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PAYOFFS IN NON-ATOMIC ECONOMIES: AN AXIOMATIC APPROACH

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PAYOFFS IN NON-ATOMIC ECONOMIES: AN AXIOMATIC APPROACH*

by

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1. Introduction

It has been a much-remarked fact that different solution concepts become equivalent in the setting of "perfectly competitive" economies (i.e., to use the modern idiom [2], economies in which the agents form a non-atomic continuum). The conjecture that the core and competitive (Walras) allocations coincide was broached as far back as 1881 by Edgeworth [10]. His insight has been vindicated in increasing generality in a spate of articles[1]([19], [9], [2], [13], [12], [7], [8], [1]) over the last two decades. More recently, it was shown by Aumann that--with a smoothness assumption on the preferences--the "value allocations"[2] also coincide with the above two.

If we restrict ourselves to the case of smooth, transferable utilities then the equivalence phenomenon turns out to be even more striking: not only do these solutions coincide but they are also unique,

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[1]This list is only meant to be indicative, and by no means exhaustive.

[2]Which are based on the Shapley value ([16], [17]), a game-theoretic concept quite different from the core.

i.e., consist of a single payoff. Our aim here is to give another view of this "coincident payoff" by putting it on an axiomatic foundation. As an upshot of our approach we get a "meta-equivalence" theorem, by way of a categorization : any solution coincides with this payoff if, and only if, it satisfies our axioms.

The transferable-utility assumption is undoubtedly restrictive. But we are encouraged by its good track record of being the precursor of the general analysis [e.g. [6] before [3]; [16] before [17]). And we hope to extend our approach to the non-transferable case, though we have not yet made significant progress on it.

Denote by M the class of non-atomic economies with transferable and differentiable utilities. Any such economy can also be viewed as a productive economy with a single consumable output. (See the discussion in Chapter 6 of [6].) We will, in fact, adopt the production interpretation for most of our discussion. But by thinking of the output as "utiles" everything we say can be translated into the exchange version as well.

The problem of determining payoffs (final distributions of the output) in these economies has been approached from many sides. Let us briefly recount some of them. First there is the classical notion of a competitive payoff which depends on prices that clear all markets, i.e., equate supply and demand. Equally well-known is the concept of the core. It is defined by the condition that no coalition of agents in the economy can, on its own, improve upon what it gets. (See Chapter 6 of [6] for a historical survey and detailed discussion of these.) Other solutions from Game Theory have also been applied to the economic model. The bargaining set [4], which contains the core, is based upon a weaker notion of stability: each "objection" can be ruled out by a "counterobjection." (In this terminology there can be, a fortiori, no objection to any payoff in the core.) Then there is the concept of the nucleolus [15]. It is derived from a criterion of equity (maximize the welfare of the worst-off coalition) and yields a point in the core. Finally, we have the Shapley value [16], which has been the focus of active, recent research (and was the starting point of this inquiry as well). It is an operator that assigns to each player of a game a number that purports to represent what he would be willing to pay in order to participate; and is uniquely determined by certain plausible conditions for all finite games [16], and a large class of non-atomic games (pNA) which include the economies in M [6]. The value thus obtained can be also interpreted from a complementary standpoint: it assigns to a player the average of his marginal contributions to coalitions he may join (in a model of random ordering of the players).

For any economy in M, all these solutions coincide and consist of a single payoff (Ch. 6 of [6], and [11]). We would like to explicate certain underlying principles which lead to this distinguished payoff. To set the stage for this, we take a map from economies to sets of payoffs, and look for a minimal list of plausible axioms that will uniquely characterize the map. Four axioms are presented which accomplish the job. The map they lead to is that of the distinguished payoff above.

This may be viewed as a meta-equivalence theorem. For instance the equivalence of core and competitive payoffs follows from our result by simply checking that the map which takes each m in M to its core (competitive) payoffs satisfies our axioms. That the value also coincides with them is immediate because it, in fact, satisfies even stronger axioms ([6], Ch. 1). In general: if any solution is a candidate for equivalence, it is both necessary and sufficient that is satisfies our axioms.

The axioms will be spelled out precisely in Section 3, but let us present them at an intuitive level now. Denote the space of agents by $[T, \mathcal{C}, \mu]$. Here T is the set of agents, $\mathcal{C}$ the sigma-algebra of coalitions, and μ a non-atomic population measure on $[T, \mathcal{C}]$. An economy is a pair of measurable functions $(\underset{\sim}{a}, \underset{\sim}{u})$ where $\underset{\sim}{a} : T \to R^n_+$ specifies the initial endowment of the n resource commodities, and $\underset{\sim}{u} : T \times R^n_+ \to R_+$ the production--alternatively, utility--functions. M is the set of all pairs $(\underset{\sim}{a}, \underset{\sim}{u})$, subject to certain conditions on $\underset{\sim}{a}$ and $\underset{\sim}{u}$ (see Section 2). For any m in M we can define an associated <u>characteristic</u> <u>function</u> (or <u>game</u>) $v_m : \mathcal{C} \to R_+$, which assigns to each coalition the maximum output that it could achieve by a reallocation of the resources of its own members, i.e.,

$$v_m(S) = \max\{\int_S \underset{\sim}{u}(t, \underset{\sim}{x}(t))d\mu(t) : \int_S \underset{\sim}{x}(t)d\mu = \int_S \underset{\sim}{a}(t)d\mu, \ \underset{\sim}{x} : T \to R^+_n\} .$$

Payoffs in $m \in M$ can be thought of as integrable functions from T to R_+ , and in turn can be identified with nonnegative countably additive measures on $(T, \mathcal{C})$ which are absolutely continuous with respect to μ . But let us make only the assumption that they lie in FA , the collection of functions from $\mathcal{C}$ to R which are finitely additive and bounded. Let $P(FA)$ be the set of all subsets of FA . Then any assignment of payoffs to economies may be represented by a map:

$$\phi : M \to P(FA) .$$

We will impose four axioms on ϕ : "continuity," "inessential economy," "anonymity," "separability." Our main result is that there is one, and only one, map which satisfies these axioms: it maps m into the (unique) competitive payoff of m .

The continuity axioms says that if the distance between two economies is small, then so is that between their sets of payoffs. It is, of course, intimately bound up with the notion of distance. The one we employ declares the distance between two economies to be zero if they yield the same characteristic function. Thus the payoffs depend on the characteristic function alone, i.e., they depend on the data $(\underset{\sim}{a},\underset{\sim}{\mu})$ of the economy only insofar as it shows up in the net production of the coalitions (if $v_m = v_{m'}$, $\phi(m) = \phi(m')$). Modulo this, however, our continuity requirement is weak. We choose a "large" norm on the characteristic functions (the bounded variation norm) and a "small" one on P(FA) (the Hausdorff distance in the bounded variation norm, which is equivalent in FA to the maximum norm).

The inessential economy axiom has to do with economies in which agents have no motivation to collude in order to increase the output. Indeed suppose that m in M is such that each coalition $S \in C$ achieves its maximum $v_m(S)$ uniquely by sticking to its initial allocation of resources. Then one would expect that no exchange, either of the inputs or the output, will occur. And this is just what the axiom says.

The anonymity axiom asserts that the labels of the agents do not matter. If we were to relabel them then this would only have the effect of relabelling their payoffs accordingly.

For the separability axiom, we must describe an economy made up of two separate, non-interacting pieces. Take m' and m" in M . Let us construct the economy m by, as it were, "collating" m' and m" . Each agent in m possesses the same initial resources that he had in m' and in m" ; also he has access to both his production functions from m' and m" . However, suppose that the input commodities of m' and m" are completely disjoint: those in m' cannot be used for production in m" , and vice versa (though the two economies produce, of course, the same output). Now consider any coalition S that forms in m . Each agent in S can send his black (white) hatted representative to m' (m") . If these two types of representatives separately maximize the output in m' and m" , then the sum of what they get back is precisely what S can obtain in m , i.e., $v_m(S) = v_{m'}(S) + v_{m''}(S)$ for all $S \in C$. Thus m , in essence, consists of operating in m' and in m" independently of each other. We require that in this case if we put together a payoff in m' with one in m" , the outcome should be feasible in m. However, we do not exclude the possibility that other payoffs may also be obtained in m . In symbols: $\phi(m') + \phi(m'') \subset \phi(m)$.

Our axiomatic approach is akin to that of [6] and invites immediate comparison. We begin with a point-to-set map (from M to FA). That $\phi(m)$ is a non-empty one-element set of FA is a deduction, not a postulate, in our case. Also note that we do not require that $\phi(m)$ consist of efficient payoffs--this, too, is deduced. Clearly separability is weaker than the linearity required in [6]. (For instance, the core satisfies separability for finite economies, but not linearity.) Continuity is closely related to the positivity axioms of [6] (Section 4 of [6] and Remark 3b). Finally, we emphasize that the axioms are invoked on the set of games that arise from M alone. This set is much smaller than the general space pNA of [6]. (Its complement in pNA is open and dense.) Thus the uniqueness of ϕ does become an issue. (Existence on the other hand is no problem: simply restrict the value on pNA to our domain.) The very question we set out with--what are payoffs in non-atomic economies? --makes it desirable that we exclude any reference to games that do not arise from M . Thus we stay within M throughout and give a self-contained analysis of it. Each axiom is cast in an economic framework and can be interpreted therein. It is fortunate that even though the scope of the axioms is diminished by this restriction of the domain, they nevertheless are sufficiently far-reaching to determine a unique map.

2. Non-Atomic Economies with Transferable, Differentiable Utilities

Let us recall more precisely the economic model presented in Chapter 6 of [6].

We begin with a measure space $[T,C,\mu]$. T is the set of agents, C the σ-algebra of coalitions, and μ the population measure. $[T,C]$ is assumed to be isomorphic to the closed unit interval $[0,1]$ with its Borel sets. μ is a finite, σ-additive, non-negative and non-atomic measure, and we assume (w.l.o.g.) that $\mu(T) = 1$.

Each agent $t \in T$ is characterized by an initial endowment of resources, $\underset{\sim}{a}^t \in R^n_+$, and a production (utility) function $\underset{\sim}{u}^t : R^n_+ \to R$. Here R^n_+ is the non-negative orthant of the Euclidean space R^n , and n is the number of (resource) commodities. Denoting the j^{th} component of $x \in R^n$ by x_j , $\underset{\sim}{a}^t_j$ is the quantity of the j^{th} commodity held by agent t , and $\underset{\sim}{u}^t(x)$ the amount of output he can produce using x . Thus the economy consists of the pair of functions $(\underset{\sim}{a},\underset{\sim}{u})$, where $\underset{\sim}{a} : T \to R^n_+$, $\underset{\sim}{u} : T \times R^n_+ \to R$ (note the identifications $\underset{\sim}{a}(t) \equiv \underset{\sim}{a}^t$; $\underset{\sim}{u}(t,x) \equiv \underset{\sim}{u}^t(x)$).

