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THE OPTIMAL PROGRESSIVE INCOME TAX

—THE EXISTENCE AND THE LIMIT TAX RATES

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by

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ABSTRACT

The purpose of this paper is to consider the problem of optimal income taxation in the domain of progressive (convex) income tax function. This paper proves the existence of an optimal tax function and that the optimal marginal and average tax rates tend asymptotically to 100 percent as income level becomes arbitrarily high.

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1. Introduction

1. It is commonly recognized that there exists a conflict between equity and efficiency in income taxation. Progressive income taxation is a device to enhance the degree of equity. A highly progressive income taxation however, obstructs labor incentives of individuals with high productivities or earning abilities, and so, it decreases the degree of efficiency. Therefore optimal income taxation should be determined to harmonize these contrary norms. To discuss this problem, Mirrlees provided a model of optimal income taxation in his pioneering work (1971). Since Mirrlees (1971), many authors have been considering this problem in variations of Mirrlees' model.

There are two directions of studies from a technical point of view. One is the approach initiated by Mirrlees himself, where income tax functions of any forms are allowed. In this approach, the method of the calculus of variations or Pontryagin's Maximum Principle is used. Another one is the approach initiated by Sheshinski (1972), where only linear tax functions are allowed. The first approach is superior to the second to consider the progressiveness of an optimal income tax function, though there are many problems arising in the optimal taxation which can be discussed in the domain of linear tax functions without losing the essence. The first approach, however, has a serious defect as Mirrlees himself pointed out. It is the lack of mathematical rigorousness. That is, although several necessary conditions for an optimal tax function are discussed under the assumption of the existence of it, the existence has never been verified in the works of this approach. The differentiability of several variables derived from an optimal tax function is also doubtful.[1]

[1] Mirrlees (1971, Sec. 4) pointed this out.

Furthermore, if we allow nonconvex tax functions, there exists multiple equilibria for each tax function with different welfare levels in general. Therefore the objective function of government becomes a set-valued function. This means that the usual maximal principle or the variational method is not applicable to the problem in this case. See Kaneko (1980b). The second approach does not have such a weak point, though we can not fully consider the progressiveness of an optimal incom tax function because of the assumption of linearity.

2. This paper considers the problem of optimal income taxation in the domain of progressive (convex) tax functions in an economy with a continuum of individuals.[2] Of course, the progressive tax functions include the linear and proportional tax functions. The model we provide is a variation of Mirrlees' model, while we take a public good--government's service--into account explicitly. One of the main results of this paper is to prove the existence of an optimal income tax function in the model. The merit of our approach is to be able not only to discuss rigorously the existence of an optimal income tax function but also to consider it's shape to a certain extent, e.g., progressiveness in a different sense broader than convexity. Since convexity of an optimal tax function is imposed as an assumption, we can not discuss it in our model. But convexity is no more than one concept of progressiveness in a weak and local sense.

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[2]Kaneko (1980b) established the existence of an optimal tax schedule in a general equilibrium context with a finite number of individuals but without convexity on the tax functions. But the problem of optimal tax rates for rich people can not be treated appropriately in a model with a finite number of individuals. For this purpose, the model with a continuum of individuals is appropriate.

Another main result of this paper is to prove that the optimal marginal and average tax rates tend asymptotically to 100 percent as income level becomes arbitrarily high. This result can be interpreted as progressiveness in a stronger sense than convexity.

Another important point of this paper is to employ the Nash social welfare function as the welfare criterion, i.e., government's objective function in the model. The Nash social welfare function has been recently discussed by many authors, e.g., Kaneko and Nakamura (1979a, 1979b), Kaneko (1980a, 1981), Brock (1979), Wagner (1980), Kim and Roush (1981) and others. One purpose of this paper is an explicit application of this concept to the income taxation problem.

This paper is written as follows. Section 2 presents a model of optimal income taxation and states the existence of an optimal tax function under relatively weak assumptions. The proof of it is given in Section 4. Section 3 provides the limit theorem that the optimal marginal and average tax rates tend asymptotically to 100 percent as income level becomes arbitrarily high. This theorem is proved in Section 5 under stronger assumptions than those used in the proof of the existence theorem.

2. Model and Problem

3. $(N, \mathcal{B}, \mu)$ is a measure space of all individuals, where N is the set of all individuals, $\mathcal{B}$ a σ-algebra of subsets of N and μ a measure on $\mathcal{B}$ with $0 < \mu(N) < +\infty$. We assume that $\{i\} \in \mathcal{B}$ and $\mu(\{i\}) = 0$ for all $i \in N$.

We assume that leisure, a consumption good and a public good enter the individuals' utility functions $U^i(a,b,Q)$ $(i \in N)$, where a denotes leisure time, b a level of the consumption good and Q a level of the public good supplied by an economic agent called "government." Every U^i $(i \in N)$ is defined on $Y \equiv [0,L] \times E^2_+$ where $L > 0$ is the initial endowment of leisure time and E^2_+ the nonnegative orthant of the 2-dimensional Euclidean space E^2. We assume:

(A) For all $i \in N$, $U^i(a,b,Q)$ is a strictly increasing and strictly quasi-concave function on $\tilde{Y} \equiv \{(a,b,Q) \in Y : b > 0\}$ with $U^i(a,0,Q) = U^i(0,0,0)$ for all $(a,0,Q) \in Y - \tilde{Y}$.

(B) For all $i \in N$, $U^i(a,b,Q)$ is a continuous function on Y.

(C) For each $(a,b,Q) \in Y$, $U^i(a,b,Q)$ is a measurable function of i.

(D) For each $i \in N$, $U^i(a,b,Q)$ is bounded, i.e., for some M^i, $U^i(a,b,Q) \leqq M^i$ for all $(a,b,Q) \in Y$.[3]

It is not necessary to explain the strict monotonicity and the strict quasi-concavity in assumption (A). The assumption that $U^i(a,0,Q) = U^i(0,0,0)$ says that if individual i's consumption level is zero, then his utility level is equal to that of the worst state $(0,0,0)$. This would not be a strange condition. Furthermore, it may be natural to assume that $U^i(0,b,Q) = U^i(0,0,0)$ for all $(0,b,Q)$, but since we do not need this assumption, we do not assume it. Assumption (B) is a technical condition. Assumption (C) is also a technical condition, which is familiar in the theory of market with a continuum of traders.[4] Assumption (D) is the boundedness of utility functions. The boundedness is justified by St. Petersburg paradox and is thought of as a natural assumption.[5]

Of course, we assume that the utility functions $U^i(a,b,Q)$ $(i \in N)$

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[3]The boundedness from below follows assumption (A).

[4]See Aumann (1964).

[5]See Owen (1968, Ch. VI) or Aumann (1977).

are measurable utility functions in the sense of von Neumann and Morgenstern (1953) or Herstein and Milnor (1953). In this paper we do not consider any probability mixtures but only "pure" states. This restriction does not lose generality.

We allow positive linear transformations of U^i $(i \in N)$. Exactly speaking, when $\alpha(i)$ and $\beta(i)$ are measurable functions of i with $\alpha(i) > 0$ for all $i \in N$, we can employ

$$V^i(a,b,Q) = \alpha(i)U^i(a,b,Q) + \beta(i) \quad \text{for all } i \in N \text{ and all } (a,b,Q) \in Y \tag{1}$$

as the same utility functions. Hence we can obtain utility functions V^i $(i \in N)$ satisfying the following (2) by certain positive linear transformations:

$$V^i(a,b,Q) - V^i(0,0,0) \text{ is uniformly bounded, i.e., there is an } M \text{ such that } V^i(a,b,Q) - V^i(0,0,0) \leqq M \text{ for all } i \in N \text{ and all } (a,b,Q) \in Y. \tag{2}$$

We shall confine ourselves to the class of utility functions satisfying (2) for the sake of integrability of the Nash social welfare function.

Furthermore we assume:

(E) For any $\varepsilon > 0$, there is a $\delta > 0$ such that $U^i(\varepsilon,\varepsilon,0) - U^i(0,0,0) > \delta$ for all $i \in N$.

This assumption is a kind of uniformness of utility functions. This assumption is not necessarily preserved for arbitrary positive linear transformations. In the following we allow only the following positive linear transformations:

$$\text{For some } \varepsilon > 0 \text{ and } M > 0 , \quad \varepsilon \leqq \alpha(i) \leqq M \text{ for all } i \in N . \tag{3}$$

That is, when $\alpha(i)$ satisfies (3), the new V^i obtained from U^i which satisfies (2) and (E) also satisfies (2) and (E).[6]

The following lemma shall be necessary to define the Nash social welfare function. The proof is standard and so, we omit it.

Lemma 1: Let $(a(i), b(i), Q(i))$ be a measurable function of i such that $(a(i), b(i), Q(i)) \in Y$ for all $i \in N$. Then $U^i(a(i), b(i), Q(i))$ is a measurable function of i.

4. Each individual $i \in N$ owns a labor production function $f^i(x)$. That is, if he works for time x, he can provide a quantity of labor $f^i(x)$. For simplicity, we assume that $f^i(x)$ coincides with the quantity of the consumption good produced by his labor $f^i(x)$, being independent of the other individuals' labors.[7] We call f^i the null labor production function iff $f^i(x) = 0$ for all $x \in [0,L]$. We assume that if f^i is nonnull, then f^i satisfies:

(F) There is an L^i $(0 < L^i < L)$ such that $f^i(x)$ is monotonically increasing on $[0, L^i]$ and is nonincreasing on $[L^i, L]$ with $f^i(0) = 0$.

