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CORES OF PARTITIONING GAMES

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## CORES OF PARTITIONING GAMES\*

by

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Abstract. A generalization of assignment games, called partitioning games, is introduced. Given a finite set  $N$  of players, there is an a priori given subset  $\pi$  of coalitions of  $N$  and only coalitions in  $\pi$  play an essential role. Necessary and sufficient conditions for the non-emptiness of the cores of all games with essential coalitions  $\pi$  are developed. These conditions appear extremely restrictive. However when  $N$  is "large," there are relatively few "types" of players, and members of  $\pi$  are "small" and defined in terms of numbers of players of each type contained in subsets, then approximate cores are non-empty.

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## 1. Introduction

In an  $n$ -person cooperative game, it may not be equally easy to form every coalition. For example, it would be very hard to form a large coalition because of coalition formation costs, and thus only small coalitions play essential roles. If a game has some special structure and even if all coalitions are permitted, it may still happen that only small coalitions play essential roles, e.g., the marriage game of Gale and Shapley [1962], the bridge game of Shubik [1971], and the assignment games of Shapley and Shubik [1972] and Kaneko [1976, 1980].<sup>1</sup> In fact, an assignment game has the special property that it always has a non-empty core independently of the payoff function (see Kaneko [1980]). We introduce concepts of partitioning games, with and without side payments, which appropriately model the situations mentioned above.

Our first result provides necessary and sufficient conditions for the non-emptiness of the cores of all  $n$ -person partitioning games with a given set of essential coalitions. While these conditions generalize the results for assignment games, they imply the strong limitation of the generalization--the property that all assignment games have non-empty cores is very special.

Our next task is to consider the above problem from the viewpoint of the approximate core theory recently developed by Wooders [1981]. That is, the approximate cores of the replica games of partitioning games are considered, and it is shown in several strong forms that the approximate cores of the replica games of partitioning games are non-empty if the number of replications is sufficiently large. This result is in complete contrast with the first result.

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<sup>1</sup>The marriage game is a special case of the assignment game without side-payments.

## 2. Partitioning Games with and without Sidepayments

Initially we consider partitioning games with sidepayments. Let  $N$  be an arbitrary finite number of players, let  $N = \{1, 2, \dots, n\}$ , and let  $\pi$  be a class of non-empty coalitions satisfying  $\{i\} \in \pi$  for all  $i \in N$ . We call  $S$  in  $\pi$  a basic coalition. For any non-empty  $S \subset N$ , we call  $p_S = \{T_1, \dots, T_k\}$  a  $\pi$ -partition of  $S$  iff

$$T_t \in \pi \text{ for all } t=1, \dots, k \text{ and } p_S \text{ is a partition of } S. \quad (1)$$

Let  $P(S)$  be the set of all  $\pi$ -partitions of  $S$ . We call a game in characteristic function form,  $(N, v)$ , a partitioning game with sidepayments iff for some real-valued function  $\bar{v}$  on  $\pi$ ,

$$v(S) = \max_{p_S \in P(S)} \sum_{T \in p_S} \bar{v}(T) \text{ for all non-empty } S \subset N. \quad (2)$$

Note that  $v$  satisfies the superadditivity property.

The core of a game  $(N, v)$  with sidepayments is the set

$\{x \in R^n : \sum_{i \in N} x_i = v(N) \text{ \& } \sum_{i \in S} x_i \geq v(S) \text{ for all non-empty } S \subset N\}$ , where  $R^n$  is the  $n$ -dimensional Euclidean space.

The basic idea of definition (2) is very simple. That is, only the basic coalitions can play essential roles in a partitioning game. The following lemma ensures that this definition is consistent with our initial description in Section 1.

Lemma 1. Let  $(N, v)$  be a partitioning game with sidepayments. Then the core coincides with the set  $\{x \in R^n : \sum_{i \in N} x_i = v(N) \text{ \& } \sum_{i \in T} x_i \geq v(T) \text{ for all } T \in \pi\} = \{x \in R^n : \sum_{i \in N} x_i = v(N) \text{ \& } \sum_{i \in T} x_i \geq \bar{v}(T) \text{ for all } T \in \pi\}$ .

Proof. Obvious.

Typical examples are Shapley's and Shubik's [1972] assignment game and Shubik's [1971] bridge game.

Example 1 (The assignment game). Let  $N = J \cup K$  and  $J \cap K = \phi$ . If  $\pi = \{\{i\} : i \in N\} \cup \{\{j,k\} : j \in J \& k \in K\}$ , then given any  $\bar{v}$  on  $\pi$ , the game  $(N,v)$  defined by (2) is called an assignment game. Shapley and Shubik prove that every assignment game has a nonempty core. Note that this proposition is independent of the choice of  $\bar{v}$ .

Example 2 (The bridge game). Let  $\pi = \{\{i\} : i \in N\} \cup \{S \subset N : |S| = 4\}$  and let  $\bar{v}$  be given as

$$\bar{v}(S) = \begin{cases} 1 & \text{if } |S| = 4 \\ 0 & \text{otherwise,} \end{cases}$$

where  $|S|$  denotes the number of members in  $S$ . Then  $(N,v)$  defined by (2) is called a bridge game. If  $n = 4m$  for some positive integer  $m$ , then the core of  $(N,v)$  is nonempty but otherwise the core is empty.

