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EXISTENCE OF EQUILIBRIUM IN A HYPERFINITE

EXCHANGE ECONOMY: I and II

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EXISTENCE OF EQUILIBRIUM IN A HYPERFINITE
EXCHANGE ECONOMY: I[*]

by

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I. Introduction

The purpose of this paper is to propose a model of
economic exchange, a hyperfinite exchange economy, which
allows a systematic and comparative investigation of eco-
nomies where there may be a finite, or measure space of
traders and a finite or denumerable number of commodities.

Our main result is a metatheorem on existence theorems
in several different types of standard exchange economies.

We shall show, using nonstandard analysis, that the
theorems of Gale-Mas-Colell (4), (5), Peleg-Yaari (6),
and Bewley (2) on the existence of a competitive equilibrium
in exchange economies with convex preferences, although differ-
ing in their assumptions regarding the number of agents and
commodities, are formally equivalent to the existence theorem
for a hyperfinite exchange economy.

The paper is divided into several sections. First, we
define hyperfinite exchange economies in Section II. The
essential feature of a hyperfinite exchange economy is that
it has all of the formal properties of an Arrow-Debreu exchange

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economy, even though it may have an infinite, i.e., non-standard, number of commodities or agents.

In Section III, we give a nonstandard proof of the Peleg-Yaari existence theorem for an economy with a denumerable number of commodities and a finite number of agents. Bewley's existence theorem is an immediate corollary of the Peleg-Yaari result.

Our proof consists of defining an appropriate hyperfinite exchange economy and asserting that a competitive equilibrium exists by transfer of the Gale-Mas-Colell existence theorem for exchange economies with a finite number of commodities and agents. The final step in the proof is to take the standard part of the equilibrium allocation and prices in the hyperfinite exchange economy and show that they constitute an equilibrium in the given standard economy.

Since in the standard model, we are dealing with an infinite dimensional commodity space, i.e., the space of real valued sequences, the choice of topology is crucial to our analysis. We shall use the product topology and give an economic justification for it, in terms of its myopic properties, in the final section of the paper.

In a subsequent paper, we shall show that a similar metatheorem holds for the existence theorems of Anderson (1) and Schmeidler (7), i.e., these theorems are formally equivalent to the existence theorem for a hyperfinite exchange economy, where preferences need not be convex, but there are "many more" agents than commodities.

II. <u>Hyperfinite Exchange Economies</u>

Let R_w denote the space of real valued sequences and
$*R_w$ the nonstandard extension of R_w . R_w is a linear metric
space, hence $*R_w$ is also a linear metric space.

If d is a hyperfinite integer, then $*R_d$ will denote a
d-dimensional internal linear subspace of $*R_w$. Formally $*R_d$
has all the properties of a finite dimensional subspace of R_w .
In particular, the relative topology on $*R_d$ is given by the
nonstandard extension of the Euclidean metric on the finite dimen-
sional subspaces of R_w .

Let T denote an internal initial segment of $*N$, the
nonstandard integers, and $|T| = m$.

Let P denote the internal set of binary relations
on $*R_d^+$, the nonnegative orthant of $*R_d$.

A hyperfinite exchange economy is an internal map
$\mathcal{E}: T \to P \times *R_d^+$.

For each agent t , the projection of $\mathcal{E}(t)$ on P ,
denoted $\succ_t$, is to be interpreted as the preference relation
of agent t . The projection of $\mathcal{E}(t)$ on $*R_d^+$, denoted
$e(t)$, is the endowment of agent t .

An allocation is an internal map $g: T \to *R_d^+$ such that
$$\sum_{t \in T} g(t) = \sum_{t \in T} e(t) .$$

The positive price simplex, $^0\Delta_d$, consists of $p \varepsilon *R_d^{++}$
such that $\sum_{i=1}^{d} p_i = 1$, where $*R_d^{++}$ is the positive orthant
of $*R_d$.

If $p \varepsilon^0 \Delta_d$, then the demand set of the $t^{\underline{th}}$ trader is $D(p, t) = \{x \varepsilon *R_d^+ | p \cdot x \leq p \cdot e(t) \text{ and } p \cdot y \leq p \cdot e(t) , y \varepsilon *R_d^+ \Rightarrow y \not\succ_t x\}$.