(G) $f^i(x)$ is a continuous and concave function.

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[6] Assumptions (A), (B) and (C) are preserved for arbitrary positive linear transformations.

[7] For example, labor can be measured in terms of the unit of man-power/hour.

Furthermore, we assume:

(H) For each $x \in [0,L]$, $f^i(x)$ is a measurable function of i .

(I) $\int_N f^i(L^i)d\mu < +\infty$ and for some $c_0 > 0$, $c_0 \leqq L^i \leqq L - c_0$ for all $i \in N$, where L^i is attached appropriately to every i whose f^i is null.[8]

Assumption (A) means that there is an interior point L^i at which i's labor productivity is saturated, i.e., his marginal productivity is zero. Assumption (G) is a standard condition. Assumption (H) is just a technical condition. Assumption (I) is the integrability of $f^i(L^i)$ and a kind of uniformness of L^i .

The government produces the public good using the consumption good as input. We assume that the public good is measured in terms of the consumption good needed to produce it. In other words, the cost function of the public good is $C(Q) = Q$. The government's expenditure coincides with Q when it plans to supply Q-amount of the public good. The government's revenue is determined by a tax function and a level of the public good.[9]

5. Now we are in a position to discuss the problem of optimal taxation.

A *tax function* T is a real-valued function on the set of nonnegative real numbers E_+ which satisfies

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[8] It is assured by (A), (G) and (H) that L^i and $f(x(i))$ are measurable functions of i , where $x(i)$ is a measurable function with $x(i) \in [0,L]$ for all $i \in N$.

[9] Ito and Kaneko (1978) showed that when cost functions of public goods are linearized by measuring the public goods in terms of the costs themselves, ratio equilibrium is invariant for such a transformation but not Lindahl equilibrium. The equilibrium concept of this paper has the same property, i.e., is invariant for the linearization of the cost function of the public good. Hence our assumption $C(Q) = Q$ does not lose any generality.

$T(y)$ is a monotonically nondecreasing and convex function such that $T(y) \leqq y$ for all $y \in E_+$.[10, 11] (4)

We denote, by $\mathcal{T}$, the set of all tax functions.

A tax function $T(y)$ means that when an individual i works for time x and earns income $y = f^i(x)$, he must pay income tax $T(y) = T(f^i(x))$ to the government. Hence T must satisfy $y \geqq T(y)$ for all $y \in E_+$. Note that negative taxes, i.e., subsidies are allowed. We take income redistribution directly into account besides the supply of the public good. Convexity of tax functions means just the progressiveness of income taxation in a local and weak sense. This paper allows only progressive (convex) income tax functions. Of course, $\mathcal{T}$ includes the proportional and linear tax functions.

Here we should give a comment on this formulation. Seade (1977), Ordover and Phelps (1979) and others showed that the optimal marginal tax rate is zero at the maximal income level, which implies that the optimal tax function is not convex. They assumed, however, that there exists the maximal income group, and they derived the above result from this assumption. In our model, we do not assume the existence of the maximal income group, but rather we do the nonexistence of it. Our assumption is much better to consider the optimal tax rate for "rich" people. In our formulation, rich people are described as ones with arbitrarily high incomes. If the existence of the maximal income group is assumed, it represents only "relatively" rich people in the model. Furthermore we

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[10] This condition implies that $T(y)$ is a continuous function.

[11] The monotonicity is not essential, because we proceed the following discussion replacing it by the continuity.

shall prove in Theorem II that the optimal marginal and average tax rates tend asymptotically to 100 percent as income level becomes arbitrarily high under certain assumptions. This result can be interpreted as progressiveness in a different sense broader than convexity. This would justify the assumption of progressiveness (convexity) against the criticism from Seade and others' result.

Let us suppose that the government employs a tax function T and decides to supply Q-amount of the public good. When each individual $i \in N$ decides to work for time $x(i)$, his gross income is $f^i x(i) = f^i(x(i))$ and his disposable income is $f^i x(i) - T(f^i(x(i)))$. In the following, we denote $T(f^i(x(i)))$ by $Tf^i x(i)$ for notational simplicity. In this case, the government's revenue is $\int_N Tf^i x(i) d\mu$. Since Q is the government's expenditure, it must hold that

$$Q \leqq \int_N Tf^i x(i) d\mu \; . \tag{5}$$

With above explanation in mind, we provide the following definitions. Every individual i maximizes his utility function under the assumption that T and Q are fixed. That is, individual i chooses $x(i) \in [0,L]$ such that

$$\begin{aligned} &U^i(L - x(i),\ f^i x(i) - Tf^i x(i),\ Q) \\ &\qquad \geqq U^i(L-x,\ f^i(x) - Tf^i(x),\ Q) \quad \text{for all} \quad x \in [0,L] \; . \end{aligned} \tag{6}$$

In general, $x(i)$ is not uniquely determined by (6). Let $X(i)$ is the set of all solutions for (6). Then the following lemma holds:

<u>Lemma 2</u>. $X(i)$ is singleton if and only if $f^i(x) - Tf^i(x) > 0$ for some $x \in X(i)$.

The proof is not difficult.

We will choose one function $x(i)$ from $X(i)$ by

$$x(i) = \begin{cases} x & \text{if } X(i) \text{ is singleton and } X(i) = \{x\} \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Since $f^i(x) - Tf^i(x) = 0$ and $U^i(L-x,\ f^i(x) - Tf^i(x),\ Q) = 0$ for all $x \in X(i)$ if $X(i)$ is not singleton, the above selection by (7) would be natural. We shall employ this selection as a representative of $X(i)$ in the following. We call $(x(i))_{i\epsilon N}$ the <u>labor (time) supply schedule</u> for (T,Q) iff each $x(i)$ satisfies (6) and (7).

<u>Lemma 3</u>. The labor supply schedule $(x(i))_{i\epsilon N}$ for (T,Q) is uniquely determined and is a measurable function of i.

The proof is standard and so, we omit it.

We call (T,Q) a <u>feasible tax schedule</u> iff (5) holds for $(x(i))_{i\epsilon N}$ satisfying (6) and (7). Further if $Q = \int_N Tf^i x(i) d\mu$, then we call (T,Q) an <u>equilibrium tax schedule</u>. We denote, by F, the set of all feasible tax schedules. We call T a <u>feasible tax function</u> iff for some $Q \in E_+$, (T,Q) is a feasible tax schedule.

Since $(N, \mathcal{B}, \mu)$ is a non-atomic measure space, the influence of each individual's choice on the government's revenue is negligible. Hence an equilibrium tax schedule (T,Q) determines a Nash equilibrium $(x(i))_{i\epsilon N}$, i.e., for all $i \in N$,

$$U^i(L - x(i),\ f^i x(i) - Tf^i x(i),\ \int_N Tf^i x(i) d\mu)$$
$$\geqq U^i(L-x,\ f^i(x) - Tf^i(x),\ \int_{N-\{i\}} Tf^j x(j) d\mu + Tf^i(x)\mu(\{i\}))$$
for all $x \in [0,L]$

and

$$Q = \int_N Tf^i x(i)d\mu = \int_{N-\{i\}} Tf^j x(j)d\mu + Tf^i(x)\mu(\{i\}) \ .$$

This is the reason why we call such a (T,Q) an equilibrium tax schedule.

Let T_0 be the function such that $T_0(y) = 0$ for all $y \in E_+$. Then $(T_0, 0)$ is always a feasible tax schedule and an equilibrium tax schedule. $(T_0, 0)$ means that the government neither imposes any tax nor supplies the public good. We call $(T_0, 0)$ the <u>trivial tax schedule</u>. This is a feasible behavior of the government. Hence we have:

<u>Proposition 1</u>. There exist a feasible tax schedule and an equilibrium tax schedule.

Let $(x_0(i))_{i\varepsilon N}$ be the labor supply schedule for $(T_0, 0)$. Then we assume:

(J) $\int_N x_0(x)d\mu > 0$.

This assumption means that individuals work for their consumptions if the government neither imposes any tax nor supplies the public good.

The following proposition states that if T is a feasible tax function, it is always possible to achieve the equation of the government's revenue and expenditure.

<u>Proposition 2</u>. For every feasible tax function T, there exists a $Q \varepsilon E_+$ such that (T,Q) is an equilibrium tax schedule.

Further it is not difficult to verify the following proposition.

Proposition 3. Let T be a feasible tax function. Then:

(i) If $T(0) < 0$, then $T(y) > 0$ for some $y \in E_+$.