Every assignment game has a nonempty core independently of the choice of  $\bar{v}$ . However, the nonemptiness of the core of a partitioning game depends, in general, upon  $\bar{v}$ . Therefore the property that the nonemptiness of the core of an assignment game is independent of  $\bar{v}$  is very special. The purpose of this section is to clarify this special property. Although a bridge game has a nonempty core if  $n = 4m$ , our general result implies that the nonemptiness of the core of a game with the same essential coalitions as the bridge game depends upon  $\bar{v}$  even in the case of  $n = 4m$ .

For any given  $N$  and  $\pi$  we denote, by  $GS(N,\pi)$ , the set of all partitioning games with sidepayments which have the set of players  $N$  and the set of basic coalitions  $\pi$ . Later in this section we will

determine necessary and sufficient conditions for every game in  $GS(N, \pi)$  to have a nonempty core.

Next let us define partitioning games without sidepayments. Let  $N$  and  $\pi$  be given. Let  $\bar{V}$  be a function on  $\pi$  to a class of subsets of  $R^n$  such that for all  $S \in \pi$  :

$$\bar{V}(S) \text{ is a closed set in } R^n ; \quad (3)$$

$$\text{if } x \in \bar{V}(S) \text{ and } y \in R^n \text{ with } y_i \leq x_i \text{ for all } i \in S, \text{ then } y \in \bar{V}(S); \text{ and} \quad (4)$$

$$\text{Pro}_S[\bar{V}(S) - \bigcup_{i \in S} \text{interior } V(\{i\})] \text{ is nonempty and bounded.}^2 \quad (5)$$

We define  $(N, V)$  by

$$V(S) = \bigcup_{P_S \in \mathcal{P}(S)} \bigcap_{T \in P_S} \bar{V}(T) \text{ for all nonempty } S \subset N. \quad (6)$$

This game  $(N, V)$  is called a partitioning game without sidepayments. Note that  $V$  also satisfies conditions (3)-(5). Definition (6) means that when a coalition  $S$  is formed, the players in  $S$  subdivide  $S$  into a  $\pi$ -partition and get the payoff sets guaranteed by the basic coalitions. This idea is almost the same as that of partitioning game with sidepayments.

The core of a game without sidepayments  $(N, V)$  is the set  $V(N) - \bigcup_{\substack{S \subset N \\ S \neq \emptyset}} \text{interior } V(S)$ . Parallel to Lemma 1, the following lemma holds.

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<sup>2</sup>  $\text{Pro}_S X = \{(x_i)_{i \in S} : x \in X\}$  for  $S \subset N$  and  $X \subset R^n$ .

Lemma 2. Let  $(N, V)$  be a partitioning game without sidepayments. Then the core coincides with  $V(N) - \bigcup_{S \in \pi} \text{interior } V(S) = V(N) - \bigcup_{S \in \pi} \text{interior } \bar{V}(S)$ .

Proof. Obvious.

Example 3 (The central assignment game). Let  $\pi$  be the collection given in Example 1. Then a game  $(N, V)$  defined by (6) is called a central assignment game. Kaneko [1980] proves that every central assignment game has a non-empty core.

There is, however, a minor conceptual difference between partitioning games with and without sidepayments. In a partitioning game with sidepayments, it is permitted to transfer money (transferable utility) in every coalition, but in a partitioning game without sidepayments, any transfers can only occur within basic coalitions. This difference appears as follows. A game with sidepayments  $(N, v)$  can be represented as a game without sidepayments  $(N, \hat{v})$  such that  $\hat{v}(S) = \{x \in \mathbb{R}^n : \sum_{i \in S} x_i \leq v(S)\}$  for all nonempty  $S \subset N$ . Even if  $(N, v)$  is a partitioning game with sidepayments,  $(N, \hat{v})$  is not a partitioning game without sidepayments, i.e., it does not satisfy (7). But in considering the core, this difference does not appear. To demonstrate this, we define another game without sidepayments  $(N, V_v)$  where

$$V_v(S) = \bigcup_{P_S \in \mathcal{P}(S)} \bigcap_{T \in P_S} \hat{v}(T) \quad \text{for all nonempty } S \subset N. \quad (7)$$

Of course,  $(N, V_v)$  is a partitioning game without sidepayments. Then the following lemma holds.

Lemma 3. Let  $(N, v)$  be a partitioning game with sidepayments. Then the core of  $(N, v)$  coincides with the cores of both  $(N, V_v)$  and  $(N, \hat{V})$ .

Proof. Obvious.

For any  $N$  and  $\pi$  we denote, by  $G(N, \pi)$ , the set of all partitioning games without sidepayments which have the set of players  $N$  and the set of basic coalitions  $\pi$ . Embedding  $GS(N, \pi)$  into  $G(N, \pi)$  by the mapping (7):  $v \mapsto V_v$ , we can regard  $GS(N, \pi)$  as a subset of  $G(N, \pi)$ .

We need several concepts to state the main result of this section. Consider the following system of equations:

$$\sum_{\substack{T \in \pi \\ T \ni i}} x_T = 1 \text{ for all } i \in N \text{ and } x_T \geq 0 \text{ for all } T \in \pi, \quad (8)$$

where  $(x_T)_{T \in \pi}$  is a variable. We say that the system of equations (8) has the integral property iff every extreme solution of (8) consists of integers. If (8) has the integral property, there exists a one-to-one onto mapping from the set of all  $\pi$ -partitions of  $N$  to the set of all extreme solutions of (8). This integer programming problem is usually called a set partitioning problem (see Balas and Padberg [1976] and Murty [1976]).