A competitive equilibrium for a hyperfinite exchange economy $\mathcal{E}$ consists of a price $p \varepsilon^0 \Delta_d$, an allocation g , and an internal set S that $g(t) \varepsilon D(p, t)$, for all $t \varepsilon S$, where $|S|/m \simeq 1$.[*]

Theorem (1) If $\mathcal{E}$ is a hyperfinite exchange economy satisfying the following assumptions, for all $t \varepsilon T$:

(i) irreflexivity: $x \not\succ_t x$

(ii) monotonicity: $x \geq y$, $x \neq y \Rightarrow x \succ_t y$

(iii) continuity: $\{(x, y) : x \succ_t y\}$ is relatively open in $*R_d^+$

(iv) convexity: $\{x \varepsilon *R_d^+ : x \succ_t y\}$ is convex for all $y *R_d^+$

and (v) $\sum_{t \varepsilon T} e(t) >> 0$.

Then $\mathcal{E}$ has a competitive equilibrium.

Proof: By transfer of the Gale-Mas-Colell existence theorem (4), (5) .

III. Markets with Countably Many Commodities

Peleg and Yaari proved the existence of a competitive equilibrium in a market with countably many commodities and a finite number of traders. In this section we give a nonstandard proof of their result, where we drop their assumptions that preferences are complete or transitive.

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[*]Note that for m finite, this is the "standard" definition of a competitive equilibrium in an exchange economy.

In the (P-Y) model, there are m traders whose consumption sets are R_w^+ , with the product topology.

Each trader, t , has an endowment vector $i(t) \varepsilon R_w^+$ and a utility function $U_t: R_w^+ \to R$, where U_t is assumed to be quasi-concave, strongly monotonic, and continuous with respect to the product topology.

If $T = \{1, 2, \ldots, m\}$, then an allocation is a function $g: T \to R_w^+$ such that $\sum_{t \varepsilon T} g(t) = \sum_{t \varepsilon T} i(t)$.

Let the social endowment be denoted as $\underset{\sim}{i}$, then they assume that $\underset{\sim}{i} = \sum_{t \varepsilon T} (i) > > 0$.

Peleg and Yaari restrict their attention to price sequences $p \varepsilon R_w^{++}$ having the property that $p \cdot \underset{\sim}{i} < \infty$ or equivalently $p \cdot \underset{\sim}{i} = 1$. Demand sets are defined in the obvious fashion for such price systems and a competitive equilibrium is then a price p and an allocation g such that g(t) is in the demand set of each agent t .

It is important to note that a (P-Y) competitive price system need not give a finite value to arbitrary consumption vectors $x \varepsilon R_d^+$ and may not be summable. Despite these limitations, every competitive equilibrium, in the sense of Peleg-Yaari, is Pareto optimal.

Let d be an infinite integer and $*R_d$ be the internal set of sequences x , such that $x_i = 0$ for all $i > d$. We define the hyperfinite exchange economy $\mathcal{E}$ where:

$$\text{(i)} \quad T = \{1, 2, \ldots, m\}$$

$$\text{(ii)} \quad e_j(t) = \begin{cases} *i_j(t) & \text{for } j \le d \\ 0 & \text{for } j > d \end{cases}$$

$$\text{(iii)} \quad x \succ_t y \quad \text{iff} \quad *U_t(x) > *U_t(y)$$

and $*i(t)$, $*U_t$ denote the nonstandard extensions of $i(t)$

and U_t to $*R_w$.

It is immediate that $\mathcal{E}$ satisfies the conditions of

Theorem 1. Hence $\mathcal{E}$ has a competitive equilibrium $\langle p, g \rangle$.

Let $q = \frac{1}{\alpha} p$ where $\alpha = p \cdot \sum_{t \in T} e(t)$.

Theorem (2) $\langle {}^0q, {}^0g \rangle$ exists and is a competitive equilibrium

for the given (P-Y) economy.

Proof: Let $\underset{\sim}{e} = \sum_{t \in T} e(t)$, since $|T|$ is finite we see that $\underset{\sim}{e}$

is standard. $\sum_{t \in T} g(t) = \sum_{t \in T} e(t) = \underset{\sim}{e}$, hence $g(t) \le \underset{\sim}{e}$ for

all t . Therefore, each $g(t)$ is near standard.[*]

$q \cdot \underset{\sim}{e} = 1$ implies that q is also near standard.

Consequently, $\langle {}^0q, {}^0g \rangle$ exists.

$q \cdot e(t_0) >_{\not\sim} 0$ for some $t_0 \in T$, since $q \cdot \underset{\sim}{e} = \sum_{t \in T} q \cdot e(t) = 1$.

