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APPROXIMATE CORES OF A GENERAL CLASS OF ECONOMIES:

PART II. SET-UP COSTS AND FIRM FORMATION IN

COALITION PRODUCTION ECONOMIES

by

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APPROXIMATE CORES OF A GENERAL CLASS OF ECONOMIES:

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ABSTRACT

A general model of a coalition production economy allowing set-up costs, indivisibilities, and non-convexities is developed. It is shown that for all sufficiently large replications, approximate cores of the economy are non-empty.

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## 1. Cores, Coalition Production Economies, Entry, Exit and Merger

General equilibrium theory is noninstitutional, or as Koopmans has remarked "preinstitutional." The basic description of the economy is given in terms of individuals, goods, preferences and production sets. Oligopoly theory, partial equilibrium theory and the study of game theoretic models of the economy in strategic form are institutional. Markets and firms and even money are assumed to exist. Cooperative game theory can be either. We can take as primitive concepts goods, individuals and groups of individuals, and the preferences and production sets of individuals and groups. Alternatively we can define firms and firms-in-being, specify the manner of trade in the markets, define what is meant by entry and exit and fully formulate the game in strategic form. Taking the game in strategic form as a point of departure we may then consider what coalitions can achieve and hence can calculate the characteristic function. This calculation will depend with some delicacy upon the assumptions we make about the nature of externalities caused by having to trade through markets and how we calculate the threat potential that a set of agents  $\bar{S}$  has upon the set  $S$ . The von Neumann-Morgenstern characteristic function is calculated assuming maximin behavior by  $S$  and  $\bar{S}$ . A procedure suggested by Harsanyi takes into account the costs of carrying out threats to the threatener. For a certain class of games (called by Shapley and Shubik "c-games"--see Shubik, 1982) the distinction does not matter. We make this concept clear and note below that exchange economies belong to this class, but economies with exchange and production pose difficulties.

The original approaches to studying the core of an economy with exchange only, of Edgeworth (1881), Shubik (1959) and Debreu and Scarf

(1963) were all based upon what amounted to the straightforward construction of a characteristic function directly and easily from the basic economic data. Given an economy with a set  $N$  of traders, any subset  $S \subset N$  can obtain anything feasible by trading among themselves and the membership of  $S$  can form an effective set against any imputation which offers them less. Figure 1 shows the payoffs for a three person game. The curve  $ab$  is the projection into a two dimensional payoff subspace

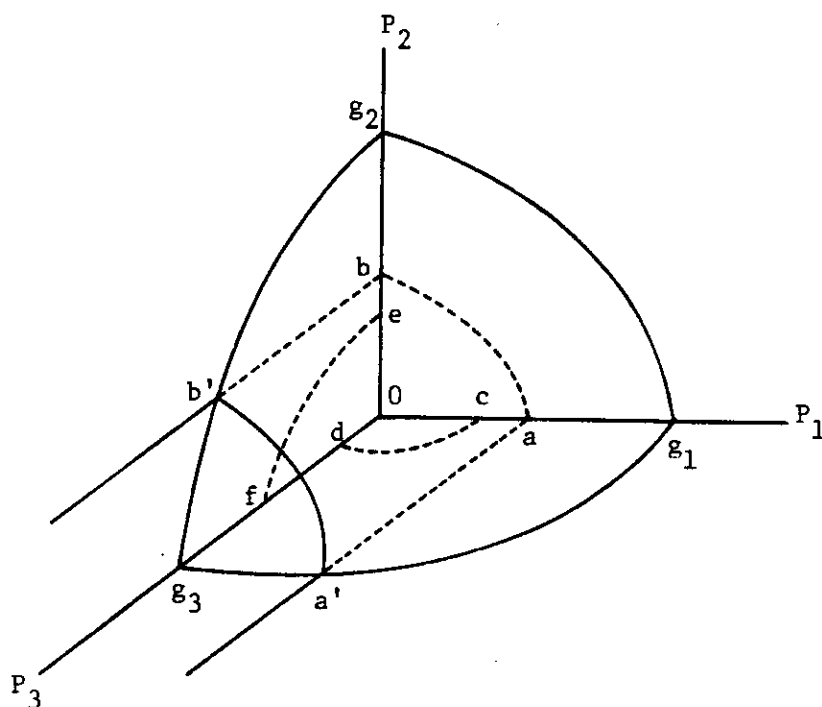


FIGURE 1

of the contract curve arising from trade between traders 1 and 2. The curves  $cd$  and  $ef$  show projections of the other contract curves. The curved surface  $g_1g_2g_3$  is the Pareto optimal surface for all three traders and the cylinder  $aba'b'$  punches out the domain of domination  $g_3a'b'$  of traders 1 and 2.

The key element to note is that the feasible set of outcomes for

a set  $S$  of traders is independent of the actions of  $\bar{S}$ . The von Neumann-Morgenstern and the Harsanyi characteristic functions will be the same. If we consider an  $n$ -person constant sum game it is reasonable to assume that the goals of  $\bar{S}$  are diametrically opposed to those of  $S$  and thus the von Neumann-Morgenstern calculation adequately reflects the cost of threat to  $\bar{S}$ . The concept of a  $c$ -game is a modeling concept. It refers to a game in cooperative form which is "adequately" represented by its von Neumann-Morgenstern characteristic function. Both  $n$  person constant sum games and "orthogonal coalition games" have this property. An orthogonal coalition game has the property that once a coalition has been formed what it obtains depends upon it alone. Market games (see Shapley and Shubik (1969)) also have this property but economies with production or organized markets do not necessarily have this property.

Figure 2 sketches some of the modeling choices.

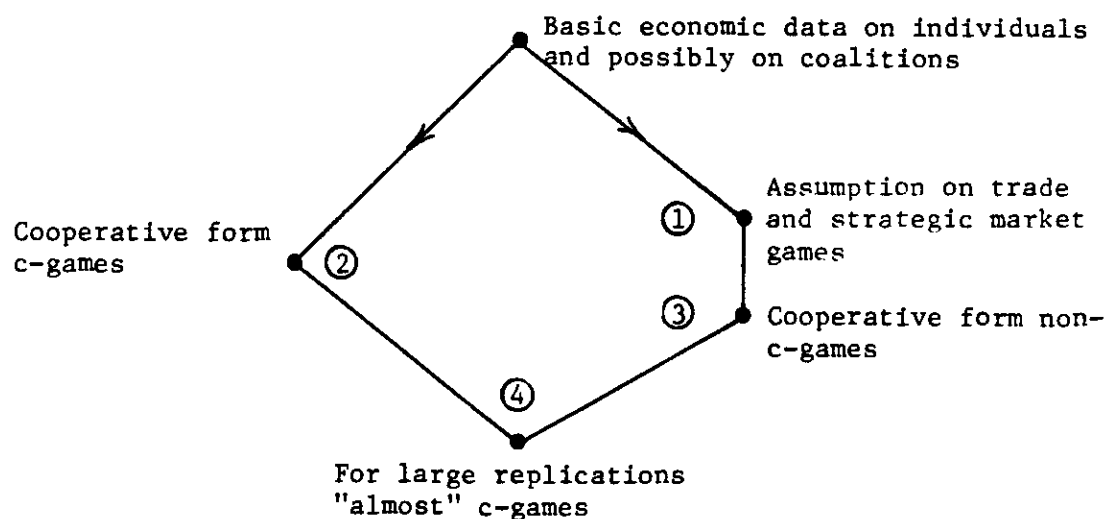


FIGURE 2

The attempts to reconcile the strategic game noncooperative approaches with Walrasian economics have concentrated on modeling and analysis involved in ①. These include Shapley and Shubik (1967), Gabzewicz and Vial (1972) for closed nonsymmetric Cournot models; Shubik (1973), Shapley and Shubik (1977) and Dubey and Shubik (1978a) for closed symmetric Cournot models; Dubey and Shubik (1978b) for closed symmetric Cournot models with trade, production, and trade in sequence; and Novshek and Sonnenschein (1978) for Nash-Cournot models with small efficient scale and set-up costs.

The results of Dubey, Mas Colell, and Shubik (1980) indicate that with a continuum of traders many different market mechanisms will lead to the same outcomes. This suggests that in large economies market structures form very weak externalities. Although to date little work has been done in ③ i.e. on games in coalitional or cooperative form derived from strategic market games, we suspect that with the appropriate conditions, for large games with many small players, for many (but not all)<sup>1</sup> analyses the distinction between a characteristic function formed directly from the basic data and one formed from a strategic market game will not matter.

Our prime concern in this paper is with games in coalitional form derived directly from the underlying economic data. Many of the basic difficulties in modeling production, indivisibilities, entry, exit and mergers can be considered directly using the coalitional form without first constructing a strategic form of the game.

The definition of the characteristic function for an exchange economy is more or less natural and in keeping with the idea that any combination of trades is possible. But when we introduce production we

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<sup>1</sup>There are some problems with voting stock and the ownership of production.

face problems in describing what a coalition can achieve. Given a set of production technologies, are they owned by society as a whole, are they individually owned, or are there shares?