(ii) If $\left.\frac{d^-T}{dy}\right|_{y_0} = 1$, then $T(y_0) > 0$.[12]

6. We employ the Nash social welfare function for a measure space of individuals of Kaneko (1981) as the government's objective function. To define the Nash social welfare function, it is necessary to set the "origin" in our context. The concept plays the most important role in the theory of the Nash social welfare function. In the context of this paper, it is natural to set the origin of Y as the "origin," i.e.,

$$O(i) = (0,0,0) . \tag{8}$$

In other words, every individual has no leisure time, no consumption and no supply of the public good. This state is equivalent to that every individual must die.[13]

Let $\tau = (T,Q) \in \mathcal{F}$ and let $(x_\tau(i))_{i\in N}$ be the labor supply schedule for (T,Q). Then the Nash social welfare function $W(\tau) = W(T,Q)$ is given as

$$W(\tau) = W(T,Q) = \int_N \log[U^i(L - x_\tau(i),\ f^i x_\tau(i) - Tf^i x_\tau(i),\ Q) - U^i(0,0,0)]d\mu . \tag{9}$$

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[12]Since T is convex and continuous with $y \geqq T(y)$ for all $y \in E_+$, T has the derivatives on the left and the right $\frac{d^-T}{dy}$ and $\frac{d^+T}{dy}$ with $\left.\frac{d^-T}{dy}\right|_y \leqq \left.\frac{d^+T}{dy}\right|_y \leqq 1$ for all $y \in E_+$.

[13]In Kaneko and Nakamura (1979a), the necessity of this setting is discussed briefly. See Kaneko and Nakamura (1979a, Sec. 4).

This integral is bounded from above for all $\tau \in \mathcal{F}$ by (2). But it may be the case that $W(\tau) = -\infty$.

We can assume $U^i(0,0,0) = 0$ for all $i \in N$ without loss of generality. For simplicity, we may write

$$G^i(L - x_\tau(i),\ f^i x_\tau(i) - Tf^i x_\tau(i),\ Q)$$
$$= \log U^i(L - x_\tau(i),\ f^i x_\tau(i) - Tf^i x_\tau(i),\ Q)$$

in the following.

The government's objective is to maximize the Nash social welfare function $W(\tau)$. That is, the government chooses a feasible tax schedule $\tau^* = (T^*, Q^*)$ such that

$$\max_{\tau \in \mathcal{F}} W(\tau) = W(\tau^*) > -\infty \tag{10}$$

We call $\tau^* = (T^*, Q^*) \in \mathcal{F}$ satisfying (10) an <u>optimal tax schedule</u>.

The purpose of this paper is to investigate the optimal tax schedules. The first result of this paper is the existence of an optimal income tax schedule. The proof of Theorem I will be provided in Section 4.

<u>Theorem I (Existence Theorem)</u>. Under assumptions (A)-(J), there exists an optimal tax schedule $\tau^* = (T^*, Q^*)$, which is an equilibrium tax schedule.

<u>7</u>. The concept of optimal tax schedule defined in the above can be interpreted as a noncooperative equilibrium point of a game in extensive form in which the government and the individuals appear as players. The game is formulated as follows. First, the government decides and announces a tax schedule (T,Q) to the individuals. Then every individual

independently decides his labor time $x(i)$. The game tree is drawn in Figure 2. The individuals' payoffs are the utilities from (T,Q) and $x(i)$'s . If the government's revenue is smaller than it's expenditure Q , then it suffers a punishment P , which is sufficiently large. The government's payoff is

$$\text{the Nash social welfare function} - \delta P \,, \tag{11}$$

where $\delta = 1$ if the revenue is smaller than Q and $\delta = 0$ otherwise. Now we have a game in extensive form. Let τ^* be an optimal tax schedule and let $(x^*(i))_{i\varepsilon N}$ be the labor supply schedule for τ^* . Then $(\tau^*, (x^*(i))_{i\varepsilon N})$ is a subgame perfect equilibrium point of the game.[14]

8. In this subsection, we provide a simple but important necessary condition for an optimal tax schedule. For any $T \varepsilon \mathcal{T}$, the <u>minimal disposable income for</u> T is given as $-T(0)$ because of (4). Let $T(0) = 0$. Then the individuals with the null labor production function can earn no disposable income. They must "live" under no consumption. Is this tax function optimal? The answer is "no" under the condition that there exist such individuals with a positive measure.

<u>Proposition 4</u>. Assume (A)-(J) and that $\mu(\{i \varepsilon N : f^i \text{ is null}\}) > 0$. Then if (T^*, Q^*) is an optimal tax schedule, then the minimal disposable income for T^* is positive.

[14]For definition of subgame perfect equilibrium, see Selten (1975).

Proof. Let $T^*(0) = 0$. Then if f^i is null, his consumption must be zero. Let $(x^*(i))_{i\varepsilon N}$ be the labor supply schedule for (T^*, Q^*) . By assumption (A), $U^i(L - x^*(i), f^i x^*(i) - T^* f^i x^*(x), Q^*) = U^i(0,0,0) = 0$ and all i such that f^i is null. Since $\mu(\{i \varepsilon N : f^i \text{ is null}\}) > 0$, then

$$\int_N \log U^i(L - x^*(i), f^i x^*(i) - T^* f^i x^*(i), Q^*) d\mu = -\infty .$$

This contradicts (10). Q.E.D.

This proposition depends crucially upon the structure of the Nash social welfare function. If we employ another social welfare function, e.g., the utilitarian or the maximim, then the above proposition could be true under ad hoc assumptions.

3. The Limit Marginal and Average Income Tax Rates

9. In the previous section we have shown the existence of an optimal income tax schedule. The next task is to investigate the shape of the optimal income tax schedule. In this section, we consider the limit optimal marginal and average tax rates when income level becomes arbitrarily high. The result which we prove is the theorem that both the optimal marginal and average tax rates tend asymptotically to 100 percent. To prove this theorem, we need new assumptions and replace some of the assumptions of the previous section by stronger ones.

Initially we make the following assumptions:

(A') For all $i \in N$, $U^i(a,b,Q)$ is a strictly increasing, strictly concave and continuously differentiable function on $\tilde{Y}$ with $U^i(a,0,Q) = U^i(0,0,0)$ for all $(a,0,Q) \in Y - \tilde{Y}$.

(K) For all $i \in N$, $U^i(a,b,Q)$ is weakly separable with respect to (a,b) and Q, i.e., there are functions g^i and h^i such that $U^i(a,b,Q) = h^i(g^i(a,b),\ Q)$ for all $(a,b,Q) \in Y$.

(L) For each $Q \geqq 0$, $U^i(L - L^i,\ b,\ Q)$ converges uniformly to $M_i = \sup_b U^i(L - L^i,\ b,\ Q)$ as $b \to \infty$, i.e., for any $\varepsilon > 0$, there is a b_0 such that $U^i(L - L^i,\ b,\ Q) \geqq M_i - \varepsilon$ for all $b \geqq b_0$ and all $i \in N$.

(M) For each $Q \geqq 0$, $\left.\frac{1}{b}\cdot\frac{U_1^i}{U_2^i}\right|_{(L-L^i,b,Q)}$ converges uniformly to 0 as $b \to \infty$, i.e., for any $\varepsilon > 0$, there is a b_0 such that

$\left.\frac{1}{b}\cdot\frac{U_1^i}{U_2^i}\right|_{(L-L^i,b,Q)} < \varepsilon$ for all $b \geqq b_0$ and all $i \in N$.

Here $U_1^i = \partial U^i/\partial a$, $U_2^i = \partial U^i/\partial b$ and $U_3^i = \partial U^i/\partial Q$.

Although assumption (A') is stronger than (A), it is not yet strong. Assumption (K) means that the individual choice of leisure and consumption is not affected by the level of the public good. Since we already take income redistribution into account by tax functions, our public good is very pure, e.g., national defense, fire fighting, etc. Hence it would be natural to assume that the level of the public good does not affect the individual choice. Assumption (L) is a kind of uniform boundedness, which is stronger than assumption (D) and (2). When the utility functions U^i can be transformed into an identical one, this assumption is, of course, true. Assumption (M) means that although the marginal rate of substitution of leisure and consumption may tend to infinity as $b \to \infty$, the order is smaller than that of b . If this is not true, it happens that the marginal rate of substitution at $(L - L^i, b, Q)$ has the magnitude of the same order or a greater order than b , which would be implausible. The utility functions which are represented as

$$g^i(a,b) = a^\alpha + b^\beta \quad \text{for all} \quad b \geqq \text{some } \tilde{b} \quad \text{and} \quad 0 < \alpha,\beta < 1$$

satisfy (M), but the Cobb-Douglas type functions $g^i(a,b) = a^\alpha b^\beta$ do not satisfy this assumption. Therefore it is fair to say that assumption (M) would be natural but restricts our consideration to a certain extent.

Next we approximate the labor production functions by piecewise linear functions. Let $n(i)$ be a measurable function from N to E_+ . We assume:

(F') For all $i \in N$, f^i satisfies

$$f^i(x) = \begin{cases} n(i)x & \text{if } x \leqq L^i \\ n(i)L^i & \text{otherwise.} \end{cases}$$

(N) If $S \subset E_+$ and $\sigma(S) > 0$, then $\mu(\{i \in N : n(i)L^i \in S\}) > 0$, and for any $\varepsilon > 0$ there is a $\beta > 0$ such that $\lim_{\alpha\to\infty} \mu(\{i \in N : n(i)L^i > \alpha+\varepsilon\})/\mu(\{i \in N : n(i)L^i > \alpha\}) = \beta$.[15]

The function $n(i)$ assigns to each individual i his marginal productivity of labor. Assumption (F') means that f^i $(i \in N)$ is approximated by a piecewise linear function. See Figure 3. This implies assumptions

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[15] σ denotes the usual Lebesgue measure on E_+ .