To spell out the conditions on $(\underset{\sim}{a},\underset{\sim}{u})$, we need some additional notation. For x , y in R^n_+ , say $x = y$ $(x \geq y$, $x >> y)$ when $x_j = y_j$ $(x_j \geq y_j$, $x_j > y_j)$ for $1 \leq j \leq n$; $x > y$ when $x \geq y$, but not $x = y$. Put $\|x\| = \max\{|x_j| : 1 \leq j \leq n\}$. Also note that R^n_+ can be regarded as a measurable space with its Borel sets. We will require that $(\underset{\sim}{a},\underset{\sim}{u})$ satisfy:

(2.1) $\underset{\sim}{a} : T \to R^n_+$ is integrable.

(2.2) $\underset{\sim}{u} : T \times R^n_+ \to R$ is measurable where $T \times R^n_+$ is equipped with the product σ-field of T and R^n_+ .

(2.3) $\underset{\sim}{u}(x) = o(\|x\|)$, as $\|x\| \to \infty$, integrably in t ; i.e., for every $\varepsilon > 0$ there is an integrable function $\eta : T \to R$ such that $|\underset{\sim}{u}^t(x)| \leq \varepsilon\|x\|$ whenever $\|x\| \geq \eta(t)$.

For almost all* $t \in T$:

(2.4) $\underset{\sim}{a}^t >> 0$ (where, without confusion, 0 also stands for the origin of R^n_+)

(2.5) $\underset{\sim}{u}^t$ is continuous and increasing (i.e., $x > y$ implies $\underset{\sim}{u}^t(x) > \underset{\sim}{u}^t(y)$)

(2.6) $\underset{\sim}{u}^t(0) = 0$

(2.7) The partial derivative $\dfrac{\partial \underset{\sim}{u}^t}{\partial x_j}$ exists and is continuous at each point where $x_j > 0$.

The collection of all pairs $(\underset{\sim}{a},\underset{\sim}{u})$ which satisfy (2.1)-(2.7) will be called M , i.e., we keep the space $[T,C,\mu]$ of agents fixed but vary their characteristics $(\underset{\sim}{a},\underset{\sim}{u})$; in particular, the number n of resource commodities can be any integer 1, 2, 3, ... As was said already in

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*i.e., for all except perhaps a μ-null set of agents.

the introduction, to each $m = (a,\mu) \in M$, we associate a game, or characteristic function, $v_m : C \to R$ by:

$$(2.8) \qquad v_m(S) = \max\{\int_S \mu^t(x^t)d\mu(t) : x : T \to R^n_+,\ x(S) = a(S)\} .$$

(For an integrable function $y : T \to R^n_+$, $y(S)$ abbreviates $\int_S y d\mu$.) That this max is attained is essentially the main theorem in [5].

FA is the collection of set functions from C to R that are finitely additive and bounded, and $P(FA)$ is the set of all subsets of FA. We are going to characterize a map $\phi : M \to P(FA)$ via axioms. It will turn out that, for any $m \in M$, $\phi(m)$ is the set of competitive payoffs in m. To remind the reader: a pair (p,x) [where $x : T \to R^n_+$ is an integrable function with $x(T) = a(T)$ and p a price vector in R^n_+] is called a transferable utility competitive equilibrium (t.u.c.e.) of the economy (a,μ) if, for almost all $t \in T$,

$$\mu^t(y) - p\cdot(y - a^t) \leq \mu^t(x^t) - p\cdot(x^t - a^t)$$

for any y in R^n_+; the corresponding competitive payoff is the measure $\nu_{p,x}$ defined by

$$\nu_{p,x}(S) = \int_S [\mu^t(x^t) - p\cdot(x^t - a^t)]d\mu$$

for $S \in C$. If we denote by $\psi(m)$ the set of competitive payoffs in m, then under the assumptions (2.1)-(2.7), $\psi(m)$ is a singleton for any $m \in M$. (See Proposition 32.3 in [6].)

3. Statement of the Theorem

In this section we prepare for and state the four axioms, as well as our main result.

A set-function v is a map from C to R such that $v(\emptyset) = 0$. It is called monotonic if $T \subset S$ implies $v(S) \geq v(T)$. The difference between two monotonic set functions is said to be of bounded variation. Let BV be the real vector space of all set-functions of bounded variation. For $v \in BV$, define the norm $\|v\|$ of v by:

$$\|v\| = \inf\{u(T) + w(T)\}$$

where the infinum ranges over all monotonic functions u and w such that $v = u - w$.

Each characteristic function v_m of a market m in M is monotonic and thus is in BV. So we can introduce the distance d on M by $d(m, m') = \|v_m - v_{m'}\|$. Also observe that $FA \subset BV$. For A and B in $P(FA)$, let $h(A,B)$ be the Hausdorff distance between A and B, i.e., $h(A,B) = \inf\{\varepsilon \in R_+ : A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\}$, where A^ε is the set $\{\alpha' \in FA : \|\alpha - \alpha'\| < \varepsilon$ for some $\alpha \in A\}$ etc; and $\inf \emptyset = \infty$. We are ready for

<u>AXIOM I (CONTINUITY)</u>. There is a constant K such that $h(\phi(m),\phi(m')) \leq Kd(m',m)$.

The next axiom requires no preparation.

<u>AXIOM II (INESSENTIAL ECONOMY)</u>. Suppose $m = (\underset{\sim}{a},\underset{\sim}{u})$ in M is such that, for each non-null set $S \in C$, $v_m(S)$ is achieved uniquely by $\underset{\sim}{a} : S \to R^n_+$ (i.e., $\underset{\sim}{a} : S \to R^n_+$ is the unique solution to the maximization problem (2.8)). Then $\phi(m)$ consists of just the payoff γ given by : $\gamma(S) = \int_S \underset{\sim}{u}^t(\underset{\sim}{a}^t)d\mu(t)$, for $S \in C$.

Let $\mathcal{Q}_\mu$ be the set of all automorphisms of $[T,\mathcal{C}]$ which preserve the measure μ, i.e., $\mathcal{Q}_\mu$ consists of bi-measurable bijections $\theta : T \to T$ such that $\mu(\theta(S)) = \mu(S)$ for all $S \in \mathcal{C}$. For $m = (\underset{\sim}{a},\underset{\sim}{u}) \in M$ and $\theta \in \mathcal{Q}_\mu$, define $\theta m = (\theta\underset{\sim}{a}, \theta\underset{\sim}{u})$ by $(\theta\underset{\sim}{a})(t) = \underset{\sim}{a}(\theta(t))$, $(\theta\underset{\sim}{u})(t,x) = \underset{\sim}{u}(\theta(t), x)$. Also, for $v \in BV$ and $\theta \in \mathcal{Q}_\mu$ define, $\theta v : \mathcal{C} \to R$ by $(\theta v)(S) = v(\theta(S))$; and for $A \subset BV$, define $\theta A = \{\theta v : v \in A\}$.

<u>AXIOM III (ANONYMITY)</u>. For any m in M, and θ in $\mathcal{Q}_\mu$, $\phi(\theta m) = \theta\phi(m)$.

Since $A \subset FA$ implies $\theta A \subset FA$, and $m \in M$ implies $\theta m \in M$, the axiom makes sense.

For our fourth and final axiom, we need to define the disjoint sum of two economies. Take $m = (\underset{\sim}{a},\underset{\sim}{u})$, $m' = (\underset{\sim}{a}', \underset{\sim}{u}')$ where $\underset{\sim}{a} : T \to R_+^\ell$ and $\underset{\sim}{a}' : T \to R_+^k$. Put $m \oplus m' = (\underset{\sim}{a} \oplus \underset{\sim}{a}', \underset{\sim}{u} \oplus \underset{\sim}{u}')$, where $(\underset{\sim}{a} \oplus \underset{\sim}{a}') : T \to R_+^{\ell+k}$ and $(\underset{\sim}{u} \oplus \underset{\sim}{u}') : T \times R_+^{\ell+k} \to R$ are given by:

$$(\underset{\sim}{a} \oplus \underset{\sim}{a}')(t) = (\underset{\sim}{a}(t), \underset{\sim}{a}'(t))$$

$$(\underset{\sim}{u} \oplus \underset{\sim}{u}')(t, (x,y)) = \underset{\sim}{u}(t,x) + \underset{\sim}{u}'(t,y) .$$

[For $x \in R_+^\ell$ and $y \in R_+^k$, (x,y) is the vector in $R_+^{\ell+k}$ whose first ℓ components are according to x, and the last k according to y.] Note that $m \oplus m' \in M$ if $m \in M$ and $m' \in M$. Also note that $A + B \in P(FA)$ if $A \in P(FA)$ and $B \in P(FA)$, where we define $A + B = \{\alpha+\beta : \alpha \in A, \beta \in B\}$.

<u>AXIOM IV (SEPARABILITY)</u>. For any m and m' in M, $\phi(m) + \phi(m') \subset \phi(m \oplus m')$.

Our main result is given by the

Theorem. There is one, and only one, map $\phi : M \to P(FA)$ that satisfies axioms I, II, III, IV. It assigns to each m in M the set consisting of the competitive payoff of m .

4. A Reformulation

Put $E = \{v \in BV : v = v_m \text{ for some } m \in M\}$, i.e., E is the set* of characteristic functions of the economies in M . Note that

(a) $w \in E , \theta \in Q_\mu \Longrightarrow \theta w \in E$

(b) $w \in E , w' \in E$, α and α' positive numbers $\Longrightarrow \alpha w + \alpha' w' \in E$.

To check (a), let $w = v_m$ for some $m \in M$. Then $\theta w = v_{\theta m}$. To check (b), let $w = v_m$ and $w' = v_{m'}$ for some $m = (\underset{\sim}{a}, \underset{\sim}{\mu})$ and $m' = (\underset{\sim}{a}', \underset{\sim}{\mu}')$ in m . Consider $\tilde{m} = (\underset{\sim}{a} \oplus \underset{\sim}{a}', \alpha\underset{\sim}{\mu} \oplus \alpha'\underset{\sim}{\mu}')$. Then $v_{\tilde{m}} = \alpha w + \alpha' w'$.

Proposition 1. Given any $\phi : M \to P(FA)$ satisfying axioms I, II, III, IV there is a $\phi^* : E \to P(FA)$ such that

(i) $\phi(m) = \phi^*(v_m)$ for any $m \in M$,

(ii) there is a K such that

$$h(\phi(w), \phi(w')) \leq K\|w - w'\| ,$$

(iii) $w \in E , \theta \in Q_\mu \Longrightarrow \phi^*(\theta w) = \theta\phi^*(w)$,

(iv) $w \in E , w' \in E \Longrightarrow \phi^*(w) + \phi^*(w') \subset \phi^*(w+w')$,

(v) $w \in E \cap FA \Longrightarrow \phi^*(w) = w$.