When production sets are general convex sets the problem of ownership becomes critical in defining the feasible production possibilities of groups. One way to model production is to imagine the existence of  $k$  production sets which are jointly owned by the  $n$  economic agents in the economy; thus as an initial endowment individual  $i$  owns  $(\theta_{i,1}, \dots, \theta_{i,k})$  shares in the production possibility sets. We must then provide a convention to decide upon what a set  $S$  can produce. The introduction of shares raises control as well as ownership problems. One somewhat pessimistic but simple way of defining the characteristic function is to assume that any coalition  $S$  is permitted to use its own resources and any production set in which it owns a controlling block of shares.

A straightforward and minimally institutional, in effect noninstitutional, way to allow for production is to define the productive capabilities of each coalition a priori. In this manner no firms are specified, but they are implicit in coalition production.

The introduction of coalition production, however, creates another set of difficulties. If we assume, as did Hildenbrand (1968), that the production correspondence, which assigns a technology to each coalition, is additive so there are no gains in production to collusion, then no problem exists. The distinction between individual and joint ownership dissolves and the  $c$ -property in defining the characteristic function is preserved. If the production correspondence is additive and convex-valued, and if there is a continuum of agents, then there is an equivalence

between the core and the equilibrium allocations.

A number of authors have subsequently considered models with coalition production where, as in Hildenbrand (1968, 1970), a production possibility set is given a priori for each coalition. While these models allow for a distinction, in general, between individually owned and collectively owned technologies, the c-property is still preserved. However, these models involve some type of "balancedness" assumption on the production correspondence. To describe these models and the balancedness assumption, let  $A = \{1, \dots, n\}$  denote the set of agents in the economy, let  $Y[S]$  denote the production possibility set for each non-empty subset  $S$  of  $A$ , and let  $\hat{Y}$  denote the "aggregate production possibility" set. It is not necessarily the case that  $\hat{Y} = Y[A]$ ; the motivation for distinguishing between  $\hat{Y}$  and  $Y[A]$  will become clear.

One approach to defining the aggregate production possibility set  $\hat{Y}$  is to have  $Y[A] = \hat{Y}$ ; then the only feasible productions for the economy are members of  $Y[A]$ . In this case the production possibility sets  $Y[S]$  are only used to define the possibilities open to a coalition  $S$  in determining the core of the economy. This is essentially the approach used by Boehm (1974b), Sondermann (1974), and Champsaur (1974).<sup>2</sup>

Another approach to defining the aggregate production possibility set is to specify that there is some given collection  $C$  of "allowable coalition structures" of  $A$  where each member  $C$  of  $C$  is a collection of subsets of  $A$  satisfying the property that  $\bigcup_{S \in C} S = A$ . Informally, if  $C$  is in  $C$ , then those subsets  $S$  in  $C$  can all coexist as

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<sup>2</sup>For the purposes of this discussion, we do not, of course, even attempt to describe the generality of the models considered by other authors. We do, however, try to capture the essence of the models and the types of assumptions made.



productive coalitions, i.e. firms. Two particular specifications of  $C$  have been studied; (1)  $C$  is a subset of the set of all partitions of  $A$  and (2)  $C$  is the set consisting of all collections  $C$  of subsets of  $A$  such that  $\bigcup_{S \in C} S = A$  ( $C$  is not necessarily a partition). In both cases the aggregate production possibility set is then defined by  $\hat{Y} = \bigcup_{C \in C} \sum_{S \in C} Y[S]$ . The first specification of  $C$  was considered by Ichiishi (1977)<sup>3</sup> and the second by Boehm (1973). Boehm, in (1973), considers  $Y[S]$  as the technology set that the agents can collectively own, and agents can be owners of several firms simultaneously. Other authors restrict the agents in the economy, in an equilibrium, to be members of one and only one productive coalition; this can be interpreted as describing a situation where an agent works full-time for the production coalition of which he is a member. We remark that in either case, we do not necessarily have  $Y[A] = \hat{Y}$ ; the set  $Y[A]$  is interpreted as the production possibility set of the agents in  $A$  acting collectively (i.e. as one firm), whereas the set  $\hat{Y}$  may include possibilities arising from other allowable coalition structures of  $A$  besides  $\{A\}$  itself.

We now consider the assumption of "balancedness." Recall that a family  $\beta$  of subsets of  $A$  is balanced if for some set of weights,  $w_S > 0$  for each  $S \in \beta$ , we have

$$\sum_{S \in \beta_i} w_S = 1 \text{ where } \beta_i = \{S \in \beta : i \in S\} \text{ for each agent } i \text{ in } A. \text{ The}$$

aggregate production possibility set  $\hat{Y}$  is balanced if for every balanced family  $\beta$  of subsets of  $A$  we have  $\sum_{S \in \beta} w_S Y[S] \subset \hat{Y}$ .<sup>4</sup>

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<sup>3</sup>Since we are considering a finite economy, while Ichiishi considered ones with general measure spaces of agents our formulations appear somewhat different from his. For our case, however, our specifications are essentially equivalent to his.

It is generally also assumed that the aggregate production possibility set  $Y$  is convex and contains the origin.

We note that, once balancedness is assumed, for any partition  $P$  of  $A$  we have

$$\sum_{S \in P} Y[S] \subset \bigcup_{C \in C} \sum_{S \in C} Y[S] \subset \hat{Y};$$

the aggregate production technology set  $Y$  contains  $\sum_{S \in P} Y[S]$  for any partition  $P$  of  $A$ .

Although numerous examples can be developed to illustrate situations where the balancedness assumption is satisfied (cf. Sondermann (1974)), it is very easy to generate examples where the aggregate production possibility set is not balanced. Some examples are contained in Section 3 of this paper. Also, it is easy to see that if for all coalitions  $S$ , either we have  $Y[S] = \{0\}$  or there are set-up costs which must be incurred before any positive amount of output can be produced, then the balancedness condition will in general not be satisfied. Consequently, although some specific examples can be developed showing balancedness of the aggregate technology set and set-up costs for some firms, it is restrictive to assume balancedness in the presence of set-up costs.

In the next section of this paper, we develop a replication model of a coalition production economy. As is standard, we take as given a basic production correspondence  $Y$  mapping subsets of agents

into production possibility sets. For each coalition  $S$  of agents we take the set of all partitions of  $S$  into non-empty subsets as the set of allowable coalition structures of  $S$ , i.e. the aggregate production possibility set for  $S$  is

$$\bar{Y}[S] = \bigcup_{P \in \mathcal{P}(S)} \sum_{S' \in P} Y[S'] \text{ where } \mathcal{P}(S) \text{ is the set of all}$$

partitions of  $S$  into non-empty subsets. In the appendix, we show that the class of allowable coalition structures can be generalized. For our theorems, the important, and essential, feature of allowable coalition structures is that superadditivity of  $\bar{Y}$  obtains; i.e. we require that given any two disjoint subsets  $S$  and  $S'$ , we have  $\bar{Y}[S] + \bar{Y}[S'] \subset \bar{Y}[S \cup S']$ . A simple and natural set of allowable coalition structures ensuring superadditivity of  $\bar{Y}$  is the set of partitions. No balancedness assumptions are made. Also, there may be set-up costs, non-convexities, and indivisibilities. Almost the only restriction made is that there is some convex cone  $Y^*$  satisfying the usual properties of a production set and each set  $Y[S]$  is contained in  $Y^*$ .

We obtain two results showing non-emptiness of approximate cores of the economy for all sufficiently large replications. One is simply a restatement of our theorem concerning non-emptiness of the weak asymptotic core of sequences of superadditive replica games, but stated in terms of the economic variables. The other is that, given  $\epsilon > 0$ , for all sufficiently large replications of the economy there are allocations which are, in per-capita terms, approximately feasible and which cannot be " $\epsilon$ -improved upon" by any coalition of agents. Informally, for the first theorem, the condition which is relaxed is that an  $\epsilon$ -core allocation cannot be  $\epsilon$ -improved upon by any coalition of agents and, for the second, it is the feasibility condition. In the literature on approxi-

mate cores and equilibria, there are both these types of results for other economic models (compare, for example, Henry (1972), Broome (1972) and Dierker (1971) with Starr (1969) and Shapley and Shubik (1966)).

In the next section of this paper, the formal model and results are developed. In Section 3 we provide examples of application of our results. Section 4 concludes Part II of the paper and Section 5 concludes the entire paper.

## 2. The Model and Results

### Preliminaries

We use the same notation and game theoretic definitions as in Part I; most of these are not repeated here. The numbering of theorems is continued from Part I.

Following the notation in Part I, the set of agents of the  $r^{\text{th}}$  economy is denoted by  $A_r = \{(t, q) : t = 1, \dots, T, q = 1, \dots, r\}$ . The set  $[t]_r$  is the set of agents of type  $t$  in the  $r^{\text{th}}$  economy; all agents of the same type will be assumed to be identical. Given a subset  $S$  of  $A_r$ , let  $s \in \mathbb{R}^T$  be the vector defined by its coordinates  $s_t = |S \cap [t]_r|$  for  $t = 1, \dots, T$ . Recall that the vector  $s$  is called the profile of  $S$  and given  $S$  with profile  $s$ , we write  $\rho(S) = s$  so  $\rho(\cdot)$  maps subsets into their profiles. Let  $I_r$  denote the collection of all profiles of  $A_r$  and let  $I = \bigcup_{r=1}^{\infty} I_r$ .

### The Model

Let  $\ell$  denote the number of private goods.