(G) and (H). Assumption (N) means that the distribution of the abilities $n(i)$ overspreads everywhere of E_+, and that the density of $n(i)$ converges asymptotically to 0 but the speed of the convergence is not too rapid. When L^i is a constant, i.e., $L^i = L^0$, this limit property is satisfied by the Pareto distributions but not by the normal distributions. Therefore it is fair to say that this assumption restricts the applicability of our theory to a certain extent.

We are now in a position to state the main result of this section. The proof of the following theorem is provided in Section 5.

Theorem II (The Limit Tax Rates Theorem). Let (T^*, Q^*) be an optimal income tax schedule. Then it holds under assumptions (A')-(N) that

$$\lim_{y\to\infty} \frac{d^- T^*(y)}{dy} = \lim_{y\to\infty} \frac{d^+ T^*(y)}{dy} = 1 \tag{12}$$

and

$$\lim_{y\to\infty} \frac{T^*(y)}{y} = 1 \,. \tag{13}$$

Theorem II says that both the optimal marginal and average tax rates tend to 100 percent as income level tend to infinity. It should be, however, noted that the disposable income $y - T^*(y)$ is always a monotonically nondecreasing function of y.

10. The result (13) follows immediately from (12). See Subsction 17. Here we should give an intuitive sketch of the exact proof of the result (12) to help us to understand the essence of the proof because it is long and complicated. Suppose that the marginal tax rate does not tend to 1 as $y \to \infty$. In this case, the disposable income $y - T^*(y)$ tends to

infinity with the same order as that of y, i.e., $\lim(y - T^*(y))/y > 0$. Then if an individual i with very high ability works for time L^i, then his disposable income $n(i)L^i - T^*n(i)L^i$ can be also very large. This observation and assumption (M) imply that individuals with sufficiently high abilities work for L^i's. In this case, if the government increases the marginal tax rate a little in a range of sufficiently high incomes, then every individual i with an income in the range still works for L^i. That is, this change does not obstruct the labor incentives. Then the government's revenue increases and so, the government can increase the level of the public good. This increase in the level of the public good makes the utility levels of individuals with low incomes rise because the tax function in the range they face does not change. Even if the utility levels of the individuals with high incomes may decline, the decrements are smaller than the increments of the utility levels of the individuals with low incomes because of the boundedness of utility functions (assumptions (D) and (L)). Therefore the government can increase the social welfare by changing a little the marginal tax rate in the range of sufficiently high incomes. This is a contradiction to the optimality of (T^*, Q^*). Thus we have the result (12) of Theorem II.

11. Theorem II is quite different from the results of the preceding studies. Mirrlees (1971) showed many possibilities of the optimal marginal tax rate but concluded that they are much less than 100 percent. Furthermore Sheshinski (1972), Fair (1971), Feldstein (1973), Kesselman (1976) and others discussed the same problem and provided different upper-bounds of the optimal marginal tax rate. But they are also smaller than 100 percent. The apparent difference between our model and their models is

that we take a public good into account explicitly but not in the others. Another difference is in welfare criteria. However, these are not very crucial. The main reason for the difference is that we made the assumption of boundedness on the utility functions but not in the other papers. The boundedness is a natural assumption, which is justified by St. Petersburg paradox. This fact can be observed in the comparison between utility increments of poor people and utility decrements of rich people in the sketch of the previous subsection.

If only linear tax functions are allowed, then proposition 3(ii) states that the 100 percent marginal tax rate is impossible. This is one reason for the difference between our result and that of Sheshinski (1972) in which linear tax functions are only allowed.

The results (12) and (13) can be regarded as progressiveness in a different sense broader than convexity. In this sense, we can say that our model does not lose the essence of the optimal taxation problem assuming convexity.

12. When T^* satisfies (12) and (13) of Theorem II, there are two possibilities such that

$$\lim(y - T^*(y)) = +\infty \tag{14}$$

and

$$\lim(y - T^*(y)) < +\infty \; . \tag{15}$$

If (15) is true, the disposable income has the upper bound. In this case, even if an individual i with very high ability works for L^i, his disposable income can not exceed the upper bound. Therefore he would not work for long time. That is, the tax function obstructs the labor incentives of individuals with high abilities. Therefore we conjecture that (14) holds but not (15). Regretfully, the author has not succeeded in proving this conjecture. This is an open problem of great importance.

4. Proof of Theorem I

13. From (2) there exists an $M > 0$ such that $U^i(a,b,Q) \leqq M$ for all $(a,b,Q) \in Y$ and $i \in N$. Hence $W(\tau) = \int_N \log U^i(L - x_\tau(i), f^i x_\tau(i) - Tf^i x_\tau(i), Q)d\mu$ is bounded from above. That is,

$$\sup_{\tau \in F} W(\tau) < +\infty . \tag{16}$$

Furthermore the following lemma holds.

Lemma 4. There exists a feasible tax schedule τ with $W(\tau) > -\infty$.

Proof. Let us consider a sequence $\{T^\nu\}$ such that $T^\nu(y) = y/\nu$ for all $y \in E_+$ and all $\nu \geqq 1$. Let $(x_\nu(i))_{i\in N}$ be the labor supply schedule for $(T^\nu, 0)$. It is easily verified that $x_\nu(i)$ converges pointwise to $x_0(i)$, where $(x_0(i))_{i\in N}$ is the labor supply schedule for the trivial tax schedule $(T_0, 0)$. By assumption (I) and Lebesgue's dominated convergence theorem, we have

$$\lim \int_N x_\nu(i)d\mu = \int_N x_0(i)d\mu .$$

Since $\int_N x_0(i)d\mu > 0$ by (J), there is an integer ν_0 such that $\int_N x_{\nu_0}(i)d\mu > 0$. It follows from this and (7) that $\int_N T^{\nu_0} f^i x_{\nu_0}(i)\, d\mu > 0$. Let us consider another sequence $\{\tilde{T}^s\}$ such that $\tilde{T}^s(y) = y/\nu_0 - 1/s$ for all $y \in E_+$ and all $s \geqq 1$. Let $(\tilde{x}_s(i))_{i\in N}$ be the labor supply schedule for $(\tilde{T}^s, 0)$. Then it is also easy to see that $\tilde{x}_s(i)$ converges pointwise to $x_{\nu_0}(i)$ as $s \to \infty$. By assumption (I) and Lebesgue's dominated convergence theorem, we have

$$\lim \int_N \tilde{x}_s(i)d\mu = \int_N x_{\nu_0}(i)d\mu \ .$$

Therefore there is an integer s_0 such that $\int_N \tilde{T}^{s_0} f^i \tilde{x}_{s_0}(i)\, d\mu > 0$ This means that $(\tilde{T}^{s_0}, 0)$ is a feasible tax schedule. Since the minimal disposable income $-\tilde{T}^{s_0}(0)$ is positive, it holds by (A), (E) and (I) that

$$W(\tilde{T}^{s_0}, 0) \geqq \int_N \log U^i(L - L^i, -\tilde{T}^{s_0}(0), 0)d\mu > -\infty \ .$$

Q.E.D.

This lemma and (16) imply that there exists a sequence $\{\tau^s\} = \{(T^s, Q^s)\}$ such that

$$(T^s, Q^s) \in F \text{ for all } s \text{ and } \{W(\tau^s)\} \text{ is a non-decreasing sequence with } \lim W(\tau^s) = \sup W(\tau) \ . \tag{17}$$

Since $Q \leqq \int_N f^i(L^i)d\mu < +\infty$ for all $(T,Q) \in F$ by assumptions (F) and (I), every Q^s $(s = 1, \ldots,)$ belongs to a compact interval. Hence there is a convergent subsequence $\{Q^{s^\nu}\}$ of $\{Q^s\}$. Since $\{(T^{s^\nu}, Q^{s^\nu})\}$ also satisfies (17), we can assume without loss of generality that $\{Q^s\}$ itself converges to Q^* .

The purpose of subsections 13 and 14 is to show that we can choose a subsequence of $\{T^s\}$ which converges in a certain sense.

<u>Lemma 5</u>. $\inf_s T^s(0) > -\infty$.

<u>Proof</u>. Suppose $\inf_s T^s(0) = -\infty$. Let $\{(x_s(i))_{i\varepsilon N}\}$ be a sequence of labor supply schedule for τ^s $(s = 1, \dots,)$, i.e., each $x_s(i)$ satisfies (6) and (7) for τ^s . Then it is clear by assumption (F) that

$$x_s(i) \leqq L^i \quad \text{for all } s \text{ and } i \in N . \tag{18}$$

Let $\tilde{T}^s(y) = y + T^s(0)$ for all $y \in E_+$. Of course, these $\tilde{T}^s$ $(s=1, \dots,)$ satisfy (4), i.e., $\tilde{T}^s \in \mathcal{T}$ for all s . Since T^s is convex and $T^s(y) \leqq y$ for all $y \in E_+$, we have

$$T^s(y) \leqq \tilde{T}^s(y) \quad \text{for all } y \in E_+ . \tag{19}$$

It follows from (18) and (19) that

$$\int_N \tilde{T}^s f^i(L^i) d\mu \geqq \int_N T^s f^i x_s(i) d\mu .$$

The left-hand term of this inequality is rewritten as

$$\int_N (f^i(L^i) + T^s(0)) d\mu = \int_N f^i(L^i) d\mu + T^s(0)\mu(N) .$$

By assumption (I), there is an s for which this value is negative. Since (T^s, Q^s) is a feasible tax schedule, it holds that

$$Q^s \leqq \int_N T^s f^i x_s(i) d\mu \leqq \int_N f^i(L^i) d\mu + T^s(0)\mu(N) < 0 .$$

This is a contradiction. Q.E.D.

