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STRONG CORE THEOREMS WITH NONCONVEX PREFERENCES

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Abstract

It is known that in large economies with strongly convex preferences, the commodity bundles agents receive at core allocations are near their demand sets. Without convexity, it is known that agents need not be near their demand sets, although they will satisfy a weaker condition. In this paper, we show that, for "most" economies (in the sense of probability and in the sense of the Baire category theorem), the stronger form of approximation holds without convexity.

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I. Introduction

It has been known for some time that in large finite economies with strongly convex preferences, core allocations can be decentralized in a strong sense: there exist prices so that individuals get commodity bundles close to their demand sets (Hildenbrand [12], Anderson [1]). However, counterexamples are known if one drops the strong convexity assumption, and the published theorems in that case give weaker results: agents get bundles which are functionally like demands, but need not be near their demand sets (Hildenbrand [12], Dierker [10], Anderson [2]). The difference is significant for the following reason. If the average distance from the core allocation to the demand sets is small, then it follows that average excess demand must also be small. Thus, core allocations will be close to approximate Walrasian equilibria. Since it could be argued that approximate equilibria are as stable as exact equilibria (after all, slight deviations in supply and demand can be accommodated through inventory adjustments), this is very nearly as strong as asserting that core allocations are near exact Walrasian equilibria. It should be emphasized, however, that not everyone shares this view, and many would consider this "strong" form of decentralization to be considerably weaker than being near an exact Walrasian equilibrium. We believe everyone, however, would consider it considerably stronger than the "functional" notion of decentralization previously known to hold without convexity.

The main thesis of this paper is that such counterexamples are rare exceptions, and that the typical economy with non-convex preferences is every bit as well-behaved as one with strongly convex preferences. We make this explicit by using both probabilistic and topological notions

of "typical" behavior.

In Section 3, we show that the strong form of convergence holds for all type sequences of economies, and in particular for replica sequences. The theorem in the replica case shows that the convexity assumption in the classic Debreu-Scarff Theorem may simply be dropped, and a conclusion nearly as strong as theirs will still hold. The proof given is completely elementary, and was obtained by translating a nonstandard proof.

In Section 4, we show that there is a residual set of economies (without convex preferences) so that the strong form of convergence holds for any purely competitive sequence converging to an economy in the residual set. Residual is a notion of a "large" class, obtained from the Baire Category Theorem; it is weaker than the notion of open and dense. The residual set contains all the economies with strongly convex preferences, so the result generalizes the previous results on core convergence with strongly convex preferences. The proof uses Nonstandard Analysis, originally introduced into the study of the core in Brown-Robinson [6], [7]. It should be noted, however, that it is possible to give a proof using the standard measure-theoretic machinery described in Hildenbrand [12]. The results of Section 4 are related to Mas-Colell [17], which notes that rate of convergence theorems of the type proven by Debreu [8] and Grodal [11] hold for sequences of economies with regular limits. However, the set of regular economies, while open in one topology and dense in another, does not appear to be residual in any topology.

In Section 5, we consider sequences of economies constructed at random by sampling from a measure describing the distribution of agents' characteristics. We show that, with probability one, all core allocations will be decentralized in the strong sense. The proof again uses Non-

standard Analysis. A general meta-theorem guarantees that a standard proof also exists, but it could be quite difficult to write it out.

Comments on this point are given in Section 5.

A few diagrams will help to clarify the intuition behind the proofs. The decentralization theorems for the nonconvex case (Anderson [1]) establish a functional relation between core allocations and demand sets. They assert that, for most traders, core allocations must lie within a relatively constrained region which grows smaller as the number of traders increases.

In Figure 1, e denotes the endowment vector, and p is an approximately decentralizing price vector. l_1 , l_2 , and l_3 are perpendicular to p ; l_2 is the boundary of the budget set with respect to p . The core allocation can be guaranteed to lie within the shaded area; as the number of traders get large, the distance between l_1 and l_3 can be guaranteed (for most traders) to shrink to 0. As this happens, the shaded region shrinks towards the demand set; this is illustrated in Figure 2 for the same preference relation used in Figure 1. It is illustrated in Figure 3 for a preference relation with multiple demands. Problems arise only if the preferences are tending to (but never quite reaching) a preference with multiple demands. In Figure 4, we illustrate such a sequence of preferences. Consider a sequence of economies in which some agents of the first economy have preference \succ_1 , some agents of the second economy have preference \succ_2 , and so on. In each case, x is the unique consumption vector in the demand set. However, consumption vectors near y could be in the core, provided the indifference curve is tending toward y at least as rapidly as l_3 is converging to l_2 . This shows how to construct examples whose core allocations do not converge