The production possibility set available to any coalition  $S$ , acting cooperatively to form a firm, is assumed to depend on the profile of  $S$ . Therefore we take as given a correspondence  $Y$  from  $I$  into closed subsets of  $\mathbb{R}^{\ell}$  where, given  $s$  in  $I$ ,  $Y[s]$  is the production

possibility set available to any coalition  $S$  with profile  $s$ . It is assumed that  $0 \in Y[s]$  for all  $s$  in  $I$ . Given a coalition  $S$  with  $\rho(S) = s$ , define  $Y[S] = Y[s]$ .

Given any  $r$  and any coalition  $S$  contained in  $A_r$ , define  $\bar{Y}[S] = \bigcup_{P \in \mathcal{P}(S)} \sum_{S' \in P} Y[S']$  where  $\mathcal{P}(S)$  is the set of all partitions of  $S$  into non-empty subsets. The correspondence  $\bar{Y}$  is the aggregate production correspondence. As indicated earlier, in the appendix we will generalize the definition of  $\bar{Y}$  so that other "coalition structures" (collections of subsets of agents which can all coexist as productive coalitions) besides partitions are allowable.

It is easy to verify that the correspondence  $\bar{Y}$  is superadditive, i.e. given any two disjoint non-empty subsets  $S$  and  $S'$  of agents,  $\bar{Y}[S] + \bar{Y}[S'] \subset \bar{Y}[S \cup S']$ . Given a non-empty subset of agents  $S$ , a production for  $S$  is a vector  $z \in \bar{Y}[S]$ .

So that feasible positive outputs do not become virtually free as the economy is replicated, we assume that there is some set  $Y^* \subset \mathbb{R}^{\ell}$  such that:

- (1)  $Y[s] \subset Y^*$  for all  $s \in I$ ;
- (2)  $Y^*$  is a closed, convex cone; and
- (3)  $Y^* \cap \mathbb{R}_+^{\ell} = \{0\}$ .

In Figure 3, we illustrate some basic production possibility sets which satisfy the required conditions where, in each diagram  $Y^*$  is the area below the dotted line.

In Figure 4, we demonstrate some possible candidates for  $\bar{Y}[s]$  for the case where all agents are identical; this illustrates the superadditivity property.

To keep this application relatively simple, our assumptions on consumption sets, preferences, and initial endowments satisfy standard

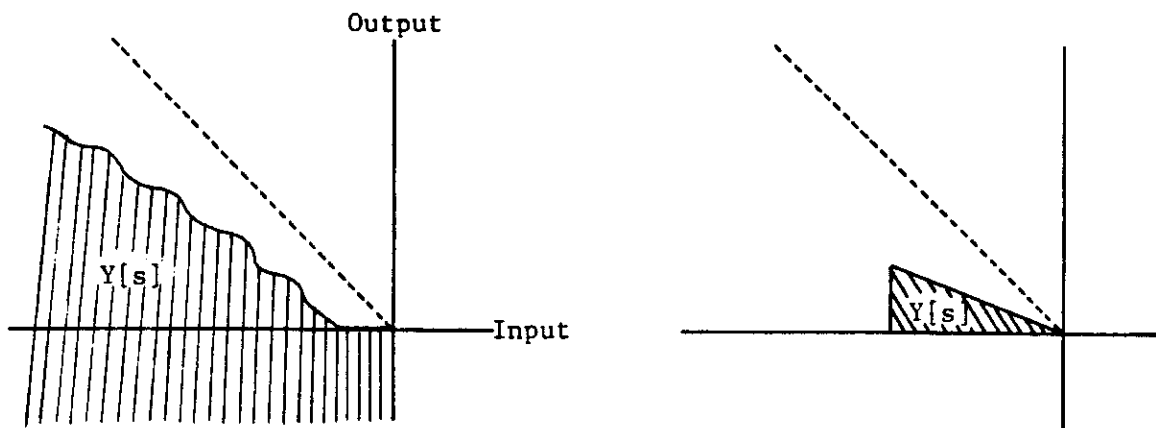
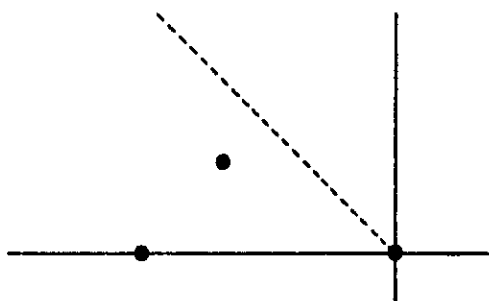
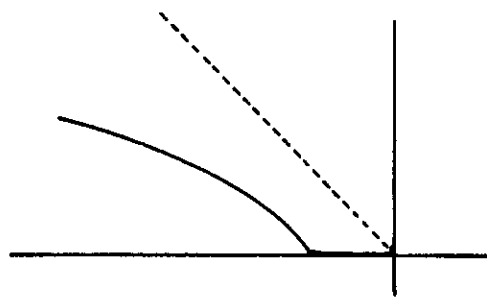


FIGURE 3



$Y[s]$  is the three dots



$Y[s]$  is the heavy line

FIGURE 4(a)

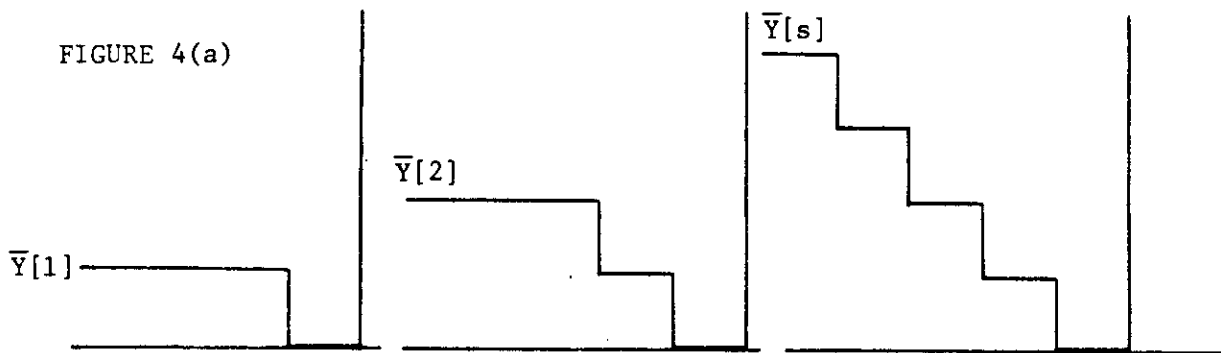
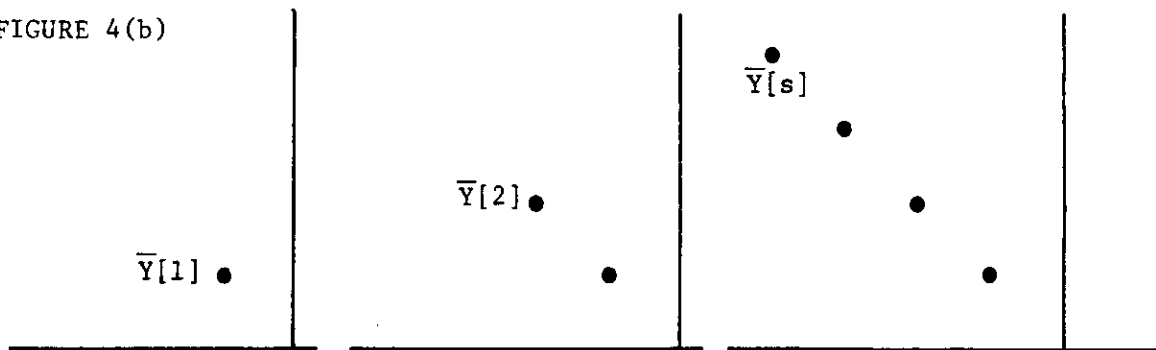


FIGURE 4(b)



properties except for possible indivisibilities. Let  $X^t$  denote the consumption set for an arbitrary agent of type  $t$ . It is assumed that  $X^t$  is closed and has a lower bound for  $\leq$ . Let  $\text{con } X^t$  denote the convex hull of  $X^t$ . For each agent of type  $t$ , there is a continuous utility function  $u^{tq}$  mapping  $\text{con } X^t$  into  $\mathbb{R}_+$ . The idea behind having the utility function of agent  $(t,q)$  defined on all of  $\text{con } X^t$  instead of  $X^t$  itself is that  $X^t$  may be restricted by technological considerations whereas the agent's preferences might ignore such considerations.<sup>6</sup> Convexity of preferences is assumed so that if  $x$  and  $x'$  are in  $\text{con } X^t$  and  $u^{tq}(x) > u^{tq}(x')$ , then for any real number  $\lambda$  where  $0 < \lambda < 1$ , we have  $u^{tq}(\lambda x + (1-\lambda)x') > u^{tq}(x')$ . Each agent  $(t,q)$  has an initial endowment  $w^{tq} \in X^t$ . For each  $t$ , all agents of type  $t$  have the same initial endowment so  $w^{tq} = w^{tq'}$  for all  $q$  and  $q'$ .

