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by

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Abstract

It is argued that although the pathological multiplicity of Nash equilibria of super games stated by the folk theorem can be removed by introducing limited observations into super games with a continuum of players, the consideration of super games in terms of the Nash equilibrium concept involves a more fundamental and conceptual difficulty.

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1. The folk theorem states that the set of all average payoffs of the Nash equilibria of a super game $\Gamma_p = (G, G, \ldots)$ coincides with the set of all feasible β-individually rational payoffs of the component game G. This implies that since the set of feasible β-individually rational payoffs is too large, this approach can not predict anything particular. Even if one could interpret meaningfully a particular equilibrium, the choice of it would be quite arbitrary. This theorem is still true even if we require subgame perfectness in Selten's sense [1975]. See Rubinstein [1979] and the Appendix of this note. Therefore this theorem embarasses some game theorists who expect that the approach in terms of super games can capture the dynamics in repeated situation.[1] However the capture of the dynamics in repeated situations would be the most important problem in game theory, economics and related fields. Thus the pathological multiplicity of equilibria stated by the folk theorem causes a great trouble to us.

This note shows that if limited observations are introduced into super games with a continuum of palyers, then the pathological multiplicity of Nash equilibria can be removed. However, a more careful consideration of these results says that the introduction of limited observations does not solve the whole difficulty. The difficulty of this approach exists on a more fundamental and conceptual level. That is, the main difficulty is the application of the Nash equilibrium concept to super games. This will be argued in Section 3.

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[1] Among some game theorists, it is believed that the Nash equilibrium concept is the most fundamental or very fundamental (in fact, the author did). See Nash [1951] and Luce and Raiffa [1957, Ch. 7]. It may be also a logical consequence that for game theorists who do not believe it the multiplicity of Nash equilibria would not be a so serious problem.

2. Consider a measure space of players $(N,\mathcal{B},\mu)$, where N is the set of players, $\mathcal{B}$ a σ-algebra with $\{i\} \in \mathcal{B}$ for all $i \in N$ and μ a measure on with $\mu(\{i\}) = 0$ for all $i \in N$ and $0 < \mu(N) < +\infty$. Every player $i \in N$ has a set of strategies S_i ($\neq \phi$) which is a subset of some set X . We assume that X is associated with some σ-algebra such that the set S of all measurable selections $s = (s_i)_{i\in N}$ from $(S_i)_{i\in N}$ is nonempty and $(s_{-i}, s_i) \in S$ for all $s_{-i} \in S_{-i}$ and all $s_i \in S_i$, where s_{-i} is the restriction of $s \in S$ on $N - \{i\}$ and S_{-i} is the set of restrictions of $s \in S$ on $N - \{i\}$. We call an $s \in S$ an outcome. Player i's $(i \in N)$ payoff function $h_i(s)$ is a real-valued function on S . We introduce the equivalence relation $\sim$ into S by $s^1 \sim s^2 \iff \mu(\{i \in N : s_i^1 = s_2^i\}) = \mu(N)$, and put $S^* = S/\sim$. We assume:

$$(1) \qquad h_i(s) \text{ is a function on } S_i \times S^* \text{ for all } i \in N .$$

Condition (1) means that player i's payoff function may depend upon his strategy and the other players' strategies but not upon any other individual player's strategy.

We have defined a strategic (or normal) form game G with a continuum of players, which is denoted by G . An outcome $\bar{s} \in S$ is said to be a Nash equilibrium in the game G iff for all $i \in N$,

$$(2) \qquad h_i(\bar{s}) \geqq h_i(\bar{s}_{-i}, s_i) \text{ for all } s_i \in S_i .$$

The super game Γ_p with perfect observations is defined as follows: Player i's super strategy σ_i is a sequence $\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots)$ such that $\sigma_i^1 \in S_i$ and for $t \geqq 2$, σ_i^t is a function from $S^{t-1} \equiv \overbrace{S \times \ldots \times S}^{t-1}$ to S_i . Let Σ_i be the set of all super strategies of player $i \in N$. Let Σ be the set of all selections $\sigma = (\sigma_i)_{i\in N}$ such that

$$(3) \qquad \sigma_i \in \Sigma_i \text{ for all } i \in N$$

$$(4) \qquad \sigma^1 = (\sigma_i^1)_{i\in N} \in S \text{ and for all } t \geqq 2, \ \sigma^t(s^1, \ldots, s^{t-1}) \in S \text{ for all } (s^1, \ldots, s^{t-1}) \in S^{t-1}.$$