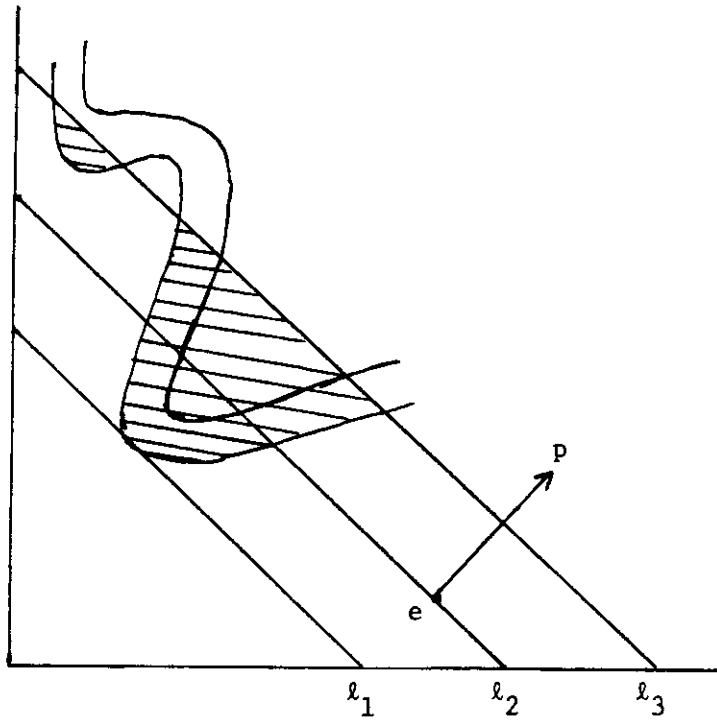


FIGURE 1

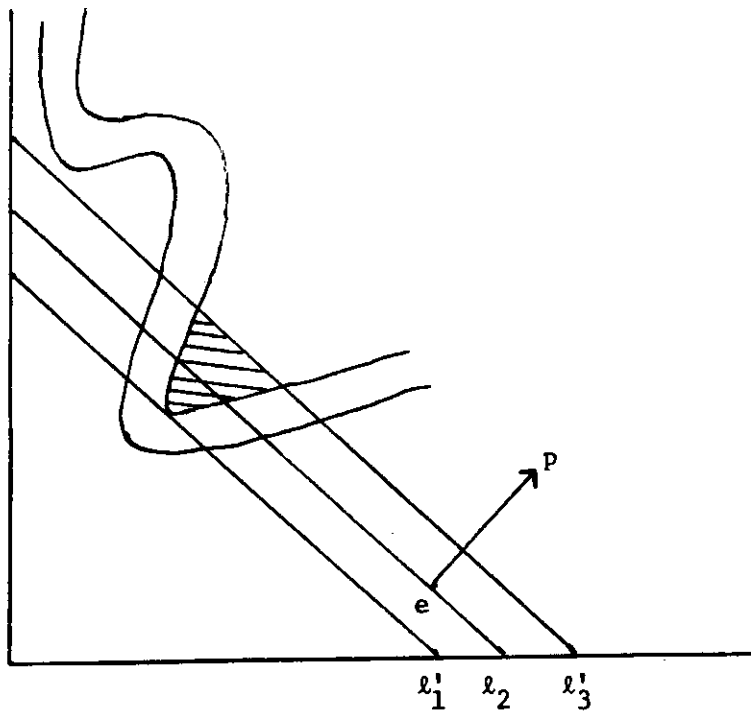


FIGURE 2

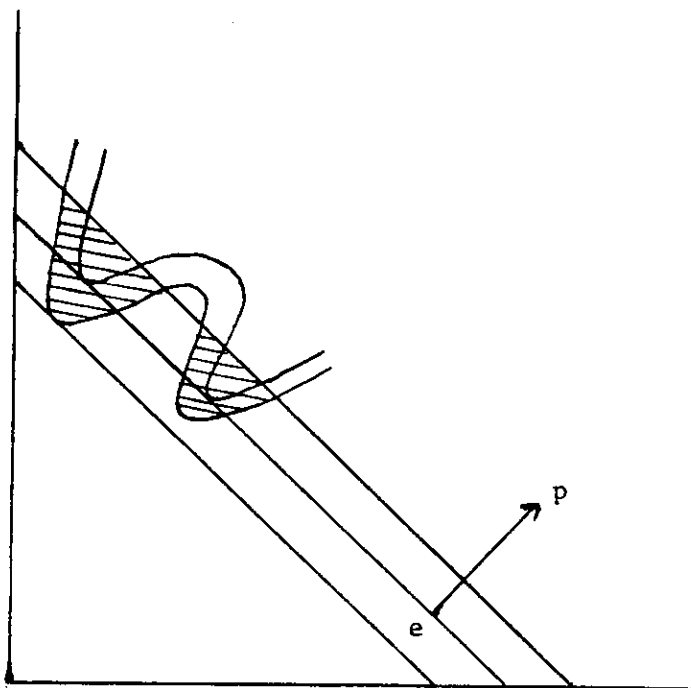


FIGURE 3

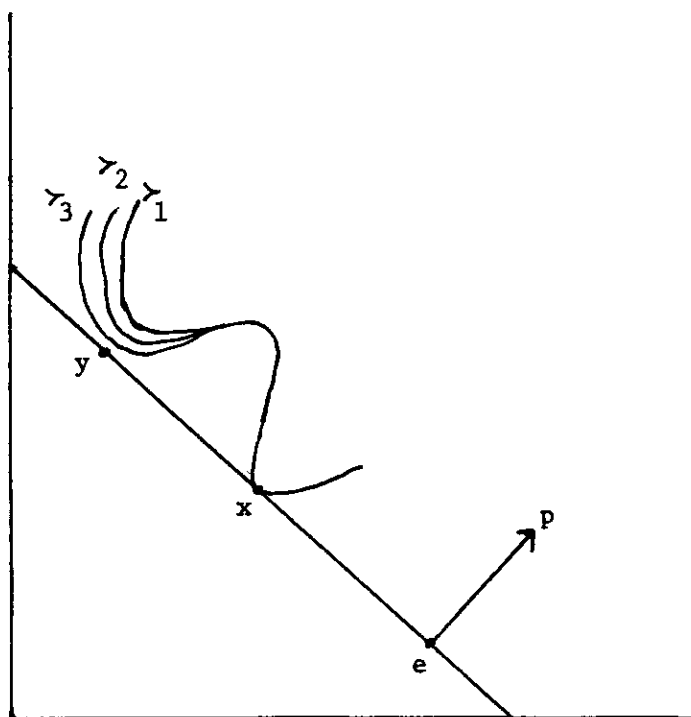


FIGURE 4

to the demand sets, but it also gives a good idea of why such examples are exceptional; the preferences must be changing in a way which is carefully matched to the number of agents.

Finally, we point out that all our theorems are actually given as limit theorems for the δ -core. The δ -core is bigger than the core, and is known to be non-empty in a more general setting than the core itself (Hildenbrand-Schmeider-Zamir [14], Kannai [15]).

2. Preliminaries

We begin with some notation and definitions which will be used throughout. Suppose $x, y \in \mathbb{R}^k$, $A \subset \mathbb{R}^k$

x^i denotes the i^{th} component of x

$x \geq y$ means $x^i \geq y^i$ for all i

$x > y$ means $x \geq y$ and $x \neq y$

$x \gg y$ means $x^i > y^i$ for all i

$$\|x\|_{\infty} = \max_{1 \leq i \leq k} |x^i|$$

$$\|x\|_1 = \sum_{i=1}^k |x^i|$$

$$\mathbb{R}_+^k = \{x \in \mathbb{R}^k : x \geq 0\} .$$

A preference is a binary relation on \mathbb{R}_+^k satisfying the following conditions

- (i) monotonicity: $x > y \implies x \succ y$
- (ii) continuity: $\{(x,y) : x \succ y\}$ is relatively open in \mathbb{R}_+^k
- (iii) free disposal: $x \gg y, y \succ z \implies x \succ z$.

Let \mathcal{P} denote the set of preferences.