Given a subset of agents  $S$ , an allocation for  $S$  is a set  $\{x^{tq} \in X^t : (t,q) \in S\}$ . A feasible state of the economy for  $S$  or simply a feasible state for  $S$  is a production for  $S$ ,  $z \in \bar{Y}[S]$ , and an allocation for the members of  $S$ ,  $\{x^{tq} \in X^t : (t,q) \in S\}$ , such that  $\sum_{tq \in S} (x^{tq} - w^{tq}) = z$ . The allocation associated with a feasible state for  $S$  is called a feasible allocation for  $S$ . When  $S = A_r$  for some  $r$ , we call a feasible state for  $S$  a feasible state of the  $r^{\text{th}}$  economy, or, when no confusion is likely to arise, simply a feasible state.

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<sup>6</sup>This approach permits indivisibilities in consumption while retaining convexity of preferences and facilitates showing the derived sequence of games is per-capita bounded. Were we to assume directly that the sequence of games derived from the sequence of economies is per-capita bounded (a reasonable assumption), then we could easily drop the assumption of convexity of preferences.

We now define the sequence of games derived from the sequence of economies. For each  $r$  and each non-empty subset  $S$  of  $A_r$ , let  $V_r(S)$  be a subset of  $\mathbb{R}^{A_r}$  such that  $\bar{u}$  is in  $V_r(S)$  if and only if for some feasible state of the economy for  $S$  with allocation  $\{x^{tq} \in X^t : (t,q) \in S\}$ , we have  $u^{tq}(x^{tq}) = \bar{u}^{tq}$  for each  $(t,q)$  in  $S$ . Coordinates of members of  $V_r(S)$  are ordered appropriately so that  $(A_r, V_r)_{r=1}^\infty$  satisfies the definition of a sequence of replica games. It is immediate that the games are superadditive.

It remains to show that  $(A_r, V_r)_{r=1}^\infty$  is per-capita bounded. To do this, we consider a sequence of games derived from another sequence of economies, identical to the given sequence except that  $Y[s] = Y^*$  for all  $s$  in  $I$  and the "technologically feasible" consumption set of an agent of type  $t$  is assumed to be  $\text{con } X^t$ ; we call this sequence of economies the \*-economies. Let  $(A_r, V'_r)_{r=1}^\infty$  be the sequence of games derived from the sequence of \*-economies. Select a real number  $K$  such that  $K > \max_{\bar{u} \in V'_1(A_1)} \max_t \bar{u}^t \bar{c}^t$ . We claim that for all  $r$  and all equal-treatment payoffs  $\bar{u}$  in  $V'_r(A_r)$ , we have  $\bar{u}^{tq} < K$  for all  $(t,q)$  in  $A_r$ . Suppose not. Then, for some  $r$ , some  $\bar{u}$  with the equal-treatment property in  $V'_r(A_r)$ , and some type  $t'$ , we have  $\bar{u}^{t'q} > K$  for all  $q$ . Since  $\bar{u}$  is in  $V'_r(A_r)$ , there is a feasible state of the  $r^{\text{th}}$  \*-economy, say  $z \in Y^*$  and  $\{x^{tq} \in \text{con } X^t : (t,q) \in A_r\}$ , with  $\sum_{tq \in A_r} (x^{tq} - w^{tq}) = z$ . Since  $Y^*$  is convex and  $0 \in Y^*$ , we have  $\frac{1}{r}z \in Y^*$ . Since  $\text{con } X^t$  is convex for each  $t$ ,  $\frac{1}{r} \sum_{q=1}^r x^{tq}$  is in  $\text{con } X^t$ . From convexity of preferences  $u^{tq}(\frac{1}{r} \sum_{q=1}^r x^{tq}) \geq \bar{u}^{tq}$  for all  $(t,q) \in A_r$ . Now the production  $\frac{1}{r}z$  is in  $Y^*$  and  $(\frac{1}{r} \sum_{q=1}^r x^{tq})$  is in  $\text{con } X^t$  for



each  $(t, q)$  in  $A_1$ . It follows that  $\max_{\bar{u}' \in V'_1(A_1)} \max_t \bar{u}'^t \geq \bar{u}'^t q > K$  which is a contradiction. Therefore the sequence  $(A_r, V'_r)_{r=1}^\infty$  is per-capita bounded. Since, for each  $r$ , the set of feasible states for our original economy is contained in the set of feasible states of the  $\star$ -economy, we have  $V_r(A_r) \subset V'_r(A_r)$  for all  $r$ ; therefore the sequence  $(A_r, V_r)_{r=1}^\infty$  is per-capita bounded.

We can now apply Theorem 1 of Part I of this paper to the sequence of derived games to conclude that the weak asymptotic core of the sequence is non-empty. Our next theorem is simply a restatement of this result in terms of the underlying economic variables and therefore no additional proof is provided.

Theorem 4. Given any  $\epsilon > 0$  and any  $\lambda > 0$  there is an  $r^*$  such that for all  $r \geq r^*$ , for some allocation for  $A_r$ , say  $\{x^{tq} \in X^t : (t, q) \in A_r\}$ , and some feasible allocation for the  $r^{\text{th}}$  economy, say  $\{x^{*tq} \in X^t : (t, q) \in A_r\}$

- (1) for all non-empty subsets  $S$  of  $A_r$ , if  $\{x^{tq} \in X^t : (t, q) \in A_r\}$  is such that  $u^{tq}(x^{tq}) > u^{tq}(x^{*tq}) + \epsilon$  for all  $(t, q) \in S$ , then  $\sum_{tq \in S} (x^{tq} - w^{tq}) \notin \bar{Y}[S]$  (no subset of agents can produce anything unanimously preferred by more than  $\epsilon$  to the given allocation), and
- (2)  $|\{(t, q) \in A_r : x^{*tq} \neq x^{tq}\}| < \lambda |A_r|$ .

In the following theorem, we use results obtained in Part I to show that given  $\epsilon > 0$ , for all  $r$  sufficiently large, there is an allocation which cannot be " $\epsilon$ -improved upon" by any coalition of agents in  $A_r$  and for any commodity the per-capita difference between the quantity "allocated" and some feasible allocation can be made arbitrarily

small.<sup>7</sup> Formally, we have the following theorem:

Theorem 5. Given any  $\epsilon > 0$ , there is a  $\Delta \in \mathbb{R}^k$  and an  $r^*$  such that for all  $r \geq r^*$ , for some allocation  $\{x^{tq} \in X^t : (t,q) \in A_r\}$  and some  $z \in \bar{Y}[A_r]$ , we have

$$(1) \quad -\Delta \leq \sum_{tq \in A_r} (x^{tq} - w^{tq}) - z \leq \Delta \quad \text{and}$$

(2) the allocation cannot be  $\epsilon$ -improved upon by any subset  $S$  of  $A_r$ , i.e. (1) of Theorem 4 is satisfied.

Proof. As in the proof of Theorem 1 there is a  $\bar{u}^* \in \mathbb{R}^T$  such that

$\prod_{i=1}^r \bar{u}^*$  is in the  $\epsilon$ -core of  $(A_r, \hat{V}_r^c)$  for all  $r$  sufficiently large,

where, as previously,  $\hat{V}_r^c$  is the balanced cover of the comprehensive cover,  $V_r^c$ , of  $V_r$ . Also, there is an  $r^0$  such that for all positive integers

$n$ , we have  $\prod_{i=1}^{nr^0} \bar{u}^*$  in the  $\epsilon$ -core of  $(A_{nr^0}, V_{nr^0}^c)$ .

For each  $n$ , let  $\bar{u}_{nr^0} \in V_{nr^0}(A_{nr^0})$  such that  $\bar{u}_{nr^0} \geq \prod_{i=1}^{nr^0} \bar{u}^*$ ; this is possible from the definition of a comprehensive cover. Denote  $\bar{u}_{nr^0}$  simply by  $\bar{u}_{r^0}$ .

For each  $t$ , let  $\bar{x}_t \in X^t$  such that for (any)  $q$ ,

$$u^{tq}(\bar{x}_t) = \min_{q'} \{ \bar{u}_{r^0}^{tq'} : (t, q') \in A_{r^0} \}.$$

Since  $\bar{u}_{r^0} \in V_{r^0}(A_{r^0})$ , it follows that there is such an  $\bar{x}_t$  for each  $t$ . Let  $\Delta \in \mathbb{R}^k$  be sufficiently large so for all  $j = 1, \dots, r^0$ , we have

$$-\Delta \leq \sum_{k=1}^j \sum_{t=1}^T (\bar{x}_t - w^t) \leq \Delta$$

where  $w^t$  is the initial endowment of an agent of type  $t$ .

In the following, for each positive integer  $r$ , let  $n$  be the largest positive integer such that  $nr^0 \leq r$  and let  $j$  satisfy  $nr^0 + j = r$ . Given  $r$ , let  $\{x^{tq} \in X^t : (t,q) \in A_r\}$  be an allocation such that

<sup>7</sup>Analogous results for exchange economies with indivisibilities and compact consumption sets has been obtained by Henry (1972) and others.