A family  $\gamma$  of non-empty coalitions of  $N$  is said to be balanced iff the system of equations

$$\sum_{S: S \ni j} \delta_S = 1 \text{ for all } j \in N, \quad (9)$$

has a nonnegative solution  $\delta = (\delta_S)_{S \in 2^N - \{\emptyset\}}$  such that  $\delta_S = 0$  iff



$S \in \gamma$ . The solution  $\delta$  is called a balancing weight vector. Games  $(N, v)$  with sidepayments and  $(N, V)$  without sidepayments are said to be balanced iff

$$\sum_{S \in \gamma} \delta_S v(S) \leq v(N) \quad \text{for any balanced family } \gamma \text{ and} \quad (10)$$

its balancing weight vector  $\delta$ ,

$$\bigcap_{S \in \gamma} v(S) \subset v(N) \quad \text{for any balanced family } \gamma, \quad (11)$$

respectively. Bondareva [1962, 1963] and Shapley [1967] show that a game with sidepayments has a nonempty core if and only if it is balanced. Scarf [1967] demonstrates that a balanced game without sidepayments has a non-empty core.<sup>3</sup>

A minimal balanced family is one that includes no other proper balanced family. A  $\pi$ -family is a subset of  $\pi$ .

Now we are in a position to state the main result of this section.

Theorem 1. The following six statements are equivalent:

- (i) The system of equation (8) has the integral property;
- (ii) Every balanced  $\pi$ -family is a union of  $\pi$ -partitions;
- (iii) Every minimal balanced  $\pi$ -family is a  $\pi$ -partition;
- (iv) Every  $(N, V)$  in  $G(N, \pi)$  is a balanced game;
- (v) Every  $(N, V)$  in  $G(N, \pi)$  has a non-empty core;
- (vi) Every  $(N, v)$  in  $GS(N, \pi)$  has a non-empty core.

Before proving this theorem, let us consider its implications.

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<sup>3</sup>Balancedness is not a necessary condition for the non-emptiness of the core of a game without sidepayments; cf. Billera [1970].

Example 4. Consider the family  $\pi$  of basic coalitions which was given in Example 2. If  $n = 8$ , the bridge game has a nonempty core. But every partitioning game with basic coalitions  $\pi$  does not necessarily have a nonempty core. For example,  $\gamma = \{\{1,2,3,4\}, \{3,4,5,6\}, \{1,2,5,6\}, \{7\}, \{8\}\}$  is a minimal balanced  $\pi$ -family but not a  $\pi$ -partition. Then Theorem 1 implies that we can find a game in  $GS(N, \pi)$  with an empty core. More concretely, the game  $(N, v)$  which is defined by  $\bar{v}$  on  $\pi$  where

$$\bar{v}(S) = \begin{cases} |S| & \text{if } S \in \gamma \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

has an empty core.

Example 5. Consider a three type assignment game, i.e.,  $N = J \cup K \cup M$  (mutually disjoint) and  $\pi = \{\{i\} : i \in N\} \cup \{\{j, k, m\} : j \in J, k \in K \& m \in M\}$ . A three-type assignment game also does not necessarily have a non-empty core. For example, let  $N = \{1, 2, \dots, 9\}$  and  $J = \{1, 2, 3\}$ ,  $K = \{4, 5, 6\}$  and  $M = \{7, 8, 9\}$ . Then  $\gamma = \{\{1, 4, 7\}, \{1, 5, 9\}, \{2, 4, 8\}, \{2, 6, 9\}, \{3, 5, 7\}, \{3, 6, 8\}\}$  is a balanced  $\pi$ -family but not a  $\pi$ -partition so we can find a game in  $GS(N, \pi)$  with an empty core. Concretely, the game defined by (12) and the  $\pi$  and  $\gamma$  of this example has an empty core (see Figure 1 below).

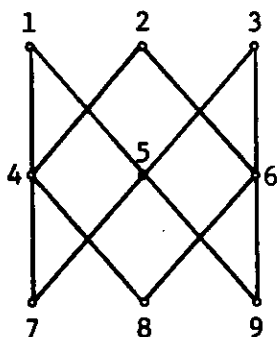


FIGURE 1

Thus, overall, Theorem 1 implies that each statement of Theorem 1 is quite strong and, it is fair to say that Theorem 1 is a negative result. That is, the special property of the assignment game (that every assignment game has a non-empty core) is hardly generalized.

However, we can find some sufficient conditions for the integral property of equation (8). Represent the system of equations (8) as the matrix form, i.e.,  $AX = e$  &  $X \geq 0$ , where  $X = (X_T)_{T \in \pi}$  and  $e$  is the vector with every component equal to 1. A sufficient condition for (8) to have the integral property is the unimodular property of  $A$ , i.e., every minor determinant of  $A$  equals 0, 1 or -1 (Hoffman and Kruskal [1956, Theorem 2]). Hoffman and Kruskal gave also several necessary and sufficient conditions and more convenient sufficient conditions for the unimodular property.

#### Proof of Theorem 1

The theorem is proved as follows:

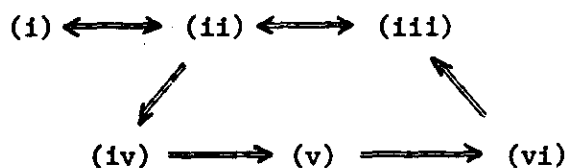


FIGURE 2

Since  $GS(N, \pi)$  is a subset of  $G(N, \pi)$ ,  $(v) \Rightarrow (vi)$  is trivial, and  $(iv) \Rightarrow (v)$  is also obvious by Scarf's theorem.