An exchange economy is a map $\varepsilon : A \rightarrow \mathcal{P} \times \mathbb{R}_+^k$, where A is a finite set. For $a \in A$, let \succ_a denote the preference of a (i.e. the projection of $\varepsilon(a)$ onto \mathcal{P}) and $e(a)$ the initial endowment of a (i.e. the projection of $\varepsilon(a)$ onto \mathbb{R}_+^k). An allocation is a map $f : A \rightarrow \mathbb{R}_+^k$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$. A coalition is a non-empty subset of A . A coalition S can improve on an allocation f if there exists $g : S \rightarrow \mathbb{R}_+^k$, $g(a) \succ_a f(a)$ for all $a \in S$, and $\sum_{a \in S} g(a) = \sum_{a \in S} e(a)$. The core of ε , $C(\varepsilon)$, is the set of all allocations which can't be improved on by any coalition. A price p is an element of \mathbb{R}_+^k with $\|p\|_\infty = 1$. S denotes the set of prices, $S^0 = \{p \in S : p \gg 0\}$. $M_\varepsilon = \max\{\|e(a_1) + \dots + e(a_\ell)\|_\infty : a_1, \dots, a_\ell \text{ are distinct elements of } A, \ell \leq k\}$. Note that in the last definition, k is the dimension of the commodity space \mathbb{R}_+^k .

Given $x \in \mathbb{R}_+^k$, $(\succ, e) \in \mathcal{P} \times \mathbb{R}_+^k$, and $p \in S$, define $\phi(p, x, (\succ, e)) = |p \cdot (x - e)| + |\inf\{p \cdot (y - e) : y \succ x\}|$. ϕ measures how far x is from being demand-like. By a slight abuse of notation, we let $\phi(p, f, a) = \phi(p, f(a), (\succ_a, e(a)))$ if f is an allocation.

The demand set for (\succ, e) , given $p \in S$, is

$$D(p, (\succ, e)) = \{x \in \mathbb{R}_+^k : p \cdot x \leq p \cdot e, y \succ x \implies p \cdot y > p \cdot e\}.$$

$D(p, (\succ, e))$ could be empty under the hypotheses we've placed on preferences.

By abuse of notation, we let $D(p, a) = D(p, (\succ_a, e(a)))$ if $a \in A$.

We shall let \mathcal{P}' denote $\{\succ \in \mathcal{P} : D(p, (\succ, e)) \neq \emptyset \text{ for all } p \in S^0 \text{ and all } e \in \mathbb{R}_+^k\}$. \mathcal{P}' includes all preferences which are irreflexive ($x \not\succeq x$) and transitive ($x \succ y, y \succ z \implies x \succ z$), as well as all preferences which are irreflexive and convex ($\{x : x \succ y\}$ is convex for

all y) (Sonnenschein [19]).

We need a way to measure the distance from $x \in \mathbb{R}_+^k$ to the demand set. For $B \in \mathbb{R}_+^k$, $\rho(x, B) = \inf\{\|x-y\|_\infty : y \in B\}$. Observe $\rho(x, D(p, (\succ, e))) = \infty$ if $D(p, (\succ, e)) = \emptyset$.

The following theorem is proved in Anderson [1], under weaker assumptions on preferences:

Theorem 2.1. Let $\varepsilon : A \rightarrow P \times \mathbb{R}_+^k$ be an exchange economy. If $f \in C(\varepsilon)$, there exists $p \in S$ such that

$$\sum_{a \in A} \phi(p, f, a) \leq 4M.$$

Finally, since the core of an exchange economy with non-convex preferences may be empty, we want to be able to deal with approximate cores.

We say an allocation $f : A \rightarrow \mathbb{R}_+^k$ can be δ -blocked if there exists $S \subset A$ and $g : S \rightarrow \mathbb{R}_+^k$ such that $g(a) \succ_a f(a)$ for all $a \in S$ and

$$\sum_{a \in S} g(a) \leq \sum_{a \in S} f(a) - (\delta|A|, \dots, \delta|A|).$$

The fat δ -core of an exchange economy is the set of all allocations which can't be δ -blocked.

The fat δ -core is similar to, but larger than, the strong δ -core and weak δ -core defined by Kannai [15], and which are proved non-empty by Kannai and Hildenbrand-Schmeidler-Zamir [14]. Observe also that, by free disposal and continuity, $\delta < \delta' \implies C_\delta(\varepsilon) \subset C_{\delta'}(\varepsilon)$, and

$$C(\varepsilon) = \bigcap_{\delta > 0} C_\delta(\varepsilon).$$

Theorem 2.2. Let $\varepsilon : A \rightarrow P \times \mathbb{R}_+^k$ be an exchange economy. If $f \in C_\delta(\varepsilon)$, there exists $p \in S$ such that

$$\sum_{a \in A} \phi(p, f, a) \leq 4(M_\varepsilon + \delta |A|) .$$

Proof. A routine modification of the Proof of Theorem 2.1, as given in [1].

3. Type Sequences

A type sequence of economies is a sequence of economies $\varepsilon_n : A_n \rightarrow T$, where T is a finite subset of $P \times \mathbb{R}_+^k$. The elements of T are called types; two individuals $a, b \in A_n$ with $\varepsilon_n(a) = \varepsilon_n(b)$ are said to be of the same type, in that they have identical characteristics. Note however that allocations (including core allocations) may give different consumption vectors in \mathbb{R}_+^k to a and b . Let M_T be the largest $\|\cdot\|_\infty$ -norm of the endowments in T .

Theorem 3.1 asserts that, under mild hypotheses, the average distance from the core allocations of individuals to their demand sets shrinks as the number of individuals grows. This theorem has been known for a long time under convexity assumptions (Debreu-Scarff [9], Hildenbrand-Kirman [13]). The closest analogue without convexity is in Brown-Robinson [7], who prove essentially the same conclusion for equal-treatment core allocations in replica sequences.

Theorem 3.1 is actually a special case of a nonstandard theorem (Theorem 5.2) which will be presented in Section 5. Indeed, our first proof of Theorem 3.1 was nonstandard, and inspired the generalization. Here, we give an elementary standard proof which we obtained by translating the original nonstandard proof; we hope it will make the ideas of

Section 5 more accessible to readers unfamiliar with the nonstandard methodology.