$u^{tq}(x_r^{tq}) = \bar{u}_{nr^0}^{tq}$  for each  $(t,q) \in A_{nr^0}$  and  $x_r^{tq} = \bar{x}_t$  for each  $(t,q)$  with  $nr^0 < q \leq r$ . Since  $\bar{u}_{nr^0} \in V_{nr^0}^{nr^0}(A_{nr^0})$ , we can select  $x_r^{tq}$  for each  $(t,q) \in A_{nr^0}$  so that there is a

$z_{nr^0} \in Y[A_{nr^0}]$  satisfying  $\sum_{tq \in A_{nr^0}} (x_r^{tq} - w^{tq}) = z_{nr^0}$ . From the construc-

tion of  $\bar{x}_t$  for each  $t$  and of  $\Delta$ , we have  $-\Delta \leq \sum_{tq \in A_r} (x_r^{tq} - w^{tq}) - z_{nr^0} \leq \Delta$ .

Also, since  $u^{tq}(x_r^{tq}) \geq \bar{u}_t^*$  for each  $(t,q) \in A_r$  and since  $\prod_{i=1}^r \bar{u}_i^*$  is in the  $\epsilon$ -core of  $(A_r, \hat{V}_r^c)$ , it follows that the allocation cannot be  $\epsilon$ -improved upon by any subset  $S$  of  $A_r$ .

Q.E.D.

This proof, given the generality of the result, is simple. This is because we employ the more difficult results obtained in Part I; especially the existence of the equal-treatment "limit payoff"

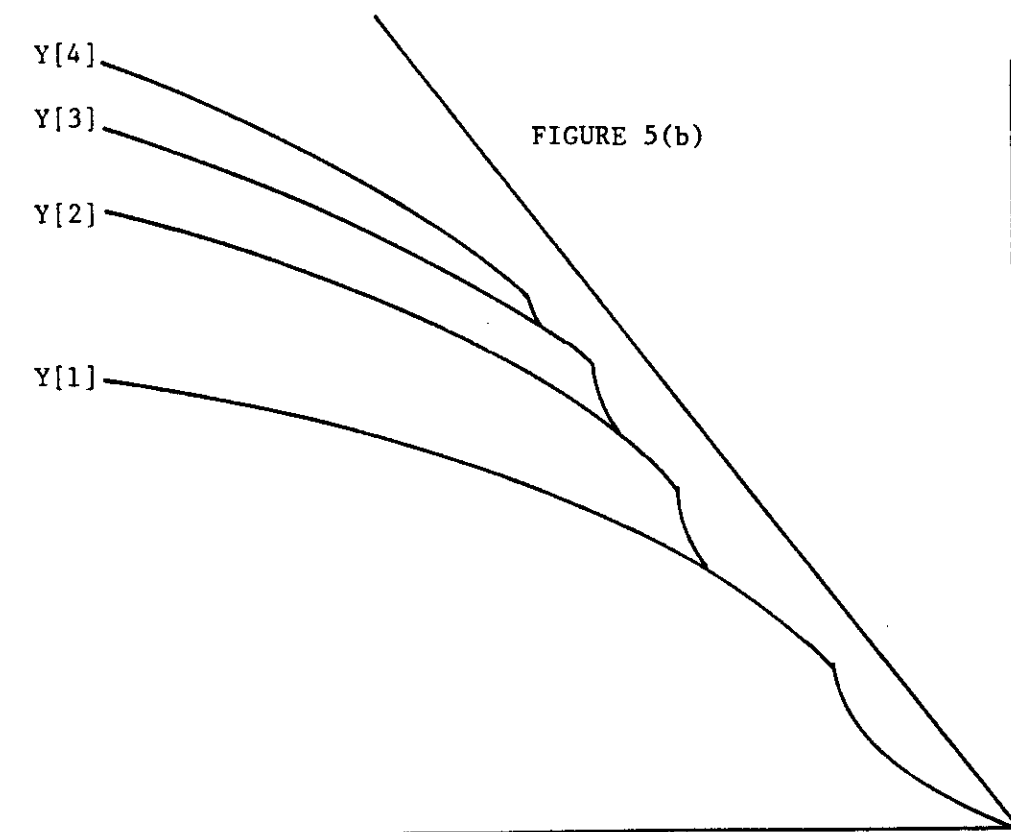
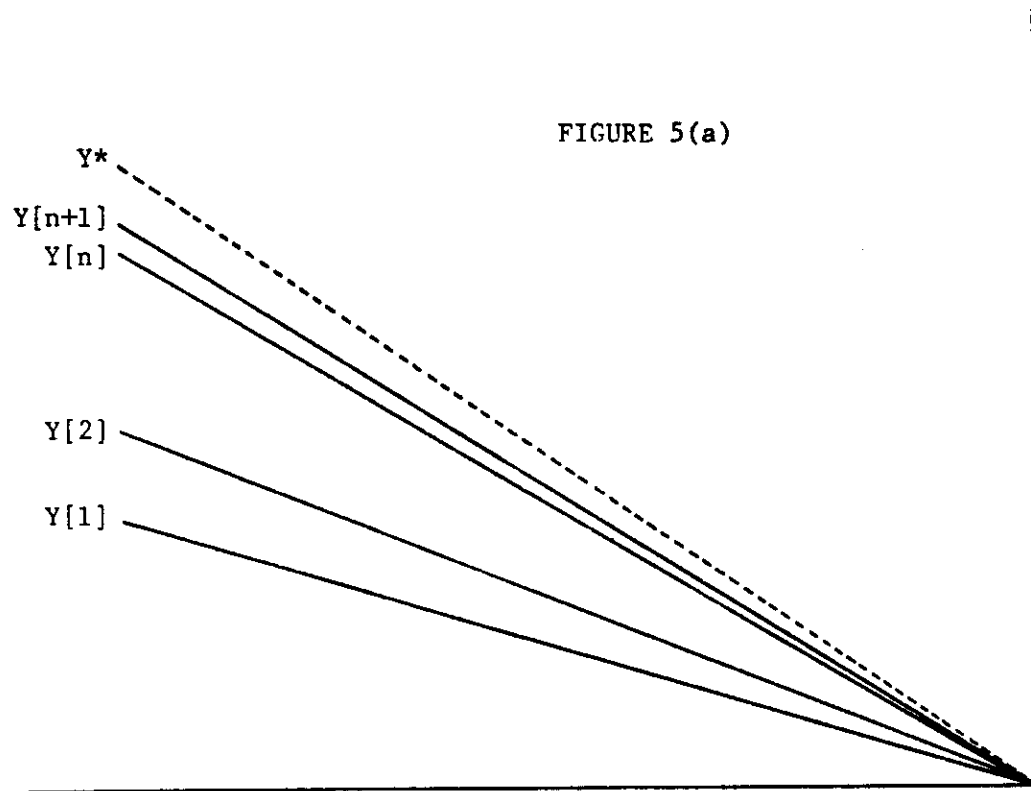
$u^*$  with the property that  $\prod_{i=1}^r u_i^*$  is in the  $\epsilon$ -core of  $(A_r, \hat{V}_r^c)$  for

all sufficiently large  $r$ , and the fact that given any  $r^0$  there is

an  $n$  such that if  $\bar{u} \in \hat{V}_{r^0}^{r^0}(A_{r^0})$ , then  $\prod_{i=1}^n \bar{u}_i \in V_{nr}^c(A_{nr})$  (Lemma 2 in

Part I). The simplicity of the result illustrates the power of Theorem 1--the non-emptiness of weak asymptotic cores.

The " $\epsilon$ -improvement" rather than simply "improvement" in the preceding theorems is in part required by the fact that no finite economy necessarily exhausts all "increasing returns to coalition size." To illustrate this, in Figure 5 we depict two simple special cases to illustrate "increasing returns to coalition size" where  $Y[S]$  depends only on the number of agents in  $S$ . In Figure 5(a) and (b),  $Y[n]$  denotes the upper boundary of the production possibility set for a coalition containing  $n$  agents. The idea is that production becomes more and more



efficient as the number of agents in the economy increases (although  $Y[n]$  remains bounded by  $Y^*$ ). The problems created by the possibility of "increasing returns to coalition size" can be eliminated by a "minimum efficient scale" assumption. We say that there is a minimum efficient scale for productive coalitions, MPC, if there is an  $r^*$  such that for all  $r \geq r^*$ ,  $\bar{Y}[A_r] \subset \bigcup_{P \in P^*(A_r)} \bigcup_{S \in P} Y[S]$  where  $P^*(A_r)$  is a partition of  $A_r$  with  $\rho(S) \leq \rho(A_{r^*})$  for all  $S$  in  $P^*(A_r)$ .

Another reason for the necessity of " $\epsilon$ -improvement" for positive  $\epsilon$  is that the sequence of derived games is not necessarily comprehensive in  $\mathbb{R}_+^{A_r}$  and does not necessarily satisfy the "quasi-transferable utility" property, QTU--the property that the boundaries of the payoff sets in  $\mathbb{R}_+^{A_r}$  do not contain line segments parallel to the coordinate planes (this is formally defined in Part I). If the derived games were comprehensive (or at least comprehensive in an appropriate subset of  $\mathbb{R}_+^{A_r}$ ), and satisfied MPC and also QTU, then it seems the theorems could be obtained for  $\epsilon$  equal to zero. One way to ensure the QTU and comprehensiveness properties would be satisfied is to assume that there is an infinitely divisible good with which everyone is initially endowed and which has "over-riding" desirability--i.e. an agent can be compensated for not getting a unit of the indivisible commodity by an increased allocation of the divisible good (see Broome (1972) for further discussion and an exact definition of "over-riding desirability").<sup>8</sup> Since the model is an example of an application of our theorem, these issues are not further investigated herein.