We call $\sigma \in \Sigma$ a <u>super strategy combination</u>. Player i's <u>super payoff function</u> $H_i(\sigma)$ is the function which is defined by

$$(5) \qquad H_i(\sigma) = \underline{\lim}_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} h_i(s^t)$$

and

$$(6) \qquad \pi(\sigma) = (s^1, s^2, \ldots),$$

where π is the function on Σ such that

$$(7) \qquad \pi(\sigma) = (s^1, \ldots), \ \sigma^1 = s^1 \text{ and for all } t \geqq 2, \ \sigma^t(s^1, \ldots, s^{t-1}) = s^t.$$

A super strategy combination $\bar{\sigma} \in \Sigma$ is said to be a <u>Nash equilibrium</u> of Γ_p iff for all $i \in N$,

$$(8) \qquad H_i(\bar{\sigma}) \geqq H_i(\bar{\sigma}_{-i}, \sigma_i) \text{ for all } \sigma_i \in \Sigma_i.$$

We can easily prove the following proposition:

<u>Proposition 1 (The Folk Theorem)</u>: Let $\bar{s}$ be a β-<u>individually rational outcome</u> of G, i.e., for all $i \in N$, for some $\hat{s}^i_{-i} \in S_{-i}$, $h_i(\hat{s}^i_{-i}, s_i) \leqq h_i(\bar{s})$ for all $s_i \in S_i$. Then there exists a Nash equilibrium $\bar{\sigma}$ of Γ_p such that $\pi(\bar{\sigma}) = (\bar{s}, \bar{s}, \ldots)$.

Remark 1. If the limit average payoff functions are replaced by discounted-sum payoff functions, i.e.,

$$H_i(\sigma) = \sum_{n=1}^{\infty} r^{n-1} h_i(s^t) \ , \quad \pi(\sigma) = (s^1, \ldots)$$

and $0 < r < 1$,

then Proposition 1 is rewritten as follows:

Proposition 1': Let $\bar{s}$ be a strongly β-individually rational outcome of G , where $\bar{s}$ is called a strongly β-individually rational outcome iff there is an $\varepsilon > 0$ such that for every $i \in N$, there is an $\hat{s}^i_{-i} \in S_{-i}$ with $h_i(\hat{s}^i_{-i}, s_i) + \varepsilon \leqq h_i(\bar{s})$ for all $s_i \in S_i$. Then for some r $(0 < r < 1)$, there is a Nash equilibrium $\bar{\sigma}$ of the super game Γ_p with perfect observations and the discounted-sum payoff functions H^r_i such that $\pi(\sigma) = (\bar{s}, \ldots)$.

If we employ the overtaking criterion as the players' preferences (see Rubinstein [1979]), then the same result holds, i.e., for any strongly β-individually rational outcome $\bar{s}$, there is a Nash equilibrium $\bar{\sigma}$ of the super game such that $\pi(\bar{\sigma}) = (\bar{s}, \ldots)$.

Proposition 1 says that every β-individually rational outcome of G is attained by a Nash equilibrium of Γ_p , in other words, the set of all Nash equilibria of Γ_p corresponds to the set of β-individually rational outcomes of G . Therefore, this proposition implies that the set of Nash equilibria of Γ_p is quite large and almost the same as the set of feasible outcomes. Even if we require subgame perfectness in Selten's sense [1975], almost the same proposition holds. See the Appendix.

However, it is assumed in this super game Γ_p that every player can observe the outcome of G precisely after the game G is played. That is, every player can know every other player's strategy which was played in the previous stage. Since G has a large number of players, this assumption is not plausible if we take observation costs into account. Rather, we have to consider opposite situations in which any player can not observe any other individual player's behavior but players can observe mass behavior of the society.

To represent this idea, we introduce an observation structure $a = (a_i)_{i \varepsilon N}$, where a_i $(i \varepsilon N)$ is a partition of S such that for all $s \varepsilon T \varepsilon a_i$ and $s' \varepsilon T' \varepsilon a_i$,

$$\text{(9)} \qquad \text{if } s_i \neq s_i' \text{ or } h_i(s) \neq h_i(s') , \text{ then } T \neq T' .$$

Let $a_i(s)$ be the function on S such that $s \varepsilon a_i(s) \varepsilon a_i$ and we call $a_i(s)$ player i's observation function. Player i's observation function $a_i(s)$ means that after G is played and player i observes the outcome s, he gets the information $a_i(s)$, i.e., if $a_i(s) \neq a_i(s')$, then he can distinguish s from s'. Condition (9) means that every player can observe at least his strategy and his payoff.

