed</u> ϕ_i, the set of strategies $S_1 \times \ldots \times S_n$ can be partitioned into E and I. Every point in E has the property that, for <u>any</u> choice of utilities, if it is an N.E. then it is perforce efficient. I is characterized by exactly the same property with "efficient" replaced by "inefficient." Therefore we call them "ultra-optimal" and "ultra-inoptimal" points. The analysis turns on the sets E, I and on showing that the set of N.E. of the game $\bar{u} = (u^1, \ldots, u^n)$ generically has a transversal intersection with E.

2. The Main Theorem

The strategy sets which occur for many classes of games are not manifolds. A standard example is the simplex in $\mathbb{R}^n$. To take this into account, we define and prove our theorems for a class of topological spaces which we call S-manifolds. It seems that most strategy sets are S-manifolds. Since the proofs for S-manifolds will be reduced to the case of manifolds, we first define our set-up for manifolds and introduce S-manifolds towards the end.

We consider games of the following type. There are n players and the j^{th} player has a strategy set S_j which is a manifold of dimension $r(j)$. Let $k(j) = \sum_{i \leq j} r(j)$. (Later S_j will be taken to be an S-manifold.) Let $X = S_1 \times \ldots \times S_n$; it is a manifold of dimension $N = k(n)$. Let $C^2(X)$ be the Banach space of functions u on X all of whose partial derivatives of order 0, 1 and 2 exist and are continuous and whose norm $\|u\|^2$, given by

$$\|u\|^2 = \sup_{\substack{x \in X \\ D}} |Du(x)|$$

where Du ranges over all partial derivatives of u of order ≤ 2 , is finite.

Let U be a Banach submanifold of $C^2(X)$. Thus U is a class of utility functions and a game consists of an element $\bar{u} = (u^1, \ldots, u^n) \in U^n$, that is, a choice of utility function for each player. A choice of strategies $\bar{s} = (s_1, \ldots, s_n) \in X$ is also an outcome and the j^{th} player's payoff is given by $u^j(\bar{s})$.

Equilibria: Assume $\bar{u} = (u^1, \ldots, u^n) \in U^n$ is fixed. Let T be a non-empty subset of $\{1, 2, \ldots, n\}$ and for $\bar{e} \in \prod_{i \in T} S_i$, let $(\bar{s}|\bar{e})$ denote the element of X obtained by replacing s^i by e^i for $i \in T$.

1) A point $\bar{s} \in X$ is called T-efficient if there is no point $\bar{e} \in \prod_{i \in T} S_i$ such that:

$$u^i(\bar{s}|\bar{e}) \geq u^i(\bar{s}) \quad \text{for all} \quad i \in T$$

$$u^j(\bar{s}|\bar{e}) > u^j(\bar{s}) \quad \text{for some} \quad j \in T$$

(the coalition of players belonging to T cannot Pareto-improve itself if the other players remain fixed).

2) A point $\bar{s} \in X$ is a Nash equilibrium if it is T-efficient for all subsets T consisting of one element.

3) A point $\bar{s} \in X$ is efficient (or Pareto optimal) if it is T-efficient for $T = \{1, 2, \ldots, n\}$.

4) A point $\bar{s} \in X$ is a strong Nash equilibrium if it is T-efficient for all subsets $T \subset \{1, \ldots, n\}$.

5) Let $N(\bar{u})$, $E(\bar{u})$, $S(\bar{u})$ denote the sets of Nash, efficient, and strong Nash points in X respectively (with respect to the utilities $\bar{u} = (u^1, \ldots, u^n)$).

The Derivative Map: We are going to define a map which will be used in the investigation of Nash and efficient points of a game. Lemma 2.2 gives the precise connection.

With notation as before, let T^* denote the cotangent bundle of X and $T^*(x)$ the fiber of T^* above $x \in X$. Thus $T^*(x)$ is the cotangent space at x (the dual of the tangent space) and if we choose local coordinates $(x_1, \ldots, x_N)$ around x , then $T^*(x)$ can be identified with $\mathbb{R}^N$. Each function $u \in C^2(X)$ defines a section ∇u of T^* ; ∇u is the gradient of u and in local coordinates $(x_1, \ldots, x_N)$ near a point $x \in X$, $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$.

Let T^*_n be the n^{th} power of T^* ; in other words, T^*_n is the vector bundle over X whose fiber at $x \in X$ is $\underbrace{T^*(x) \times \ldots \times T^*(x)}_{n\text{-times}}$ and in local coordinates we may view an element of this fiber as a matrix with n rows and N columns.

Given $\bar{u} = (u^1, \ldots, u^n) \in U^n$, we obtain a section $D(\bar{u})$ of T^*_n whose value at $x \in X$ we denote by $D(\bar{u},x)$. In local coordinates $(x_1, \ldots, x_N)$ near x ,

$$D(\bar{u},x) = \begin{bmatrix} \frac{\partial u^1}{\partial x_1} & \cdots & \frac{\partial u^1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial u^n}{\partial x_1} & & \frac{\partial u^n}{\partial x_N} \end{bmatrix} \in T_n^*(x) \ .$$

The map $D(\bar{u},x)$ from $U^n \in X$ to T_n^* is called the <u>derivative</u> <u>map</u>.

We will <u>always</u> choose local coordinates $(x_1, \ldots, x_N)$ which are compatible with the product structure $X = S_1 \times \ldots \times S_n$; that is, we always choose $(x_1, \ldots, x_N)$ so that $(x_1, \ldots, x_{k(1)})$ are local coordinates for S_1 , $(x_{k(1)+1}, \ldots, x_{k(2)})$ are local coordinates for S_2 , etc.

Let S_j' be a submanifold of S_j for $j = 1, \ldots, n$ and put $X' = S_1' \times \ldots \times S_n'$; let $d(j) = \dim(S_j')$, $a = \sum_{j=1}^{n} d(j)$, and let T'^* (resp. $T_n'^*$) be the cotangent bundle of X' (resp. product of T'^* with itself n times). Every cotangent vector $v \in T^*(x)$ defines, by restriction, a cotangent vector in $T'^*(x)$, for $x \in X'$. To see this, note that the tangent space $T'(x)$ to x in X' is a subspace of the tangent space $T(x)$ to x in X and hence elements of the dual space of $T(x)$ (namely, cotangent vectors) restrict to give elements of the dual space of $T'(x)$. We shall assume that $d(j) > 0$ for all j .