Proof of (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii): Since every  $\pi$ -partition is a minimal balanced family and a balanced family is the union of the minimal balanced families that it contains (Shapley [1967, p. 457, Corollary]), the equivalence of (ii) and (iii) is true.

With any solution  $(X_S)_{S \in \pi}$  of (8), we associate  $\delta^X = (\delta_S^X)_{S \in 2^N - \{\emptyset\}}$  such that  $\delta_S^X = X_S$  for all  $S \in \pi$  and  $\delta_S^X = 0$  otherwise. Then  $\delta^X$  is a solution of (9), and it is easy to see that  $\delta^X$  is an extreme point of (9) iff  $X$  is an extreme point of (8). Shapley's lemma [1967, Lemma 2] states that  $\delta^X$  is an extreme point of (9) iff the balanced family  $\gamma^X \equiv \{S : \delta_S^X > 0\}$  is minimal. Therefore  $X$  is an extreme point of (8) iff  $\gamma^X$  is a minimal balanced family. Then it is clear that  $X$  is an integral solution iff  $\gamma^X$  is a  $\pi$ -partition.

Proof of (ii)  $\Rightarrow$  (iv): (a) Let  $\gamma$  be a balanced  $\pi$ -family. We show that if  $x \in \bigcap_{T \in \gamma} V(T)$ , then  $x \in V(N)$ . Since  $\gamma$  includes a  $\pi$ -partition

$$p_N, \quad x \in \bigcap_{T \in \gamma} V(T) \subset \bigcap_{T \in p_N} V(T) \subset \bigcup_{p'_N \in \mathcal{EP}(N)} \bigcap_{T \in p'_N} V(T).$$

(b) Let  $\gamma$  be a balanced family which is not a  $\pi$ -family. Suppose  $x \in \bigcap_{T \in \gamma} V(T)$ . If  $S \in \gamma$  does not belong to  $\pi$ , then there is a  $\pi$ -partition  $p_S^*$  of  $S$  with  $x \in \bigcap_{T \in p_S^*} V(T)$  by (6). For  $T \in \pi$ , let  $\hat{\gamma}_T = \{S : S \in \gamma, S \notin \pi \text{ \& } T \in p_S^*\}$ . We define  $\hat{\gamma}$  and  $\hat{\delta}$  by

$$\hat{\gamma} = \{T : T \in \gamma \text{ \& } T \in \pi\} \cup \left( \bigcup_{\substack{S \in \gamma \\ S \notin \pi}} p_S^* \right),$$

$$\hat{\delta}_T = \begin{cases} \delta_T + \sum_{S \in \gamma_T} \delta_S & \text{if } T \in \gamma \text{ and } T \in \pi \\ \sum_{S \in \gamma_T} \delta_S & \text{if } T \notin \gamma \text{ and } T \in \pi \\ 0 & \text{otherwise,} \end{cases}$$

where  $\delta$  is a balancing weight vector for  $\gamma$ . It is easily verified that this  $\hat{\gamma}$  is a balanced  $\pi$ -family with the balancing weight vector  $\hat{\delta}$ . Since  $x \in V(T)$  for all  $T \in \hat{\gamma}$ , the above argument (9) is applicable to this case.

Proof of (vi)  $\Rightarrow$  (iii): Suppose that there is a minimal balanced  $\pi$ -family  $\gamma$  which is not a  $\pi$ -partition. Consider the game  $(N, v)$  with sidepayments which is defined by

$$\bar{v}(S) = \begin{cases} |S| & \text{if } S \in \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\sum_{S \in \gamma} \delta_S v(S) = \sum_{i \in N} \sum_{\substack{S \in \gamma \\ S \ni i}} \delta_S = |N|$ , where  $\delta$  is a balancing weight vector

for  $\gamma$ . But since  $\gamma$  does not include any  $\pi$ -partition,

$v(N) = \max_{P_N \in \mathcal{P}(N)} \sum_{T \in P_N} v(T) < |N|$ . By Bondareva-Shapley's theorem, the core

of  $(N, v)$  is empty.

### 3. Approximate Cores of Partitioning Games

Obviously, the conditions stated for the non-emptiness of the cores of partitioning games are extremely restrictive and, without some very special structure on the collection of basic coalitions, we would not expect these conditions to be met. In this section, however, we show that, given  $N$  and  $\pi$ , the replications of games in both  $GS(N, \pi)$  and  $G(N, \pi)$  will have non-empty approximate cores if the number of replications is sufficiently large. The results we will obtain depend only on  $N$  and  $\pi$  but not on the particular games  $(N, v)$  or  $(N, V)$ .

Formally, given the set of players  $N = \{1, \dots, i, \dots, n\}$ , for each positive integer  $r$ , define  $N_r = \{(i, q) : i = 1, \dots, n \text{ and } q = 1, \dots, r\}$ . The set  $N_r$  is called the set of players of the  $r^{\text{th}}$  replication of  $N$ . For each  $i \in N$ , the set  $\{(i, q) : q = 1, \dots, r\}$  is called the set of players of type  $i$  of the  $r^{\text{th}}$  replication of  $N$ . Given any subset  $S$  of  $N$ , let  $s = (s_1, \dots, s_n)$  be defined by its

coordinates  $s_i = |S \cap \{(i,q) : q = 1, \dots, r\}|$  where  $|\cdot|$  denotes the cardinal number of a set. Then  $s$  is called the profile of  $S$  and is simply a list of the numbers of players of each type contained in  $S$ . Given  $S$ , define  $\rho(S) = s$ , so  $\rho(\cdot)$  maps subsets of  $N_r$  into their profiles.