We denote the super game with an observation structure $a = (a_i)_{i \varepsilon N}$ by Γ_a. In Γ_a, player i's super strategy σ_i must satisfy

$$\text{(10)} \qquad \text{for any } (s^1, \ldots, s^{t-1}) \text{ and } (\hat{s}^1, \ldots, \hat{s}^{t-1}) \varepsilon S^{t-1}$$
$$(t \geqq 2) , \text{ if } a_i(s^\nu) = a_i(\hat{s}^\nu) \text{ for all } \nu = 1, \ldots, t-1 ,$$
$$\text{then } \sigma_i^t(s^1, \ldots, s^{t-1}) = \sigma_i^t(\hat{s}^1, \ldots, \hat{s}^{t-1}) .$$

We denote by Σ_i^a the set of all super strategies of player i with condition (10). Let Σ^a be the subset of Σ whose σ satisfies condition (10) for all $i \varepsilon N$. Of course we regard Γ_a as a game in extensive form. The supposition of (10) that $a_i(s^\nu) = a_i(\hat{s}^\nu)$ for $\nu = 1, \ldots, t-1$ means that the positions $(s^1, \ldots, s^{t-1})$ and $(\hat{s}^1, \ldots, \hat{s}^{t-1})$ are in the same information set of player i. Therefore condition (10) requires that player i's local strategy is the same at every position in an information set. The payoff functions of Γ_a are the same as those of Γ_p.

If $a = (a_i)_{i \in N}$ consists of the finest partitions, i.e., $a_i = \{\{s\} : s \in S\}$ for all $i \in N$, then the game Γ_a is Γ_p.

Player i is said to have a <u>positive informational influence</u> in Γ_a iff there are sequences $\{s^1, \ldots, s^k\}$ and $\{\hat{s}^1, \ldots, \hat{s}^k\}$ such that

(11) $$s_j^1 \neq \hat{s}_j^1 \iff j = i$$

(12) $$\text{for any } t \leqq k-1, \; a_j(s^\nu) = a_j(\hat{s}^\nu) \text{ for all } \nu = 1, \ldots, t \text{ implies } s_j^{t+1} = \hat{s}_j^{t+1},$$

(13) $$\mu(\{j \in N : s_j^k \neq \hat{s}_j^k\}) > 0 .$$ [2]

If a is the finest partitions, then every player has a positive informational influence. Conversely if a consists of the coarsest partitions, i.e.,

$$a_i = \{\{s \in S : s_i = \bar{s}_i \;\&\; h_i(s) = h_i(\bar{s})\} : \bar{s} \in S\} \text{ for all } i \in N,$$

then any player has no positive informational influence by condition (1). Of course we can find more interesting examples for this case. Let us consider this case a little further. This means that any individual player

[2] It is easily verified that this definition is equivalent to that of "a positive informational influence" in Kaneko [1981].

cannot affect positive measure players through the observation structure a. This is equivalent to say that almost all players can not observe any particular player's behavior. Therefore this condition represents the idea of limited observations mentioned in a previous section.

A sequence $(\bar{s}^1, \bar{s}^2, \ldots) \in S^\infty$ is said to be an N.E. sequence iff $\bar{s}^t$ is a Nash equilibrium in G for all $t \geq 1$. A sequence $(\bar{s}^1, \bar{s}^2, \ldots) \in S^\infty$ is said to be an almost N.E. sequence iff for all $i \in N$,

$$\underline{\lim}\, \frac{1}{n} \sum_{t=1}^{n} h_i(\bar{s}_i^t) \geq \underline{\lim}\, \frac{1}{n} \sum_{t=1}^{n} h_i(\bar{s}_{-i}^t, s_i^t) \quad \text{for all} \quad (s_i^1, s_i^2, \ldots) \in S_i^\infty . \tag{14}$$

Of course an N.E. sequence is an almost N.E. sequence. If an almost N.E. sequence $(\bar{s}^1, \bar{s}^2, \ldots)$ is stationary, i.e., $\bar{s}^1 = \bar{s}^2 = \ldots$, then s^1 is a Nash equilibrium of G. Roughly speaking, an almost N.E. sequence consists of Nash equilibria of G except negligible part.