Theorem 3.1. Let $\varepsilon_n : A_n \rightarrow T$ be a type sequence of exchange economies satisfying $|A_n| \rightarrow \infty$, $\inf_n |\varepsilon_n^{-1}(t)| / |A_n| > 0$ for each $t \in T$, and

$\sum_{(\gamma, e) \in T} e \gg 0$. Then

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{f \in C_{\delta}(\varepsilon_n)} \inf_{p \in S} \frac{1}{|A_n|} \sum_{a \in A_n} \rho(f(a), D(p, a)) = 0 .$$

In particular,

$$\lim_{n \rightarrow \infty} \sup_{f \in C(\varepsilon_n)} \inf_{p \in S} \frac{1}{|A_n|} \sum_{a \in A_n} \rho(f(a), D(p, a)) = 0 .$$

Remark. If we recall the definition of ρ from the end of Section 2, we see that if $C(\varepsilon_n)$ is non-empty for all n , then the demand set $D(p, (\gamma, e))$ is non-empty, at least for some $p \in S$. This is somewhat surprising, since our assumptions on preferences are not sufficient to guarantee non-empty demand sets. An analogous phenomenon occurs in Aumann's equivalence theorem [12].

Proof of Theorem 3.1. If the theorem is false, we may find $\delta_n \rightarrow 0$, $\sigma > 0$ and a subsequence (still denoted ε_n) such that for some $f_n \in C_{\delta_n}(\varepsilon_n)$, $\frac{1}{|A_n|} \sum_{a \in A_n} \rho(f_n(a), D(p_n, a)) > \sigma$ for all n and all $p_n \in S$. We shall show this leads to a contradiction.

We may find prices $q_n \in S$ such that

$$\sum_{a \in A_n} \phi(q_n, f_n, a) \leq 4(M_{\varepsilon_n} + \delta_n |A_n|) , \text{ by Theorem 2.2.}$$

Since $M_{\varepsilon_n} \leq kM_T$, $\frac{1}{|A_n|} \sum_{a \in A_n} \phi(q_n, f_n, a) \leq 4 \left(\frac{kM_T}{|A_n|} + \delta_n \right) \rightarrow 0$. By passing to a subsequence, we may assume $q_n \rightarrow q \in S$.

The bulk of the proof will be devoted to showing $q \gg 0$. This is by now a routine argument (see Anderson [2] or Hildenbrand [12] for similar ideas). If q is not strictly positive, we may assume without loss of generality that $q^1 = 0$.

Let $\alpha = \inf_{n,i} \frac{1}{|A_n|} \sum e_n^i(a) > 0$. $\frac{1}{|A_n|} \sum_{a \in A_n} q_n \cdot f_n(a) = q_n \cdot \frac{1}{|A_n|} \sum_{a \in A_n} e_n(a) \geq \alpha$.

Hence, there exists j_n such that $\frac{1}{|A_n|} \sum_{a \in A_n} q_n^{j_n} \cdot f_n^{j_n}(a) \geq \alpha/k$. Passing

to a subsequence, we may assume j_n does not depend on n . Relabelling coordinates, we may take $j_n = 2$. Thus, $\frac{1}{|A_n|} \sum_{a \in A_n} q_n^2 \cdot f_n^2(a) \geq \alpha/k$, for all n . Observe $q_n^2 \rightarrow q^2$, and hence $q^2 > 0$.

For any $S_n \subset A_n$,

$$\begin{aligned} \frac{1}{|A_n|} \sum_{a \in S_n} q_n^2 \cdot f_n^2(a) &\leq \frac{1}{|A_n|} \sum_{a \in S_n} q_n \cdot f_n(a) \\ &\leq \frac{1}{|A_n|} \left(\sum_{a \in S_n} q_n \cdot (f_n(a) - e_n(a)) + \left\| \sum_{a \in S_n} e_n(a) \right\|_{\infty} \right) \\ &\leq \frac{4kM_T}{|A_n|} + 4\delta_n + \frac{|S_n|}{|A_n|} M_T. \end{aligned}$$

Letting $S_n = \{a \in A_n : f_n^2(a) \geq \alpha/2k\}$,

$$\begin{aligned}
\frac{|S_n|}{|A_n|} &\geq \left[\frac{1}{|A_n|} \sum_{a \in S_n} q_n^2 \cdot f_n^2(a) - \frac{4kM_T}{|A_n|} - 4\delta_n \right] / M_T \\
&\geq \left[\frac{\alpha}{k} - \frac{\alpha}{2k} - \frac{4kM_T}{|A_n|} - 4\delta_n \right] / M_T \\
&\geq \frac{\alpha}{4M_T k} \text{ for } n \text{ sufficiently large.}
\end{aligned}$$

On the other hand, $\frac{1}{|A_n|} \sum_{a \in A_n} \|f_n(a)\|_\infty \leq \frac{k}{|A_n|} \|\sum_{a \in A_n} e_n(a)\|_\infty \leq kM_T$,

so $\left| \left\{ a \in A_n : \|f_n(a)\|_\infty \leq kM_T / \left(\frac{\alpha}{8kM_T} \right) \right\} / |A_n| \right| \geq 1 - \frac{\alpha}{8kM_T}$. In

other words, we've found $\beta > 0$, $\gamma \in \mathbb{R}$, and $\delta > 0$ such that

$$|\{a \in A_n : f_n^2(a) \geq \beta, \|f_n(a)\|_\infty \leq \gamma\}| / |A_n| \geq \delta.$$

There exists $\lambda > 0$ such that for all preferences \succ in T and all x with $\|x\|_\infty \leq \gamma$, $x^2 \geq \beta$, $x + (1, -\lambda, 0, \dots, 0) \succ x$ (if not, take a subsequence converging to x_0 , and use monotonicity and continuity).

Suppose $a \in A_n$ satisfies $f_n^2(a) \geq \beta$, $\|f_n(a)\|_\infty \leq \gamma$. $\phi(q_n, f_n, a) \geq \lambda q_n^2 - q_n^1$. Therefore $\frac{1}{|A_n|} \sum_{a \in A_n} \phi(q_n, f_n, a) \geq \delta(\lambda q_n^2 - q_n^1) \rightarrow \delta \lambda q^2 > 0$. This contradiction shows $q \gg 0$.

We claim that, if $\phi(q_n, x_n, (\gamma, e)) \rightarrow 0$, then $\rho(x_n, D(q, (\gamma, e))) \rightarrow 0$. If not, we can pass to a subsequence and find $x_n \rightarrow x$, $\phi(q_n, x_n, (\gamma, e)) \rightarrow 0$, $\delta > 0$, $\rho(x_n, D(q, (\gamma, e))) > \delta$. But $\phi(q, x, (\gamma, e)) = \lim_{n \rightarrow \infty} \phi(q_n, x_n, (\gamma, e)) = 0$. Thus $q \cdot x = q \cdot e$ and $\inf\{q \cdot y : y \succ x\} = q \cdot e$. By continuity, $y \succ x \implies q \cdot y > q \cdot x$, so $x \in D(q, (\gamma, e))$. But then $\|x_n - x\|_\infty \rightarrow 0$, contradicting $\rho(x_n, D(q, (\gamma, e))) > \delta$.