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<sup>8</sup>The over-riding desirability of an infinitely divisible good has become standard, cf. Mas-Collel (1975) and Kahn and Yamazaki (1981).

Before concluding this section, we remark that an obvious extension to the model would be to allow the preferences of agents to depend on the profiles of the set of agents with whom joint production is undertaken. The only difficulty in extending Theorem 4 to this case is in determining conditions under which utility functions are per-capita bounded, i.e. we would need to rule out the possibility that equal-treatment utilities could become arbitrarily large because of the increasing coalition size. Of course, Theorem 4 would then be stated in terms of both an allocation of goods and an admissible coalition structure since utilities would be, in part, determined by the coalition structure. The extension of Theorem 5 would require the specification of a "coalition structure" of a set of agents with "approximately" the same number of agents of each type as actually in the economy.

### 3. Some Simple Examples

We present simple examples in this section to illustrate our model applied to set-up costs, entry, exit and merger.

#### Model 1. Set-up costs with decreasing returns<sup>9</sup>

Consider a set  $A$  of  $n$  individuals each with a utility function of the form

$$(1) \quad U^i = \min[x^i, y^i]$$

and initial endowments  $(2,0)$  for each agent. Any coalition  $S$  containing  $s$  members has available  $s$  production functions of the form:

$$(2) \quad y = \max[0, (a\sqrt{-z} - 1)]$$

where  $z$  is the (negative) input of the first good. The production

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<sup>9</sup> See also Joseph (1938).

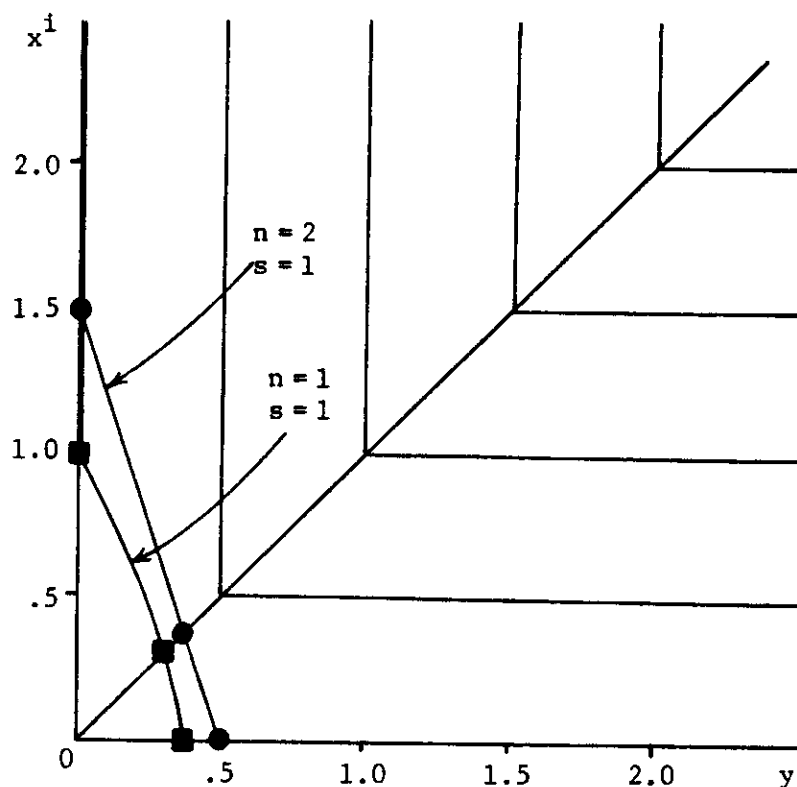


FIGURE 6

function has a set-up cost of 1 unit of input. Furthermore as is shown in Figure 6, the symmetric optimization can be represented on a diagram showing the preferences of a representative individual. The curved lines represent the production possibilities frontiers. The characteristic function is easy to calculate as is noted below.

Given the simplicity of the utility functions, at a symmetric equilibrium we require  $x^i = y^i$ ; thus inputs will be  $n(2-x^i)$  and if  $s$  "plants" or production functions are used

$$(3) \quad nx = as\sqrt{\frac{2n}{s} - \frac{nx}{s}} - s$$

(the superscript  $i$  is dropped, but implicit) or

$$(4) \quad (nx+s) = a\sqrt{sn}(\sqrt{2-x})$$

giving

$$(5) \quad x = \frac{-s(2+a^2) + a\sqrt{4s^2 + s^2a^2 + 8sn}}{2n}$$

The  $n$  individuals acting together must select an optimal number of plants to operate. Utilizing equation (5) we must optimize over  $s$ , for integral values of  $s$ . Per-capita payoffs are illustrated for  $a = 1$

n \ s	1	2	3	4	5	10
1	.3028	.3956	.3975	.3853	.3708	.3110
2	---	.3028	.3744	.3956	.4000	.3708
3	---	---	.3028	.3593	.3845	.3941
4	---	---	---	.3028	.3492	.4000
5	---	---	---	---	.3028	.3956
6	---	---	---	---	---	.3845

When  $a = 1$  the optimum is with 40 percent of the firms (production functions) active. Thus for  $n = 5k$ , the optimal number of firms is  $k$  where  $k = 1, 2, \dots$ . The characteristic function for any game with  $n \leq 5$  can be constructed from this table.

It is straightforward to observe that for  $a > 1$  the optimal coalition size grows smaller. Fewer active plants per-capita are needed.

Returning to  $a = 1$ , games with  $n = 1, 2, 3$  and  $n = 0 \pmod{5}$  have cores. All others have  $\epsilon$ -cores for sufficiently large positive  $\epsilon$  and  $\epsilon$  can be allowed to become arbitrarily small as  $n \rightarrow \infty$ .

### Model 2. Set-up costs, constant returns and capacity

Figure 7(a) shows the production function utilized in Model 1. The presence of decreasing returns to scale plus a set-up cost produces a "U-shaped" average cost-curve and leads automatically to the definition of a minimum efficient size for the firm. If constant returns are assumed,



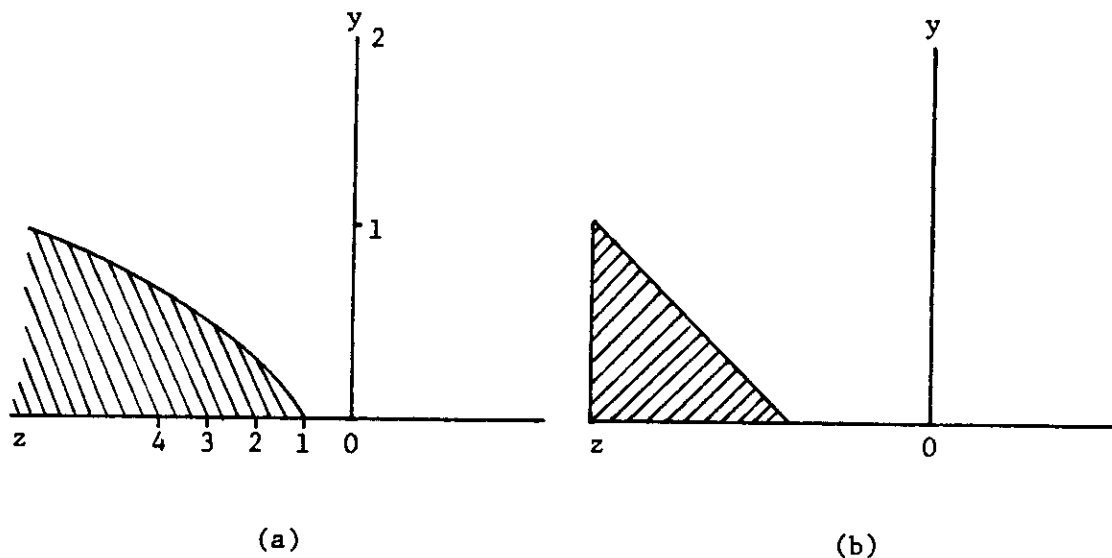


FIGURE 7

apart from the set-up cost, then without an exogenously introduced capacity as is shown in Figure 7(b), the firm would have unlimited (though diminishing) increasing returns to scale as it spreads overheads over larger production. Figure 8 shows this for the production function given by

$$(6) \quad y = \max[0, -az - 1]$$

with no capacity constraint. It is drawn for  $a = 1$  where individual resources are  $(2,0)$ . The three production conditions illustrated  $AC_1P_1$ ,  $AC_2P_2$ , and  $AP_\infty$ , show the utilization of one firm by 1, 2 and then a continuum of agents. The well known problem of pricing with increasing returns is illustrated. If price is set at  $p = 1$  only variable costs are covered, excluding set-up costs. In the continuum case the set-up costs are spread so thin that they are zero per-capita and  $p = 1$  covers full costs as is indicated by  $AP_\infty$ .