Proof. Define $\phi^* : E \to P(FA)$ by: $\phi^*(w) = \phi(m)$ for some $m \in M$ such that $v_m = w$. Clearly such an m exists, and since ϕ satisfies axiom I, ϕ^* is well defined independent of the choice of m . Now (ii), (iii), and (iv) follow easily from the fact that ϕ satisfies axioms I, III and IV. It remains to check (v). Let $w \in E \cap FA$. Then w is countably

*Adding the set-function $v = 0$ would make E a cone.

additive and absolutely continuous w.r.t. μ, hence there is an integrable $\eta : T \to R_+$ such that $w(S) = \int_S \eta(t)d\mu$. Consider $m = (\underset{\sim}{a}, \underset{\sim}{u})$ where $\underset{\sim}{a}^t = \eta(t)$ [i.e. there is only one resource commodity] and $\underset{\sim}{u}^t : R_+ \to R_+$ is a strictly concave, strictly increasing function chosen so that $(d\underset{\sim}{u}^t/dx)(\eta(t)) = p > 0$ for some p, and $\underset{\sim}{u}^t(\eta(t)) = \eta(t)$. By Proposition 36.4 of [6] $v_m(S)$ is achieved uniquely by $\eta : S \to R_+$ for each S in C; so $v_m(S) = \int_S \underset{\sim}{u}^t(\eta(t))d\mu = \int_S \eta(t)d\mu = w(S)$. But, by axiom II, $(\phi(m))(S) = \int_S \eta(t)d\mu = w(S)$. Thus $\phi^*(w) = \phi(m) = w$, proving (v). Q.E.D.

5. Proof of the Theorem

The proof is broken into two parts. First, we show that there is a subset M^* of M, containing economies of a special kind, which is dense in M (section B). Then we prove that for $m \in M^*$, $\phi(m)$ is a non-empty, one-element set which is uniquely determined by axioms I, II, III, IV (Sections C, D). Uniqueness of ϕ on all of M then follows from the continuity axiom I.

A. Preliminaries

We will compile in this section a miscellany of definitions and facts.

Let F denote the set of all real-valued functions f on* R_+^n that are continuous, non-decreasing (i.e., $x \geq y$ implies $f(x) \geq f(y)$) and satisfy

(i) $f(x) = o(\|x\|)$ as $\|x\| \to \infty$.

*Thus F depends on the dimension n, and we should be writing F^n; but the space R_+^n used for defining F will always be clear from the context.

For any f in the vector space generated by $\mathcal{F}$ (i.e., $f \in \mathcal{F}-\mathcal{F}$), put* $\|f\| = \sup\{|f(x)|/(1+\Sigma x) : x \in R^n_+\}$. Then $\| \ \|$ constitutes a norm.

$$\hat{\mathcal{F}} = \left\{ f \in \mathcal{F} : \frac{\partial f}{\partial x_j} \text{ exists and is continuous at each } x \in R^n_+ \text{ for which } x_j > 0 \right\}$$

$$\mathcal{F}^1 = \left\{ f \in \mathcal{F} : \frac{\partial f}{\partial x_j} \text{ exists and is continuous on all of } R^n_+ \right\}$$

$$\mathcal{F}^0 = \{f \in \mathcal{F} : f(0) = 0\}$$

$$\mathcal{F}^C = \{f \in \mathcal{F} : f \text{ is concave}\}$$

$$\mathcal{F}^L = \{f \in \mathcal{F} : f \text{ is Lipschitz, i.e., there is a } K > 0 \text{ such that for } x, y \text{ in } R^n_+ \ |f(x) - f(y)| \leq K\|x-y\|\}$$

$$\overline{\mathcal{F}} = \{f \in \mathcal{F} : f \text{ is strictly increasing, i.e., } x > y \text{ implies } f(x) > f(y)\}$$

$$\mathcal{F}^* = \mathcal{F}^1 \cap \mathcal{F}^0 \cap \mathcal{F}^C \cap \mathcal{F}^L \cap \overline{\mathcal{F}} .$$

2) Finite-type economies

An economy $(\underset{\sim}{a}, \underset{\sim}{u})$ in M is called finite utility-type if there is a finite partition of T into measurable sets $T^1, \ldots, T^p$ and functions $u^1, \ldots, u^p$ (where $u^i : R^n_+ \to R$), such that $\underset{\sim}{u}^t = u^i$ if $t \in T^i$, $1 \leq i \leq p$. In this case we write $(\underset{\sim}{a}, u^1, \ldots, u^p)$ in place of $(\underset{\sim}{a}, \underset{\sim}{u})$. Finite endowment-type economies are defined similarly, and we use the notation $(a^1, \ldots, a^p, \underset{\sim}{u})$ for them.

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* Σx denotes $\sum_{j=1}^{n} x_j$.

3) <u>δ-approximations of utilities</u>

Let U be the set of all functions from $R^n_+ \times T$ to R which satisfy conditions (2.2), (2.3), (2.5)-(2.7). (Note that if $\mu \in U$, then $\mu^t \in F^0 \cap \bar{F} \cap \hat{F}$ for all $t \in T$.) For $\delta > 0$, a δ-approximation of μ is defined to be a member $\hat{\mu}$ of U such that $\|\hat{\mu}^t - \mu^t\| \leq \delta$ for all t except possibly a set of μ-measure $\leq \delta$, in which $\hat{u}^t(x) = \sqrt{\Sigma x}$. (Here $\| \ \|$ is the norm defined in (1).)

4) <u>The bounded variation norm</u>

A non-decreasing sequence Ω of measurable sets $S_o \subset S_1 \subset \ldots \subset S_m$ (each $S_i \in \mathcal{C}$) will be called a chain. The variation of the set function v over the chain Ω is defined by

$$\|v\|_\Omega = \sum_1^m |v(S_i) - v(S_{i-1})|$$

It can be shown that for any $v \in BV$, $\|v\| = \sup \|v\|_\Omega$, where the supremum is taken over all chains Ω. (See Proposition 4.1 in [6].)

B. <u>An Approximation Lemma (Existence of M^*)</u>

M^* is the set of all m in M which satisfy

(i) $m = (a^1, \ldots, a^p; u^1, \ldots, u^p)$ is made up of p types (p any positive integer) such that all agents of type i have the common characteristics (a^i, u^i); each type has μ-measure $1/p$; each $u^i \in F^*$ $(1 \leq i \leq p)$.

(ii) For $1 \leq i \leq p$, there is a $y^i \in R^n_+$ and a neighborhood N^i (in R^n_+) of y^i and constants $c_1, \ldots, c_n$ in R_+ such that:

$$\sum_{i=1}^{p} y^i = \sum_{i=1}^{p} a^i ;$$

$$v_m(T) = (1/p) \sum_{i=1}^{p} u^i(y^i) ;$$

$$u^i(z) - u^i(y^i) = \sum_{j=1}^{n} c_j (z - y^i)_j \quad \text{for} \quad 1 \leq i \leq p , \quad \text{and all} \quad z \in N^i .$$

Our aim in this subsection is

Lemma I. M^* is dense in M .

The proof will be built up from several propositions, some of which may be of interest in themselves.

Proposition 2. Consider an $m = (\underset{\sim}{a}, \underset{\sim}{\mu}) \in M$, and a sequence m^ℓ in M , with $m^\ell = (\underset{\sim}{a}^\ell, \underset{\sim}{\mu})$ and $\int_T \|\underset{\sim}{a}_\ell - \underset{\sim}{a}\| \underset{\ell}{\to} 0$. Then $d(m^\ell, m) \underset{\ell}{\to} 0$.

Proof. First we establish the result labelled (*)

(*) For every $\varepsilon > 0$, there is a $\delta > 0$, such that if *

① $\int_T \|\underset{\sim}{a} - \hat{\underset{\sim}{a}}\| < \delta$

② $S_1 \subset \ldots \subset S_m = T$, $\int_{S_1} \hat{\underset{\sim}{a}} > 2\varepsilon e$, $\mu(S_{j+1} \setminus S_j) < \delta$

then

$$\sum_{k=1}^{m} |v(S_{k+1}) - \hat{v}(S_{k+1}) - (v(S_k) - \hat{v}(S_k))| < \varepsilon ,$$

where** $v(S) = u_S(\int_S \underset{\sim}{a})$, $\hat{v}(S) = u_S(\int_S \hat{\underset{\sim}{a}})$.

---

* e^j is the vector in R^n whose j^{th} component is 1 and all others 0, and $e = e^1 + \ldots + e^n$.

** $u_S(x) = \max\{\int_S \underset{\sim}{u}^t(\underset{\sim}{\chi}^t) d\mu(t) : \underset{\sim}{\chi} : T \to R^n_+, \underset{\sim}{\chi}(S) = x\}$.

To begin with fix k. Put $A_k = v(S_{k+1}) - \hat{v}(S_{k+1}) - (v(S_k) - \hat{v}(S_k))$, $S_{k+1} = S$, $S_k = S_0$, $u_{S_{k+1}} = w$, $u_{S_k} = w_0$, $\int_{S_{k+1}} \underset{\sim}{a} = b$, $\int_{S_k} \underset{\sim}{a} = b_0$, $\int_{S_{k+1}} \hat{\underset{\sim}{a}} = \hat{b}$, $\int_{S_k} \hat{\underset{\sim}{a}} = \hat{b}_0$. Suppose $u_S(b)$ is attained at $\underset{\sim}{y}$, i.e., $u_S(b) = \int_S \underset{\sim}{u}^t(\underset{\sim}{y}(t))d\mu(t)$. $\hat{\underset{\sim}{y}}$, is defined similarly for $\hat{b}$.

Let $V = S \backslash S_0$, $\Delta = \int_V (\underset{\sim}{a} - \underset{\sim}{y})$, $\hat{\Delta} = \int_V (\hat{\underset{\sim}{a}} - \hat{\underset{\sim}{y}})$. Also introduce*

$$I_1 = w'(\hat{b}) \cdot [\int_V \underset{\sim}{a} - \hat{\underset{\sim}{a}}]$$

$$\hat{I}_1 = w'(b) \cdot [\int_V \hat{\underset{\sim}{a}} - \underset{\sim}{a}]$$

$$I_2 = [w'(b) - w'(\hat{b})] \cdot \Delta$$

$$\hat{I}_2 = [w'(\hat{b}) - w'(b)] \cdot \hat{\Delta} .$$

By (40.2) of lemma 40.1 in [6], there exist $c \in (\int_{S_0} \underset{\sim}{y}, \int_{S_0} \underset{\sim}{a})$ and $\hat{c} \in (\int_{S_0} \hat{\underset{\sim}{y}}, \int_{S_0} \hat{\underset{\sim}{a}})$ such that **

$$v(S) - v(S_0) = w(b) - w_0(b_0) = [\int_V u(\underset{\sim}{y})] + w_0'(c) \cdot \Delta ,$$

$$\hat{v}(S) - \hat{v}(S_0) = w(\hat{b}) - w_0(\hat{b}_0) = [\int_V u(\hat{\underset{\sim}{y}})] + w_0'(\hat{c}) \cdot \hat{\Delta} ;$$

and also, by Proposition 38.5 in [6],

$$\boxed{*} \quad \underset{\sim}{u}(t, \hat{\underset{\sim}{y}}(t)) - \underset{\sim}{u}(t, \underset{\sim}{y}(t)) \le w'(b) \cdot (\hat{\underset{\sim}{y}}(t) - \underset{\sim}{y}(t))$$

$$\boxed{**} \quad \underset{\sim}{u}(t, \underset{\sim}{y}(t)) - \underset{\sim}{u}(t, \hat{\underset{\sim}{y}}(t)) \le w'(\hat{b}) \cdot (\underset{\sim}{y}(t) - \hat{\underset{\sim}{y}}(t))$$

---

* $w'(b)$ is the gradient of w at b, etc.