We define two subsets E'' and N'' of $T_n'^*$ as follows. Let $(x_1, \ldots, x_a)$ be local coordinates for X' around a point $x \in X'$, chosen so that $(x_1, \ldots, x_{d(1)})$ are local coordinates for S_1' , $(x_{d(1)+1}, \ldots, x_{d(1)+d(2)})$ are local coordinates for S_2' , etc. With respect to these coordinates, an element of $T_n'^*(x)$ is represented as a matrix with n rows and a columns. Define:

$$E'' = \{(x,V) \in T_n'^* : \ V \text{ has linearly dependent rows}\} \ .$$

(Here $(x,V) \in T_n'^*$ means that V is an element of the fiber $T_n'^*$ represented as a matrix in local coordinates.) Define:

$$N'' = \{(x,V) \in T_n'^* : \ V_{ij} = 0 \ \text{ for } \ 1 + \sum_{\ell=1}^{i-1} d(\ell) \leq j \leq \sum_{\ell=1}^{i} d(\ell)\}$$

where V_{ij} denotes the (i,j)-entry of V. In other words, $(x,V) \in N''$ if V has the form:

$$V = \left.\begin{bmatrix} \boxed{0 \ \dots \ 0} & & & * \\ & \boxed{0 \ \dots \ 0} & & \\ & & \ddots & \\ & * & & \boxed{0 \ \dots \ 0} \end{bmatrix}\right\} n$$

$$\underbrace{\qquad\quad}_{d(1)} \ \underbrace{\qquad\quad}_{d(2)} \qquad \underbrace{\qquad\quad}_{d(n)}$$

Define E' (resp. N') to be the set of elements $(x,V) \in T_n^*$ such that:

1) $x \in X'$

2) the restriction of V to $T_n'^*(x)$ lies in E'' (resp. N'').

It is obvious that N'' is a closed subset of $T_n'^*$ (which is itself a manifold). It is also obvious that E'' is closed in $T_n'^*$ and this can be checked as follows. Because $T_n'^*$ is a locally trivial bundle, it will suffice to show that if $\{A_m\}$ is a sequence of $n \times a$ matrices with linearly dependent rows such that $\lim_{m\to\infty} A_m = A$, then A has linearly dependent rows. If A_m has linearly dependent rows, there is a non-zero vector $v_m \in \mathbb{R}^n$ such that $v_m A_m = 0$ and, multiplying v_m by a scalar if necessary, we may assume that $\|v_m\| = 1$. Since the unit sphere in $\mathbb{R}^n$ is compact, there is a subsequence of the v_m which converges to a vector $v \in \mathbb{R}^n$ such that $\|v\| = 1$ and it is clear that $vA = 0$. Hence A has dependent rows.

In the next lemma, we somewhat pedantically prove some statements that will be needed to apply the transversal density theorem and ask the reader for whom these statements are obvious to bear with us.

Lemma 2.1: (i) N' is a submanifold of T_n^* of codimension N.
(ii) E' is a finite union of submanifolds of T_n^* of codimension greater than or equal to $N-n+1$.
(iii) $E' \cap N'$ is a finite union of submanifolds of T_n^* of codimension greater than or equal to $N+1+(a-n)$.

Proof: Because T_n^* is a locally trivial bundle, every point $x \in X$ has an open neighborhood $V \subseteq X$ and local coordinates $(x_1, \ldots, x_N)$ on V such that the restriction of T_n^* to V is isomorphic to $V \times \mathrm{Mat}(n,N)$ where $\mathrm{Mat}(n,N)$ denotes the set of matrices with n rows, N columns, and real entries. Furthermore, we may choose the coordinates $(x_1, \ldots, x_N)$ so that $(x_{k(i-1)+1}, \ldots, x_{k(i-1)+d(i)})$ are local coordinates for S_i'. It is clear that: 1) to prove that N', E', and $E' \cap N'$ are unions of submanifolds, it suffices to show that N'', E'', and $N'' \cap E''$ are; and 2) $\dim(N') = \dim(N'') + (N-a)n$, $\dim(E') = \dim(E'') + (N-a)n$, and $\dim(N' \cap E') = \dim(N'' \cap E'') + (N-a)n$ where $\dim(E')$ and $\dim(E' \cap N')$ denotes the maximum dimension of the submanifolds whose union is E', $E' \cap N'$, etc. Therefore we may as well assume that $X = X'$ and that $a = N$. Then (i) is obvious.

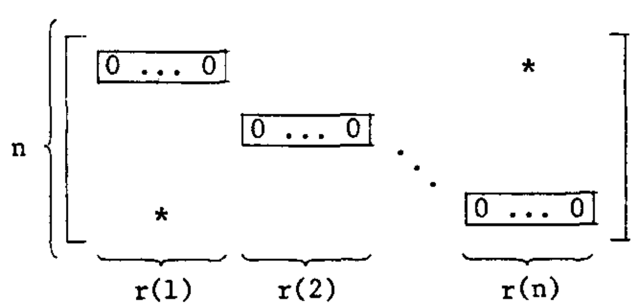
To prove (ii), it will suffice, in view of the local triviality of T_n^*, to show that the set Δ of $n \times N$ matrices with linearly dependent rows is a finite union of submanifolds of dimension $\leq (n-1)(N+1)$. Let T be a proper subset of $\{1, 2, \ldots, n\}$ and let Δ_T be the subset of $\mathrm{Mat}(n,N)$ of matrices A such that if A_j denotes the j^{th} row of A, then the vectors $\{A_j : j \in T\}$ are linearly independent and each A_k for $k \notin T$ is dependent on the set $\{A_j : j \in T\}$. We show that Δ_T is a submanifold of $Mat(n,N)$ and compute its dimension. Since $\Delta = \bigcup_T \Delta_T$, this will show that Δ is a finite union of submanifolds.

We may assume that $T = \{1, \ldots, t\}$ without loss of generality. Let S_t be the set of elements in $Mat(t,N)$ with linearly dependent rows. We have shown that S_t is a proper closed subset and hence the set

$$(Mat(t,N) - S_t) \times Mat(n-t,\ t)$$

is a manifold of dimension $tN + (n-t)t$. We construct an embedding Φ from $(Mat(t,N) - S_t) \times Mat(n-t,\ t)$ onto Δ_T as follows: for $B \in (Mat(t,N) - S_t)$ with rows $B_1, \ldots, B_t$ and $C = (C_{ij}) \in Mat(n-t,\ t)$, let $\Phi(B,C)$ be the matrix whose i^{th} row is B_i for $1 \le i \le t$ and whose i^{th} row is $\sum_{j=1}^{t} C_{ij}B_j$ for $t < i \le n$. This shows that Δ_T is a manifold of dimension $tN + (n-t)t$ and it is easy to see that the largest of these dimensions $(1 \le t \le n-1)$ is the case $t = n-1$, i.e., the largest dimension is $(N+1)(n-1)$, and it follows that E' is a finite union of submanifolds of dimension less than or equal to $N + (N+1)(n-1)$.