14. Let $K = \inf_s T^s(0)$. Let k be an arbitrary positive integer. Then we define $C[0,k]$ by

$$C[0,k] = \{t : t \text{ is a continuous, convex and non-decreasing function on the interval } [0,k] \text{ which has } \left.\frac{d^-t}{dy}\right|_k \leqq 1 \text{ and satisfies } K \leqq t(y) \leqq y \text{ for all } y \in [0,k]\} . \tag{20}$$

It is not difficult to verify the following lemma.

Lemma 6. Let t^s be the restriction of T^s on $[0,k]$ $(s = 1, \ldots,)$. Then every t^s belongs to $C[0,k]$.

We introduce the topology defined by the distance $d(t_1, t_2) = \sup_{y\varepsilon[0,k]} |t_1(y) - t_2(y)|$ into $C[0,k]$. Then the following lemma holds:

Lemma 7. $C[0,k]$ is a compact set for any k .

Proof. By Ascoli's theorem (Simmons (1963, p. 126, Theorem C)), it is sufficient to show that $C[0,k]$ is closed, bounded and equicontinuous.

Initially we show that $C[0,k]$ is bounded and equicontinuous. It is clear by definition that

$$K \leqq t(y) \leqq k \quad \text{for all } y \in [0,k] \text{ and all } t \in C[0,k] .$$

Hence $C[0,k]$ is bounded. Since any $t \in C[0,k]$ is convex and nondecreasing with $\left.\frac{d^-t(y)}{dy}\right|_k \leqq 1$, we have $0 \leqq \frac{d^-t(y)}{dy}$, $\frac{d^+t(y)}{dy} \leqq 1$ for all $y \in [0,k]$ and $t \in C[0,k]$. This implies

$$t(y+\varepsilon) \leqq t(y) + \varepsilon \quad \text{and} \quad t(y-\varepsilon) \geqq t(y) - \varepsilon$$
$$\text{for all } y-\varepsilon,\ y+\varepsilon \in [0,k] . .$$

Hence for any $\varepsilon > 0$, let $\delta = \varepsilon$, and so, it holds that for all $y, y' \in [0,k]$ and all $t \in C[0,k]$,

$$|y - y'| \leqq \delta \text{ implies } |t(y) - t(y')| \leqq \varepsilon .$$

This means that $C[0,k]$ is equicontinuous.

Next we show that $C[0,k]$ is a closed set. Let $\{t^s\}$ be a convergent sequence of functions in $C[0,k]$. Let $t^0 = \lim_{s\to\infty} t^s$. Then it is easily verified that t^0 is continuous, convex and nondecreasing. Since t^s is convex with $\left.\frac{d^- t^s(y)}{dy}\right|_k \leqq 1$, we have

$$\frac{t^s(k-\varepsilon) - t^s(k)}{-\varepsilon} \leqq 1 \text{ for all } s \text{ and all } \varepsilon > 0 .$$

Hence we have

$$\frac{t^0(k-\varepsilon) - t^0(k)}{-\varepsilon} \leqq 1 \text{ for all } \varepsilon > 0 .$$

This implies $\left.\frac{d^- t^0(y)}{dy}\right|_k \leqq 1$. Thus we have shown that $t^0 \in C[0,k]$.

Q.E.D.

Let t^s be the restriction of T^s on $[0,1]$ $(s = 1, \ldots,)$. Then it follows from Lemma 7 that $\{t^s\}$ has a convergent subsequence $\{\tilde{t}^{1s}\}$. Let T^{1s} be the original tax function on E_+ in $\{T^s\}$ corresponding to $\tilde{t}^{1s}$. Then $\{T^{1s}\}$ converges uniformly on $[0,1]$. Let t^{1s} be the restriction of T^{1s} on $[0,2]$ $(s = 1, \ldots,)$. In the same way let $\{\tilde{t}^{2s}\}$ be a subsequence of $\{t^{1s}\}$ such that $\{t^{2s}\}$ converges. Let T^{2s} be the original tax function on E_+ in $\{T^s\}$ corresponding to $\tilde{t}^{2s}$. Then $\{T^{2s}\}$ converges uniformly on $[0,2]$. Continueing this process, we get an array of sequences of the form:

$$\{T^s\} = \{T^1, T^2, \ldots, \} ,$$
$$\{T^{1s}\} = \{T^{11}, T^{12}, \ldots, \} ,$$
$$\{T^{2s}\} = \{T^{21}, T^{22}, \ldots, \} ,$$
$$\{T^{3s}\} = \{T^{31}, T^{32}, \ldots, \} ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

in which each sequence is a subsequence of the one directly above it, and for each k the sequence $\{T^{ks}\}$ has the property that $\{T^{ks}\}$ converges uniformly on $[0,k]$. If we define $\tilde{T}^1, \tilde{T}^2, \ldots,$ by $\tilde{T}^1 = T^{11}$, $\tilde{T}^2 = T^{22}$, $\tilde{T}^3 = T^{33}$, $\ldots,$ then the sequence $\{\tilde{T}^s\}$ is the "diagonal" subsequence of $\{T^s\}$. It is clear from this construction that for any k , $\{\tilde{T}^s\}$ converges uniformly on $[0,k]$. Hence $\{\tilde{T}^s\}$ also converges pointwise everywhere on E_+ . We can define the function T^* on E_+ by

$$T^*(y) = \lim_{s\to\infty} \tilde{T}^s(y) \quad \text{for all} \quad y \in E_+ . \tag{21}$$

<u>Lemma 8</u>. $T^* \in \mathcal{T}$.

<u>Proof</u>. Since each $\tilde{T}^s$ is convex and nondecreasing with $\tilde{T}^s(y) \leqq y$ for all $y \in E_+$, it holds that for any y_1, y_2 $(0 \leqq y_1 \leqq y_2)$ and $\alpha \in [0,1]$,

$$\tilde{T}^s(\alpha y_1 + (1-\alpha)y_2) \leqq \alpha\tilde{T}^s(y_1) + (1-\alpha)\tilde{T}^s(y_2) ,$$
$$\tilde{T}^s(y_1) \leqq \tilde{T}^s(y_2) \quad \text{and} \quad \tilde{T}^s(y_1) \leqq y_1 \quad \text{for all} \quad s .$$

We have, by (21)

$$T^*(\alpha y_1 + (1-\alpha)y_2) \leqq \alpha T^*(y_1) + (1-\alpha)T^*(y_2) ,$$
$$T^*(y_1) \leqq T^*(y_2) \quad \text{and} \quad T^*(y_1) \leqq y_1 .$$

Q.E.D.

Let $\{\tilde{Q}^s\}$ be the subsequence of $\{Q^s\}$ corresponding to $\{\tilde{T}^s\}$. This $\{\tilde{Q}^s\}$ also converges to Q^*. Hence we have shown the following lemma.

Lemma 9. $\{\tau^s\} = \{(T^s, Q^s)\}$ has a subsequence $\{(\tilde{T}^s, \tilde{Q}^s)\}$ which satisfies

(i) $\lim_{s\to\infty} \tilde{Q}^s = Q^* \geqq 0$, $\lim_{s\to\infty} \tilde{T}^s(y) = T^*(y)$ for all $y \in E_+$ and $T^* \in \mathcal{T}$.

(ii) $\{\tilde{T}^s\}$ converges uniformly to T^* on $[0,k]$ for any k.

15. The purpose of this subsection is to prove that (T^*, Q^*) is a feasible tax schedule and

$$W(\tau^*) = W(T^*, Q^*) = \sup_{\tau\in\mathcal{F}} W(\tau), \tag{22}$$

which means that $\tau^* = (T^*, Q^*)$ is an optimal tax schedule. That is, if we show the feasibility and (22), we complete the proof of the existence of an optimal tax schedule.

To show the feasibility, it is sufficient to prove

$$Q^* \leqq \int_N T^* f^i x^*(i)d\mu, \tag{23}$$

where $(x^*(i))_{i\in N}$ is the labor time supplies for (T^*, Q^*).

Lemma 10. Let $\{(\tilde{T}^s, \tilde{Q}^s)\}$ be the sequence given in Lemma 9 and let $(\tilde{x}_s(i))_{i\in N}$ be the labor supply schedule for $(\tilde{T}^s, \tilde{Q}^s)$ $(s = 1, \ldots,)$. Then $\{\tilde{x}_s(i)\}$ converges to $x^*(i)$ for each $i \in N$.

The proof is not difficult.