A subset  $S \subset N_r$  is called a basic coalition (of  $N_r$ ) iff  $\rho(S) = \rho(S')$  for some basic coalition  $S' \in \pi$  of the set  $N$ . This definition allows all subsets of  $N_r$  which are identical in terms of their profiles to some basic coalition in  $N (= N_1)$  to be basic coalitions of  $N_r$ . Let  $\pi_r$  be the set of all non-empty basic coalitions of  $N_r$ . For any non-empty  $S \subset N_r$ , we call  $p_S = \{T_1, \dots, T_k\}$  a  $\pi_r$ -partition of  $S$  iff  $T_t \in \pi_r$  for all  $1, \dots, k$  and  $p_S$  is a partition of  $S$ . Let  $P_r(S)$  be the set of all  $\pi_r$ -partitions of  $S$ .

For a given partitioning game  $(N, v) \in GS(N, \pi)$ , we define the  $r^{\text{th}}$  replica game  $(N_r, v_r)$  generated by  $(N, v)$  by:

$$\begin{aligned} \text{for all } T \in \pi_r, \quad v_r(T) &= v(T'), \quad \text{where } T' \in \pi \\ &\text{with } \rho(T) = \rho(T'); \quad \text{and} \end{aligned} \tag{13}$$

$$\text{for all } T \in \pi_r, \quad v_r(T) = \max_{p_S \in P_r(S)} \sum_{T \in p_S} v_r(T). \tag{14}$$

For a given partitioning game  $(N, V) \in G(N, \pi)$ , we define the  $r^{\text{th}}$  replica game  $(N_r, V_r)$  generated by  $(N, V)$  by:

$$\begin{aligned} \text{for all } S \in \pi_r, \quad V_r(S) &= R^{N_r - N^S} \times V(S'), \quad \text{where} \\ &S' \in \pi \text{ with } \rho(S) = \rho(S'); \quad \text{and} \end{aligned} \tag{15}$$

$$\text{for all } S \in \pi_r, \quad V_r(S) = \bigcup_{p_S \in P_r(S)} \bigcap_{T \in p_S} V_r(T), \tag{16}$$

where  $N^S$  is a subset of  $N_r$  such that  $S \subset N^S$  and  $\rho(N^S) = (1, \dots, 1)$ .

We remark that in both the sidepayments and no-sidepayments cases, the games  $(N_r, v_r)$  and  $(N_r, V_r)$  are partitioning games.

Example 6. Consider a three-type assignment game as given in Example 5. The  $r^{\text{th}}$  replication is defined as follows:

$$N_r = \{(i, q) : i \in N, q = 1, \dots, r\},$$

and  $N_r$  is divided into

$$J_r = \{(j, q) : j \in J, q = 1, \dots, r\},$$

$$K_r = \{(k, q) : k \in K, q = 1, \dots, r\},$$

$$M_r = \{(m, q) : m \in M, q = 1, \dots, r\}.$$

The collection of basic coalitions  $\pi_r$  is given as

$$\begin{aligned} \pi_r = \{ & \{(i, q)\} : (i, q) \in N_r\} \cup \{(j, q), (k, q'), (m, q'')\} : \\ & (j, q) \in J_r, (k, q') \in K_r, (m, q'') \in M_r \}. \end{aligned}$$

In particular, let us consider the three-type assignment game with an empty core given in Example 5. Let  $r = 2$ . Then it holds that

$$\begin{aligned} v_2(N_2) = & v_2(\{(1,1), (4,1), (7,1)\}) + v_2(\{(2,1), (6,1), (9,1)\}) \\ & + v_2(\{(3,1), (5,1), (7,2)\}) + v_2(\{(2,2), (4,2), (8,1)\}) \\ & + v_2(\{(1,2), (5,2), (9,2)\}) + v_2(\{(3,2), (6,2), (8,2)\}) = 18. \end{aligned} \quad (17)$$

The partition associated with  $v_2(N_2)$  is described by Figure 3. Hence vector  $(1, 1, \dots, 1)$  is feasible and further it is easy to see that

no coalition can improve upon this vector. Therefore  $(N_2, v_2)$  already has a non-empty core. This property is generalized in Theorem 2.

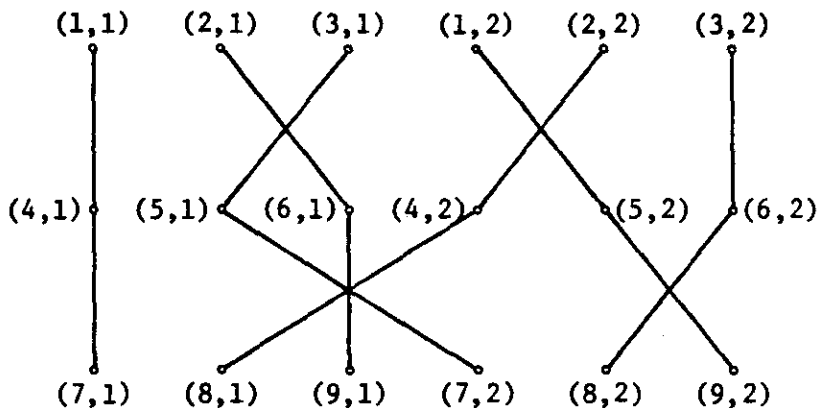


FIGURE 3

Before starting and proving our theorems, we require the concept of the balanced cover of a game.