It remains to prove (iii). Again by the local triviality of T_n^* and the above remarks, it is enough to show that the set Ω of $n \times N$ matrices of the form



with linearly independent rows is a finite union of submanifolds of dimension $\leq N(n-1) - (N-n+1)$. Let T be a proper subset of $\{1, \ldots, n\}$ and let Ω_T be the subset of Ω of matrices A with rows A_j such that $\{A_j : j \in T\}$ is a linearly independent set of matrices. We show that Ω_T is a submanifold of $\mathrm{Mat}(n,N)$ and compute its dimension; we may assume that $T = \{1, \ldots, t\}$ with $t < n$. Let Ω'_T be the set of t-tuples of row vectors of length N $(A_1, \ldots, A_t)$ such that the A_j are linearly independent and A_j forms the j^{th} row of some matrix in Ω , i.e., A_j has zeroes in the appropriate places. As before, Ω'_T is a manifold. Consider the map:

$$\psi : \Omega'_T \times \mathrm{Mat}(n-t,\ t) \to \mathbb{R}^{r(t+1)} \times \ldots \times \mathbb{R}^{r(n)}$$

defined as follows. Let $(A_1, \ldots, A_t) \times (C_{ij}) \in \Omega'_T \times \mathrm{Mat}(n-t,\ t)$ and consider the row vectors $\sum_{j=1}^{t} C_{ij}A_j$ for $i = 1, \ldots, n-t$. Let v_i be the vector consisting of the $k(t+i-1) + 1$ to $k(t+i)$ entries of the vector $\sum_{j=1}^{t} C_{ij}A_j$; if v_i is zero, then the vector $\sum_{j=1}^{t} C_{ij}A_j$ qualifies as the $(i+t)^{th}$ row of a matrix in Ω . Define $\psi((A_1, \ldots, A_t) \times (C_{ij})) = (v_1, v_2, \ldots, v_{n-t})$. On the other hand, we have a map from $\psi^{-1}((0, \ldots, 0))$ to Ω_T : send $(A_1, \ldots, A_t) \times (C_{ij}) \in \psi^{-1}((0, \ldots, 0))$ to the matrix whose first t rows are $A_1, \ldots, A_t$ and whose i^{th} row for $t < i \leq n$ is $\sum_{j=1}^{t} C_{(i-t)j}A_j$. If we show that $\psi^{-1}((0, \ldots, 0))$ is a manifold, it will follow that Ω_T is, and to do this, we need only check that ψ is transverse to $(0, \ldots, 0)$. We omit the straightforward verification. It also follows that Ω_T is a manifold and since the codimension of $\psi^{-1}(0, \ldots, 0)$ in $\Omega'_T \times \mathrm{Mat}(n-t,\ t)$ is the same as that of $(0, \ldots, 0)$ in $\mathbb{R}^{k(t+1)} \times \ldots \times \mathbb{R}^{k(n)}$,

$$\dim \Omega_T = (t-1)N + (n-t)t = -t^2 + (N+n)t - N .$$

Here $1 \leq t \leq n-1$ and the maximum value of this dimension occurs for $t = n-1$ and we get $(n-2)N + (n-1) = N(n-1) - (N-n+1)$. This proves (iii).

The transversal density theorem will be applied to the map

$$D : U^n \times X \to T_n^*$$

$$(\overline{u},x) \mapsto D(\overline{u},x) .$$

We recall the definition of transversality. Let $\phi : X \to Y$ be a differentiable map between two manifolds X and Y and let $W \subset Y$ be a submanifold. Let T_xX (resp. T_xY , T_xW) be the tangent space to x in X (resp. Y , W) for $x \in X$ (resp. Y , W) . Then ϕ is said to be transverse to W at a point $x \in X$ if either $\phi(x) \notin W$ or $\phi(x) \in W$ and $T_{\phi(x)}Y = T_{\phi(x)}W + d\phi(T_xY)$. This is written as $\phi \pitchfork_x W$. If ϕ is transverse to W for all $x \in X$, we say that ϕ is transverse to W and write $\phi \pitchfork W$. If $\phi \pitchfork W$, then $\phi^{-1}(W)$ is a submanifold of X ([1]).

<u>Lemma 2.2</u>: Let $X' = S_1' \times \ldots \times S_n'$ as before. Suppose that the j^{th} player is constrained to pick his strategy from S_j' and let $\overline{u} = (u^1, \ldots, u^n) \in U^n$ be a choice of utilities. In other words, utilities come from U but the strategy sets are reduced to S_j' . Then

(i) If $x \in X'$ is a Nash equilibrium for this game, then $D(\overline{u},x) \in N'$.

(ii) If $x \in X'$ is an efficient point for this game, then $D(\overline{u},x) \in E'$ (this condition was used by Smale [6]).

(iii) If $x \in X'$ and $x \in N(\overline{u})$, the x is a Nash equilibrium for this restricted game.

Proof: (i) If $x \in X'$ is a Nash equilibrium, it follows that for all i, the point x is a local maximum for u^i with respect to the S_i'-variables, hence the corresponding partial derivatives must vanish.

(ii) If the projection of the rows of $D(\bar{u},x)$ onto $T_n'^*(x)$ are linearly independent, then there is a vector $v \in \mathbb{R}^a$ such that $D'(\bar{u},x)v$ has positive entries, where $D'(u,x)$ denotes the projection of $D(\bar{u},x)$ onto $T_n'^*(x)$. Then v defines a direction in X' along which each u^i is increasing. Hence x is not an efficient point for this game. (iii) is clear.

We will say that the space $U \subseteq C^2(X)$ satisfies condition T.C. (transversality condition) if for all $u \in U$, $x \in X$, and $v \in T^*(x)$, there exists a path $\xi(t)$ in U such that $\xi(0) = u$ and $\frac{d}{dt}(D(\xi(t), x))\big|_{t=0} = v$. Thus condition T.C. implies that the map

$$D : U^n \times X \to T_n^*$$

is transverse to all submanifolds of T_n^*. It also implies that the map $D' : U^n \times X' \to T_n'^*$ for X' as before is transverse to all submanifolds of $T_n'^*$.