Let individual $i \in N$ be arbitrarily fixed. By the continuity of f^i, we have $\lim_{s\to\infty} f^i\tilde{x}_s(i) = f^i x^*(i)$. Since $0 \leqq f^i\tilde{x}_s(i) \leqq f^i(L^i)$ for all s, we have $\lim_{s\to\infty} \tilde{T}^s f^i x_s(i) = T^* f^i x^*(i)$, because

$|\tilde{T}^s f^i \tilde{x}_s(i) - T^* f^i x^*(i)| \leqq |\tilde{T}^s f^i \tilde{x}_s(i) - T^* f^i \tilde{x}_s(i)| + |T^* f^i \tilde{x}_s(i) - T^* f^i x^*(i)|$

$\to 0 \quad (s \to \infty)$ by Lemma 9(ii).[16]

Since $\tilde{T}^s f^i \tilde{x}_s(i) \leqq f^i(L^i)$ for all s, all $i \in N$ and $f^i(L^i)$ is integrable by assumption (I), we have, by Lebesgue's dominated convergence theorem,

$$\lim_{s\to\infty} \int_N \tilde{T}^s f^i \tilde{x}_s(i) d\mu = \int_N T^* f^i x^*(i) d\mu .$$

Since $(\tilde{T}^s, \tilde{Q}^s)$ is feasible, it holds that

$$\tilde{Q}^s \leqq \int_N \tilde{T}^s f^i \tilde{x}_s(i) d\mu \quad \text{for all } s .$$

Hence we have

$$Q^* = \lim_{s\to\infty} \tilde{Q}^s \leqq \lim_{s\to\infty} \int_N \tilde{T}^s f^i \tilde{x}_s(i) d\mu = \int_N T^* f^i x^*(i) d\mu ,$$

which is the feasibility of (T^*, Q^*).

<u>Lemma 11</u>. (22) holds.

<u>Proof</u>. Since $\{(\tilde{T}^s, \tilde{Q}^s)\}$ is a subsequence of $\{(T^s, Q^s)\}$ and $\lim_{s\to\infty} W(T^s, Q^s) = \sup_{\tau\in F} W(\tau)$, it holds that $\lim_{s\to\infty} W(\tilde{T}^s, \tilde{Q}^s) = \sup_{\tau\in F} W(\tau)$.
For each $i \in N$, $U^i(L - \tilde{x}_s(i), f^i \tilde{x}_s(i) - \tilde{T}^s f^i \tilde{x}_s(i), Q^s)$
$\to U^i(L - x^*(i), f^i x^*(i) - T^s f^i x^*(i), Q^*)$ as $s \to \infty$ because of assumptions (A), (G), Lemma 9 and Lemma 10. Since U^i is uniformly bounded by (2) and $\{W(\tilde{T}^s, \tilde{Q}^s)\}$ is nondecreasing by (17), we have, by Lebesgue's dominated

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[16] Note that T^* is a continuous function because $T^* \in \mathcal{T}$.

convergence theorem,[17]

$$\lim_{s\to\infty} \int_N \log U^i(L-\tilde{x}_s(i),\ f^i\tilde{x}_s(i)-\tilde{T}f^i\tilde{x}_s(i),\ \tilde{Q}^s)d\mu$$

$$\leqq \int_N \log U^i(L-x^*(i),\ f^i x^*(i)-T^*f^i x^*(i),\ Q^*)d\mu\ .$$

That is, $\sup_{\tau\varepsilon F} W(\tau) = \lim_{s\to\infty} W(\tilde{T}^s, \tilde{Q}^s) \leqq W(T^*, Q^*)$. Q.E.D.

<u>16</u>. The purpose of this subsection is to show that (T^*,Q^*) is an equilibrium tax schedule. Suppose $Q^* < \int_N T^*f^i x^*(i)d\mu$.

<u>Lemma 12</u> Let $\{q^s\}$ be a sequence which is decreasing and converges to Q^* . Let $(\bar{x}_s(i))_{i\varepsilon N}$ be the labor supply schedule for (T^*, q^s) $(s = 1, \ldots,)$. Then $\{\bar{x}_s(i)\}$ converges to $x^*(i)$ for each $i \varepsilon N$.

The proof is not difficult.

Hence we have, by Lebesgue's dominated convergence theorem,

$$\lim_{s\to\infty} \int_N T^*f^i\bar{x}_s(i)d\mu = \int_N T^*f^i x^*(i)d\mu > Q^* = \lim_{s\to\infty} q^s\ .$$

Therefore there is an integer s such that $\int_N T^*f^i\bar{x}_s(i)d\mu > q^s > Q^*$. This means that (T^*, q^s) is a feasible tax schedule. For this s , it holds by assumption (A) that for all $i \varepsilon N$ with $f^i x^*(i) - T^*f^i x^*(i) > 0$,

$$U^i(L-x^*(i),\ f^i x^*(i)-T^*f^i x^*(i),\ Q^*)$$

$$< U^i(L-x^*(i),\ f^i x^*(i)-T^*f^i x^*(i),\ q^s)$$

$$\leqq U^i(L-\bar{x}_s(i),\ f^i\bar{x}_s(i)-T^*f^i\bar{x}_s(i),\ q^s)\ .$$

[17] Strictly speaking, we use the following variation of Lebesgue's theorem: Let $\{f_s\}$ be uniformly bounded from above, i.e., for some M, $f_s(i) \leqq M$ for all s and all $i \varepsilon N$. Then $\overline{\lim} \int_N f_s(i)\, d\mu \leqq \int_N \overline{\lim}\, f_s(i)\, d\mu$.

Thus we have

$$-\infty < W(T^*, Q^*) = \int_N \log U^i(L - x^*(i),\ f^i x^*(i) - T^* f^i x^*(i),\ Q^*)d\mu$$

$$< \int_N \log U^i(L - \bar{x}_s(i),\ f^i \bar{x}_s(i) - T^* f^i \bar{x}_s(i),\ q^s)d\mu = W(T^*, q^s)\ .$$

This is a contradiction to the optimality of (T^*, Q^*) .

5. Proof of Theorem II

17. Throughout this section we assume that (T^*, Q^*) is an optimal tax schedule. Since T^* is a convex function with $T^*(y) \leqq y$ for all $y \in E_+$, it holds that for all $y > 0$ and $\varepsilon > 0$,

$$\left.\frac{d^+T^*}{dy}\right|_{y-\varepsilon} \leqq \left.\frac{d^-T^*}{dy}\right|_{y} \leqq \left.\frac{d^+T^*}{dy}\right|_{y} \leqq \left.\frac{d^-T^*}{dy}\right|_{y+\varepsilon} \leqq 1\ . \tag{24}$$

Hence we have $\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_y = \lim_{y\to\infty} \left.\frac{d^+T^*}{dy}\right|_y$.

Suppose $\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_y = 1$. Since T^* is convex, it holds that

$$\left.\frac{d^+T^*}{dy}\right|_{y} \leqq T^*(y+1) - T^*(y) \leqq \left.\frac{d^-T^*}{dy}\right|_{y+1} \quad \text{for all } y \in E_+\ .$$

This implies $\lim_{y\to\infty}(T^*(y+1) - T^*(y)) = 1$. Using the following familiar lemma (Lemma 13), we have

$$\lim_{y\to\infty} \frac{T^*(y)}{y} = \lim_{y\to\infty}(T^*(y+1) - T^*(y)) = 1\ . \tag{25}$$

Hence it is sufficient to show that $\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_y = 1$.

Lemma 13. (Komatsu (1962, Theorem 40,5)). If $f(x)$ and $g(x)$ are defined on E_+ and $g(x)$ is monotonically increasing with $g(x) \to \infty$ $(x \to \infty)$, and if $(f(x+1) - f(x))/(g(x+1) - g(x)) \to \alpha$ $(x \to \infty)$, then $f(x)/g(x) \to \alpha$ $(x \to \infty)$, where α is a real number.

In the following we suppose

$$\lim_{y\to\infty} \left.\frac{d^-T^*}{dy}\right|_y = \lim_{y\to\infty} \left.\frac{d^+T^*}{dy}\right|_y \equiv a < 1 . \tag{26}$$

We shall derive a contradiction from the supposition (26) in the following two subsections.

18. We define a sequence of tax functions $\{T^s\}$ by

$$T^s(y) = \begin{cases} T^*(y) & \text{if } y \leqq sL \\ (a+\delta)(y-sL) + T^*(sL) & \text{if } y \geqq sL , \end{cases} \tag{27}$$

where δ is a real number such that $a < a+\delta < 1$. It is clear that every T^s belongs to $\mathcal{T}$. See Figure 4.

Lemma 14. There is an integer n_0 such that for all i with $n(i) \geqq n_0$,

$$1 - a - \delta > \frac{1}{n(i)} \cdot \left.\frac{U_1^i}{U_2^i}\right|_{(L-L^i,\, n(i)L^i - T^* n(i)L^i,\, Q^*)} \tag{28}$$

$$1 - a - \delta > \frac{1}{n(i)} \cdot \left.\frac{U_1^i}{U_2^i}\right|_{(L-L^i,\, n(i)L^i - T^s n(i)L^i,\, Q^*)} \tag{29}$$

for all $s \geqq 1$.