Proposition 2 (The Anti-Folk Theorem): Assume that any player has no positive informational influence in Γ_a. Then a super strategy combination $\bar{\sigma}$ is a Nash equilibrium of Γ_a if and only if $\pi(\bar{\sigma}) = (\bar{s}^1, \bar{s}^2, \ldots)$ is an almost N.E. sequence.

Remark 2. Since there is no proper subgame in Γ_a under the assumption of Proposition 2, a Nash equilibrium is also a subgame perfect equilibrium.

Remark 3. If the limit average payoff functions are replaced by discounted-sum payoff functions or the overtaking criterion, then Proposition 2 holds in the following stronger form:

Proposition 2': Under the assumption of Proposition 2, it holds that $\bar{\sigma}$ is a Nash equilibrium of Γ_a if and only if $\pi(\bar{\sigma}) = (\bar{s}^1, \bar{s}^2, \ldots)$ is an N.E. sequence.

Proof of Proposition 2. Necessity: Let $\pi(\bar{\sigma}) = (\bar{s}^1, \bar{s}^2, \ldots)$ be not an almost N.E. sequence. Then there is a sequence $(\hat{s}_i^1, \ldots)$ such that

$$\underline{\lim}\ \frac{1}{n}\sum_{t=1}^{n} h_i(\bar{s}^t) < \underline{\lim}\ \frac{1}{n}\sum_{t=1}^{n} h_i(\bar{s}_{-i}^t, \hat{s}_i^t) .$$

Let $\hat{\sigma}_i$ be the super strategy such that

$$(15) \qquad \hat{\sigma}_i^1 = \hat{s}_i^1 \text{ and for } t \geqq 2 , \quad \hat{\sigma}_i^t(s^1, \ldots, s^{t-1}) = \hat{s}_i^t \quad \text{for all } (s^1, \ldots, s^{t-1}) \in S^{t-1} .$$

Let $\pi(\bar{\sigma}_{-i}, \hat{\sigma}_i) = (\tilde{s}^1, \tilde{s}^2, \ldots)$. Note $\tilde{s}_i^t = \hat{s}_i^t$ for all $t \geqq 1$. It follows from the assumption of the proposition and condition (10) that for any $t \geqq 1$,

$$\mu(\{j \in N : a_j(\bar{s}^t) = a_j(\tilde{s}^t)\}) = \mu(N)$$
$$\mu(\{j \in N : \bar{s}_j^t = \tilde{s}_j^t \}) = \mu(N) .$$

Therefore it follows from condition (1) that

$$H_i(\bar{\sigma}_{-i}, \hat{\sigma}_i) = \underline{\lim}\ \frac{1}{n}\sum_{t=1}^{n} h_i(\tilde{s}^t) = \underline{\lim}\ \frac{1}{n}\sum_{t=1}^{n} h_i(\bar{s}_{-i}^t, \hat{s}_i^t)$$
$$> \underline{\lim}\ \frac{1}{n}\sum_{t=1}^{n} h_i(\bar{s}^t) = H_i(\bar{\sigma}) .$$

That is, $\bar{\sigma}$ is not a Nash equilibrium of Γ_a , which is a contradiction.

Sufficiency: Let player i change his strategy $\bar{\sigma}_i$ to $\hat{\sigma}_i$. Let $\pi(\bar{\sigma}_{-i}, \hat{\sigma}_i) = (\hat{s}^1, \hat{s}^2, \ldots)$. By the assumption of this proposition and (10), we have

$$\mu(\{j \in N : \bar{s}_j^t = \hat{s}_j^t\}) = \mu(N) \quad \text{for all } t \geqq 1 .$$

Therefore, by (1), $h_i(\hat{s}^t) = h_i(\bar{s}^t_{-i}, \hat{s}^t_i)$ for all $t \geqq 1$. This implies

$$H_i(\bar{\sigma}_{-i}, \hat{\sigma}_i) = \underline{\lim} \frac{1}{n} \sum_{t=1}^{n} h_i(\hat{s}^t) = \underline{\lim} \frac{1}{n} \sum_{t=1}^{n} h_i(\bar{s}^t_{-i}, \hat{s}^t_i)$$

$$\leqq \underline{\lim} \frac{1}{n} \sum_{t=1}^{n} h_i(\bar{s}^t) = H_i(\bar{\sigma}) .$$

Q.E.D.