Now we define S-manifolds. Let M be an n-dimensional manifold. If X is a subset of M, the interior X^o of X is the set of points $x \in X$ such that there exists an open neighborhood of x in M contained in X. A subset X of M is called an S-manifold if

(i) $\overline{X^o} = X$ (where $\overline{X^o}$ = the closure of X^o in M)

(ii) $X - X^o$ is a union $\bigcup_{j=1}^{s} X^j$ where X^j is a submanifold of M and $\overline{X^j}$ is a submanifold with boundary.

The X^j $(j = 0, 1, \ldots, s)$ are called components of X.

If X is a closed subset of a manifold M, we define $C^2(X)$ to be the set of restrictions to X of C^2-functions f with bounded norm on some open neighborhood of X in M and identify two functions on an open neighborhood of X if they agree on X. Set

$$\|f\|^2 = \sup_{\substack{x \in X \\ D}} |Df(x)|$$

where Df ranges over all partial derivatives of f of order ≤ 2. Using the Whitney extension theorem ([1]) and an argument involving partitions of unity, we can choose one open neighborhood $\tilde{X}$ of X in M such that every $f \in C^2(X)$ is the restriction to X of an element of $C^2(\tilde{X})$. Let U be a submanifold of $C^2(\tilde{X})$ with the norm $\|\ \|^2$ defined above and let $\|\ \|_0^2$ be the usual C^2-norm on $C^2(\tilde{X})$. It is clear that $\|f\|^2 \leq \|f\|_0^2$ for all $f \in C^2(\tilde{X})$ and so a dense subset of $C^2(\tilde{X})$ in the norm $\|\ \|_0^2$ is, a fortiori, dense under the norm $\|\ \|^2$.

Now assume that the strategy sets S_j are S-manifolds (defined as subsets of a manifold M_j of dimension $r(j)$) with open neighborhoods $\tilde{S}_j \subset M_j$ as in the previous paragraph. Let $X = S_1 \times \ldots \times S_n$, and let $\tilde{X} \subseteq \tilde{S}_1 \times \ldots \times \tilde{S}_n$, $C^2(\tilde{X})$ the space of C^2-functions under the norm $\|\cdot\|^2$, as in the previous paragraph. Let $S_j^{i_j}$ be a component of S_i and set $X_\alpha = S_1^{i_1} \times \ldots \times S_n^{i_n}$ where $\alpha = (i_1, \ldots, i_n)$; let A be the set of such α for which $\dim(S_j^{i_j}) > 0$ for all j. Let N'_α and E'_α be the subsets of $\tilde{T}_n^*$ associated to X_α, where $\tilde{T}_n^*$ is the n^{th}-power of the cotangent bundle of $\tilde{X}$ (these sets are defined as in Lemma 1 and the paragraph preceding it).

Theorem 2.3: Assume that U satisfies T.C. with respect to the map $\tilde{D} : U^n \times \tilde{X} \to \tilde{T}^*_n$. Then there is a dense set $U_0 \subset U^n$ (with respect to the norm $\| \|^2$) such that for all $\bar{u} \in U_0$:

(i) $D(\bar{u})(X) \cap (E'_\alpha \cap N'_\alpha) = \phi$ for all $\alpha \in A$

(ii) If X is compact, then a U_0 satisfying (i) may be chosen open and dense and such that for all $\bar{u} \in U_0$: $D(\bar{u})(X) \cap N'_\alpha$ is finite for all α and $\operatorname{codim}(D(\bar{u})^{-1}(E'_\alpha) = 1 + N - n$.

Here $D(\bar{u})(X)$ denotes the image of X under the map $D(\bar{u}) = D(\bar{u},x)$.

Proof: We apply the transversal density theorem to the derivative map $\tilde{D} : U^n \times \tilde{X} \to \tilde{T}^*_n$. Since U satisfies T.C., this theorem implies that there is a dense set $U_0 \subset U^n$ such that for all $\bar{u} = (u^1, \ldots, u^n) \in U^n$, the map

$$D(\bar{u}) : X \to \tilde{T}^*_n$$

is transverse to N'_α for $\alpha \in A$ and each of the finite number of submanifolds whose union is $N'_\alpha \cap E'_\alpha$. From Lemma 2.1 we have

(a) $\dim(X) = \operatorname{codim}(N'_\alpha)$ for $\alpha \in A$

(b) $\dim(X) < \operatorname{codim}(N'_\alpha \cap E'_\alpha)$ for $\alpha \in A$

where $\operatorname{codim}(N'_\alpha \cap E'_\alpha)$ denotes the smallest of the codimensions of the submanifolds whose union is $N'_\alpha \cap E'_\alpha$. Part (i) follows immediately from (b) and (a) implies that the Nash set $N(\bar{u})$, which is contained in $D(\bar{u})^{-1}(\bigcup_{\alpha \in A} N'_\alpha)$, is a zero-dimensional submanifold of X . In particular, if $\tilde{X}$ is compact, $D(\bar{u})^{-1}(\bigcup_{\alpha \in A} N'_\alpha)$ is a finite set.

Suppose that X is compact. Since each $\bar{S}_j^{i_j}$ is a manifold with boundary, we can, by considering the boundary components of $S_j^{i_j}$ separately, choose U_0 so that $D(\bar{u})(X) \cap (\bar{N}'_\alpha \cap \bar{E}'_\alpha) = \phi$ for all $\alpha \in A$ and $\bar{u} \in U_0$.

Here $\overline{N}'_\alpha$ and $\overline{E}'_\alpha$ denote the closures of N'_α and E'_α. Furthermore, the implicit function theorem and the transversality property of $D(\overline{u})$ for $\overline{u} \subset U_0$ show that for $\overline{u}'$ sufficiently close to $\overline{u}$, the set $D(\overline{u}')^{-1}(\bigcup_{\alpha \in A} N'_\alpha)$ is also finite and varies continuously for $\overline{u}'$ near $\overline{u}$. This also shows that for $\overline{u}'$ close to $\overline{u}$, $D(\overline{u}')(X) \cap (\overline{N}'_\alpha \cap \overline{E}'_\alpha) = \phi$ for all $\alpha \in A$ hence we may assume U_o open. Furthermore, E''_α is closed in $T^{\alpha*}_n$ for all $\alpha \in A$ and so, by considering the restriction of $D(\overline{u})$ to the X_α, the transversal density theorem shows that there is an open set $U_1 \subset U^n$ such that $D(\overline{u})$ is transverse to E'_α for all $\alpha \in A$. Replacing U_o by $U_o \cap U_1$ proves (ii).