** For any integrable $\underset{\sim}{x} : T \longrightarrow R_+^n$ and $W \in \mathcal{C}$, we denote $\int_W \underset{\sim}{u}(t, \underset{\sim}{x}(t))d\mu(t)$ by $\int_W u(\underset{\sim}{x})$ throughout this proof.

for almost all $t \in S$.

With c and $\hat{c}$ as above, finally put

$$I_3 = [w_0'(c) - w'(b)]\cdot\Delta$$

$$\hat{I}_3 = [w_0'(\hat{c}) - w'(\hat{b})]\cdot\hat{\Delta} \ .$$

Now

$$A_k = [\int_V u(\underset{\sim}{y})] + w_0'(c)\cdot\Delta - \{[\int_V u(\hat{\underset{\sim}{y}})] + w_0'(\hat{c})\cdot\hat{\Delta}\}$$

$$= [\int_V u(\underset{\sim}{y})] + w'(b)\cdot\Delta + [w_0'(c) - w'(b)]\cdot\Delta$$

$$- \int_V [u(\hat{\underset{\sim}{y}}) - w'(\hat{b})\cdot\hat{\underset{\sim}{y}} + w'(\hat{b})\cdot\hat{\underset{\sim}{a}}] - [w_0'(\hat{c}) - w'(\hat{b})]\cdot\hat{\Delta}$$

$$\leq [\int_V u(\underset{\sim}{y})] + w'(b)\cdot\Delta + I_3 - \int_V [u(\underset{\sim}{y}) - w'(\hat{b})\cdot\underset{\sim}{y} + w'(\hat{b})\cdot\underset{\sim}{a}]$$

$$+ w'(\hat{b})\cdot[\int_V (\underset{\sim}{a} - \hat{\underset{\sim}{a}})] - \hat{I}_3$$

$$= I_1 + I_2 + I_3 - \hat{I}_3 \ .$$

(The inequality follows by integrating **[**]** over V.) Similarly, using **[*]**,

$$-A_k \leq \hat{I}_1 + \hat{I}_2 + \hat{I}_3 - I_3 \ .$$

Consider the measure generated by $\hat{\underset{\sim}{a}}$. It is absolutely continuous (component-wise) w.r.t. μ. So there is an $\hat{\varepsilon} > 0$ such that $\int_A \hat{\underset{\sim}{a}} > 2\varepsilon e$ implies $\mu(A) > \hat{\varepsilon}$, for any $A \in \mathcal{C}$. Then by Lemma 37.8 in [6] (taking $\hat{\varepsilon}$ for ε, and $\hat{\alpha} = 1 + \max\{\sum\int_T \underset{\sim}{a}, \sum\int_T \hat{\underset{\sim}{a}}\}$ for α), there is a μ-integrable function $\underset{\sim}{\eta}$ such that $\underset{\sim}{a}$, $\hat{\underset{\sim}{a}}$, $\underset{\sim}{y}$, , $\hat{\underset{\sim}{y}}_0$ are each $\leq \underset{\sim}{\eta}e$ (for any choice of S_k, S_{k+1} in the chain).

Now take the measure generated by $\underset{\sim}{\eta}$. It is also absolutely continuous w.r.t. μ . Hence for some $\delta_1 > 0$, $\mu(V) < \delta_1$ implies $\int_V \underset{\sim}{\eta} e < \frac{1}{2}\varepsilon e$. Note also that if $\delta < \varepsilon$, then $\int_S \underset{\sim}{a} > \varepsilon e$ (since $\int_S \hat{\underset{\sim}{a}} > 2\varepsilon e$). Thus if we choose $\delta < \min[\varepsilon, \delta_1]$, then we have $\int_{S_0} \underset{\sim}{y} = \int_{S_0} \underset{\sim}{y} + \int_V \underset{\sim}{y} - \int_V \underset{\sim}{y}$ $= \int_S \underset{\sim}{a} - \int_V \underset{\sim}{y} \geq \varepsilon e - \int_V \underset{\sim}{\eta} e \geq \frac{1}{2}\varepsilon e$; similarly $\int_{S_0} \hat{\underset{\sim}{y}} \geq \frac{1}{2}\varepsilon e$. This permits us to apply (40.3) of Lemma 40.1 in [6] to get $w'(b) = w_0'(\int_{S_0} \underset{\sim}{y})$, i.e., $|w_0'(c) - w'(b)| = |w_0'(c) - w_0'(\int_{S_0} \underset{\sim}{y})|$. Similarly $|w_0'(\hat{c}) - w'(\hat{b})| = |w_0'(\hat{c}) - w_0'(\int_{S_0} \hat{\underset{\sim}{y}})|$.

Observe that $c - \int_{S_0} \underset{\sim}{y} = \theta \int_{S_0} \underset{\sim}{y} + (1-\theta)b_0 - \int_{S_0} \underset{\sim}{y} = (1-\theta)(b_0 - \int_{S_0} \underset{\sim}{y})$ for some $0 < \theta < 1$; but $b_0 + \int_V \underset{\sim}{a} = \int_{S_0} \underset{\sim}{y} + \int_V \underset{\sim}{y}$, thus $b_0 - \int_{S_0} y = -\Delta$, i.e., $c - \int_{S_0} \underset{\sim}{y} = -(1-\theta)\Delta$, and so $\|c - \int_{S_0} \underset{\sim}{y}\| \leq \|\Delta\| \leq \|\int_V \underset{\sim}{\eta} e\|$. Similarly $\|\hat{c} - \int_{S_0} \hat{\underset{\sim}{y}}\| \leq \|\int_V \underset{\sim}{\eta} e\|$.

Now clearly c , $\hat{c}$, $\int_{S_0} \underset{\sim}{y}$, $\int_{S_0} \hat{\underset{\sim}{y}}$, b , $\hat{b}$, b_0 , $\hat{b}_0$ all lie in $A(\varepsilon^*, \hat{\alpha}) = \{x \in \mathbb{R}^n_+ : x \geq \varepsilon^* e, \sum x \leq \hat{\alpha}\}$, whenever $0 < \varepsilon^* < \min\left[\frac{1}{2}\varepsilon, \hat{\varepsilon}\right]$. Apply proposition 38.7 in [6]. It says that, for an arbitrarily small $0 < \varepsilon^* < \min\left[\frac{1}{2}\varepsilon, \hat{\varepsilon}\right]$, we can find a δ_2 such that if $\|c - \int_{S_0} \underset{\sim}{y}\|$, $\|\hat{c} - \int_{S_0} \hat{\underset{\sim}{y}}\|$, $\|b - \hat{b}\|$, is each $< \delta_2$, then the norms of the coefficients multiplying Δ and $\hat{\Delta}$ (in I_1 , $\hat{I}_1$) will each be less than ε^* . To ensure that the three terms will be less than δ_2 , invoke (again!) the absolute continuity of the measure generated by $\underset{\sim}{\eta}$ w.r.t. μ . There is a $\delta_3 > 0$ such that if $\mu(V) < \delta_3$ then $\|\int_V \underset{\sim}{\eta} e\| < \delta_2$ (and we have shown: $\max\{\|c - \int_{S_0} \underset{\sim}{y}\|$, $\|\hat{c} - \int_{S_0} \hat{\underset{\sim}{y}}\|\} \leq \|\int_V \underset{\sim}{\eta} e\|$). On the other hand, obviously

$\|b - \hat{b}\| \le \int_T \|\underset{\sim}{a} - \underset{\sim}{\hat{a}}\| < \delta$. Thus if δ is smaller than both δ_2 and δ_3 (in addition to being smaller than δ_1 and ε), the coefficients multiplying Δ and $\hat{\Delta}$ in I_1 , $\hat{I}_1$ will each be smaller than ε^* .

Choose a positive $\varepsilon^* < \min\left\{\frac{1}{2}\varepsilon,\ \hat{\varepsilon},\ (6n\int_T \underset{\sim}{\eta})^{-1}\varepsilon\right\}$. Then choose δ_2 and δ_3 (which depend on ε^*) as above. Put $M = \sup\{\|u_S'(b)\| : \mu(S) \ge \varepsilon^*,\ b \in A(\varepsilon^*, \hat{\alpha})\}$. (That M is finite is guaranteed by (38.8) in [6].) Now pick a positive $\delta < \min\{\varepsilon, \delta_1, \delta_2, \delta_3, (2nM)^{-1}\varepsilon\}$. Then for any k ,

$$|A_k| \le nM(\int_V \|\underset{\sim}{a} - \underset{\sim}{\hat{a}}\|) + 3\varepsilon^* n\|\int_V \underset{\sim}{\eta} e\| .$$

Hence

$$\sum_{k=1}^{m} |A_k| \le nM(\int_T \|\underset{\sim}{a} - \underset{\sim}{\hat{a}}\|) + 3\varepsilon^* n\|\int_T \underset{\sim}{\eta} e\|$$

$$\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$

$$= \varepsilon ,$$

proving (*).

To complete the proof note that, by Proposition 37.13 in [6], u_T is continuous at $0 \in \mathbb{R}^n_+$ (indeed on all of $\mathbb{R}^n_+$). Also clearly $u_T(0) = 0$. Hence for any $\frac{1}{4}\varepsilon > 0$ there is a $\gamma > 0$ such that $\|a\| < 2\gamma$ implies $u_T(a) < \frac{1}{4}\varepsilon$. Since $\underset{\sim}{a} > 0$ we can pick a $0 < \gamma < \int_T \underset{\sim}{a}_1$, and an $\varepsilon_1 > 0$, to ensure

(a) $\int_S \underset{\sim}{a}_1 \le \gamma \Longrightarrow \int_S \underset{\sim}{a} \le \gamma e$

(b) $\int_S \underset{\sim}{a}_1 \ge \gamma \Longrightarrow \int_S \underset{\sim}{a} \ge 2\varepsilon_1 e$.

Put $\varepsilon^* = \min(\varepsilon_1, \frac{1}{2}\varepsilon)$ and choose δ^* to correspond to ε^* in accordance with (*). Let $\delta = \min[\gamma, \delta^*]$. Since $\int_T \|\underset{\sim}{a}^\ell - \underset{\sim}{a}\| \xrightarrow[\ell]{} 0$, there is an N such that if $\ell \ge N$, then $\int_T \|\underset{\sim}{a}^\ell - \underset{\sim}{a}\| < \delta$. We will show that if

$\ell \geq N$, $\|v^{\ell} - v\| < \varepsilon$, where v^{ℓ} abbreviates $v_{m^{\ell}}$.