For arbitrarily given games  $(N, v)$  and  $(N, V)$  with and without sidepayments, the balanced cover games  $(N, \tilde{v})$  and  $(N, \hat{V})$  are defined as follows:

$$\tilde{v}(S) = \begin{cases} v(S) & \text{if } S \neq N \\ \max\left\{ \sum_{T \in \gamma} \delta_T v(T) : \gamma \text{ is a balanced family} \right. & \text{if } S = N \\ \left. \text{with a balancing weight vector } \delta \right\} & \end{cases} \quad (18)$$

$$\hat{V}(S) = \begin{cases} V(S) & \text{if } S \neq N \\ \bigcup_{\gamma \in \beta} \bigcap_{T \in \gamma} V(T) & \text{if } S = N, \end{cases} \quad (19)$$

where  $\beta$  is the set of all balanced families.

Our next theorem will be used in the proof of the following theorems and is of some interest itself.



Theorem 2. Let  $(N, \pi)$  be given. Then there is an integer  $m^0$  such that for any positive integer  $k$  and any  $(N, v) \in GS(N, \pi)$  and  $(N, V) \in G(N, \pi)$ , the replica games  $(N_r, v_r)$  and  $(N_r, V_r)$  have non-empty cores, where  $r = km^0$ .

Proof. Let  $B$  denote the collection of all minimal  $\pi$ -balanced families. Given  $\gamma \in B$ , observe that the (unique) balancing weights  $\delta_S$  for  $S \in \gamma$  are all rational numbers, because  $\delta$  is the solution of linear equations with integral coefficients. Therefore there is an  $m^0$  which satisfies the requirement that  $m^0 \delta_S$  is an integer for all  $S \in \gamma$  and for all  $\gamma \in B$ . We claim that this  $m^0$  satisfies the requirements of the theorem.

This  $m^0$  is also the integer given in Lemma 5 of Wooders [1981], i.e.,

$$\text{if } x \in \hat{V}(N), \text{ then } \prod_{i=1}^{m^0} x \in V_{m^0, m^0}(N_{m^0}). \quad (20)$$

Also from Lemma 3 of Wooders [1981] we have:

$$\begin{aligned} \text{for all positive integer } k, \text{ if } \prod_{i=1}^{m^0} x \in V_{m^0, m^0}(N_{m^0}), \\ \text{then } \prod_{i=1}^{km^0} x \in V_{km^0, km^0}(N_{km^0}). \end{aligned} \quad (21)$$

Let  $x$  be in the core of  $(N, \hat{V})$ ; from Scarf's theorem [1967], there

is such an  $x$ . We show that for all positive integer  $k$ ,  $\prod_{i=1}^{km^0} x$  is

in the core of  $(N_r, V_r)$ , where  $r = km^0$ . From (20) and (21),

$\prod_{i=1}^r x \in V_r(N_r)$ . Therefore it is sufficient to show that for any  $S \subset N_r$ ,

$\prod_{i=1}^r x$  does not belong to  $\text{int } V_r(S)$ .

Suppose for some non-empty subset  $S \subset N_r$ ,  $\prod_{i=1}^r x_i \in \text{int } V_r(S)$ .

From the definition of the game  $(N_r, V_r)$ , there is an  $\pi_r$ -partition

$p_S$  of  $S$  such that  $\prod_{i=1}^r x_i \in \text{int } \bigcap_{T \in p_S} V_r(T)$ , i.e.,  $\prod_{i=1}^r x_i \in \text{int } V_r(T)$

for all  $T \in p_S$ . Given  $T \in p_S$ , there is a  $T^* \in \pi$  by (16) such that

$\rho(T^*) = \rho(T)$  and  $V_r(T) = R^{N_r - N^T} \times V(T^*)$ , where  $N^T \subset N_r$  such that  $T \subset N^T$  and  $\rho(N^T) = (1, \dots, 1)$ . This implies  $x \in \text{int } V(T^*)$ . This is a contradiction to the choice of  $x$ .

The above proof also applies to partitioning games with side-payments.<sup>6</sup>

Q.E.D.

The above theorem states that given any  $(N, V)$  and  $(N, v)$  in  $G(N, \pi)$  and  $GS(N, \pi)$  respectively, there are subsequences of the generated sequences of replica games such that all games in the subsequences have non-empty cores. This type of property was noted by Shubik for his Bridge Game Example and our result generalized Shubik's observation. Just as Shubik's Bridge Game has a non-empty core for all numbers of players such that the set of players can be partitioned into groups of four, our result shows that any partitioning game has a non-empty core if the set of players can be partitioned into basic coalitions associated with a payoff  $x^r$  in the core of the balanced cover of the (unreplicated) game. In the following we introduce concepts of approximate cores, one for partitioning games with sidepayments and a more restrictive one for par-

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<sup>6</sup>This theorem can be easily extended to show that for some  $m^0$ , for all positive integers  $k$ , the games  $(N_r, V_r)$  and  $(N_r, v_r)$  where  $r = km^0$  are "totally balanced" in the sense that all subgames of the games  $(N_r, V_r)$  and  $(N_r, v_r)$  have non-empty cores.

partitioning games without sidepayments. We show that independently of  $V$  (or  $v$ ), all sufficiently large replications of partitioning games have non-empty approximate cores.

For games without sidepayments, we have the following result (this also applies to games with sidepayments).