We reformulate Theorem 2.3 in game-theoretic language. Let S_j be the strategy set of the j^{th} player; since it is an S-manifold, $S_j = S_j^0 \cup (\bigcup_{i=1}^{m_j} S_j^i)$ where the S_j^i are manifolds. A point $x \in S_j$ will be called a <u>vertex</u> if it is a zero-dimensional component of S_j under a minimal such decomposition of S_j, i.e., if $x = S_j^i$ for some i and m_j is minimal in the above decomposition.

<u>Theorem 2.4</u>: Assume that U satisfies T.C. and that X is compact. Then there is an open dense set $U_0 \subset U$ such that for all $\overline{u} \in U_0$:

1) $N(\overline{u})$ is a finite set which varies continuously for $\overline{u} \in U_0$.
2) If $x = (s_1, \ldots, s_n) \in X$ and $x \in N(\overline{u}) \cap E(\overline{u})$, then some S_j is a vertex. In particular, if the S_j have no vertices, $N(\overline{u}) \cap E(\overline{u}) = \phi$.
3) If $x \in S(\overline{u}) \cap E(\overline{u})$, then at most one S_j is not a vertex.
4) $E(\overline{u})$ is a finite union of "submanifolds" of X of codimension at least $1 + (N-n)$. (Here "submanifold" of X means the intersection of X with a submanifold of $\tilde{X}$.)

Proof: This is essentially a restatement of Theorem 3 together with some remarks made in the proof. Let $M \subset \{1, \ldots, n\}$ be a subset and look at the finite number of "subgames" among the players in M obtained by placing all other players on one of their vertices. If $\mathrm{card}(M) > 1$ our argument shows that there is an open dense set U_M such that if $\bar{u} \in U_M$, and x is a Nash Equilibrium of the subgame $\bar{u}$, then it is not M-efficient unless one of the players in M is also at a vertex. Take U_0 to be the intersection of all such U_M.

Finally, we remark that condition T.C. is satisfied if $U = C^2(X)$ or if U is any linear subspace of $C^2(X)$ such that for all $x \in X$, there exist N functions in U which provide local coordinates near x. For example, if $X \subseteq \mathbb{R}^n$, then a linear subspace U satisfies T.C. if it contains the linear functions on $\mathbb{R}^n$. Furthermore, there are many well-known theorems which guarantee the existence of Nash equilibria for various open classes of functions, e.g. functions satisfying convexity conditions, etc.

3. Multi-matrix Games

These were introduced by Nash in [5]. Each player i has a finite set K_i of "pure strategies" which we number for convenience as follows:

$$K_1 = \{1, \ldots, k(1)\}$$
$$\vdots$$
$$K_i = \{k(i-1)+1, \ldots, k(i)\}$$
$$\vdots$$
$$K_n = \{k(n-1)+1, \ldots, k(n)\} .$$

Each K_i is now enlarged to a set X_i of "mixed strategies," which are simply probability distributions on K^i :

$$X_i = \{x \in \mathbb{R}^{K_i} : \sum_{j \in K_i} x_j = 1,\ x_j \geq 0\} .$$

By $\mathbb{R}^{K_i}$ we mean the Euclidean space of dimension $\text{card}(K_i)$ whose axes are indexed by the elements of K_i . We identify K_i with the set of vertices of X_i by associating $j \in K_i$ with the point $(\underbrace{0, \ldots, 0,}_{j} 1, 0, \ldots, 0) \in \mathbb{R}^{K_i}$.

Let $K = K_1 \times \ldots \times K_n$. A multi-matrix game is specified by payoffs: $a^1 \in \mathbb{R}^K, \ldots, a^n \in \mathbb{R}^K$. For any $k \in K$, a^i_k is the payoff to i if the n-tuple of pure strategies given by k are used. Given $a^1, \ldots, a^n$ we now define the payoffs $\Pi_{a^1}, \ldots, \Pi_{a^n}$ on $X = X_1 \times \ldots \times X_n$ as the expectation of the pure strategy payoffs. Let $Z_i = \left\{x \in \mathbb{R}^{K_i} : \frac{1}{2} < \sum_{j \in K_i} x_j < 1\frac{1}{2},\ |x_j| < 2\right\}$ i.e., Z_i is an open set in $\mathbb{R}^{K_i}$ which contains the simplex X_i . Put $Z = Z_1 \times \ldots \times Z_n$. For $a \in \mathbb{R}^K$, define $\Pi_a : Z \to \mathbb{R}$ by

$$\Pi_a(x) = \sum_{k \in K} x_k a_k$$

where x_k denotes $x_{j(1)} \cdot \ldots \cdot x_{j(n)}$ for $k = (j(1), \ldots, j(n))$. Then if $a^i \in \mathbb{R}^K$ is the payoff of i in the pure-strategy game, Π_{a^i} restricted to X gives his payoff in its "mixed extension."

To apply Theorem 1 to this context it will suffice to check that $U = \{\Pi_a : a \in \mathbb{R}^K\}$ satisfies the T.C. condition for any $z \in Z$. Put $L = K_1 \cup \ldots \cup K_n = \{1, \ldots, k(n)\}$. For any $j \in L$, let $K^{-j} = K_1 \times \ldots \times K_{i-1} \times K_{i+1} \times \ldots \times K_n$ where i is such that $j \in K_i$. (Since L is a disjoint union, this is well-defined.) Also for any $q = (\ell(1), \ldots, \ell(i-1), \ell(i+1), \ldots, \ell(n))$ in K^{-j} , denote the element

$(\ell(1), \ldots, \ell(i-1), j, \ell(i+1), \ldots, \ell(n))$ of K by (q,j). With this notation, we see that $\frac{\partial \Pi_a}{\partial x_j}(z) = \sum_{k \in K^{-j}} z_q \, a_{(q,j)}$ where $z_q = z_{\ell(1)} \cdots z_{\ell(i-1)} \cdot z_{\ell(i+1)} \cdots z_{\ell(n)}$ for $q = (\ell(1), \ldots, \ell(i-1), \ell(i+1), \ldots, \ell(n))$. Take any $v \in R^L$. For each $j = 1, \ldots, k(n)$ there is a $q(j) \in K^{-j}$ such that $z_{q(j)} \neq 0$ because $\sum_{j \in K^i} z_j \neq 0$ for $i = 1, \ldots, n$. Now consider the path $\Pi_{a^t}\big|_{t=0}^{1}$ where:

$$a_k^t = \begin{cases} a_k + (tv_j / z_{q(j)}) & \text{if } k = (q(j), j) \\ a_k & \text{otherwise} \end{cases}$$

Then $\frac{\partial}{\partial t}((D\Pi_{a^t})(z)) = v$. This shows that U satisfies T.C. at any $z \in Z$.