Let $w = v - v^{\ell}$, and let $\phi = S_0 \subset S_1 \subset \ldots \subset S_{m+1} = T$ be a chain. Then, by Lyapunov's theorem, it is possible to insert finitely many additional sets $S_{01}, S_{02}, \ldots, S_{11}, S_{12}, \ldots, S_{m1}, S_{m2}, \ldots$ to get a refined chain $S_0 \subset S_{01} \subset S_{02} \subset \ldots \subset S_1 \subset S_{11} \subset S_{12} \subset \ldots \subset S_m \subset S_{m1} \subset S_{m2} \subset \ldots \subset S_{m+1}$ so that if we relabel the new sequence $U_0, \ldots, U_{p+1}$ we have (i) $\mu(U_{k+1} \setminus U_k) < \delta$ for all k, (ii) $\int_{U_q} \mathfrak{a}^1 = \gamma$ for some q. Then

$$\sum_{k=0}^{m} |w(S_{k+1}) - w(S_k)| \leq \sum_{k=0}^{p} |w(U_{k+1}) - w(U_k)|$$

$$= \sum_{k=0}^{q} + \sum_{k=q+1}^{p} .$$

Now

$$\sum_{k=0}^{q} = \sum_{k=0}^{q} |v(U_{k+1}) - v(U_k) - (v^{\ell}(U_{k+1}) - v^{\ell}(U_k)|$$

$$\leq \sum_{k=0}^{q} |v(U_{k+1}) - v(U_k)| + \sum_{k=0}^{q} |v^{\ell}(U_{k+1}) - v^{\ell}(U_k)|$$

$$= \sum_{k=0}^{q} v(U_{k+1}) - v(U_k) + \sum_{k=0}^{q} v^{\ell}(U_{k+1}) - v^{\ell}(U_k)$$

$$= v(U_q) + v^{\ell}(U_q) .$$

But $\int_{U_q} \mathfrak{a} \leq \delta e$; and since $\int_T \|\mathfrak{a} - \mathfrak{a}^{\ell}\| < \delta < \delta$, $\int_{U_q} \mathfrak{a}^{\ell} \leq 2\delta e$. Therefore $v(U_q) \leq u_T(\int_{U_q} \mathfrak{a}) \leq \frac{1}{4}\varepsilon$, $v^{\ell}(U_q) \leq u_T(\int_{U_q} \mathfrak{a}^{\ell}) \leq \frac{1}{4}\varepsilon$. Hence $\sum_{k=0}^{q} \leq \frac{1}{2}\varepsilon$. On the other hand, by (*), $\sum_{k=q+1}^{p} \leq \varepsilon^* \leq \frac{1}{2}\varepsilon$. Consequently

$$\sum_{k=0}^{m} |w(S_{k+1}) - w(S_k)| \leq \varepsilon .$$

Q.E.D.

Remark. Notice that Proposition 1 says that the characteristic function of a market is continuous in its endowments, thus complementing Proposition 40.24 of [6] which showed continuity in utilities.

For $\delta > 0$, let $I(x,\delta) = \{y : x \le y \le x + \delta e\}$. We define an operator $A^\delta : F \to F$ where, for $f \in F$, the function $A^\delta f : R^n_+ \to R$ is given by

$$(A^\delta f)(x) = \delta^{-n}\int_{I(x,\delta)} f(y)dy \equiv \delta^{-n}\int_{I(0,\delta)} f(x+y)dy .$$

Proposition 3. For any $\delta > 0$:

(i) $f \in F \Longrightarrow A^\delta f \in F^1$

(ii) $f \in \overline{F} \Longrightarrow A^\delta f \in \overline{F}$

(iii) $f \in F^L \Longrightarrow A^\delta f \in F^L$

(iv) $f \in F^C \Longrightarrow A^\delta f \in F^C$

(v) $f \in F \Longrightarrow \|A^\delta f - f\| \to 0$ as $\delta \to 0$.

Proof. If x, z are in R^n_+, $(A^\delta f)(z) - (A^\delta f)(x) = \delta^{-n}\int_{I(0,\delta)} (f(z+y) - f(x+y))dy$, from which (ii) and (iii) are easily deduced. Similarly $(A^\delta f)(z) + (A^\delta f)(x) - 2(A^\delta f)\left(\frac{x+z}{2}\right) = \delta^{-n}\int_{I(0,\delta)} \left\{f(x+y) + f(z+y) - 2f\left(\frac{x+z}{2} + y\right)\right\}dy$, which implies (iv).

Let $I^i(x,\delta)$ denote the face $\{y \in R^n_+ : x \le y \le x + \delta(e - e^i)\}$ of the cube $I(x,\delta)$. Then, for $0 < \Delta < \delta$,

$$\frac{(A^\delta f)(x+\Delta e^i) - (A^\delta f)(x)}{\Delta} = (\delta^{-n}\int_{I(x+\Delta e^i,\delta)} f(y)dy - \delta^{-n}\int_{I(x,\delta)} f(y)dy)/\Delta$$

$$= (\delta^{-n}\int_{I_1} f(y)dy)/\Delta - (\delta^{-n}\int_{I_2} f(y)dy)/\Delta ,$$

where $I_1 = I(x+\Delta e^i, \delta)\backslash I(x,\delta) = I^i(x+\delta e^i, \delta) + \{t\Delta e^i : 0 \le t \le 1\}$ and $I_2 = I(x,\delta)\backslash I(x+\Delta e^i, \delta) = I^i(x,\delta) + \{t\Delta e^i : 0 \le t \le 1\}$. The function f is continuous on R^n_+. In particular it is continuous on the compact subset $I(x, 2\delta)$ of R^n_+ and thus uniformly continuous on $I(x, 2\delta)$. So for every $\alpha > 0$ there is Δ_0 such that: z in $I^i(x+\delta e^i, \delta)$ or in $I^i(x,\delta)$, $0 \le t \le 1$ and $\Delta \le \Delta_0 \Longrightarrow |f(z+t\Delta e^i)-f(z)| \le \alpha$. This shows that $((A^\delta f)(x+\Delta e^i) - (A^\delta f)(x))/\Delta$ converges, as $\Delta > 0$ goes to 0, to $\delta^{-n}\int_{I^i(x+\delta e^i,\delta)} f(y)dy - \delta^{-n}\int_{I^i(x,\delta)} f(y)dy$ where by dy here we mean $dy_1 \cdot dy_2 \cdots\cdots dy_{i-1} \cdot dy_{i+1} \cdots\cdots dy_n$. Continuity of f implies that both $\int_{I^i(x,\delta)} f(y)dy$ and $\int_{I^i(x+\delta e^i,\delta)} f(y)dy$ are continuous in x. Therefore the one-sided derivative $\lim_{\substack{\Delta>0\\ \Delta\to 0}} ((A^\delta f)(x+\Delta e^i) - (A^\delta f)(x))/\Delta$ exists and is continuous at each $x \in R^n_+$. Similarly if $x_i > 0$, the derivative $\lim_{\Delta\to 0} ((A^\delta f)(x+\Delta e^i) - (A^\delta f)(x))/\Delta$ exists and is continuous. This proves (i).

Take $\alpha > 0$. Since $f(x) = o(\|x\|)$ as $\|x\| \to \infty$, there is a $K > n$ such that $|f(x)| < \alpha(\Sigma x)/4$ whenever $\Sigma x \ge K$. Clearly $f(x) \le (A^\delta f)(x) \le f(x+\delta e)$, hence

$$|(A^\delta f)(x) - f(x)| \le |f(x+\delta e) - f(x)| .$$

Therefore, on the one hand, for x in R^n_+ with $\Sigma x \ge K$, $|(A^\delta f)(x) - f(x)|/(1+\Sigma x) \le |f(x)|/(1+\Sigma x) + 2(|f(x+\delta e)|/(1+\Sigma(x+\delta e)))$ $\le \alpha/4 + 2(\alpha/4) = (3/4)\alpha$. On the other hand, f is continuous on the compact subset $\{x \in R^n_+ : \Sigma x \le K+n\}$, and thus there is $0 < \delta_0 \le 1$ such that for $\delta \le \delta_0$ and x in R^n_+ with $\Sigma x \le K$,

$$|(A^{\delta}f)(x) - f(x)| \le |f(x+\delta e) - f(x)| \le \alpha/2 \ .$$

Altogether $\|A^{\delta}f - f\| = \sup_{x \in R^n_+} |(A^{\delta}f)(x) - f(x)|/(1 + \Sigma x) < \alpha$, which proves (v).

Q.E.D.

For any $K > 0$, define $p_K : R^n_+ \to [0,K]^n$ by $(p_K x)_i = \min\{x_i, K\}$. For $f \in F$ define $L^K f : R^n_+ \to R$ by $L^K f = f \circ p_K$ (i.e. $(L^K f)(x) = f(p_K(x))$ for all x). Observe that $L^K f \in F$ (the non-decreasingness of $L^K f$ follows from: $x \ge y \Longrightarrow p_K x \ge p_K y$). Thus L^K is an operator from F to F .

<u>Proposition 4</u>. For any $K > 0$,

(i) $f \in F^1 \Longrightarrow L^K f \in F^L$

(ii) $f \in F^C \Longrightarrow L^K f \in F^C$

(iii) $f \in F \Longrightarrow \|L^K f - f\| \to 0$ as $K \to \infty$.

<u>Proof</u>. (i) To check that $L^K f \in F^L$ observe that $L^K f = f$ on $[0,K]^n$, and f is continuously differentiable on $[0,K]^n$. Let A be a maximum of all the partial derivatives of f on $[0,K]^n$. Then $|(L^K f)(x) - L^K f(x)| \le A\|x-y\|$ when x and y are in $[0,K]^n$. For any x , y in R^n_+ , $\|p_K x - p_K y\| \le \|x-y\|$, thus the inequality follows for any x , y in R^n_+ .

(ii) Take any x and y in R^n_+ , and $z = \lambda x + (1-\lambda)y$ where $0 \le \lambda \le 1$. Since the function $\xi : R_+ \to R_+$ given by $\xi(x) = \min\{x,K\}$ is concave, we see that $p_K(z) \ge \lambda p_K(x) + (1-\lambda)p_K(y)$. But then $(L^K f)(z) = f(p_K(z)) \ge f(\lambda p_K(x) + (1-\lambda)p_K(y)) \ge \lambda f(p_K(x)) + (1-\lambda)f(p_K(y))$ $= \lambda(L^K f)(x) + (1-\lambda)(L^K f)(y)$. (The first " $\ge$ " follows from the fact that, being in F , f is non-decreasing; the second " $\ge$ " holds because f is concave on $[0,K]^n$.)