Suppose $q_i \underset{\sim}{\le} 0$ for all finite i . Let $y = {}^0g(t_0)$ and

$x = y + (1, 0, 0, \ldots)$, then $x \succ_{t_0} y$ by monotonicity of $\succ_{t_0}$.

Since $\succ_{t_0}$ is continuous in the product topology, which is

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[*]Throughout this paper, near standard will be defined with
respect to the product topology.

strongly myopic (see Section IV), there is some n such that
$\overset{v}{x}_n \succ_{t_0} y$,

where $(\overset{v}{x}_n)_i = \begin{cases} x_i & \text{for } i \leq n \\ \\ 0 & \text{for } i > n \end{cases}$

Again by continuity of $\succ_{t_0}$ in the product topology,
$\overset{v}{x}_n \succ_{t_0} g(t_0)$, but $q \cdot \overset{v}{x}_n \underset{\sim}{\sim} 0$, hence is in the budget set of

trader t_0, which contradicts the maximality of $g(t_0)$. There-

fore, for some finite i , say $i = 1$, $q_i \not\sim 0$.

Since $\underset{\sim}{e}_1 \not\sim 0$, some trader, say t_1 , has a noninfinite-
simal amount of the first good.

Therefore, $q \cdot g(t_1) = q \cdot e(t_1) \not\sim 0$. Suppose for some finite

$j \neq 1$, that $q_j \underset{\sim}{\sim} 0$, say $j = 2$. Let $z = {}^0 g(t_1)$ and $v = z +$

$(0, 1, \ldots 0, 0, \ldots)$, then $\overset{v}{v}_m \succ_{t_1} z$ by the same argument as

above. But $q \cdot \overset{v}{v}_m \underset{\sim}{\leq} q \cdot e(t_1)$, hence there is a $u \epsilon R_w^+$ such that

$q \cdot u \underset{\sim}{\lessgtr} q \cdot e(t_1)$ and $\overset{v}{u}_m \succ_{t_1} g(t_1)$, by the continuity of $\succ_{t_1}$, in

the product topology. This contradicts the maximality of $g(t_i)$.

Hence $q_i \not\sim 0$ for all finite i .

We now show that $\langle {}^0 q, {}^0 g \rangle$ is a (P-Y) equilibrium.

$${}^0 \left(\sum_{t \epsilon T} g(t) \right) = {}^0 \left(\sum_{t \epsilon T} e(t) \right) \text{, and } {}^0 \left(\sum_{t \epsilon T} g(t) \right) = \sum_{t \epsilon T} {}^0 g(t) \text{,}$$

$${}^0 \left(\sum_{t \epsilon T} e(t) \right) = \sum_{t \epsilon T} {}^0 e(t) \text{ since } |T| \text{ is finite. Hence}$$

$$\sum_{t \epsilon T} {}^0 g(t) = \sum_{t \epsilon T} i(t) \text{ . Suppose for some } t_0 \text{, } {}^0 q \cdot {}^0 g(t_0) <$$

$^0q \cdot i(t)$, then by monotonicity, continuity of $\succ_{t_0}$ in product topology, and the strong myopia of the product topology there is for some n , and $x \varepsilon R_w^+$, $\overset{v}{x}_n \succ_{t_0} g(t_0)$ where $^0q \cdot \overset{v}{x}_n < {}^0q \cdot i(t)$.

Hence $q \cdot \overset{v}{x}_n < q \cdot e(t_0)$, a contradiction. Hence $^0q \cdot {}^0g(t) \geq {}^0q \cdot i(t)$ for all t . Suppose for some t_1 that $^0q \cdot {}^0g(t) > {}^0q \cdot i(t)$. Then $\sum_{t \varepsilon T} {}^0q \cdot {}^0g(t) > \sum_{t \varepsilon T} {}^0q \cdot i(t)$, which contradicts

$$\sum_{t \varepsilon T} {}^0g(t) = \sum_{t \varepsilon T} i(t) .$$

Therefore $^0q \cdot {}^0g(t) = {}^0q \cdot i(t)$, for all $t \varepsilon T$. Suppose for some t_1 , that $^0q \cdot y \leq {}^0q \cdot i(t_1)$, $y \varepsilon R_w^+$, and $y \succ_{t_1} {}^0g(t_1)$. Again invoking the continuity of $\succ_{t_1}$ in the product topology and the strong myopia of the product topology, there is an m such that $\overset{v}{y}_m \succ_{t_1} g(t_1)$. Hence there is a $u \varepsilon R_w^+$ such that

$\overset{v}{u}_m \succ_{t_1} g(t_1)$ and $^0q \cdot \overset{v}{u}_m < {}^0q \cdot \dot{e}(t_1)$, by continuity of $\succ_{t_1}$ in product topology. That is, $q \cdot \overset{v}{u}_m \leq q \cdot e(t_1)$ and $\overset{v}{u}_m \succ_{t_1} g(t_1)$, a contradiction. Therefore, $^0g(t)$ is in the demand set of trader t .