Since $\frac{1}{|A_n|} \sum_{a \in A_n} \phi(q_n, f_n, a) \rightarrow 0$, we find $\tau_n \rightarrow 0$ such that, with

$S_n = \{a \in A_n : \phi(q_n, f_n, a) < \tau_n\}$, $|S_n|/|A_n| \rightarrow 1$. If we choose $a_n \in S_n$ maximizing $\rho(f_n(a_n), D(q, a_n))$, then $\rho(f_n(a_n), D(q, a_n)) \rightarrow 0$ because of the last paragraph and because T is finite. Thus

$$\sup_{a \in S_n} \rho(f_n(a), D(q, a)) \rightarrow 0.$$

Since $|S_n|/|A_n| \rightarrow 1$ and $\inf_{n,t} |\varepsilon_n^{-1}(t)|/|A_n| > 0$, S_n contains at least one trader of each type when n is sufficiently large. Hence, $D(q, a)$ is nonempty. Thus, $\rho(f_n(a), D(q, a)) \leq \max\{\|f_n(a)\|_\infty, \|D(a, q)\|_\infty\}$

$$\leq \frac{(2k+1)M_T}{\min\{q_n^1, \dots, q_n^k\}} + \frac{kM_T}{\min\{q^1, \dots, q^k\}} + \frac{(3k+1)M_T}{\min\{q^1, \dots, q^k\}}. \text{ Therefore,}$$

$$\frac{1}{|A_n|} \sum_{a \in A_n} \rho(f_n(a), D(q, a)) \leq \frac{|S_n|}{|A_n|} \sup_{a \in S_n} \rho(f_n(a), D(q, a)) + \left[1 - \frac{|S_n|}{|A_n|}\right] \cdot \left[\frac{(2k+1)M_T}{\min\{q_n^1, \dots, q_n^k\}} + \frac{kM_T}{\min\{q^1, \dots, q^k\}} \right] \rightarrow 0.$$

Letting $p_n = q$, we obtain a contradiction proving the theorem.

4. The Topological Approach

In this section, we show that most sequences of economies, in the sense of the Baire Category Theorem, exhibit strong core convergence, without assuming convex preferences. The theorem includes all sequences of economies with strongly convex preferences. We shall require preferences to lie in P' , and thus have non-empty demand sets.

The statement of the theorem depends on notions of weak convergence (see Hildenbrand [12]). The proof we give uses nonstandard measure theory (Loeb [16], Anderson and Rashid [4]). It is not difficult to give a standard proof using the measure theoretic methods and core convergence results described in Hildenbrand [12]. Giving an elementary statement and proof of the theorem, not using measure theory or nonstandard analysis

appears to be a more difficult task. This is possible in principle by translating the nonstandard statement and proof; however, the elementary theorems we have found in this way so far have proved too complicated to be interesting.

We begin by summarizing some facts from Hildenbrand [12]. P' can be made into a Borel subset of a compact metric space; the topology this metric generates is called the topology of closed convergence. Hence, $P' \times \mathbb{R}_+^k$ is a Borel subset of a complete separable metric space. Let M be the space of Borel probability measures μ on $P' \times \mathbb{R}_+^k$, with the property $0 << \int_{\mathbb{R}_+^k} d\mu_e << \infty$, where μ_e is the marginal distribution of μ on \mathbb{R}_+^k . We can give M the topology generated by sets of the form

$$\{ \mu : | \int_{P' \times \mathbb{R}_+^k} F d\mu - \alpha | < \beta, | \int_{\mathbb{R}_+^k} d\mu_e - \gamma | < \delta \}, \alpha, \beta, \gamma, \delta \in \mathbb{R}, F \in C(P' \times \mathbb{R}_+^k) \} .$$

In other words, $\mu^n \rightarrow \mu$ if it converges weakly and $\int_{\mathbb{R}_+^k} d\mu_e^n \rightarrow \int_{\mathbb{R}_+^k} d\mu_e$.

This topology is generated by a separable metric.

Definition. Given $\mu \in M$, we say $p \in S$ is an equilibrium price for μ if there is a probability space (Ω, B, P) and a measurable map $\epsilon : \Omega \rightarrow P \times \mathbb{R}_+^k$ such that $\mu(A) = P \circ \epsilon^{-1}(A)$ for all Borel subsets A of $P' \times \mathbb{R}_+^k$, and such that p is an equilibrium price of the exchange economy ϵ , i.e. $\exists f : \Omega \rightarrow \mathbb{R}_+^k, \int_{\Omega} f dP = \int_{\Omega} \epsilon dP$, and $f(\omega) \in D(p, \omega)$ for almost all $\omega \in \Omega$. Notations such as ϵ , D and so on defined for finite exchange economies in Section 3 have the analogous meanings for exchange economies with a measure space of traders.

We say μ is dispersed if, for every equilibrium price p of μ ,

$\mu(\{(\lambda, e) : D(p, (\lambda, e)) \text{ contains more than one point}\}) = 0$. Let \mathcal{D} be the set of dispersed measures $\mu \in M$.

Remark. We call measure $\mu \in \mathcal{D}$ dispersed because "few" traders have multiple demands for any given equilibrium price. Of course, every trader might have a multiple demand at some point. But our assumption says that traders are not so much alike that a set of traders of positive measure all have multiple demands at the same equilibrium price. This is thus a natural dispersion notion.

A much smaller set, $N = \{\mu \in M : \mu(\{(\lambda, e) : D(p, (\lambda, e)) \text{ contains more than one point}\}) = 0 \text{ for all } p \in S^0\}$ is a residual set in M . A set is called residual if it is the complement of a countable union of nowhere dense sets. While "residual" is a weaker notion than "open and dense," the Baire Category Theorem justifies it as a notion of a large set, and it is frequently interpreted this way in analysis. Mas-Colell and Neufeind demonstrated (in a slightly different setting) that M is topologically complete and N is residual in M ; their proof can be adapted to our setting without difficulty.