By Theorem 1 there is an open dense set V_1 of $(\mathbb{R}^K)^n$ such that if $(a^1, \ldots, a^n) = a \in V_1$ then (a) the N.E. of $\Pi_a = (\Pi_{a^1}, \ldots, \Pi_{a^n})$ are finite in number, (b) if an N.E. of Π_a is efficient, there is at least one player who uses a pure strategy, (c) if an N.E. of Π_a is strong, then at most one player's strategy is possibly not pure. To sharpen (c), let V_2 be the subset of U^n given by

$$V_2 = \{(a^1, \ldots, a^n) \in U^n : a_k^\ell \neq a_{k'}^{\ell'} \text{ if either } \ell \neq \ell' \text{ or } k \neq k'\} .$$

V_2 is open and dense in U^n. Moreover if $x = (x_1, \ldots, x_n) \in X$ is an N.E. of Π_a for $a \in V_2$, and if all but one of the players use pure strategies at this N.E., then clearly so does the remaining player. Let $V = V_1 \cap V_2$. We have proved

<u>Theorem 3.1</u>: There is an open dense set V of U^n such that, if $a \in V$,

(a) the N.E. of Π_a are finite in number,

(b) if an N.E. of Π_a is efficient, then at least one player uses a pure strategy,

(c) if an N.E. of Π_a is strong, then each player uses a pure strategy.

4. An Example

We present the following example because it is particularly simple and illustrates with maximum clarity all of the features of the general case.

Consider a game with two players where the strategy set of each player is the interval $[0,1]$. The payoff functions are then functions on the square $[0,1] \times [0,1]$, which we call X ; a point in X is denoted by (x_1, x_2) where x_j is the j^{th} player's strategy choice.

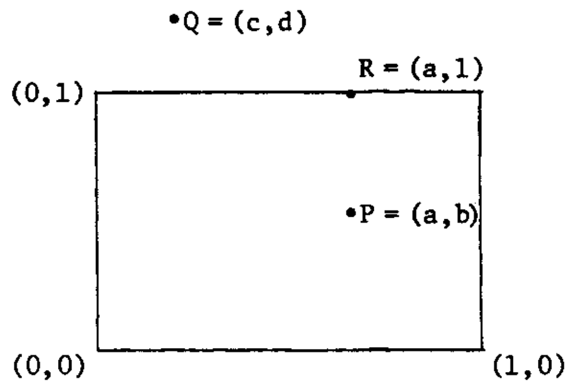
For each point $P = (a,b) \in \mathbb{R}^2$, let u_P be the function

$$u_P(x,y) = (x-a)^2 + (y-b)^2$$

i.e., $u_P(x,y)$ is the square of the distance from (x,y) to P . Let U be the set $\{u_P : P \in \mathbb{R}^2\}$ of all such functions. Then U is a submanifold of $C^2(\mathbb{R}^2)$ and is isomorphic as a manifold to $\mathbb{R}^2$ itself.

We want to examine the Nash and efficient sets of the games where each player's payoff function is selected from U . A game of this type is determined by assigning a point $P = (a,b)$ to player 1 and a point $Q = (c,d)$ to player 2, so that their payoff functions are respectively u_P and u_Q . We denote this game to be (P,Q) .

To find the Nash equilibria of the game (P,Q) , note that player 1's best response to <u>any</u> strategy choice of player 2 is the point in $[0,1]$ closest to a ; thus his best response is a if $a \in [0,1]$, 0 is $a < 0$, and 1 if $a > 1$. Similarly player 2's best response to any strategy choice of player 1 is the point in $[0,1]$ closest to d .



In the figure on the left, the point R is the Nash equilibrium of the game (P,Q) . In particular, we see that every game (P,Q) has a unique Nash equilibrium.

To describe the efficient set of the game (P,Q) , denoted by E(P,Q) , we need a definition. Given any closed convex set $C \subseteq \mathbb{R}^2$ and a point $M \in \mathbb{R}^2$, there is a unique closest point to M in C . We denote this point by $r_C(M)$ and call it the retraction of M into C . Thus

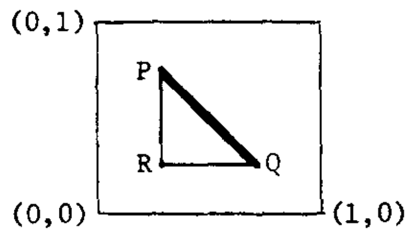
$$\text{distance}(M, r_C(M)) = \min_{P \in C}(\text{distance}(M,P))$$

and r_C defines a continuous map of $\mathbb{R}^2$ onto C such that $r_C(P) = P$ if $P \in C$.

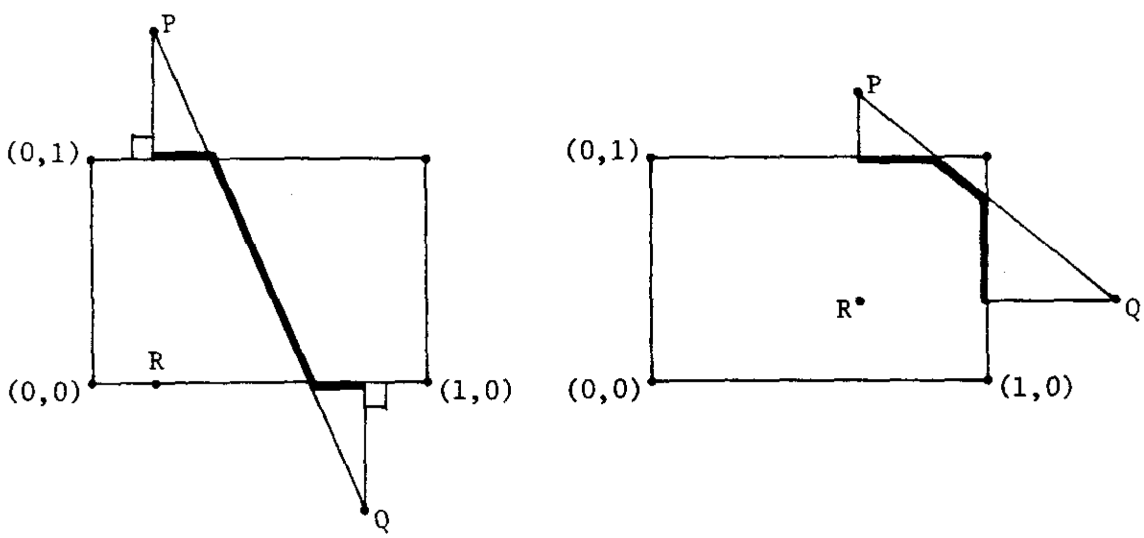
Lemma 4.1: For all $P, Q \in \mathbb{R}^2$, E(P,Q) is equal to the retraction into X of the line segment joining P and Q . In other words, if L(P,Q) is the line segment joining P and Q , then $E(P,Q) = r_X(L(P,Q))$.