It is doubtful if the original argument of (P-Y) can be extended to unordered preferences, since their proof used the equal treatment property of core allocations in a replicated economy and Scarf's existence theorem for the core of an N-person game, where in both instances agents are assumed to have quasi-concave utility functions.

Bewley's existence theorem for an exchange economy with a finite number of agents, where the commodity space is ℓ_∞ , the space of bounded real valued sequences, with the Mackey topology for the pairing $\langle \ell_\infty, \ell_1 \rangle$ and ℓ_1 is the space of summable real valued sequences, is an immediate consequence of the (P-Y) existence theorem.

Bewley assumes that each agent has ℓ_∞^+ , the space of nonnegative bounded real valued sequences, as his consumption set; that the social endowment $\underset{\sim}{i}$ is uniformly bounded away from 0 and is in ℓ_∞^+ ; that preferences are given by quasi-concave, strongly monotonic, and Mackey continuous utility functions. He then demonstrates the existence of a competitive equilibrium with strictly positive prices in ℓ_1 .

Since $\underset{\sim}{i} \varepsilon \ell_\infty^+$, we can assume that each agent's consumption set is norm bounded, but on norm bounded subsets of ℓ_∞ , the Mackey topology agrees with the product topology. Hence by the (P-Y) existence theorem there exists an allocation g and a price $p \varepsilon R_w^{++}$ such that g(t) is in the demand set of each agent t and $p \cdot \underset{\sim}{i} = 1$. The allocation g takes values in ℓ_∞^+ , since $g(t) \leq \underset{\sim}{i}$ for each t . Let u = (1, 1, ...1, ...) , then $\underset{\sim}{i} \geq \beta u$ where $\beta > 0$. Hence $\beta q \cdot u \leq q \cdot \underset{\sim}{i} = 1$, and $q \cdot u \leq 1/\beta$. That is, $q \varepsilon \ell_1$.

III. A Myopic Characterization of the Product Topology on R_w

A topology $\mathcal{J}$ on R_w is said to be strongly myopic; if for all $x\epsilon R_w$, $\overset{v}{x}_n$ converges to x with respect to $\mathcal{J}$. The fact that the product topology, $\mathcal{J}_p$, is strongly myopic played a crucial role in our proof of the (P-Y) existence theorem. It is conceivable that there are other topologies on R_w , e.g., the uniform topology or the box topology, which are finer than the product topology and are strongly myopic, in which case, those topologies would admit a larger family of strongly myopic preferences than the product topology.

In this section, we shall show that if these topologies are required to be locally convex (linear) topologies on R_w then this is not the case. That is, the product topology is the finest locally convex (linear) topology on R_w which is strongly myopic.

If R_∞ is the set of real valued sequences which are nonzero in at most a finite number of places, then the product topology is the Mackey topology for the duality $\left\langle R_w, R_\infty \right\rangle$. (See example 3 on page. 249 in (8)). Suppose $\mathcal{J}$ is a strongly myopic topology on R_w , then every $\mathcal{J}$-continuous linear functional L can be represented as a sequence p . This can be seen by considering $u^i = (0, \ 0, \ \ldots, \ 0, \overset{ith}{1}, \ 0, \ \ldots)$ and letting $p_i = L(u^i)$, for $i = 1, 2, \ldots$. Then $L(\overset{v}{x}_n) = \sum\limits_{i=1}^{n} p_i x_i$, but $\overset{v}{x}_n$ converges to x with respect to $\mathcal{J}$ and L is $\mathcal{J}$-continuous. Therefore, $L(x) = \sum\limits_{i=1}^{\infty} p_i x_i = p \cdot x$. Hence, $p\epsilon R_\infty$

Let $\mathcal{J}_{sm}$ be the finest locally convex (linear) topology on R_w which is strongly myopic. $\mathcal{J}_{sm}$ exists since it is generated by the family of seminorms $||\ \ ||_\alpha$, where for each $x \varepsilon R_w$, $||\overset{v}{x}_n - x||_\alpha$ converges to 0 . Note that every $p \varepsilon R_\infty$ defines such a seminorm, $|p \cdot x|$. Hence every linear functional defined by a $p \varepsilon R_\infty$ is $\mathcal{J}_{sm}$ continuous. Therefore $\mathcal{J}_{sm} \subseteq \mathcal{J}_p$. But $\mathcal{J}_p$ is strongly myopic, hence $\mathcal{J}_p \subseteq \mathcal{J}_{sm}$.