Note finally that if μ is concentrated on $\{(\lambda, e) : \lambda \text{ is strongly convex}\}$, then $\mu \in \mathcal{D}$. We thus have three reasons to regard \mathcal{D} as a natural class of measures on $P \times \mathbb{R}_+^k$: (i) they exhibit a natural

dispersion property, (ii) they are a residual ("big") subset of M , and (iii) they include all the measures which are concentrated on the strongly convex preferences.

We can now state the main result of this section. If $\epsilon : A \rightarrow P' \times \mathbb{R}_+^k$ is a finite exchange economy, we define $\mu^\epsilon(B) = |\epsilon^{-1}(B)|/|A|$, for any Borel subset B of $P' \times \mathbb{R}_+^k$. Note that $\mu^\epsilon \in M$.

Theorem 4.1. Let $\epsilon_n : A_n \rightarrow P' \times \mathbb{R}_+^k$ be a sequence of exchange economies satisfying

- (i) $|A_n| \rightarrow \infty$
- (ii) $\mu^{\epsilon_n} \rightarrow \mu$, for some $\mu \in \mathcal{D}$.

Then

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{f \in C_\delta(\epsilon_n)} \inf_{p \in S} \frac{1}{|A_n|} \sum_{a \in A_n} \rho(f_n(a), D_a(p)) = 0.$$

In particular,

$$\lim_{n \rightarrow \infty} \sup_{f \in C(\epsilon_n)} \inf_{p \in S} \frac{1}{|A_n|} \sum_{a \in A_n} \rho(f_n(a), D_a(p)) = 0.$$

Proof. Suppose we're given $\delta_n \rightarrow 0$. Choose $n \in \mathbb{N} - \mathbb{N}$, and let $\epsilon = \epsilon_n$, $A = A_n$, etc. Select $f \in C_\delta(\epsilon)$. Transferring Theorem 2.2, there exists

$p \in S$ such that $\frac{1}{|A|} \sum_{a \in A} \phi(p, f, a) \leq 4 \left(\frac{kM_T}{|A|} + \delta \right)$. As in Hildenbrand

[12], $\mu^{\epsilon_n} \rightarrow \mu$ implies the sequence of endowments of ϵ_n is uniformly

integrable. Thus, in particular, $\frac{M_{\epsilon_n}}{|A_n|} \rightarrow 0$, so $\frac{M_\epsilon}{|A|} \simeq 0$. Since

$$\delta \simeq 0, \quad \frac{1}{|A|} \sum_{a \in A} \phi(p, f, a) \simeq 0.$$

A by-now routine argument shows ${}^0p \gg 0$ (see Anderson [2]). We claim 0p is an equilibrium price for μ . To see this, let ν be the normalized counting measure $\nu(B) = |B|/|A|$ defined on the algebra A of internal subsets of A . ${}^0\nu$ has a unique extension to a standard probability measure $\bar{\nu}$ defined on \bar{A} , the σ -algebra generated by A (Loeb [16]). \bar{A} is called the Loeb measure generated by $\bar{\nu}$. Define $\bar{\varepsilon} : (A, \bar{A}, \bar{\nu}) \rightarrow P' \times \mathbb{R}_+^k$ by $\bar{\varepsilon}(a) = (\bar{\gamma}_a, \bar{e}(a)) = ({}^0\gamma_a, {}^0e(a))$. Since $\int \bar{\varepsilon}^n \rightarrow \mu$, $\bar{\nu}(\bar{\varepsilon}^{-1}(B)) = \mu(B)$ for all Baire subsets B of $P \times \mathbb{R}_+^k$ (Anderson and Rashid [4]). Since $P' \times \mathbb{R}_+^k$ is separable metric, all Borel subsets are Baire sets (Billingsley [5]).

Hence, $\bar{\varepsilon}$ is an exchange economy with a standard measure space of traders, and the distribution of $\bar{\varepsilon}$ is μ . Moreover, ${}^0f(a) \in D({}^0p, \bar{\varepsilon}(a))$ for $\bar{\nu}$ -almost all a . To see this, note $\int_A |{}^0p \cdot ({}^0f(a) - \bar{e}(a))| d\bar{\nu} \lesssim \frac{1}{|A|} \sum_{a \in A} |p \cdot (f(a) - e(a))|$ (Anderson [3]) $\simeq 0$, so ${}^0p \cdot {}^0f(a) = {}^0p \cdot \bar{e}(a)$ for $\bar{\nu}$ -almost all a . If ${}^0p \cdot \bar{e}(a) > 0$, $x \in \mathbb{R}_+^k$, $x \bar{\gamma}_a > {}^0f(a)$, and ${}^0p \cdot x \leq {}^0p \cdot \bar{e}(a)$, then by continuity of $\bar{\gamma}_a$, $\exists y \in \mathbb{R}_+^k$, ${}^0p \cdot y < {}^0p \cdot \bar{e}(a)$, $y \bar{\gamma}_a > {}^0f(a)$. Hence, $y \bar{\gamma}_a > f(a)$, so $\phi(p, a) \not\geq 0$. Moreover, if ${}^0p \cdot \bar{e}(a) = 0$, then either $f(a) \not\geq 0 \not\geq D({}^0p, \bar{e}(a))$, or else $\phi(p, a) \not\geq 0$. Thus we have either ${}^0f(a) \in D({}^0p, \bar{\varepsilon}(a))$, or else $\phi(p, a) \not\geq 0$. But $\phi(p, a) \not\geq 0$ for $\bar{\nu}$ -almost all a , so ${}^0f(a) \in D({}^0p, \bar{\varepsilon}(a))$ for $\bar{\nu}$ -almost all a . Thus, 0p is an equilibrium price for μ .

Since $\mu \in \mathcal{D}$, we know $\mu(\{(\gamma, e) : D({}^0p, (\gamma, e)) \text{ has more than one element}\}) = 0$. Thus $\bar{\nu}(\{a : D({}^0p, \bar{\varepsilon}(a)) \text{ has more than one element}\}) = 0$. Since $\bar{\gamma}_a \in P'$, $D(p, \varepsilon(a)) \neq \phi$. Since $\varepsilon(a) \simeq \bar{\varepsilon}(a)$ for $\bar{\nu}$ -almost all a , and the demand correspondence is upper semi-continuous, ${}^0D(p, \varepsilon(a)) \in D({}^0p, \bar{\varepsilon}(a))$. We saw also ${}^0f(a) \in D({}^0p, \bar{\varepsilon}(a))$. Since

$D(\bar{p}, \bar{\epsilon}(a))$ is a singleton for $\bar{\nu}$ -almost all a , $f(a) \underline{\nu} D(p, \epsilon(a))$ for $\bar{\nu}$ -almost all a .