Proof: We leave this as a simple exercise for the reader. Note that for all $x \in X$, a point y is a Pareto improvement on x if y lies on the perpendicular drawn from x to the line through P and Q .

Examples: (i) if P and Q both lie in X , then E(P,Q) is L(P,Q) .



(ii) if P and Q lie outside of X , then E(P,Q) may look like the following (the bold line is E(P,Q)):

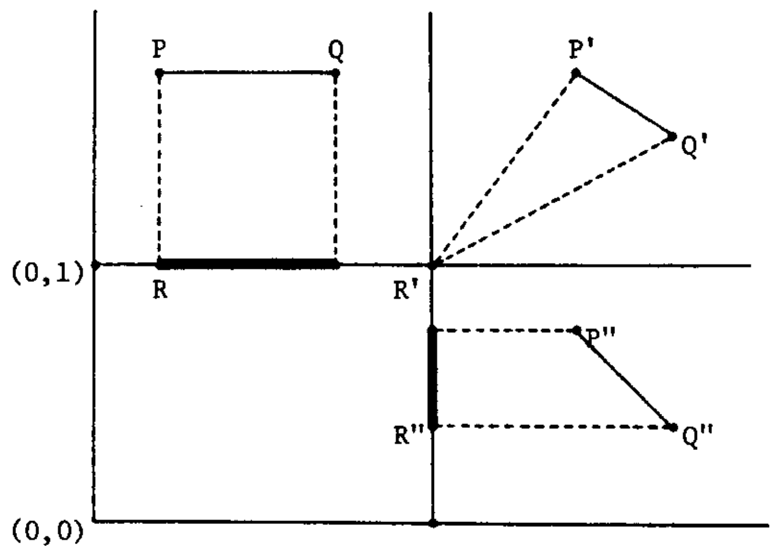


In the examples (i) and (ii), the point R is the unique Nash equilibrium and in both cases it is inefficient.

Lemma 4.2: Let R be the Nash equilibrium of the game (P,Q) where P = (a,b) and Q = (c,d) . Then

(a) If R does not lie on the boundary of X (that is, if neither player is on a vertex of his strategy set at R), then R is efficient if and only if a = c and/or b = d . The Nash equilibria of nearby games are generically inefficient.

(b) In the figure below, consider the three games (P,Q) , (P', Q') , and (P'', Q'') , with Nash equilibria R , R' , and R'' respectively. In these three cases, the Nash equilibrium is efficient and the Nash equilibria of all nearby games remain efficient. In all three cases, at least one player is at a vertex and in the game (P', Q') , the efficient set is reduced to a point.



Proof: This is easy to check using Lemma 4.1.

This example illustrates the following main points of the general theorem:

(1) Nash equilibria are finite in number and vary continuously in u.

(2) The efficient set is contained in a union of submanifolds of codimension at least $1 + N - n$ (equal to 1 in the above example).

(3) Efficient Nash equilibria at which no player is on a vertex of his strategy set are not robust.

(4) Robust examples of efficient Nash equilibria where at least one player is on a vertex of his strategy set exist.

Note that the submanifold of functions U satisfies condition T.C., as is easily checked, and hence the above results are consequences of Theorem 2.4. Also, the same game can be played in n-dimensions with n players with similar results.

5. Varying Outcome Functions

Suppose now that utilities $u_1, \ldots, u_n$ are defined on an outcome space Y and that an outcome function $\phi : X = S_1 \times \ldots \times S_n \to Y$ is given, so that the j^{th} player's payoff on a choice $\bar{s} = (s_1, \ldots, s_n) \in X$ of strategies is $u^j(\phi(\bar{s}))$. In this section, we consider the question "to what extent do the conclusions of Theorem 2.4 remain true in this setting?" and try to give a qualitative answer. No attempt is made at covering a "general case" and we therefore make some technical assumptions to simplify matters.

Assume that each strategy set S_j is an open set in $\mathbb{R}^{r(j)}$, Y is an open subset of $\mathbb{R}^M$, and that the closures of the S_j and Y are compact. Let U be the space of C^2-functions defined on some fixed neighborhood $\tilde{Y}$ of Y. We also fix a neighborhood $\tilde{X}$ of X in $\mathbb{R}^N$, where $N = \sum_{j=1}^{n} r(j)$, and let Ω be the set of smooth maps from X to Y obtained by restricting to X smooth maps from $\tilde{X}$ to $\tilde{Y}$.

The choice of a map $\phi \in \Omega$ and utilities $\bar{u} = (u^1, \ldots, u^n) \in U^n$ defines a game. Let $N_\phi(u)$ and $E_\phi(u)$ the subsets of X of Nash and efficient points respectively.

<u>Theorem 5.1</u>: Assume that $2 \leq n \leq M \leq N$ and suppose that the following inequality holds:

$$(*) \qquad n - 1 < \frac{1}{2}\left(N + M - \sqrt{(N-M)^2 + 4N}\right).$$

Then there is an open dense set $\Omega_0 \in \Omega$ such that: for all $\phi \in \Omega_0$, there is an open dense set $U_\phi \subset U$ such that

(i) $N_\phi(\overline{u})$ is finite for all $\overline{u} \in U_\phi$.

(ii) $E_\phi(\overline{u}) \cap N_\phi(\overline{u}) = \emptyset$ for all $\overline{u} \in U_\phi$.

<u>Remarks</u>: (i) Inequality $(*)$ is satisfied if $n-1 < M - \sqrt{N}$.

(ii) If $M = N$, inequality $(*)$ becomes $n-1 < N - \sqrt{N}$. For example, if $M = N \geq 2$ and each strategy set has dimension at least two, it is satisfied.