For the unrestricted capacity the characteristic function can be

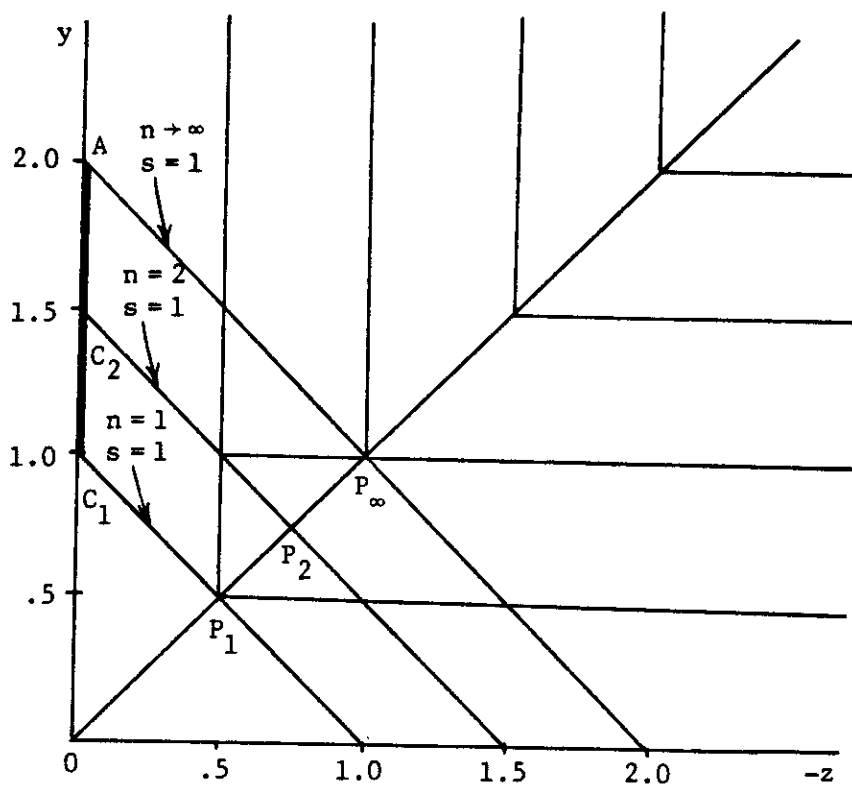


FIGURE 8

written as<sup>9</sup>

$$(7) \quad f(0) = 0$$

$$f(s) = \frac{s}{a+1} \left( 2a - \frac{1}{s} \right), \quad s = 1, \dots, n.$$

This is totally balanced; cores exist for games of any size.

If a capacity constraint is imposed this is not so. We illustrate this with  $a = 1$  and a capacity on each individual firm of  $k = 3/2$

$$(8) \quad f(0) = 0$$

$$f(s) = \frac{3}{4}(s-1) + \frac{1}{2}, \quad s = 1, 3, 5, 7, \dots$$

$$f(s) = \frac{35}{4}, \quad s = 2, 4, 6, 8, \dots$$

<sup>9</sup>As only numbers, not identity, count here we use  $f(s)$  rather than  $v(S)$  or  $V(S)$ .

It can be seen immediately that all efficient coalitions are of size 2. All games for  $n$  even have cores. Furthermore

$$\lim_{s \rightarrow \infty} \frac{f(s)}{s} = \frac{3}{4} \left( \frac{s-1}{s} \right) + \frac{1}{2s} \rightarrow \frac{3}{4} \quad \text{for } s = 1, 3, 5, \dots ;$$

hence if we were to commence with an economy with an even number of agents,  $g$  and replicate (or alternatively "fractionate" the agents) then for any economy with  $gt$  agents for  $t = 1, 2, \dots$  the associated game  $(gt, f)$  will have a core and in the limit the core may be interpreted in terms of a price system. For  $gt$  odd, only an  $\epsilon$ -core exists although in the limit the same price system emerges.

### Model 3. Externalities and industry size

There are at least three distinct economic problems wrapped up in the discussion of entry. They are (1) indivisibilities and set-up costs; (2) entry, exit, merger and organizational costs and timing and (3) longer run externalities to the firm and even the industry which directly influence productivity. The first we have dealt with in Models 1 and 2; the second is considered in the following subsection and the third is considered here.

Marshall (1890) in his partial equilibrium description of the firm and industry painted a picture giving a broad sweep of nonspecific industrial dynamics and organization. This included items such as better education, communications and other aspects of the social infrastructure which might improve owing to the growth of industry and feed back to improve productivity. The other possibilities include overcrowding, growth of crime, clogging of transportation and so forth. An example of a production technology reflecting these economies is exhibited in (9) where

$$(9) \quad y = \max[0, (-ase^{-s}z - 1)]$$

productivity is first favorably then unfavorably influenced by coalition size.<sup>10</sup> Explanations of capacity, innovation, productivity and industrial organization call for factors which are usually not described in general equilibrium models and are treated either verbally or (correctly so) in an ad hoc manner in studies of industrial organization.

Many of the factors to be considered are like local public goods where externalities to the individual are a function of population size and composition. An example of this variety has been presented in Part I of this paper; hence we do not develop this one further.

#### 4. A Note on Entry, Exit and Merger

Most of the writings on general equilibrium have been concerned with statics. Much of the work on oligopoly theory utilizing the Cournot-Nash equilibria has also been static. Yet at the least the analysis of entry, exit or merger calls for comparative statics. A desire to reconcile Cournot-Nash and Walrasian equilibria calls for building closed strategic market games. The problems of nonconvexity or, more generally, non-balancedness, caused by set-up costs, indivisibilities and other factors are related to but different from the problems of entry, exit and merger. With the former nonconvexity is the key factor. With the latter, capital, time, information and flexibility are central.

It is suggested here that the examples we have presented in Models 1 and 2 and models such as that of Novshek and Sonnenschein (1978) provide reasonably adequate models of set-up costs but do not

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<sup>10</sup>This function could easily be modified to depend on profiles for the case where the number of types is greater than one.

catch the fundamental nonsymmetry between a firm in existence and a firm-in-being, or potential entrant.

The conventions of accounting make a distinction between the capital account and current account that is not made in most equilibrium models, under the implicit or explicit claim that it is more general and less arbitrary to ignore this distinction. Yet, in general, entry and exit costs are capital account items and set-up costs and overheads are current account items. This distinction in its most elementary form can be made in the characteristic function; and we could make it in a further model.

The accounting distinction between current and capital accounts, which undoubtedly has a great deal of arbitrariness to it, may nevertheless be looked at as an important way of modeling an economy that is closed in space and agents, but open in time. In particular, somehow or other, capital stock needs to be represented in at least some aggregated form.

An adequate study of entry, exit and merger, calls for a specification of the game in extensive form, or at the least in strategic form. A strategic description of the basic aspects of threats, timing, signalling and even explicit negotiation involved in merger activities or in studies of entry and exit is beyond the scope of the type of modeling done at the level of generality of general equilibrium. Dealing with the extensive form we can more or less manage an entry move or set-up expenditure, purchase of materials, production and sales. But it is hard to remain general and fully specify a game of any complexity in strategic form.

In this paper we have adopted a finesse of institutional detail and even the description of moves by going directly to a coalitional form

without specification of individual strategies. Our defense of this approach is that by merely using some reasonably plausible assumptions on a large replica economy we can obtain limit results of considerable generality and interest without committing ourselves to strategic detail.

The merger, exit and entry of the firms are all implicit in the coalition production conditions at a level sufficient to illustrate properties of the  $\epsilon$ -core and to construct examples where the utilization of production sets by various coalitions can be interpreted as the endogenous formation of firms. Yet we wish to warn against the dangers of making easy analysis where the bridging of the gap between verbal description and the mathematics requires large flights of imagination. We regard both our work and the work utilizing games in strategic form as having taken only a rudimentary step in reconciling game theoretic models of closed economies with the Walrasian system.

The stress in this paper has been upon nonbalancedness and nonconvexities caused by indivisibilities, externalities and group effects. This extra economic information provides a basis for being able to calculate which of the players will be active in production. As such our theory includes firm distribution and size as endogenous variables. This is a first step towards capturing the dynamics of entry, exit and merger.

Finally, as remarked in the introduction, recently there has been renewed interest in showing that Nash-Cournot equilibria converge to Walrasian equilibria as an economy becomes large. In particular, in a model with endogenous firm formation Novshek and Sonnenshein (1978) have shown that when firms have minimum efficient scale bounded away from zero<sup>11</sup> then a Nash-Cournot equilibrium will exist for all sufficiently

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<sup>11</sup>And, of course, some other conditions are satisfied.

large economies and these equilibria converge to competitive equilibria of the limit economy. The results obtained herein are only for approximate cores of the class of economies considered; no limit economy nor equilibria of the limit economy have been defined. If we showed the allocations in the approximate cores converge to competitive equilibria, then our results would be analogous to those of Novshek and Sonnenschein but for the cooperative solution of the core rather than the non-cooperative Nash-Cournot equilibrium. While demonstrating such convergence is clearly beyond the scope of this paper, there is reason to believe the convergence result can be obtained.

## 5. Conclusions

In this paper, Parts I and II, we have developed a theorem stating quite general conditions under which large replica games have non-empty approximate cores and demonstrated that the theorem can be applied to games derived from a diverse collection of economic models. In particular, the theorem can be applied to economic situations where, previously, almost all<sup>12</sup> results showing non-emptiness of cores have relied on "balancedness" assumptions as in the coalition production economy literature or on a number-theoretic assumption (which is actually a "balancedness"-type assumption).<sup>13</sup> Also, by making appropriate, additional restrictions on the production correspondence, it is obvious that the results in

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<sup>12</sup>An exception is Wooders (1980); however, the result in that paper depends on the fact that preferences and/or production possibilities depend only on the number of agents in a jurisdiction instead of, more generally, profiles of jurisdiction.