Since the endowment sequence is uniformly integrable, e is S -integrable (Anderson [3]). For any internal $E \subset A$ such that $\nu(E) \underline{\nu} 0$,

$$p \cdot \int_E f d\nu \leq p \cdot \int_E e d\nu + \frac{4M}{|A|} \epsilon + \delta \underline{\nu} p \cdot \int_E e d\nu \underline{\nu} 0. \quad \text{Thus } \left\| \int_E f d\nu \right\|_{\infty} \\ \lesssim \frac{1}{\min\{p^1, \dots, p^k\}} p \cdot \int_E f d\nu \underline{\nu} 0, \quad \text{so } f \text{ is } S\text{-integrable. The demand is}$$

S -integrable for similar, but simpler, reasons. Hence

$$\frac{1}{|A|} \sum_{a \in A} * \rho(f(a), D(p, a)) \underline{\nu} \int_A (* \rho(f(a), D(p, a))) d\bar{\nu} = 0.$$

$$\sup_{f \in C_{\delta}(\epsilon)} \inf_{p \in *S} \frac{1}{|A|} \sum_{a \in A} * \rho(f(a), D(p, a)) \underline{\nu} 0, \quad \text{since } C_{\delta}(\epsilon) \text{ is internal.}$$

Thus, by transfer,

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{f \in C_{\delta}(\epsilon)} \inf_{p \in S} \frac{1}{|A|} \sum_{a \in A} \rho(f(a), D(p, a)) = 0.$$

5. Random Sequences of Economies

The purpose of this section is to show that strong core convergence holds for almost all sequences of economies drawn at random. The key observation in the proof is that sequences of economies drawn at random from a given distribution of agents' characteristics converge in a stronger sense than weak convergence. The proof is really an elaboration of the original nonstandard proof of Theorem 3.1.

Suppose $\mu \in M$, the class of Borel probability measures on $P^1 \times \mathbb{R}_+^k$ with finite mean endowment, as defined in Section 4. We may think of μ as describing the underlying distribution of characteristics of "all possible people," and construct sequences of finite economies by sampling from μ with replacement. Specifically, we take Ω to be the countable product $(P^1 \times \mathbb{R}_+^k)^{\mathbb{N}}$, with the countable product measure $P = \mu^{\mathbb{N}}$. Any

$\omega \in \Omega$ is a sequence $(\omega_1, \omega_2, \dots)$, $\omega_i \in P' \times \mathbb{R}_+^k$ of characteristics. Given such an ω , we form a sequence of economies

$$\varepsilon_n^\omega : A_n \rightarrow P \times \mathbb{R}_+^k,$$

where $A_n = \{1, \dots, n\}$, $\varepsilon_n^\omega(i) = \omega_i$. In other words, ε_n^ω is the economy whose agents have characteristics $\omega_1, \omega_2, \dots, \omega_n$.

Hildenbrand [12] notes that, for almost all ω , the sequence of distributions of characteristics converges weakly to μ . However, it converges to μ in a stronger sense, which we shall exploit in a critical way.

Theorem 5.1. Suppose $\mu \in M$. Then for almost all ω ,

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{f \in C_\delta(\varepsilon_n^\omega)} \inf_{p \in S} \frac{1}{n} \sum_{a=1}^n \rho(f(a), D_a(p)) = 0.$$

In particular,

$$\lim_{n \rightarrow \infty} \sup_{f \in C(\varepsilon_n^\omega)} \inf_{p \in S} \frac{1}{n} \sum_{a=1}^n \rho(f(a), D_a(p)) = 0.$$

Remark. As we've indicated above, our proof will use Nonstandard Analysis.

A general metatheorem guarantees the existence of a standard proof, but it could in principle be exceedingly complex. We know of no tractable standard proof in the general case presented here. It is not terribly difficult, however, to give a standard (though hardly elementary) proof if there are only a finite or countable number of equilibrium prices for μ . Our first result is a result for Nonstandard exchange economies, which implies both Theorem 5.1 and Theorem 3.1. The nonstandard core

$C_0(\varepsilon)$ we define in the theorem is essentially the same as the Nonstandard Core defined in Brown-Robinson [6].

Theorem 5.2. Suppose $e : A \rightarrow {}^*(P \times \mathbb{R}_+^k)$ is an internal exchange economy and

- (i) $|A| \in {}^*N - N$.
- (ii) the induced measure $\nu(B) = |\varepsilon^{-1}(B)|/n$ is standardly concentrated (Anderson [3, Section 8]).
- (iii) $M_\varepsilon/|A| \simeq 0$.
- (iv) ${}^o\left(\frac{1}{n} \sum_{a \in A} e(a)\right) \gg 0$.

Let $C_0(\varepsilon) = \bigcap_{\substack{\delta > 0 \\ \delta \neq 0}} {}^*C_\delta(\varepsilon)$. If $f \in C_0(\varepsilon)$, there exists $p \in S$ such that

${}^*\rho(f(a), D_a(p)) \simeq 0$ for $L(\nu)$ -almost all a . If in addition

- (v) the endowment map $e : A \rightarrow {}^*\mathbb{R}_+^k$ is S -integrable (Anderson [3]),
- and (vi) $e : A \rightarrow {}^*(P' \times \mathbb{R}_+^k)$, then

$$\frac{1}{|A|} \sum_{a \in A} {}^*\rho(f(a), D_a(p)) \simeq 0.$$

Proof. Let $\mu(B) = L(\nu)(st^{-1}(B))$ for Borel $B \subset P \times \mathbb{R}_+^k$, where $L(\nu)$ is the Loeb measure generated by ν [16]. By Anderson [3, Proposition 8.4(ii)], μ is a Radon Probability measure. Given $p \in S$, $p \gg 0$, and $0 < \varepsilon$, $\delta \in \mathbb{R}$, define

$$B_{p, \varepsilon, \delta} = \{(\succ, e) \in P \times \mathbb{R}_+^k : \phi(p, x, (\succ, e)) < \delta \implies \rho(x, D(p, (\succ, e))) < \varepsilon\}.$$

For fixed ε and p , $\bigcap_{\delta > 0} B_{p, \varepsilon, \delta} = P \times \mathbb{R}_+^k$, by the same argument as

in the proof of Theorem 3.1. Thus, given ε , there exists δ such that

$\mu(B_{p,\varepsilon,\delta}) > 1-\varepsilon$. Therefore, $\nu(*B_{p,\varepsilon,\delta}) > 1-\varepsilon$.