<u>Proof</u>: For fixed $\phi \in \Omega$, consider the map

$$D_\phi : U^n \times X \to \mathrm{Mat}(n,N)$$

$$((u^1, \ldots, u^n), x) \mapsto \left[\frac{\partial u^i}{\partial s_j}\right] \quad (\text{an } n \times N \text{ matrix})$$

where the s_j are Cartesian coordinates on $\mathbb{R}^N$. Let E' be the subset of $\mathrm{Mat}(n,N)$ of matrices with dependent rows and let N' be the subset of matrices of the form

$$n\left\{\left[\begin{array}{cccc} \overbrace{\boxed{0 \ldots 0}}^{r(1)} & \overbrace{}^{r(2)} & * & \overbrace{}^{r(n)} \\ & \boxed{0 \ldots 0} & & \\ & & \ddots & \\ & * & & \boxed{0 \ldots 0} \end{array}\right]\right.$$

$$\underbrace{\hspace{6cm}}_{N}$$

As in Section 2, if a point $x \in X$ lies in $N_\phi(u)$ (resp. $E_\phi(u)$) then $D(\bar{u},x) \in N'$ (resp. E'). By Lemma 2.1, $\mathrm{codim}(N') = N$ and $\mathrm{codim}(E' \cap N') > N$.

For $x \in X$, let $d\phi_x$ be the derivative of ϕ at x , i.e., $d\phi_x = \left[\frac{\partial\phi_i}{\partial s_j}(x)\right]$ where $\phi((x_1, \ldots, x_N)) = (\phi_1(x_1, \ldots, x_N), \ldots, \phi_M(x_1, \ldots, x_N))$. For $\bar{u} \in U^n$ and $y \in Y$, let $A_{\bar{u}}(y) = \left[\frac{\partial u^i}{\partial y_j}\right]$ (an $n \times M$ matrix). By the chain rule:

$$D_\phi(\bar{u},x) = A_{\bar{u}}(\phi(x)) \cdot d\phi_x \ .$$

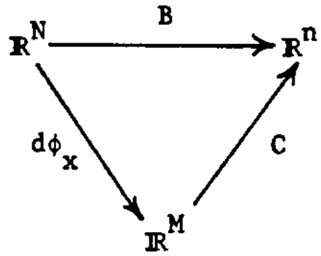
Conclusions (i) and (ii) of Theorem 5.1 hold if the map $D_\phi(\bar{u},x)$ is transverse to N' and $E' \cap N'$, as in Section 2. For fixed ϕ , the open dense set $U_\phi \subset U$ satisfying (i) and (ii) exists if the map

$$D_\phi : U^n \times X \to \mathrm{Mat}(n,N)$$

is transverse. The first observation is that D_ϕ is transverse if the rank of $d\phi_x$ is greater than or equal to n for all $x \in X$. To prove this, let $B \in \mathrm{Mat}(n,N)$. It suffices to show that for all $x \in X$ and $\bar{u} \in U^n$, there is a path $\bar{u}_t \in U^n$ such that $\bar{u}_0 = \bar{u}$ and:

$$\left\{\frac{d}{dt}(A_{\bar{u}_t}(\phi(x)))\Big|_{t=0}\right\} \cdot d\phi_x = B \ .$$

It is clear that for all $C \in \mathrm{Mat}(n,M)$, there is a path $\bar{u}_t$ such that $\bar{u}_0 = \bar{u}$ and $\frac{d}{dt}(A_{\bar{u}_t}(\phi(x)))\Big|_{t=0} = C$. Hence we must see that if $\mathrm{rank}(d\phi_x) \geq n$, then there exists a C such that $C \cdot d\phi_x = B$. This is obvious if we consider B , C , and $d\phi_x$ as linear maps and the equality $C \cdot d\phi_x = B$ as the commutativity of the diagram



If the dimension of the image of $d\phi_x$ is at least n, then C can be chosen so that $C \cdot d\phi_x = B$

To prove the theorem, it remains only to find conditions on M, N and n which guarantee the existence of an open dense set $\Omega_0 \subset \Omega$ such that for all $\phi \in \Omega_0$, $\text{rank}(d\phi_x) \geq n$ for all $x \in X$.

<u>Lemma 5.2</u>: Let $f(a) = -a^2 + (N+M)a - N(M-1)$. For all integers $0 \leq a \leq M$, there is an open dense set $\Omega^a \subset \Omega$ such that for all $\phi \in \Omega_a$, $\{x \in X : \text{rank}(d\phi_x) = a\}$ is a finite union of submanifolds of X of dimension $f(a)$ (if $f(a) < 0$, this is taken to mean that $\{x \in X : \text{rank}(d\phi_x) = a\}$ is empty).

Before proving Lemma 5.2, we show that it implies Theorem 5.1. Let $\Omega_0 = \bigcap_{a \leq n-1} \Omega^a$. We need only check that **$f(a) < 0$ for $a \leq n-1$ if the inequality (*)** of Theorem 5.1 is satisfied. Note that $f(a)$ is increasing on the interval $[0,M]$ and the smallest zero of $f(a)$ lies in that interval. In fact, the smallest zero is equal to $\frac{1}{2}\left((N+M) - \sqrt{(N-M)^2 + 4N}\right)$ and $f(n-1)$ is therefore negative if n satisfies (*).

Lemma 5.2 is also a consequence of the transversal density theorem. Consider the map

$$\Delta : \Omega \times X \to \text{Mat}(M,N)$$

$$(\phi, x) \mapsto d\phi_x$$

We have seen, in the proof of Lemma 2.1, that the set R_a of a $M \times N$ matrices of rank a $(0 \leq a \leq M)$ is a finite union of closed submanifolds of $\mathrm{Mat}(M,N)$ of dimension $aN + (M-a)a$. Furthermore, Δ is obviously transverse and so there is an open dense set $\Omega^a \subset \Omega$ such that $\Delta(\phi,x)$ is transverse to R_a for all $\phi \in \Omega^a$ (openness follows since the definition of Ω implies that all $\phi \in \Omega$ are restrictions to X of maps defined on a compact set containing X). In particular, the codimension of $\Delta(\phi)^{-1}(R_a)$ is equal to the codimension of R_a in $\mathrm{Mat}(M,N)$. A short calculation yields Lemma 5.2.

Corollary 5.3: If $M \geq n$, there is a non-empty open (though not necessarily dense) set Ω_0 in Ω such that the conclusion of Theorem 5.1 holds for all $\phi \in \Omega_0$.

Proof: This follows from the above proof and the remark that the set $\{\phi \in \Omega : \mathrm{rank}(d\phi_x) \geq n \text{ for all } x \in X\}$ is open and non-empty in Ω .

What happens if N , M and n do not satisfy inequality (*) of Theorem 5.1? Any one of the following conclusions may hold robustly, though not generically, depending on the conditions placed on ϕ : (a) the conclusion of Theorem 5.1, (b) all Nash equilibria are efficient (the case $Y = \{pt.\}$), (c) the efficient Nash equilibria are a union of submanifolds of the Nash set of codimension at least one.

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