(iii) Take any $\varepsilon > 0$. For K large enough, $(|f(x)|/(1+\Sigma x)) < \varepsilon/2$ if $\Sigma x \geq K$ (because f is in F). Hence for such K , $\|L^K f - f\| = \sup\{|(L^K f)(x) - f(x)|/(1+\Sigma x) : x \in R^n_+\} = \sup\{|(L^K f)(x) - f(x)|/(1+\Sigma x) : x \in R^n_+, \Sigma x \geq K\} \leq \varepsilon$.

Q.E.D.

<u>Proposition 5</u>. F^* is dense in $F^0 \cap F^C$.

<u>Proof</u>. Let $f \in F^0 \cap F^C$ and $\varepsilon > 0$. By Propositions 3 and 4, we see that if $\delta > 0$ is sufficiently small and $K > 0$ is sufficiently large, the function $f^1 = L^K(A^\delta f)$ is in F^L and satisfies

(i) $\|f^1 - f\| < \varepsilon$

(ii) $0 \leq f^1(0) < \varepsilon$

Let $\eta \in \overline{F} \cap F^0 \cap F^L \cap F^C$. Then the function $f^2 = f^1 + (\varepsilon\eta/\|\eta\|)$ is in $F^C \cap F^L \cap \overline{F}$ and satisfies

(iii) $\|f^2 - f\| < 2\varepsilon$

(iv) $0 \leq f^2(0) = f^1(0) < \varepsilon$.

Next, for $\gamma > 0$, put $f^3 = A^\gamma f^2$. Then $f^3 \in F^C \cap F^L \cap \overline{F} \cap F^1$ and, for sufficiently small $\gamma > 0$, we also have

(v) $\|f^3 - f^2\| < \varepsilon$

(vi) $f^2(0) \leq f^3(0) < 2\varepsilon$.

Let $g = f^3 - f^3(0)$. Then $g \in F^*$ and $\|g-f\| \leq \|g - f^3\| + \|f^3 - f^2\| + \|f^2 - f\| \leq 2\varepsilon + \varepsilon + 2\varepsilon = 5\varepsilon$. Since ε was arbitrary, this completes the proof.

Q.E.D.

Proposition 6. Let $f \in F^*$, $z \in R^n_+$ and $\hat{\delta} > 0$. Suppose $h : R^n_+ \to R$ is an affine function such that

(i) $h(z) = f(z)$

(ii) $h(x) \geq f(x)$ for all $x \in R^n_+$.

Then there is a $g \in F^*$ such that

(iii) $g-h$ is a constant in some neighborhood of z

(iv) $\|g-h\| < \hat{\delta}$.

Proof. We will construct g in three steps. First put $g^1 = \min\{f, h-(\hat{\delta}/4)\}$. Then $g^1 \in F^C \cap \overline{F} \cap F^L$. To see this observe that $h-\varepsilon \in F^C \cap \overline{F} \cap F^L$ ($h-\varepsilon \in \overline{F}$ follows from $f \in \overline{F}$, and (i), (ii) above), and that the mimimum of two concave (increasing, Lipschitz) functions is also a concave (increasing, Lipschitz) function. Observe that $f(x) - (\hat{\delta}/4) \leq g^1(x) \leq f(x)$, and thus

(v) $-\hat{\delta}/4 \leq g^1(0) \leq 0$

(vi) $\sup\{|g^1(x) - f(x)| : x \in R^n_+\} \leq \hat{\delta}/4$.

As both f and h are continuous,

(vii) $h - g^1 = \hat{\delta}/4$ in some neighborhood N_1 of z.

Next put $g^2 = A^\delta g^1$. By Proposition 3, $g^2 \in F^C \cap \overline{F} \cap F^L \cap F^1$.

If we choose δ small enough we also have

(viii) $|g^2(0)| < \hat{\delta}/2$ (using (v))

(ix) $\|g^2 - g^1\| < \hat{\delta}/4$

(x) $h - g^2$ is a constant in some neighborhood N_2 of z (using (vii)).

Finally put $g = g^2 - g^2(0)$. Clearly $g \in F^*$. Moreover, from (vi), (viii) and (ix), and the obvious equality $\|g - g^2\| = |g^2(0)|$, we get

$$\|g-f\| \le \|g-g^2\| + \|g^2-g^1\| + \|g^1-f\| \le \hat{\delta}/2 + \hat{\delta}/4 + \hat{\delta}/4 = \hat{\delta} .$$

Clearly $h-g$ is a constant in the neighborhood N_2 of z .

Q.E.D.

Now we turn to the proof of Lemma I. It will be convenient to regard the first half of it as an independent

Proposition 7. For any $\varepsilon > 0$ and $m \in M$, there is an economy $\tilde{m} = (a^1, \ldots, a^p; f^1, \ldots, f^p)$ of p types such that (i) $d(m,\tilde{m}) < \varepsilon$; (ii) all agents of type i have the common characteristics (a^i, f^i) ; (iii) each type has μ-measure $1/p$; (iv) each $f^i \in F^*$.

Proof

STEP 1. There is a finite utility-type market $m^1 = (\underset{\sim}{a}, u^1, \ldots, u^r)$ such that (i) $d(m^1, m) < \varepsilon/3$, (ii) each of the r utility-types has μ-measure $1/r$.

By Proposition 35.6 of [6], for any $\delta > 0$ there is a finite-type $(\hat{u}^1, \ldots, \hat{u}^s) \in \mathcal{U}$, which is a δ-approximation on $\underset{\sim}{\mu}$. Pick an integer r sufficiently large to ensure that $s(1/r) < \delta$. Divide T_i arbitrarily into disjoint sets $T_i^1, \ldots, T_i^{s_i}$ such that $\mu(T_i^j) = 1/r$ for each j , and $\mu(T_i \setminus \bigcup_{j=1}^{s_i} T_i^j) < 1/r$. Let $T_0 = T \setminus \bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} T_i^j$. Then $\mu(T_0) = k/r$ for some integer $k \le s$. Define the finite-type $u^* = (u^1, \ldots, u^r) \in \mathcal{U}$ by:

$$u^*(t) = \begin{cases} \hat{u}^i(t) & \text{if } t \in T_i^j \text{ for } 1 \le j \le s_i \\ \sqrt{\Sigma x} & \text{otherwise (i.e., if } t \in T_0 \text{).} \end{cases}$$

Then u^* is a 2δ-approximation to $\underset{\sim}{\mu}$, and satisfies (ii). Since δ was arbitrary, the conclusion now follows from Proposition 40.24 in [6].

STEP 2. There is an economy $m^2 = (a^1, \ldots, a^p; \hat{u}^1, \ldots, \hat{u}^p)$ of p types such that (i) $d(m^1, m^2) < \varepsilon/3$; (ii) all agents of type i have the common characteristics $(a^i, \hat{u}^i)$; (iii) each type has μ-measure $1/p$.

This follows from STEP 1 and Proposition 2.

STEP 3. Replace each $\hat{u}^i$ by its concavification u^i . Let $m^3 = (a^1, \ldots, a^p; u^1, \ldots, u^p)$. Since each $\hat{u}^i \in F^0 \cap \bar{F} \cap \hat{F}$ we get $u^i \in F^0 \cap \bar{F} \cap \hat{F} \cap F^C$ (by Lemma 39.5 in [6]), and thus $m \in M$. By Proposition 36.3 in [6], $d(m^2, m^3) = 0$.

STEP 4. Take any $\delta > 0$. By Proposition 5, we can find $f^i \in F^*$ such that $\|f^i - u^i\| < \delta$ for $1 \leq i \leq p$. Let $\tilde{m} = (a^1, \ldots, a^p; f^1, \ldots, f^p)$. Then by Proposition 40.24, we get $d(\tilde{m}, m^3) < \varepsilon$ if δ is chosen suitably small. Hence $d(m,\tilde{m}) < \varepsilon$.

Q.E.D.

Proof of Lemma 1. Consider the $\tilde{m}$ of Proposition 7. By Proposition 36.4 of [6] there exist $z^1, \ldots, z^p$ in R^n_+ such that

$$\sum_{i=1}^{p} z^i = \sum_{i=1}^{p} a^i ,$$

$$\nu_m(T) = \frac{1}{p}\sum_{i=1}^{p} f^i(z^i) .$$

Also, by Proposition 36.4 in [6], we know that there are non-negative constants $c_1, \ldots, c_n$ and affine functions $h^i : R^n_+ \to R_+$ which satisfy:

(i) $h^i(z^i) = f^i(z^i)$

(ii) $\partial h^i/\partial x_j = c_j$ for $1 \leq j \leq n$

(iii) $h^i(x) \geq f^i(x)$ for any x in R^n_+ .

Take any $\hat{\delta} > 0$. By Proposition 6, there exist $f^i_* \in F^*$ $(1 \leq i \leq n)$

and neighborhoods N^i of z^i such that

(iv) $\|f^i_* - f^i\| < \hat{\delta}$

(v) $f^i_* - h^i$ is a constant in N^i , and thus $f^i_*(z) - f^i_*(z^i)$

$= \sum_{j=1}^{n} c_j(z - z^i)_j$ for all z in N^i .

For any $\varepsilon > 0$, if we choose $\hat{\delta}$ sufficiently small in defining f^i_* then we get, by Proposition 40.24 in [6], that $d(\tilde{m}, m^*) < \varepsilon$, where $m^* = (a^1, \ldots, a^p; f^1_*, \ldots, f^p_*)$. By Proposition 7, $d(m,\tilde{m}) < \varepsilon$, hence $d(m, m^*) < 2\varepsilon$. To complete the proof observe that

$$f^i_*(x) - f^i_*(z^i) \leq c\cdot(x - z^i)$$

for all $x \in R^n_+$ and $1 \leq i \leq p$, by the very construction of f^i_* . Therefore, by Proposition 36.4 of [6],

$$v_{m^*}(T) = \sum_{i=1}^{p} f^i_*(z^i) .$$

Q.E.D.

C. <u>The characteristic functions arising from M^*</u>

<u>Lemma II</u>. The characteristic function of any economy in M^* is of the form $f\circ(\mu_1, \ldots, \mu_p)$ where p is a positive integer, $(\mu_1, \ldots, \mu_p)$ is a vector of mutually singular non-atomic probability measures on $\{T,C\}$ with $\sum_{i=1}^{p} \mu_i = p\mu$, and f is a function from $[0,1]^p$ (the range of the vector measure $(\mu_1, \ldots, \mu_p)$) to R that is

(i) Lipschitz on $[0,1]^p$

(ii) linear on a conical neighborhood of the diagonal $[0, (\mu_1, \ldots, \mu_p)(T)]$ (i.e., there is an $\varepsilon > 0$ and $\alpha_1, \ldots, \alpha_p$ in R , such that if $x \in \{y \in [0,1]^p : |y_i - y_j| \leq \varepsilon \sum_{\ell=1}^{p} y_\ell$ for every $1 \leq i, j \leq p\}$, then $f(x) = \sum_{i=1}^{p} \alpha_i x_i$.)