Proof. It follows from assumption (M) that there is a b_0 such that

$$\frac{1}{b}\cdot\frac{U_1^i}{U_2^i}\bigg|_{(L-L^i,b,Q^*)} < \frac{1-(a+\delta)}{L} \quad \text{for all} \quad i \in N \quad \text{and} \quad b \geqq b_0 .$$

Hence we have

$$\frac{1}{n(i)L^i - T^1 n(i)L^i}\cdot\frac{U_1^i}{U_2^i}\bigg|_{(L-L^i,n(i)L^i-T^1n(i)L^i,Q^*)} < \frac{1-(a+\delta)}{L}$$

$$\text{for all} \quad i \quad \text{with} \quad n(i)L^i - T^1 n(i)L^i \geqq b_0 .$$

Since

$$\frac{n(i)L^i - T^1 n(i)L^i}{n(i)} = L^i\left(1 - \frac{T^1 n(i)L^i}{n(i)L^i}\right) \leqq L ,$$

it holds that for all i with $n(i)L^i - T^1 n(i)L^i \geqq b_0$,

$$\frac{1}{n(i)}\cdot\frac{U_1^i}{U_2^i}\bigg|_{(L-L^i,n(i)L^i-T^1n(i)L^i,Q^*)}$$

$$\leqq L\cdot\frac{1}{n(i)L^i - T^1 n(i)L^i}\cdot\frac{U_1^i}{U_2^i}\bigg|_{(L-L^i,n(i)L^i-T^1n(i)L^i,Q^*)} < 1-(a+\delta) .$$

Since $\lim\limits_{n(i)\to\infty}\left(1 - \frac{d^- T^1}{dy}\bigg|_{n(i)L^i}\right) = 1 - a - \delta > 0$, it holds that $n(i)L^i - T^1 n(i)L^i \to \infty$ as $n(i) \to \infty$. Therefore we can choose an n_0 such that

$$n(i)L^i - T^1 n(i)L^i \geqq b_0 \quad \text{for all} \quad i \quad \text{with} \quad n(i) \geqq n_0 .$$

It is easily verified that $n(i)L^i - T^{s+1}n(i)L^i \geqq n(i)L^i - T^s n(i)L^i$ for all s and all i. This implies $n(i)L^i - T^s n(i)L^i \geqq b_0$ for all i with $n(i) \geqq n_0$ and all s. Hence (29) holds for this n_0. In the above argument, T^1 can be replaced by T^*. Then we get (28). Q.E.D.

Lemma 15. Let $(x^*(i))_{i \varepsilon N}$ and $(x_s(i))_{i \varepsilon N}$ be the labor supply schedule for (T^*, Q^*) and (T^s, Q^s) $(s = 1, \ldots,)$ respectively, where Q^s is any nonnegative real number. Then it holds that

$$x^*(i) = x_s(i) \quad \text{for all} \quad i \varepsilon N \quad \text{and all} \quad s \geqq n_0 , \tag{30}$$

$$x^*(i) = L^i \quad \text{for all} \quad i \quad \text{with} \quad n(i)x^*(i) \geqq n_0 L , \tag{31}$$

where n_0 is the integer given in Lemma 14.

Proof. Note that the labor supply schedule for a tax schedule (T,Q) does not depend upon Q by assumption (K), that is, if $(x_1(i))_{i \varepsilon N}$ and $(x_2(i))_{i \varepsilon N}$ are the labor supply schedule for tax schedules (T, Q_1) and (T, Q_2) respectively, then $x_1(i) = x_2(i)$ for all $i \varepsilon N$.

Let $s \geqq n_0$. Let i be an individual such that $n(i)x^*(i) < sL$. Since $T^s(y) = T^*(y)$ for all $y \leqq sL$ by (27) and T^s is convex, $x^*(i)$ also satisfies

$$\begin{aligned} &U^i(L - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^*) \\ &\quad \geqq U^i(L - x,\ n(i)x - T^s n(i)x,\ Q^*) \quad \text{for all} \quad x \leqq L^i . \end{aligned}$$

In fact, if this inequality is not true, there is an $x^0 \leqq L^i$ such that

$$\begin{aligned} &n(i)x^0 \geqq sL \quad \text{and} \\ &U^i(L - x^0,\ n(i)x^0 - T^s n(i)x^0,\ Q^*) \\ &\quad > U^i(L - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^*) . \end{aligned}$$

This implies $U^i(L-x,\ n(i)x - T^s n(i)x,\ Q^*) > U^i(L - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^*)$ for all x $(x^*(i) < x \leqq x^0)$ by the concavity of U^i, which is a contradiction. Hence we have shown that $x^*(i) = x_s(i)$ for all i with $n(i)x^*(i) < sL$.

Let i be an individual such that $n(i)x^*(i) \geqq sL$. Since $n(i) > n(i)x^*(i)/L \geqq s \geqq n_0$, (29) of Lemma 14 is true, i.e.,

$$1 - a - \delta > \frac{1}{n(i)} \cdot \left.\frac{U_1^i}{U_2^i}\right|_{(L-L^i,n(i)L^i-T^s n(i)L^i,Q^*)} .$$

Using $\left.\frac{d^- T^s}{dy}\right|_{n(i)x^*(i)} \leqq a+\delta$ by (27), it follows that

$$\left.\left[-U_1^i + U_2^i\, n(i)\left(1 - \frac{d^- T^s}{dy}\right)\right]\right|_{(L-L^i,n(i)L^i-T^s n(i)L^i,Q^*)} > 0 ,$$

which is equivalent to $\left.\frac{d^-}{dx}U^i(L-x,\ n(i)x - T^s n(i)x,\ Q^*)\right|_{x=L^i} > 0$. This and assumptions (A'), (F') imply $x_s(i) = L^i$. Analogously we can prove that $x^*(i) = L^i$ for all i with $n(i)x^*(i) \geqq n_0 L$. This is (31). Hence it holds that $x_s(i) = x^*(i) = L^i$ for all i with $n(i)x^*(i) \geqq sL$.

Q.E.D.

Lemma 16. If the government employs T^s $(s \geqq n_0)$ in place of T^*, the increment of the government's revenue is not smaller than $\delta \cdot \mu(F(sL+1))$, i.e.,

$$\int_N T^s n(i)x_s(i)d\mu - \int_N T^* n(i)x^*(i)d\mu \geqq \delta \cdot \mu(F(sL+1)) , \tag{32}$$

where $F(sL+1) = \{i \in N : n(i)L^i \geqq sL+1\}$.

Proof. Note that $x^*(i) = x_s(i)$ for all $i \in N$. Since $T^s n(i)x^*(i) - T^*n(i)x^*(i) = (a+\delta)(n(i)x^*(i) - sL) + T^*(sL) - T^*n(i)x^*(i) \geqq 0$ for all $i \in N$ with $n(i)x^*(i) \geqq sL$ by (27), it holds that

$$\int_{F(sL,sL+1)} [T^s n(i)x^*(i) - T^*n(i)x^*(i)]d\mu \geqq 0 , \tag{33}$$

where $F(sL, sL+1) = \{i \in N : sL \leqq n(i)x^*(i) < sL+1\}$.

Since $T^*n(i)x^*(i) \leqq a(n(i)x^*(i) - sL) + T^*(sL)$ for all i with $n(i)x^*(i) \geqq sL$ by the definition of a , it holds by (27) and (31) that

$$\int_{F(sL+1)} [T^s n(i)x^*(i) - T^*n(i)x^*(i)]d\mu \tag{34}$$

$$\geqq \int_{F(sL+1)} [(a+\delta)(n(i)x^*(i) - sL) + T^*(sL) - [a(n(i)x^*(i) - sL) + T^*(sL)]]d\mu$$

$$= \int_{F(sL+1)} \delta[(n(i)x^*(i) - sL)]d\mu = \int_{F(sL+1)} \delta(n(i)L^1 - sL)d\mu$$

$$\geqq \int_{F(sL+1)} \delta\cdot 1d\mu = \delta\cdot\mu(F(sL+1)) .$$

Since $T^s n(i)x^*(i) = T^*n(i)x^*(i)$ for all i with $n(i)x^*(i) \leqq sL$ by (27), we have

$$\int_{N-F(sL)} [T^s n(i)x^*(i) - T^*n(i)x^*(i)]d\mu = 0 . \tag{35}$$

Hence it follows from (33), (34) and (35) that

$$\int_N [T^s n(i)x^*(i) - T^*n(i)x^*(i)]d\mu$$

$$= \int_{F(sL,sL+1)} [T^s n(i)x^*(i) - T^*n(i)x^*(i)]d\mu$$

$$+ \int_{F(sL+1)} [T^s n(i)x^*(i) - T^*n(i)x^*(i)]d\mu$$

$$\geqq \delta\cdot\mu(F(sL+1)) .$$

Q.E.D.

19. In the following we assume $s \geqq n_0$ and use Lemma 15 without any remark, where n_0 is the integer given in Lemma 14.

Since $G_3^i(a,b,Q) = \left.\frac{U_3^i}{U^i}\right|_{(a,b,Q)} > 0$ for all $i \in N$ and all $(a,b,Q) \in \tilde{Y}$ by (A') and $\mu(\{i \in N : U^i(L - x^*(i), n(i)x^*(i) - T^*n(i)x^*(i), Q^*) = 0\}) = 0$ by (10), there is a positive number $\tilde{b}$ such that

$$\mu(H) \geqq \frac{1}{2}\mu(F(0, n_0L)) \tag{36}$$

$$H = \{i \in F(0, n_0L) : G_3^i(L - x^*(i), n(i)x^*(i) - T^*n(i)x^*(i), Q^*) \geqq \tilde{b}\} ,$$

where $F(0, n_0L) = \{i \in N : 0 \leqq n(i)x^*(i) < n_0L\}$.