This result complements our work in (3) , where we showed that the strict topology is the finest locally convex (linear) topology on ℓ_∞ which is strongly myopic.

In (3), we explored the economic interpretation of myopia, when the $x \varepsilon \ell_\infty$ were interpreted as state-contingent claims. We argued that myopia resolved the St. Petersburg paradox in the sense of Buffon, i.e., agents "neglect" sufficiently unlikely states. This resolution has been justly criticized as assuming away the essential feature of the St. Petersburg paradox. That is, $x \varepsilon R_w$ and $x \notin \ell_\infty$ is the interesting case. The myopic characterization of the product topology, given above, is a response to this criticism.*

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*We owe this observation to John Geanakoplos.

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EXISTENCE OF EQUILIBRIUM IN A HYPERFINITE
EXCHANGE ECONOMY: II[*]

by

D. J. Brown and L. M. Lewis

I. Introduction

Hyperfinite exchange economies were defined in (1), where
it was shown that the existence theorems of Gale-Mas-Colell (4) and
Peleg-Yaari (7) are formally equivalent to the existence of a com-
petitive equilibrium in a hyperfinite exchange economy with convex
preferences.

Here we extend that analysis to the nonconvex case. Our
first result is a new proof of the existence of a near standard
competitive equilibrium in a nonconvex hyperfinite exchange economy,
where there are "many more" agents than commodities, first established
by Lewis in (5) .

With this proof in hand, it is relatively easy to demonstrate
the formal equivalence of the existence theorems of Anderson (2) and
Schmeidler (9).

Our proof consists of first proving the existence of a com-
petitive equilibrium in a hyperfinite exchange economy with noncon-
vex preferences, where the ratio of the number of commodities to the
number of agents is infinitesimal, by transferring Anderson's exis-
tence theorem for approximate equilibria in a finite economy with

nonconvex preferences. In theorem (2), we establish the existence
of a near standard competitive equilibrium where the initial endow-
ment is integrable and agent's preferences lie in the nonstandard
extension of a compact set of preferences, each of which is contin-
uous in the product topology.   Then, following Rashid (8), we
construct a measure-theoretic exchange economy from the hyperfinite
exchange economy and demonstrate that Schmeidler's theorem holds by
taking the standard part of a near standard competitive equilibrium
and showing that it is a measure-theoretic competitive equilibrium.

Finally, using Rashid's construction, we prove our major result,
the existence of a competitive equilibrium in an exchange economy with
a measure space of agents and a countable number of commodities, where
agents have nonconvex and incomplete preferences, which are continuous
in the product topology. This result extends the Peleg-Yaari theorem
for a finite number of agents to a measure space of agents.

II. Hyperfinite Exchange Economies

Let R_w denote the space of real valued sequences and $^*R_w$
the nonstandard extension of R_w . R_w is a linear metric space,
hence $^*R_w$ is also a linear metric space.

If d is a hyperfinite integer, then $^*R_d$ will denote a
d-dimensional internal linear subspace of $^*R_w$. Formally $^*R_d$
has all the properties of a finite dimensional subspace of R_w .
In particular, the relative topology on $^*R_d$ is given by the non-
standard extension of the Euclidean metric on the finite dimen-
sional subspaces of R_w .

Let   T   denote an internal initial segment of $*N$, the nonstandard integers, and $|T| = m$.

Let   P   denote the internal set of binary relations on $*R_d^+$, the nonnegative orthant of $*R_d$.

A hyperfinite exchange economy is an internal map $\mathcal{E}: T \rightarrow P \times *R_d^+$.

For each agent   t , the projection of $\mathcal{E}(t)$ on   P , denoted $\succ_t$, is to be interpreted as the preference relation of agent   t .   The projection of $\mathcal{E}(t)$ on $*R_d^+$, denoted   e(t) , is the endowment of agent   t .