Proof. Let $m = (\underset{\sim}{a}, \underset{\sim}{u})$ be in M^*. Then (recall the definition of M^* from section B) there is a partition $\{T_1, \ldots, T_p\}$ of T into p measurable sets with $\mu(T_i) = 1/p$, and (for $1 \leq i \leq p$) $0 < a^i \in R^n_+$ and $u^i \in F^*$ $(u^i : R^n_+ \to R)$ such that $\underset{\sim}{a}^t = a^i$ and $\underset{\sim}{u}^t = u^i$ if $t \in T_i$.
Define $G : R^p_+ \times R^n_+ \to R$ by (for x in R^p_+ and a in R^n_+):

$$G(x,a) = \max\{\sum_{i=1}^{p} x_i u^i(z^i) : z^1, \ldots, z^p \text{ are in } R^n_+ \text{ and } \sum_{i=1}^{p} x_i z^i \leq a\} .$$

Then G is non-decreasing, concave, homogeneous of degree 1, and

$$v_m(S) = G(\nu(S), \underset{\sim}{a}(S)) , \quad \text{all } S \in C ,$$

where $\nu = (\nu_1, \ldots, \nu_p)$ is the non-atomic vector measure given by $\nu_j(S) = \mu(S \cap T_j)$.

Define $F : R^p_+ \to R$ by:

$$F(x) = G(x, \sum_{i=1}^{p} x_i a^i) .$$

Now $v_m(S) = F(\nu(S))$ for all $S \in C$, i.e., $v_m = F \circ \nu$. Also, since the map $x \to (x, \sum_{i=1}^{p} x_i a^i)$ is monotonic and linear, F inherits from G the nondecreasingness, concavity and homogeneity of degree 1.

Observe that in order to prove our Lemma it is sufficient to show that F is Lipschitz on $[0, 1/p]^p$ and linear on a conical neighborhood of $[0, (1/p, \ldots, 1/p)]$ (simply set $\mu_i = p\nu_i$ and $f(x) = F(x/p)$). First let us establish the Lipschitz property of F. As F is concave and homogeneous of degree 1, it will suffice to show that F is Lipschitz on the strip $C = \{y \in R^p_+ : 1/2 \leq \sum_{i=1}^{p} y_i \leq 1\}$. This, in turn, will follow if we prove that: $G(x,a)$ is Lipschitz in a everywhere; and Lipschitz in x whenever $x \in C$ and $a \in D = \{y \in R_+^n : y \leq p\underset{\sim}{a}(T) = \sum_{i=1}^{p} a^i\}$.

To show that G is Lipschitz in a, first consider a, b in R_+^n with $a \leq b$, and any x in R_+^p. Then for any $z^1, \ldots, z^p$ in R_+^n with $\sum_{i=1}^{p} x_i z^i \leq b$, we can find $\bar{z}^1, \ldots, \bar{z}^p$ in R_+^n such that: $\sum_{i=1}^{p} x_i \bar{z}^i \leq a$; $\bar{z}^i \leq z^i$; and $\sum_{i=1}^{p} x_i(z^i - \bar{z}^i) \leq b-a$ (which implies also $\sum_{i=1}^{p} x_i \|z^i - \bar{z}^i\| \leq \|b-a\|p$). Therefore if $G(x,b)$ is attained at $z^1, \ldots, z^p$ we have $0 \leq G(x,b) - G(x,a) \leq \sum_{i=1}^{p} x_i u^i(z^i) - \sum_{i=1}^{p} x_i u^i(\bar{z}^i)$ $= \sum_{i=1}^{p} x_i(u^i(z^i) - u^i(\bar{z}^i)) \leq K\|b-a\|p$ where K is a Lipschitz constant for each of $u^1, \ldots, u^p$. Now take arbitrary a and b in R_+^n. Let $a \vee b$ be the vector whose j^{th} component is $\max\{a_j, b_j\}$, and observe $|G(x,b) - G(x,a)| \leq |G(x, a \vee b) - G(x,a)| + |G(x,b) - G(x, a \vee b)|$ $\leq K(\|(a \vee b) - a\| + \|(a \vee b) - b\|)p \leq 2K\|b - a\|p$.

Next we must show that G is Lipschitz in x, for $x \in C$ and $a \in D$. Put $\hat{C} = \{y \in R_+^p : 1/2 \leq \sum_{i=1}^{p} y_i \leq p\}$. Note that, by Lemma 37.8 of [6], there is a $b \in R_+^n$ such that:

$$x \in \hat{C}, \quad a \in D \Longrightarrow G(x,a) \text{ is attained at}$$
$$z^1, \ldots, z^p \ (z^i \text{ in } R_+^n) \text{ with each } z^i \leq b .$$

Let K_1 be an upper bound for $u^i(z)$ where $1 \leq i \leq p$ and $z \leq b$. Let $a \in D$ and first take $x \in C \subset \hat{C}$, $\bar{x} \in C$, $\bar{x} \leq x$. Then $G(x,a)$ is attained at $\bar{z}^1, \ldots, \bar{z}^p$ with each $\bar{z}^i \leq b$, and $\sum_{i=1}^{p} x_i \bar{z}^i \leq a$. Obviously we have $\sum_{i=1}^{p} \bar{x}_i \bar{z}^i \leq a$, therefore $G(\bar{x},a) \geq \sum_{i=1}^{p} \bar{x}_i u^i(z^i)$. As G is non-decreasing it follows that $0 \leq G(x,a) - G(\bar{x},a) \leq \sum_{i=1}^{p} x_i u^i(z^i)$ $- \sum_{i=1}^{p} \bar{x}_i u^i(z^i) \leq K_1 \sum_{i=1}^{p} |x_i - \bar{x}_i|$. Now consider arbitrary $\bar{x}$ and x in

C , and note that $\overline{x} \vee x \in \hat{C}$. Thus the fact that $|G(x,a) - G(\overline{x},a)| \leq K_1 \|x - \overline{x}\| p$ can be established as before.

Finally, it remains to show that F is linear in a conical neighborhood of the diagonal $[0, (1/p, \ldots, 1/p)]$. Given the homogeneity of F this will follow if we can show that F is linear, for some $\varepsilon > 0$, on the set $\{x \in [0, 1/p]^p : |x_i - 1/p| < \varepsilon ,\ \text{all } i\}$.

Since m is in M^* there exist (recall again the definition of M^* from section B), for $1 \leq i \leq p$: $y^i \in R^n_+$, a neighborhood N^i of y^i in R^n_+ , and constants $c_1, \ldots, c_n$ in R_+ such that

$$\sum_{i=1}^{p} y^i = \sum_{i=1}^{p} a^i ;$$

$$v_m(T) = (1/p) \sum_{i=1}^{p} u^i(y^i) ;$$

$$u^i(z) - u^i(y^i) = \sum_{j=1}^{n} c_j (z - y^i)_j , \quad \text{for all } 1 \leq i \leq p , \text{ and } z \in N^i .$$

Since $0 < (1/p)a^1 + \ldots + (1/p)a^p \in (1/p)N^1 + \ldots + (1/p)N^p$, there is an $\varepsilon > 0$ such that if $\|(x_1, \ldots, x_p) - (1/p, \ldots, 1/p)\| < \varepsilon$, there exist $z^1 \in N^1, \ldots, z^p \in N^p$ for which $x_1 z^1 + \ldots + x_p z^p = x_1 a^1 + \ldots + x_p a^p$. To translate this to our situation: if $\|\nu(S) - (1/p, \ldots, 1/p)\| < \varepsilon$ we can find vectors $z^1_S \in N^1, \ldots, z^p_S \in N^p$ such that $\sum_{i=1}^{p} \nu_i(S) z^i_S = \sum_{i=1}^{p} \nu_i(S) a^i$. As each u^i is concave

$$u^i(x) - u^i(z^i_S) \leq c \cdot (x - z^i_S)$$

for all $x \in R^n_+$ and $1 \leq i \leq p$. But then, by Proposition 36.4 in [6],

$F(\nu(S)) = v_m(S) = \sum_{i=1}^{p} \nu_i(S)u^i(z_S^i)$ whenever $\|\nu(S) - (1/p, \ldots, 1/p)\| < \varepsilon$.
Each $u^i(x) = b_i + c\cdot x$, for $x \in N^i$, where b_i is some constant;
thus $F(\nu(S)) = \sum_{i=1}^{p} b_i\nu_i(S) + c\cdot \sum_{i=1}^{p} \nu_i(S)z_S^i = \sum_{i=1}^{p} b_i\nu_i(S) + c\cdot \sum_{i=1}^{p} \nu_i(S)a^i$
for such $S \in \mathcal{C}$. This reveals that F is linear on a neighborhood of $(1/p, \ldots, 1/p)$, hence by homogeneity, on a conical neighborhood of $[0, (1/p, \ldots, 1/p)]$.

Q.E.D.

D. Uniqueness of ϕ on M^*

This will follow from Lemma II and

Lemma III. Let $\mu_1, \ldots, \mu_p$, be mutually singular non-atomic probability measures with $\sum_{i=1}^{p} \mu_i = p\mu$ and let $f : [0,1]^p \to R$ be a function such that $f \circ (\mu_1, \ldots, \mu_p)$ is a characteristic function of an economy m in M . Then if f is linear on a conical neighborhood of the diagonal and Lipschitz on $[0,1]^p$, $\phi(m) = \{\sum_{i=1}^{p} \alpha_i\mu_i\}$.

Proof. Let $\nu = \mu_1 + \ldots + \mu_p$ and let $f_i = d\mu_i/d\nu$. Then $0 \leq f_i \leq 1$ a.e. and thus $f_i \in L_2(\nu)$ and $\|f_i\|_2^2 = \langle f_i, f_i\rangle = \int f_i(s)f_i(s)d\nu(s) = 1$, and also $\|f_i - f_j\|^2 = 2$ for $i \neq j$. For f and g in the Hilbert space $L_2(\nu)$ let $\langle f,g\rangle$ denote the inner product $\int f(t)g(t)d\nu(t)$.
Let θ be a μ-preserving transformation such that for all $1 \leq i < j \leq p$ $((1/\sqrt{2})(f_i - f_j) \circ \theta^k)_{k=0}^{\infty}$ is an orthonormal sequence in $L_2(\nu)$.