3. Since the set of Nash equilibria of a strategic form game is usually not large, the anti-folk theorem implies that the pathological multiplicity of Nash equilibria stated by the folk theorem can be removed by the introduction of limited observations into super games. When the number of players is not small, it is quite natural to assume limited observations. Therefore it looks as if the problem is solved at least in games with a large number of players.

However, this result does not mean that we have solved the whole difficulty in super games. The difficulty of this approach is on a more fundamental and conceptual level. In fact, the application of the Nash equilibrium concept to super games itself involves a conceptual problem.

Now we are going to reflect upon the foundation (or the meaning) of the Nash equilibrium concept. In the following, two possible interpretations of it are provided but it is argued that both of them fail to explain fully the Nash equilibrium concept in super games.

The first one is the complete information interpretation:

(1) The game is completely one-shot game. Every player knows the rules of the game and the utility functions of all the players, and he knows the others know these and so on. That

is, the rules of game and the utility functions are common knowledge among the players. Under this common knowledge assumption, every player calculates a Nash equilibrium as a possible solution before the game is actually played.

From this viewpoint, a super game is regarded as a large one-shot game when the Nash equilibrium concept is applied to the super game.[3] Furthermore, from this viewpoint, the Nash equilibrium makes sense only if it is unique or if the game has a solvability property in some sense (see Nash [1951] or Luce and Raiffa [1957, Ch. 7]). That is, this argument requires that every player calculate the Nash equilibrium as the solution on condition that every other player also calculate the Nash equilibrium as the solution, and that every player reach the same (or in a weak sense) conclusion. Therefore if the game has no solvability property, then any player can reach no definite conclusion. The folk theorem implies that the super game with perfect observations does not satisfy any solvability property. Therefore the Nash equilibrium concept does not make sense in super games with perfect observations from the viewpoint of the complete information interpretation. Next, since the set of Nash equilibria of the component game G is usually not large, the anti-folk theorem implies that the super game with limited observations does not necessarily violate the solvability requirement. But the problem is the common knowledge assumption. The common knowledge assumption is quite strong. How do the players acquire the common knowledge? Is it a gift of God? No, the only meaningful answer is that after the game has been played many times, they acquire the common knowledge from the history.[4] However, the anti-folk

[3] See Aumann-Maschler [1972, p. 58].

[4] Of course, the author does not think that it is inadequate to assume the

theorem requires that every player can get only aggregated information, not precise information. In this case, the players can not acquire the common knowledge from the history even after the game has been played many times, because of limited observations. The common knowledge assumption is more compatible with the case of perfect observations. But we argued, in that case the folk theorem denied the application of the Nash equilibrium concept.

One may insist that if some stronger and appropriate condition was imposed on the Nash equilibrium concept in super games, then the multiplicity of Nash equilibria could be removed even in the case of perfect observations. But this is not an essential solution. Because even if we succeeded in it, the common knowledge assumption would be quite strong and still remain an unexplained assumption.

The second one is the naive interpretation:

(2) The game is repeated many times and every player has learned the society's responses to his actions. According to this knowledge, every player tries to maximize his utility. A Nash equilibrium is a possible stationary state in such a repeated situation.

From this viewpoint, we may understand the Nash equilibrium concept in the component game G but can not do it in the super game Γ_a (or Γ_p). If we interpret the Nash equilibrium in Γ_a from this viewpoint, we have to think that Γ_a is also repeated. Then we have the 2nd order super game $\Gamma^2 = (\Gamma_a, \Gamma_a, \ldots)$. Furthermore if we apply the Nash equilibrium

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the common knowledge assumption in a very simple game, in which the number of players is very small and the rules of the game are also very simple. However the author does not think that the common knowledge assumption implies necessarily the Nash equilibrium. In a forthcoming paper, the author will discuss it.

to Γ^2 , then we have to think of the 3rd order super game $\Gamma^3 = (\Gamma^2, \Gamma^2, \ldots)$. Clearly this argument is nonsense. From this viewpoint, the Nash equilibrium concept is not applicable to the super game Γ_a (Γ_p) .