<sup>13</sup>In Wooders (1978, 1980a), the number-theoretic assumption ensures that agents can be appropriately partitioned into jurisdictions for the non-emptiness of the core to obtain.

Part II apply to standard Arrow-Debreu economic models of private goods economies with production. Other examples, of non-emptiness of weak asymptotic cores of partitioning games (generalized assignment games), are provided in Kaneko and Wooders (1982) and for these games strong forms of the non-emptiness of weak asymptotic cores are demonstrated.

The generality of our results, however, has been achieved at some cost. In particular, we have adopted a replication approach whereas techniques currently in use to show non-emptiness of approximate cores of private goods economies do not rely on replication, cf. Khan and Yamazaki (1981). However, as the example in the appendix to Part I illustrates, economies where partitioning of agents into groups for the purposes of joint production and/or consumption within the groups create different difficulties than ones with private goods only. Moreover, as our other applications illustrate, our game-theoretic results can be applied to a broad class of economic models.



## APPENDIX

In this appendix, we show that the set of "allowable coalition structures" can be generalized.

Given any non-empty subset of agents  $S$  we take as given a priori a non-empty family  $C(S)$  where each member of  $C(S)$  is a collection of subsets of  $S$ . It is assumed that

(1) given any  $C \in C(S)$ , we have  $S \subset \bigcup_{S' \in C} S'$  (allowable coalition structures of  $S$  cover  $S$ );

(2) if  $C \in C(S)$  and  $C' \in C(S')$  where  $S$  and  $S'$  are non-empty, disjoint subsets of agents, then  $\{S'' : S'' \in C \cup C'\} \in C(S \cup S')$ ;

and (3) if  $S$  and  $S'$  are non-empty subsets of agents where  $\rho(S) = \rho(S')$ , then there is a one to one mapping, say  $\psi$ , of  $C(S)$  onto  $C(S')$  such that if  $\psi(C) = C'$ , then the collection of profiles of members of  $C$  equals those of  $C'$ .

For the purposes of this appendix, given any non-empty subset of agents  $S$ , we define  $\bar{Y}[S] = \bigcup_{C \in C(S)} \sum_{S' \in C} Y[S']$ . We need to show that  $\bar{Y}$  satisfies two properties:  $\bar{Y}$  is superadditive and, if  $S$  and  $S'$  have the same profiles, then  $\bar{Y}[S] = \bar{Y}[S']$ . It is then obvious that the theorems of Section 2 of this paper can be applied to the model where the aggregate production possibility set is  $\bar{Y}$  as defined in this appendix.

To show  $\bar{Y}$  is superadditive, let  $S$  and  $S^*$  be two disjoint non-empty subsets of agents. Let  $z$  and  $z^*$  be in  $\bar{Y}[S]$  and  $\bar{Y}[S^*]$  respectively. Let  $C \in C(S)$  such that  $z \in \sum_{S' \in C} Y[S']$  and let  $C^* \in C(S^*)$  be

such that  $z^* \in \sum_{S' \in C^*} Y[S']$ . From assumption (2) on allowable coalition structures we have

$z + z^* \in \sum_{S' \in C} Y[S'] + \sum_{S' \in C^*} Y[S'] = \sum_{S' \in C \cup C^*} Y[S'] \in \bar{Y}[S \cup S^*]$ . Therefore  $\bar{Y}$  is superadditive correspondence.

Let  $S$  and  $S^*$  be non-empty subsets of agents where  $\rho(S) = \rho(S^*)$ . Let  $C \in C(S)$  and let  $\{s_1, \dots, s_k, \dots, s_k\}$  denote the collection of profiles (not necessarily distinct) of members of  $C$ . From assumption (3) there is a  $C^* \in C(S^*)$  such that the collection of profiles of members of  $C^*$  equals that of  $C$ . Since, for any  $S'$ ,  $Y[S']$  depends only on the profile of  $S'$ , we have  $\sum_{S' \in C} Y[S'] = \sum_{k=1}^K Y[s_k] = \sum_{S' \in C^*} Y[S']$ . It follows that  $\bar{Y}[S] = \bar{Y}[S^*]$ .

## REFERENCES

- V. Bořhm, "The Limit of the Core of an Economy with Production," International Economic Review 15 (1974), 143-148. (a)
- \_\_\_\_\_, "The Core of an Economy with Production," Review of Economic Studies 41 (1974), 429-436. (b)
- \_\_\_\_\_, "Firms and Market Equilibria in a Private Ownership Economy," Zeitschrift fur Nationalokonomie 33 (1973), 87-102.
- J. Broome, "Approximate Equilibrium in Economies with Indivisible Commodities," Journal of Economic Theory 5 (1972), 224-249.
- P. Champsaur, "Note Sur Le Noyau d'une Economie Avec Production," Econometrica 42 (1974), 933-945.
- G. Debreu and H. Scarf, "A Limit Theorem on the Core of an Economy," International Economic Review 4 (1963), 235-246.
- E. Dierker, "Equilibrium Analysis of Exchange Economies with Indivisible Commodities," Econometrica 39 (1971), 997-1008.
- P. Dubey and M. Shubik, ""The Noncooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies," Journal of Economic Theory 17, 1 (Feb. 1978), 1-20. (a)
- \_\_\_\_\_ and \_\_\_\_\_, "A Closed Economic System with Production and Exchanged Modelled as a Game of Strategy," Journal of Mathematical Economics 4, 1 (March 1978), 253-287. (b)
- P. Dubey, A. Mas-Colell and M. Shubik, ""Efficiency Properties of Strategic Market Games: An Axiomatic Approach," Journal of Economic Theory 22, 2 (1980), 339-362.
- F. Y. Edgeworth, Mathematical Psychics. London: Paul Kegan, 1881.
- J. J. Gabszewicz and J. P. Vial, "Oligopoly 'à la Cournot' in General Equilibrium Theory," Journal of Economic Theory 4 (1972), 381-400.
- C. Henry, "Market Games with Indivisible Commodities and Nonconvex Preferences," Review of Economic Studies 39 (1972), 73-76.
- W. Hildenbrand, "The Core of an Economy with a Measure Space of Agents," Review of Economic Studies 35 (1968), 443-452.
- \_\_\_\_\_, "Existence of Equilibria for Economies with Production and a Measure Space of Consumers," Econometrica 38 (1970), 608-623.
- T. Ichiishi, "Coalition Structure in a Labour-managed Market Economy," Econometrica 45 (1977), 341-360.
- J. Joseph, "A Discontinuous Cost Curve and the Tendency to Increasing Returns". Economic Journal. XLIII, pp. 390-398. (1938).

- M. Kaneko and M. Wooders, "Cores of Partitioning Games", Cowles Foundation Discussion Paper No. 620, (forthcoming, International Journal of Mathematical Social Sciences, 1982).
- M. A. Khan and A. Yamazaki, "On the Cores of Economies with Indivisible Commodities and a Continuum of Players," Journal of Economic Theory 22 (1981), 218-225.
- A. Mas-Colell, "Indivisible Commodities and General Equilibrium Theory," Journal of Economic Theory 16 (1977), 443-456.
- A. Marshall, Principles of Economics. London: Macmillan Publishing Co., 1890.
- W. Novshek and H. Sonnenschein, "Cournot and Walras Equilibria," Journal of Economic Theory 19 (1978), 223-266.
- L. S. Shapley and M. Shubik, "Quasi-Cores in a Monetary Economy with Nonconvex Preferences," Econometrica 34 (1966), 805-827.
- \_\_\_\_\_ and \_\_\_\_\_, "Concepts and Theories of Pure Competition," in M. Shubik (ed.), Essays in Mathematical Economics in Honor of Oskar Morgenstern, Princeton: Princeton University Press, (1967), 63-79.
- \_\_\_\_\_ and \_\_\_\_\_, "On Market Games," Journal of Economic Theory, 1 (1969), 9-25.
- \_\_\_\_\_ and \_\_\_\_\_, "Trade Using One Commodity as a Means of Payment," The Journal of Political Economy, 85, 5, October (1977), 937-968.
- M. Shubik, "Edgeworth Market Games," in R. D. Luce and A. W. Tucker, eds., Contributions to the Theory of Games, IV, Annals of Mathematical Studies 40, Princeton, Princeton University Press, 1959, 267-278.
- \_\_\_\_\_, "Commodity Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model," Western Economic Journal, 11, 1 (1973), 24-38.
- \_\_\_\_\_, Game Theory in the Social Sciences, Vol. 1. Cambridge, Mass.: MIT Press, 1982.
- D. Sondermann, "Economies of Scale and Equilibria in Coalition Production Economies," Journal of Economic Theory 8 (1974), 259-271.
- R. M. Starr, "Quasi-Equilibria in Markets with Non-convex Preferences," Econometrica 37 (1969), 25-38.

M. Wooders, "Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good," Journal of Economic Theory 18 (1978), 328-348.

\_\_\_\_\_, "The Tiebout Hypothesis; Near Optimality in Local Public Good Economies," Econometrica 48 (1980), 1467-1485.

\_\_\_\_\_, "Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good; A Correction," Journal of Economic Theory 25 (1981), 144-151.