Theorem 3. For any  $\lambda > 0$ , there is an  $r^*$  such that for all  $r \geq r^*$ , and for any  $(N, V) \in G(N, \pi)$ , there is a vector  $\bar{x}^r$  in the core of the balanced cover game  $(N_r, \bar{V}_r)$  of the  $r^{\text{th}}$  replica game generated by  $(N, V)$  and a vector  $x^r$  in  $V_r(N_r)$  such that

$$|\{(i, q) \in N_r : \bar{x}_{iq}^r \neq x_{iq}^r\}| < \lambda r. \quad (22)$$

Informally, Theorem 3 states that given a game  $(N, V)$ , for sufficiently large replications  $r$ , it is possible to find vectors  $x^r$  in  $V_r(N_r)$  which "approximate" some vector  $\bar{x}^r$  in the core of the balanced cover game in the sense that the percentage of players whose payoff  $x_{iq}^r$  differs from  $\bar{x}_{iq}^r$  can be made arbitrarily small. Moreover, "close" approximations can be obtained simultaneously for all  $(N, V)$  in  $G(N, \pi)$  by the appropriate choice of  $r^*$  and  $\lambda$ .<sup>7</sup>

Theorem 3 differs from a related theorem of Shubik and Wooders [1982] in that they use a slightly different, less restrictive, concept of approximate core than we do. We are able to obtain our stronger result because sequences of replica games generated by a given game satisfy properties not required by Shubik and Wooders (see Shubik and Wooders [1982] and also Wooders [1981]).

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<sup>7</sup> Similar theorems concerning approximate cores of games derived from exchange economies are well-known (cf. Henry [1972], Dierker [1971], and Broome [1972]).

Proof of Theorem 3. Let  $m^0$  be as defined in the proof of Theorem 2.

Let  $r^*$  be sufficiently large so that  $m^0|N|/r^* < \lambda$ . For any

$(N, V) \in G(N, \pi)$ , select any  $y$  in the core of the balanced cover game

$(N, \hat{V})$  of  $(N, V)$ . For each  $r$ , let  $\tilde{x}^r = \prod_{i=1}^r y$ . It follows that

$\tilde{x}^r$  is in the core of  $(N_r, \hat{V}_r)$  for each  $r$  (the proof is essentially

the same as the proof that  $\prod_{i=1}^r x$  is in the core of  $(N_r, V_r)$  in the

proof of Theorem 2).

Given  $r \geq r^*$ , let  $k$  be the largest integer such that  $km^0 \leq r$

and let  $j = r - km^0$ . Arbitrarily select  $z \in V(N)$ . Let  $x^r = \prod_{i=1}^{km^0} y \times \prod_{i=1}^j z$ .

From superadditivity,  $x^r \in V_r(N_r)$  since  $\prod_{i=1}^{km^0} y \in V_{km^0}(N_{km^0})$  and

$\prod_{k=1}^j z \in V_j(N_j)$ . Then it holds that

$$|\{(t, q) \in N_r : \tilde{x}_{tq}^r \neq x_{tq}^r\}| = j|N| \leq m^0|N| \leq \lambda r^* \leq \lambda r.$$

Q.E.D.

Now we consider non-emptiness of approximate cores of partitioning games with sidepayments. The approximate core concept used in Theorem 4, however, is the Shapley-Shubik [1966] weak  $\epsilon$ -core. To enable us to state the theorem independently of the function  $v$ , we normalize the games in  $GS(N, \pi)$ . Let  $GS^*(N, \pi)$  denote the class of partitioning games with sidepayments, normalized so that for all  $(N, v) \in GS^*(N, \pi)$ ,

$$v(N) \leq |N|^8. \quad (23)$$

<sup>8</sup>Note that this does not allow  $\bar{v}(T) < 0$  for all  $T \in \pi$ . Therefore it is not true that every game  $(N, \pi) \in GS(N, \pi)$  can be normalized into  $GS^*(N, \pi)$  only by a parallel transformation.

Theorem 4. For any  $\epsilon > 0$ , there is an  $r^*$  such that for all  $r \geq r^*$  and for any  $(N, v) \in GS^*(N, \pi)$ , the  $r^{\text{th}}$  replica game generated by  $(N, v)$  has a non-empty  $\epsilon$ -core, i.e., there is a vector  $x^r$  such that

$$\sum_{(i,q) \in N_r} x_{iq}^r \leq v_r(N_r) \quad (24)$$

$$\sum_{(i,q) \in S} x_{iq}^r \geq v_r(S) - \epsilon |S| \quad \text{for all } S \subset N_r. \quad (25)$$

Example 7. Consider the three-type assignment game with an empty core given in Example 5. In Example 6, it is shown that  $(N_2, v_2)$  already has a non-empty core. Furthermore it is easily verified that

$$v_r(N_r) = \begin{cases} 9r & \text{if } r \text{ is even} \\ 9(r-1) + 6 & \text{if } r \text{ is odd.} \end{cases}$$

and, if  $r$  is even, then the core of  $(N_r, v_r)$  is non-empty. If  $r$  is odd, then the vector  $x^r$  such that

$$x_{iq}^r = 1 - \frac{1}{3r} \quad \text{for all } (i,q) \in N_r$$

is feasible. Since  $1/3r \rightarrow 0$  ( $r \rightarrow \infty$ ), this vector is in the  $\epsilon$ -core for all sufficiently large  $r$ . Note that Theorem 4 states that it is possible to choose  $r$  independently of a particular game in  $GS(N, \pi)$ .