Thus the real difficulty of the folk theorem is the application of the Nash equilibrium concept to super games as a solution concept. We can conclude from the above consideration that the Nash equilibrium concept is not the most fundamental solution concept which is applicable to every game situation, but one which is applicable to restricted and appropriate situations. One consistent solution to the above difficulty is to construct a new solution theory for repeated situations. It is desirable that the new theory can explain the Nash equilibrium of the component game in some sense. One example for such theories is provided by Kaneko [1981], which deals with the 2nd viewpoint and is closely related to von Neumann and Morgenstern's "standards of behavior," and which shows that the Nash equilibrium makes sense in the component game under a very restricted informational condition.

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APPENDIX

Let us consider the super game Γ_p with perfect observations and let $\Gamma_p^{h_t}$ be the subgame of Γ_p which is determined by a <u>history</u> $h_t = (\hat{s}^1, \ldots, \hat{s}^t)$. Note that every subgame of Γ_p is isomorphic to Γ_p , i.e., it has the same structure as Γ_p as a game in extensive form. For a given super strategy combination $\bar{\sigma}$ of Γ_p , we define the <u>induced super strategy combination</u> $\hat{\sigma}$ of $\bar{\sigma}$ to $\Gamma_p^{h_t}$ by

$$\hat{\sigma} = (\hat{\sigma}^{t+1}, \hat{\sigma}^{t+2}, \ldots) ,$$

$$\hat{\sigma}^{t+1} = \bar{\sigma}^{t+1}(\hat{s}^1, \ldots, \hat{s}^t) ,$$

$$\text{for all } \nu \geqq 2 , \quad \hat{\sigma}^{t+\nu}(s^{t+1}, \ldots, s^{t+\nu-1})$$
$$= \bar{\sigma}^{t+\nu}(\hat{s}^1, \ldots, \hat{s}^t, s^{t+1}, \ldots, s^{t+\nu-1})$$
$$\text{for all } (s^{t+1}, \ldots, s^{t+\nu-1}) \in S^{\nu-1} .$$

A super strategy combination $\bar{\sigma}$ is called a <u>subgame perfect equilibrium</u> of Γ_p iff for any $t \geqq 0$ and any history $h_t = (s^1, \ldots, s^t)$, the induced super strategy combination $\hat{\sigma}$ of $\bar{\sigma}$ to the subgame $\Gamma_p^{h_t}$ is a Nash equilibrium of $\Gamma_p^{h_t}$. Here if $t = 0$, then $\Gamma_p^{h_t}$ is Γ_p itself.

<u>Proposition 1" (The Perfect Folk Theorem)</u>: Assume that there are m and M such that $m \leqq h_i(s) \leqq M$ for all $s \in S$ and all $i \in N$. Let $\bar{s}$ be a strongly β-individually rational outcome of G . Then there exists a subgame perfect equilibrium $\bar{\sigma}$ of Γ_p such that $\pi(\bar{\sigma}) = (\bar{s}, \bar{s}, \ldots)$.

This proposition is slightly different from Rubinstein's [1979]. For completeness we also prove this proposition, but the following proof is a slight modification of Rubinstein's.

Proof. Since $\bar{s}$ is strongly β-individually rational, there is an $\varepsilon > 0$ such that for every $i \in N$, there is an $\hat{s}^i_{-i} \in S_i$ with $h_i(\hat{s}^i_{-i}, s_i) + \varepsilon \leqq h_i(\bar{s})$ for all $s_i \in S_i$. Let K be an integer such that $\varepsilon \cdot K \geqq M-m$.

We define $\bar{\sigma} = (\bar{\sigma}^1, \ldots)$ as follows:

$$\bar{\sigma}^1 = \bar{s}\ , \quad \text{and for } t \geqq 2 \text{ and } j \in N$$

$$\bar{\sigma}^{t+1}_j(s^1, \ldots, s^t) = \begin{cases} \hat{s}^i_j & \text{if there is a } k \text{ with } t-K+1 \leqq k \leqq t \\ & \text{such that for some } i \neq j\ ,\ s^k_{-i} = \bar{s}_{-i}\ \& \\ & s^k_i \neq \bar{s}_i \text{ and } s^{k'} = \bar{s} \text{ for all } k' \text{ with} \\ & t-K+1 \leqq k' < k\ , \\ \bar{s}_j & \text{otherwise.} \end{cases}$$

It is clear that $\pi(\bar{\sigma}) = (\bar{s}, \bar{s}, \ldots)$.