Since $(x^*(i))_{i\in N}$ is invariant for Q by assumption (K), we can write

$$J^i(Q) = G^i(L - x^*(i), n(i)x^*(i) - T^*n(i)x^*(i), Q) \tag{37}$$
$$\text{for all } i \in N .$$

It holds by assumption (A') that for all $i \in N$,

$$J^i(Q^* + \Delta Q) = J^i(Q^*) + J^{i'}(Q^*)\Delta Q + \Delta Q \cdot \varepsilon_{\Delta Q}^i \tag{38}$$

and

$$\varepsilon_{\Delta Q}^i \to 0 \quad \text{as} \quad \Delta Q \to 0 ,$$

where $J^{i'} = (dJ^i/dQ)$.

Lemma 17. Suppose $\int_H J^{i'}(Q^*)d\mu < +\infty$.[18] Let $\{q_k\}$ be a decreasing sequence such that $\lim_{k\to\infty} q_k = 0$. Then it holds that

$$\int_H \varepsilon^i_{q_k} d\mu \to 0 \quad \text{as} \quad k \to \infty \ . \tag{39}$$

Proof. Let i be fixed. $(J^i(q^* + q_k) - J^i(Q^*))/q_k$ is the inclination of the line connecting two points $(Q^*, J^i(Q^*))$ and $(Q^* + q_k, J^i(Q^* + q_k))$. Since $J^i(Q)$ is a concave function by assumption (A'), this inclination is a nondecreasing function of k , i.e.,

$$(J^i(Q^* + q_k) - J^i(Q^*))/q_k \leqq (J^i(Q^* + q_{k+1}) - J^i(Q^*))/q_{k+1}$$
$$\text{for all} \quad k \ .$$

Further it holds that $(J^i(Q^* + q_k) - J^i(Q^*))/q_k \leqq dJ^i(Q^*)/dQ$ for all k . Hence it follows that $|\varepsilon^i_{q_k}| = |(J^i(Q^* + q_k) - J^i(Q^*))/q_k - dJ^i(Q^*)/dQ|$ is nonincreasing function of k . Since $(J^i(Q^* + q_k) - J^i(Q^*))/q_k - dJ^i(Q^*)/dQ$ is an integrable function of $i \in H$ and since $\varepsilon^i_{q_k} \to 0$ as $k \to \infty$ for all $i \in H$, we have, by Lebesgue's dominated convergence theorem,

$$\int_H \varepsilon^i_{q_k} d\mu \to 0 \quad \text{as} \quad k \to \infty \ .$$

Q.E.D.

Since $n(i)L^i - T^s n(i)L^i \to \infty$ as $n(i)L^i \to \infty$ and $y - T^s(y) \leqq y - T^{s+1}(y)$ for all $y \in E_+$ and all s , there is an $s \geqq n_0$ by assumption (L) such that

$$\bar{M}_1 - \frac{\tilde{b}\cdot\delta\mu(F(0,\ n_0L))c_1}{4} \leqq G^i(L - L^i,\ n(i)L^i - T^s n(i)L^i,\ Q^*) \tag{40}$$
$$\text{for all} \quad i \quad \text{with} \quad n(i)L^i \geqq sL \ ,$$

[18] It is easily verified that dJ^i/dQ is a measurable function of i .

where $\bar{M}_i = \log M_i$ (M_i's are given in (L)) and c_1 is the positive number such that

$$c_1 = \lim_{s\to\infty} \mu(F(sL+1))/\mu(F(sL)) \ . \tag{41}$$

Note that the positiveness of c_1 is ensured by assumption (N).

We are now in a position to evaluate the value:

$$W(T^s,\ Q^* + \delta\mu(F(sL+1))) - W(T^*,\ Q^*) \tag{42}$$

$$= \int_N G^i(L - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^* + \delta\mu(F(sL+1)))d\mu$$

$$- \int_N G^i(L - x^*(i),\ n(i)x^*(i) - T^* n(i)x^*(i),\ Q^*)d\mu$$

$$= \int_{F(0,sL)} [G^i(1 - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^* + \delta\mu(F(sL+1)))$$

$$- G^i(L - x^*(i),\ n(i)x^*(i) - T^* n(i)x^*(i),\ Q^*)]d\mu$$

$$+ \int_{F(sL)} [G^i(L - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^* + \delta\mu(F(sL+1)))$$

$$- G^i(L - x^*(i),\ n(i)x^*(i) - T^* n(i)x^*(i),\ Q^*)]d\mu \ .$$

For a sufficiently large s , the second term of the right hand side can be rewritten as follows:

$$\int_{F(sL)} [G^i(L - x^*(i),\ n(i)x^*(i) - T^s n(i)x^*(i),\ Q^* + \delta\mu(F(sL+1))) \tag{43}$$

$$- G^i(L - x^*(i),\ n(i)x^*(i) - T^* n(i)x^*(i),\ Q^*)]d\mu$$

$$\gtrapprox \int_{F(sL)} \left[\bar{M}_i - \frac{\tilde{b}\delta\mu(F(0,\ n_0L))c_1}{4} - \bar{M}_i\right]d\mu$$

$$= - \frac{\tilde{b}\delta\mu(F(0,\ n_0L)c_1\mu(F(sL))}{4} \ .$$

Let us suppose that $\int_H J^{i'}(Q^*)d\mu < +\infty$. Since $T^*(y) = T^s(y)$ for all $y \leqq sL$ by (27), the first term of the right hand side of (42) can be rewritten by (36) as follows:

$$\int_{F(0,sL)} [J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)]d\mu \tag{44}$$

$$\geqq \int_H [J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)]d\mu$$

$$\geqq \int_H \tilde{b}\delta\mu(F(sL+1))d\mu + \int_H \varepsilon_s^i d\mu \cdot \delta \cdot \mu(F(sL+1))$$

$$\geqq \tilde{b}\delta\mu(F(sL+1))\mu(F(0,\ n_0L))/2 + \int_H \varepsilon_s^i d\mu \cdot \delta \cdot \mu(F(sL+1)) \ ,$$

where $\varepsilon_s^i = \varepsilon^i_{\delta\mu(F(sL+1))}$ and $\tilde{b}$ is the number defined by (36). Hence it holds for a sufficiently large s that

$$W(T^s,\ Q^* + \delta\mu(F(sL+1))) - W(T^*,\ Q^*) \tag{45}$$

$$\geqq \tilde{b}\delta\mu(F(sL+1))\mu(F(0,\ n_0L))/2 + \int_H \varepsilon_s^i d\mu \cdot \delta \cdot \mu(F(sL+1))$$

$$- \tilde{b}\delta\mu(F(sL))c_1\mu(F(0,\ n_0L))/4 \ .$$

Since $c_1 = \lim_{s\to\infty} \mu(F(sL+1))/\mu(F(sL)) > 0$ and $\int_H \varepsilon_s^i d\mu \to 0$ as $s \to \infty$ by Lemma 16, the right hand side of (45) becomes positive for a sufficiently large s . This is a contradiction to the optimality of (T^*, Q^*) .

When $\int_H J^{i'}(Q^*)d\mu = +\infty$, (45) can be rewritten as follows:

$$W(T^s,\ Q^* + \delta\mu(F(sL+1))) - W(T^*,\ Q^*) \tag{46}$$

$$\geqq \int_H [J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)]d\mu$$

$$- \tilde{b}\delta\mu(F(sL))c_1\mu(F(0,\ n_0L))/4 \ .$$

Since $\dfrac{\int_H [J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)]d\mu}{\delta\mu(F(sL+1))} \to \infty$ as $s \to \infty$ by the following Lemma 18 and $\int_H J^{i'}(Q^*)d\mu = \infty$, and since $\lim_{s\to\infty} \mu(F(sL+1))/\mu(F(sL)) = c_1 > 0$ by (N), it holds for a sufficiently large s that

$$\frac{\int_H [J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*)]d\mu}{\delta\mu(F(sL+1))} - \tilde{b}\cdot\frac{\mu(F(sL))}{\mu(F(sL+1))}c_1\cdot(F(0,n_0L))/4 > 0 .$$[19]

This means that the right hand side of (46) is positive for such an s. This is a contradiction to the optimality of (T^*, Q^*).

Lemma 18. Let $\{f_s\}$ be a sequence of measurable functions on H such that $0 \leqq f_s(i) \leqq f_{s+1}(i)$ for all s and all $i \in H$. If there is a measurable function f such that $\lim_{s\to\infty} f_s(i) = f(i)$ for all $i \in H$ and $\int_H f(i)d\mu = \infty$, then $\lim_{s\to\infty} \int_H f_s(i)d\mu = \infty$.

We can prove this lemma in the standard way.

[19] Since $J^i(Q)$ is convex and monotonically increasing with respect to Q for all $i \in H$, $(J^i(Q^* + \delta\mu(F(sL+1))) - J^i(Q^*))/\delta\mu(F(sL+1))$ is positive and nondecreasing with respect to s for each i.

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