Let $u \in \phi(m)$ and let $w = \sum \alpha_i \mu_i$. By [14] there is a universal constant K such that for every positive integer ℓ , there are real numbers a_i , with $|a_i| = 1$, $1 \leq i \leq \ell$ with $\|\sum_1^\ell a_i(\theta^i)\ (u-w)\| \geq 2K\sqrt{\ell}\|u-w\|$. By the triangle inequality it follows that there are nonnegative real numbers $a_i \in \{0,1\}$, $1 \leq i \leq \ell$ with

$$(6.2) \qquad \|\sum_1^\ell a_i(\theta^i)\ (u-w)\| > K\sqrt{\ell}\|u-w\| .$$

Let $\overline{\mu} = (\mu_1, \ldots, \mu_p)$ and for $\alpha > 0$ let

$$\mathcal{D}_\alpha = \{S \in \mathcal{C} : |\mu_i(S) - \mu_j(S)| \leq \alpha\nu(S) \text{ for all } 1 \leq i, j \leq p\} .$$

Then $S \in \mathcal{D}_\varepsilon$ implies $(f\circ\mu)(S) = \sum \alpha_i \mu_i(S)$. Observe that $S \in \mathcal{D}_\alpha$ iff $|\langle\chi_S, f_i - f_j\rangle| \leq \alpha\nu(S)$ for all $1 \leq i < j \leq p$ and that $\theta^k S \in \mathcal{D}_\alpha$ iff $|\langle\chi_S, (f_i - f_j)\circ\theta^k\rangle| \leq \alpha\nu(S)$. By the Parserval's inequality,

$$\sum_{0\leq k} \langle\chi_S, (f_i - f_j)\circ\theta^k\rangle^2 \leq 2\|\chi_S\|_2^2 = 2\nu(S)$$

and therefore

$$(6.3) \qquad \{k : \theta^k S \notin \mathcal{D}_{\varepsilon/2}\} \leq \frac{p(p-1)}{2}\,\frac{8}{\varepsilon^2}/\nu(S) .$$

Observe that there is $\beta < 1$ s.t.

$$(6.4) \qquad \text{if } \theta^k S \in \mathcal{D}_{\varepsilon/2} \text{ and } S' \subset S \text{ with } \nu(S') \geq \beta\nu(S) \text{ then } \theta^k S' \in \mathcal{D}_\varepsilon .$$

Fix $N > 0$ and let $\ell = N^6$. We will show that for sufficiently large N ,

$$(6.5) \qquad \|\sum_{i<\ell} a_i(\theta^i)\ (f\circ\mu - \sum\alpha_i\mu_i)\| \leq N^2 = o(N^3) = o(\sqrt{\ell})$$

for every set of numbers a_i with $|a_i| \leq 1$. To prove (6.5) let Ω be a fixed chain. We have to prove that $\|\sum_{i<\ell} a_i(\theta^i)\ (f\circ\mu - \sum\alpha_i\mu_i)\|_\Omega \leq N^2$. Without loss of generality we may assume that for every $0 \leq m \leq N$ there is S in Ω with $\nu(S) = \beta^m\nu(I) = p\beta^m$.

For $0 \leq m < N$, let Ω_m be the subchain of Ω of all S in Ω with $\beta^{m+1}\nu(I) \leq \nu(S) \leq \beta^m\nu(I)$ and let Ω_N be the subchain of Ω of all S in Ω with $\nu(S) \leq \beta^N\nu(I)$. Denote by g the function $g : R^p \to R$ given by $g(x_1, \ldots, x_p) = \sum\alpha_i x_i$. As $f-g$ is a Lipschitz function, there is $A > 0$ such that for all $i < \ell$,

$$\|a_i(\theta^i)\ (f\circ\bar{\mu} - g\circ\bar{\mu})\|_{\Omega_N} \leq pA\beta^N$$

and $$\|a_i(\theta^i)\ (f\circ\bar{\mu} - g\circ\bar{\mu})\|_{\Omega_m} \leq pA(\beta^m - \beta^{m+1}) \quad \text{for } m < N,\ i < \ell .$$

Thus

$$\|\sum a_i(\theta^i)\ (f\circ\bar{\mu} - g\circ\bar{\mu})\|_\Omega = \sum_{m\leq N} \|\sum_{i<\ell} a_i(\theta^i)\ (f\circ\bar{\mu} - g\circ\bar{\mu})\|_{\Omega_m}$$

$$\leq \sum_{m<N} \sum_{i<\ell} \|(\theta^i)\ (f\circ\bar{\mu} - g\circ\bar{\mu})\|_{\Omega_m} + p\ell A\beta^N .$$

For each fixed $m < N$ let S be the maximal element in Ω_m; in particular $\nu(S) = p\beta^m$. By (6.3)

$$\#\{k : \theta^k S \notin \mathcal{D}_{\varepsilon/2}\} \leq \frac{4(p-1)}{\varepsilon^2}\beta^{-m}$$

and for each k with $\theta^k S \in \mathcal{D}_{\varepsilon/2}$, (6.4) implies that $\Omega_m \subset \mathcal{D}_\varepsilon$ (i.e., $\forall S \in \Omega_m$, $S \in \mathcal{D}_\varepsilon$) and thus $\|(\theta^k)\ (f\circ\bar{\mu} - \Sigma\alpha_i\mu_i)\|_{\Omega_m} = 0$. Hence

$$\sum_{k<\ell} \|(\theta^k)\ (f\circ\bar{\mu} - \Sigma\alpha_i\mu_i)\|_{\Omega_m} \leq \frac{4(p-1)}{\varepsilon^2}\beta^{-m} pA(\beta^m - \beta^{m+1}) = \frac{4p(p-1)}{\varepsilon^2}A(1-\beta) .$$

Altogether,

$$\|\sum_{i<\ell} a_i(\theta^i)\ (f\circ\bar{\mu} - \Sigma\alpha_i\mu_i)\|_\Omega \leq pA\ell\beta^N + \frac{4p(p-1)}{\varepsilon^2}A(1-\beta)N .$$

As $N^6\beta^N \to 0$ as $N \to \infty$, we conclude that for sufficiently large N,

$$\|\sum_{i<\ell} a_i(\theta^i)\ (f\circ\bar{\mu} - \Sigma\alpha_i\mu_i)\|_\Omega \leq N^2$$

which proves (6.5).

Let $a_k \in \{0,1\}$ be given so as to satisfy (6.2), and consider the two market games

$$v_\ell = \sum_{k<\ell} a_k(\theta^k)\ (f\circ\bar{\mu})$$

and

$$w_\ell = \sum_{k<\ell} a_k(\theta^k)\ (\Sigma\alpha_i\mu_i) .$$

The market game w_ℓ is the characteristic function of an inessential market and thus $\phi w_\ell = \{w_\ell\}$. By the symmetry axiom $(\theta^k)\ u \in \phi((\theta^k)\ (f\circ\bar{\mu}))$ and together with the separability axiom $\Sigma a_k(\theta^k)\ u \in \phi u_\ell$. Therefore the two inequalities $\|v_\ell - w_\ell\| = o(\sqrt{\ell})$, and $\|\sum_{k<\ell} a_k(\theta^k)\ u - w_\ell\| > k\sqrt{\ell}\|u-w\|$ contradicts the continuity axiom unless $u = w$ which completes the proof of Lemma III.

Q.E.D.

E. Proof of the Theorem

By Lemma I, M^* is dense in M. By Lemmas II and III, $\phi(m)$ is uniquely determined for $m \in M^*$ and is a one-element set. By axiom I (continuity of ϕ), $\phi(m)$ is also uniquely determined for any m in M, and is a one-element set. Thus ϕ, if it exists, is unique, and mapps M into singletons in $P(FA)$.

Put $\phi(m)$ = the set consisting of the Shapley value of v_m (see Chapter 1 of [6] for the definition). Then this is also the set consisting of the unique competitive payoff in m (Theorem J and Proposition 32.5 in [6]). Since the set $\{v_m : m \in M\} \subset pNA$ (Theorem J in [6]), and the Shapley value satisfies all our axioms on pNA (Theorem B in [6]), we are done.

Q.E.D.

6. Remarks

(1) In our definition of M the population measure μ was held fixed. If we let it vary over all possible non-atomic measures to obtain the larger class of economies N $(M \subset N)$, then the same theorem holds on the domain N. (This follows trivially from our theorem.) Let G denote the set of all characteristic functions that arise from N, and A the set of all automorphisms (i.e., bimeasurable bijections) on $(T, \mathcal{C})$. Our theorem in particular implies that there is* a unique map $\phi : G \to FA$ which satisfies (for any v,w in G; θ in A; α,β in R_+):

(i) $\phi(\alpha v + \beta w) = \alpha\phi(v) + \beta\phi(w)$

(ii) $\phi(\theta v) = \theta\phi(v)$

(iii) $(\phi v)(T) = v(T)$

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*Dropping any one of (iii), (iv) would not affect this result.

(iv) $v \in Q \cap FA \Longrightarrow \phi(v) = v$

(v) ϕ is continuous

(vi) $v-w$ monotonic $\Longrightarrow \phi(v) - \phi(w)$ monotonic

(Observe that $\alpha v + \beta w \in G$, and $\theta G = G$, so that the (i), (ii) make sense.) Though this is a weaker result, we should point out that we are unable to prove it in a significantly simpler way.

(2) What if we drop the differentiability assumption (iv) on the utilities and look at the larger domain M' of economies $(M \subset M')$? Is there a map from M' into FA satisfying axioms (i)-(vi)? This had been an open question, until recently, when J. F. Mertens showed the existence of a continuous* value on a large space of games which includes M'. Consider the two maps on M':

$$m \to \{\text{the Mertens-value of } v_m\}$$

$$m \to \text{core } v_m .$$

It is well-known that, for $m \in M'$, core v_m is no longer necessarily a singleton-set (though it still coincides with the competitive payoffs). Furthermore it can be checked that the core map also satisfies our axioms. We conclude that our theorem (the uniqueness part) breaks down on M'. It is still an open problem whether there is a unique, continuous value on the space generated by characteristic functions arising from M'.

(3)(a) The continuity axiom has been used to obtain (where $m, \bar{m} \in M$, and $\{m_k\}$ is a sequence in M): (i) $\phi(m) \neq \emptyset$, (ii) if $d(m_k, m) \to 0$, $|\phi(m_k)| = 1$, and $\lim \phi(m_k)$ exists $m \in M \Longrightarrow \phi(m) = \lim \phi(m_k)$,

*Actually a value of norm 1.

(iii) for every $\varepsilon > 0$, there is a $\delta > 0$, such that: $d(m,\bar{m}) < \delta$, $\bar{m}$ is an inessential economy $\Longrightarrow d(\phi(m), \phi(\bar{m})) < \varepsilon$. Thus if one assumes that $\phi(m)$ is non-empty, the continuity axiom can be weakened in various obvious ways just to guarantee (ii) and (iii).

(b) For m and m' in M, say " $m \succ m'$ " if the "marginals" in m are at least as big as in m', i.e., for every S_1 and S_2 in $\mathcal{C}$, if $S_1 \subset S_2$, then $v_m(S_2) - v_m(S_1) \geq v_{m'}(S_2) - v_{m'}(S_1)$. A map $\phi : M \to P(FA)$ satisfies the positivity axiom if: $m \succ m' \Longrightarrow \phi(m) \subset \phi(m') + FA_+$. It satisfies the efficiency axiom if: $\alpha \in \phi(m) \Longrightarrow \alpha(T) = v_m(T)$. The mapping ϕ from M to $P(FA)$ that was uniquely characterized by axioms I - IV turns out be also efficient and positive.

In the presence of separability and efficiency, continuity is closely related to positivity. It is possible that our theorem is still true if one replaces continuity by positivity and efficiency.

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