Let us consider a subgame $\Gamma_p^{h_t}$ $(h_t = (\hat{s}^1, \ldots, \hat{s}^t))$, and let $\hat{\sigma}$ be the induced super strategy combination of $\bar{\sigma}$ to $\Gamma_p^{h_t}$. It is easily verified that $\pi(\hat{\sigma}) = (\hat{s}^{t+1}, \hat{s}^{t+2}, \ldots)$ satisfies

$$\hat{s}^\nu = \bar{s} \quad \text{for all } \nu \geqq t+K+1\ .$$

Therefore player i's induced super payoff function of $\Gamma_p^{h_t}$ is

$$\hat{H}_i(\hat{\sigma}) = \underline{\lim}\, \frac{1}{n} \sum_{\nu=1}^{n} h_i(\hat{s}^{t+\nu}) = h_i(\bar{s})\ .$$

Suppose that a player i changes his super strategy $\hat{\sigma}_i$ to $\tilde{\sigma}_i$ in $\Gamma_p^{h_t}$.

Let $\pi(\hat{\sigma}_{-i}, \tilde{\sigma}_i) = (\tilde{s}^{t+1}, \tilde{s}^{t+2}, \ldots)$. Then there are two cases:

(A) there is a finite ν_0 such that

$$\tilde{s}^{\nu} = \bar{s} \quad \text{for all} \quad \nu \geqq \nu_0 .$$

(B) there is an infinite sequence $\{\nu^{\tau}\}$ such that

$$\nu^{\tau+1} \geqq \nu^{\tau} + K + 1 \quad \text{for all} \quad \tau \geqq 1 ,$$

$$\tilde{s}_{-i}^{\nu^{\tau}} = \bar{s}_{-i} \quad \text{and} \quad \tilde{s}_i^{\nu^{\tau}} \neq \bar{s}_i \quad \text{for all} \quad \tau \geqq 1 ,$$

$$\text{for all} \quad \tau \geqq 1 , \quad \tilde{s}_{-i}^{\nu} = \hat{s}_{-i}^{i} \quad \text{for all} \quad \nu \quad \text{such that} \quad \nu^{\tau}+1 \leqq \nu \leqq \nu^{\tau}+K ,$$

$$\text{for all} \quad \tau \geqq 1 , \quad \tilde{s}^{\nu} = \bar{s} \quad \text{for all} \quad \nu \quad \text{such that} \quad \nu^{\tau}+K+1 \leqq \nu \leqq \nu^{\tau+1}-1 .$$

In the case (A) it is clear that $\hat{H}_i(\hat{\sigma}_{-i}, \tilde{\sigma}_i) = h_i(\bar{s}) = \hat{H}_i(\hat{\sigma})$.

Consider the case (B). $\hat{H}_i(\hat{\sigma}_{-i}, \tilde{\sigma}_i)$ is calculated as follows:

$$\hat{H}_i(\hat{\sigma}_{-i}, \tilde{\sigma}_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} h_i(s^{t+\nu}) = \lim_{n\to\infty} \frac{1}{\nu^{n+1}-t} \left[\sum_{\nu=t+1}^{\nu^1-1} h_i(\tilde{s}^{\nu}) + \sum_{\tau=1}^{n} \sum_{\nu=\nu^{\tau}}^{\nu^{\tau+1}-1} h_i(\tilde{s}^{\nu}) \right]$$

$$= \lim_{n\to\infty} \frac{1}{\nu^{n+1}-t} \sum_{\tau=1}^{n} \left[h_i(\tilde{s}^{\nu^{\tau}}) + \sum_{t=\nu^{\tau}+1}^{\nu^{\tau}+K} h_i(\tilde{s}^{t}) + \sum_{t=\nu^{\tau}+K+1}^{\nu^{\tau+1}-1} h_i(\tilde{s}^{t}) \right]$$

$$\leqq \lim_{n\to\infty} \frac{1}{\nu^{n+1}-t} \sum_{\tau=1}^{n} [h_i(\bar{s}) + M - m + K(h_i(\bar{s}) - \varepsilon) + (\nu^{\tau+1} - 1 - \nu^{\tau} - K)h_i(\bar{s})]$$

$$\leqq \lim_{n\to\infty} \frac{1}{\nu^{n+1}-t} \sum_{\tau=1}^{n} (\nu^{\tau+1} - \nu^{\tau})h_i(\bar{s}) = h_i(\bar{s}) = \hat{H}_i(\hat{\sigma}) .$$

Therefore $\hat{\sigma}$ is a Nash equilibrium in $\Gamma_p^{h_t}$.

Q.E.D.