Suppose $f \in C_0(\varepsilon)$. For all $\delta \not\approx 0$, $\sum_{a \in A} \phi(p, f, a) \leq 4(M_\varepsilon + \delta|A|)$, for some $p \in *S$, by transferring Theorem 2.2. Since $M_\varepsilon/|A| \approx 0$ and δ can be any noninfinitesimal positive number, $\frac{1}{|A|} \sum_{a \in A} \phi(p, f, a) \approx 0$ for some $p \in *S$. Therefore, $\phi(p, f, a) \approx 0$ for $L(\nu)$ -almost all a .

Yet again, we see ${}^o p \gg 0$ (Anderson [2]). If $\phi(p, f, a) \approx 0$, ${}^o e(a) \ll \infty$, and $(\not\approx_a, e(a)) \in \bigcup_{\delta \not\approx 0} {}^o B_{p,\varepsilon,\delta}$, then $\phi({}^o p, f, a) \approx 0$ and so $\rho(f(a), D({}^o p, a)) < \varepsilon$. $L(\nu)(\bigcup_{\delta \not\approx 0} {}^o B_{p,\varepsilon,\delta}) = 1$, so $\rho(f(a), D({}^o p, a)) < \varepsilon$ for $L(\nu)$ -almost all a . Therefore, $(f(a), D({}^o p, a)) \approx 0$ for $L(\nu)$ -almost all a .

If e is S -integrable, then $|D({}^o p, a)| \leq |e(a)|/\min\{p^1, \dots, p^k\}$, and $|\sum_{a \in S} f(a)| \leq |\sum_{a \in S} e(a)| + 2M_\varepsilon/\min\{p^1, \dots, p^k\}$, so f and D are S -integrable. If all preferences are in $*P'$, $D({}^o p, a)$ is non-empty. Therefore, $\frac{1}{|A|} \int * \rho(f(a), D({}^o p, a)) \approx \int_A ({}^o \rho(f(a), D({}^o p, a))) dL(\nu) = 0$, since the integrand is 0 almost everywhere.

Proof of Theorem 5.1. By the Strong Law of Large Numbers,

$L(P)(\{\omega : \int e d\nu_n^\omega \approx \int e d\mu \text{ for all infinite } n\}) = 1$. By Anderson [3, Theorem 8.7(iii)], $L(P)(\{\omega : L(\nu_n^\omega)st^{-1} = \mu \text{ for all infinite } n\}) = 1$. For any ω satisfying these two conditions, e is S -integrable with respect to ν_n^ω for all infinite n . Moreover, by Anderson [3, Theorem 8.7(i)], there exists an internal $\Omega'_n \subset \Omega^*$ such that $*P(\Omega'_n) \geq 1 - 2ne^{-\sqrt{n}/4}$ and ν_n^ω is standardly concentrated for all $\omega \in \Omega'_n$.

Hence, given $n \in *N - N$, we can find an internal set $\Omega_n \in * \Omega$ such that $*P(\Omega_n) \approx 1$, and such that for all $\omega \in \Omega_n$, $\int e d\nu_m^\omega \approx \int e d\mu$, $L(\nu_m^\omega)st^{-1} = \mu$ (and thus e is S -integrable with respect to ν_m^ω), and ν_m^ω is standardly concentrated for all $m \geq n$. Hence, for all $\delta \approx 0$,

$$\sup_{\omega \in \Omega_n} \sup_{m > n} \sup_{f \in *C_\delta(\epsilon_m^\omega)} \inf_{p \in *\Delta} \frac{1}{m} \sum_{a \in A_m} *p(f(a), D(p, a)) \simeq 0 .$$

If $\gamma \in (0, 1)$, then

$$\sup_{\omega \in \Omega_n} \sup_{m > n} \sup_{f \in *C_\delta(\epsilon_m^\omega)} \inf_{p \in *\Delta} \frac{1}{m} \sum_{a \in A_m} *p(f(a), D(p, a)) \leq \gamma$$

for all $\delta \simeq 0$, and thus this statement is true for some $\delta \in (0, 1)$.

In other words, given $\gamma \in (0, 1)$ we've found $\delta \in (0, 1)$ such that

$$*P(\{\omega : \sup_{m > n} \sup_{f \in *C_\delta(\epsilon_m^\omega)} \inf_{p \in *\Delta} \frac{1}{m} \sum_{a \in A_m} *p(f(a), D(p, a)) \leq \gamma\}) \simeq 1 .$$

Hence, given $\sigma \in (0, 1)$ there exists $n \in \mathbb{N}$ such that

$$*P(\{\omega : \sup_{m > n} \sup_{f \in *C_\delta(\epsilon_m^\omega)} \inf_{p \in *\Delta} \frac{1}{m} \sum_{a \in A_m} *p(f(a), D(p, a)) \leq \gamma\}) \geq 1 - \sigma .$$

By Transfer,

$$P(\{\omega : \sup_{m > n} \sup_{f \in C_\delta(\epsilon_m^\omega)} \inf_{p \in \Delta} \frac{1}{m} \sum_{a \in A_m} \rho(f(a), D(p, a)) \leq \gamma\}) \geq 1 - \sigma .$$

Thus,

$$\forall \gamma \exists \delta \overline{P(\{\omega : \lim_{n \rightarrow \infty} \sup_{f \in C_\delta(\epsilon_n^\omega)} \inf_{p \in \Delta} \frac{1}{n} \sum_{a \in A_n} \rho(f(a), D(p, a)) \leq \gamma\})} = 1) .$$

By considering a countable sequence of γ 's tending to 0, we get

$$P(\{\omega : \forall \gamma \exists \delta \overline{\lim_{n \rightarrow \infty} \sup_{f \in C_\delta(\epsilon_n^\omega)} \inf_{p \in \Delta} \frac{1}{n} \sum_{a \in A_n} \rho(f(a), D(p, a)) \leq \gamma\}) = 1 .$$

In other words,

$$P(\{\omega : \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \sup_{f \in C_{\delta}(\varepsilon_n^{\omega})} \inf_{p \in \Delta} \frac{1}{n} \sum_{a \in A_n} \rho(f(a), D(p, a)) = 0\}) = 1 ,$$

as was to be proven.

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