Proof of Theorem 4. From a result established in the proof of Theorem 3, we have the result that if  $x$  is in the core of the balanced cover game  $(N, \tilde{v})$  of  $(N, v) \in GS^*(N, \pi)$ , then  $\prod_{i=1}^r x$  is in the core of the balanced cover game  $(N_r, \tilde{v}_r)$  of the  $r^{\text{th}}$  replica game  $(N_r, v_r)$  of  $(N, v)$ . Therefore

$$\tilde{v}_r(N_r) = rv(N) \quad \text{for all } r \geq 1. \quad (26)$$

Let  $m^0$  be as defined in the proof of Theorem 2. Then

if  $r = km^0 + j$  for some integers  $k$  and  $j$ , then we have

$$\tilde{v}_r(N_r) = \tilde{v}_{km^0}(N_{km^0}) + \tilde{v}_j(N_j). \quad (27)$$

For any  $(N, v) \in GS^*(N, \pi)$ , we have

$$\tilde{v}(N) = \max_{S \in \gamma} \left\{ \sum_{S \in \gamma} \delta_S v(S) : \gamma \text{ is a balanced family with balancing weight vector } \delta \right\} \quad (28)$$

$$\leq v(N) \sum_{S \in \gamma} \delta_S \leq |N|v(N) \leq |N|^2.$$

For  $r \geq m^0$ , let  $k$  be the largest integer such that  $km^0 \leq r$  and let  $j = r - km^0$ . Given any  $(N, v) \in GS^*(N, \pi)$ , from Theorem 2 and the Bondareva-Shapley theorem

$$\text{for all positive integer } k, \quad \tilde{v}_{km^0}(N_{km^0}) = v_{km^0}(N_{km^0}). \quad (29)$$

Then it follows from (26), (27), (28), (29) and superadditivity that

$$\begin{aligned} \tilde{v}_r(N_r) - v_r(N_r) &\leq \tilde{v}_{km^0}(N_{km^0}) + \tilde{v}_j(N_j) - [v_{km^0}(N_{km^0}) + v_j(N_j)] \\ &= \tilde{v}_j(N_j) - v_j(N_j) \leq j\tilde{v}(N) - jv(N) \leq j|N|^2 \leq m^0|N|^2. \end{aligned} \quad (30)$$

Now select  $r^*$  sufficiently large so that  $m^0|N|^2/r^* < \epsilon$ . Given any  $(N, v) \in GS^*(N, \pi)$  and any  $r \geq r^*$ , we have, by (30), and

$$\tilde{v}_r(N_r) - v_r(N_r) = m^0|N|^2 < r\epsilon. \quad (31)$$

Let  $x^r$  be in the core of  $(N_r, \check{v}_r)$ . Then we define  $y^r$  by

$$y_{tq}^r = x_{tq}^r - \varepsilon \text{ for all } (t,q) \in N_r.$$

It follows from (31) that this  $y^r$  satisfies (25) and (26).

Q.E.D.

Remarks: An Extension

Although Wooders' result [1981] applies to a larger class of sequences of games than those constructed from given games with sidepayments, Theorem 4 suggests another theorem which, for the sidepayments case, is stronger than the result in Wooders [1981].

In this extension, we will define a more general class of sequences of replica games than previously considered herein and obtain a result analogous to Theorem 4 for this class.

Given  $n$ , let  $I$  denote the  $n$ -fold Cartesian product of the non-negative integers, called the set of profiles. Given any  $s \in I$  and  $s' \in I$  where  $s' \leq s$ , the profile  $s'$  is called a subprofile of  $s$ .

Let  $v$  denote a superadditive function mapping  $I$  into  $R$ , where  $v(0) = 0$ . Define  $N_r = \{(i,q) : i = 1, \dots, n \text{ and } q = 1, \dots, r\}$  and  $v_r$ , a function mapping subsets of  $N_r$  into  $R$ , so that  $v_r(S) = v(s)$  when  $\rho(S) = s$ .<sup>9,10</sup> Then  $(N_r, v_r)$  is a game and the sequence  $(N_r, v_r)_{r=1}^{\infty}$  is called a sequence of replica games. Note that we do not necessarily have  $\check{v}(N_r) = r\check{v}(N_1)$ , a property of replica games generated by a given game.

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<sup>9</sup>Here again  $\rho(S)$  is the profile of  $S$ ; i.e., the  $i^{\text{th}}$  coordinate of  $\rho(S)$  is  $|S \cap \{(i,q) : q = 1, \dots, r\}|$ .

<sup>10</sup>The profile of  $N_1$  is the vector  $(1, \dots, 1) \in R^n$ . This does not rule out the possibility that members of  $N_1$  are substitutes for each other.

Let  $G^*S$  denote the set of all sequences of replica games  $(N_r, v_r)_{r=1}^{\infty}$  normalized so that for all  $S \subset N_r$ ,  $0 \leq v_r(S)$  and  $\limsup_{r \rightarrow \infty} \frac{v(N_r)}{r} \leq |N_1|$ . We then have the following theorem which is

stated without proof since the result is an easy extension of Theorem 4 and results in Wooders [1981] applied to sidepayment games.

Theorem 4'. Given any  $\epsilon > 0$  there is an  $r^*$  such that for any function  $v$  defined as above, and for all  $r \geq r^*$  the  $r^{\text{th}}$  replica game  $(N_r, v_r)$  has a non-empty  $\epsilon$ -core, i.e., there is an  $x^r \in R^{N_r}$  satisfying  $\sum_{(i,q) \in N_r} x_{iq}^r \leq v_r(N_r)$  and  $\sum_{(i,q) \in S} x_{iq}^r \geq v_r(S) - \epsilon|S|$  for all  $S \subset N_r$ .



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