An allocation is an internal map $g: T \rightarrow *R_d^+$ such that

$$\sum_{t \in T} g(t) = \sum_{t \in T} e(t) .$$

The positive price simplex $^0\Delta_d$, consists of   p $*R_d^{++}$ such that $\sum_{i=1}^{d} p_i = 1$, where $*R_d^{++}$ is the positive orthant of $*R_d$.

If $p \varepsilon {}^0\Delta_d$, then the demand set of the $t^{\underline{th}}$ trader is

$$D(p, t) = \{x \varepsilon *R_d^+ | p \cdot x \leq p \cdot e(t) \text{ and } p \cdot y \leq p \cdot e(t) , y \varepsilon *R_d^+ \Rightarrow y \not\succ_t x\}.$$

A competitive equilibrium for a hyperfinite exchange economy $\mathcal{E}$ consists of a price $p \varepsilon {}^0\Delta_d$, an allocation   g , and an internal set   S   such that $g(t) \varepsilon D(p, t)$, for all $t \varepsilon S$, where $|S|/m \underset{\sim}{\cdot} 1$.

<u>Theorem (1)</u>   If $\mathcal{E}$ is a hyperfinite exchange economy satisfying the following assumptions, for all $t \varepsilon T$:

       (i)   irreflexivity: $x \not\succ_t x$

       (ii) monotonicity: $x \geq y$, $x \neq y \Rightarrow x \succ_t y$

(iii)   continuity: $\{(x, y): x \succ_t y\}$ is relatively open

in $*R_d^+$

(iv)   transivity: $x \succ_t y$, $y \succ_t z \Rightarrow x \succ_t z$

and   (v) $\sum_{t \in T} e(t) >> 0$.

(vi) $d/m \sim 0$

Then $\mathcal{E}$ has a competitive equilibrium.

<u>Proof</u>: By transfer of the Anderson existence theorem (2).

A vector $x \in *R_d$ is said to be near standard if for all finite

i , x_i is finite, i.e., x is near standard in the product topo-

logy.[*] $x >_{\chi} > 0$ means $x_i >_{\chi} 0$ for all finite i .

If d is finite, then a family of preference relations on

$*R_d^+$ is said to be near standard if they lie in the nonstandard ex-

tension of a family of preference relations on R_d^+ , denoted $\mathcal{P}$,

which is compact in the topology of closed convergence.

If d is infinite, then a family of preference relations on

$*R_d^+$ is said to be near standard if they lie in the nonstandard ex-

tension of a family of preference relations on R_w^+ , denoted $\mathcal{P}$,

which is compact with respect to the Hausdorff uniformity on closed

subsets of $R_w^+ \times R_w^+$, where R_w^+ has the product topology.

A near standard competitive equilibrium for a hyperfinite

exchange economy $\mathcal{E}$ is a competitive equilibrium $\langle p, g, S \rangle$ such

that p is near standard and "almost all" traders in S have near

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[*]If x is a near standard vector in $*R_d$, then ${}^0x$, the standard
part of x , is the sequence ${}^0x_i$, the standard part of x_i , for
all finite i .

standard allocations. That is, for every $\epsilon >_\chi 0$, there exists an internal set of traders, V_ϵ , such that $\{t\epsilon S | g(t)$ is not near standard$\}$ $\subseteq V_\epsilon$ and $|V_\epsilon|/m < \epsilon$.

If $h(t)$ is an internal map from T into $^*R_d$, then $h(t)$ is said to be integrable if for all finite i , $(\frac{1}{m} \sum_{t\epsilon T} h(t))_i$ is integrable, i.e., $(\frac{1}{m} \sum_{t\epsilon T} h(t))_i$ is finite and $(\frac{1}{m} \sum_{t\epsilon V} h(t))_i \stackrel{\sim}{=} 0$, if $|V|/m \stackrel{\sim}{=} 0$, for all finite i .

For the remainder of this paper, we shall assume that preferences are near standard, i.e., lie in $^*\mathscr{P}$, where each preference relation in $\mathscr{P}$ is irreflexive, strongly monotonic, convex, and continuous in the product topology.

<u>Theorem (2)</u>: If $\mathscr{E}$ is a hyperfinite exchange economy satisfying the following assumptions:

       (i)   preferences, $>_t$, are near standard

      (ii) $\sum_{t\epsilon T} e(t) >> 0$

     (iii) $e(t)$ is integrable

     (iv) $\frac{1}{m} \sum_{t\epsilon T} e(t) >_\chi > 0$,

then $\mathscr{E}$ has a near standard competitive equilibrium $\langle q , g , s \rangle$, where $q >_\chi > 0$.

<u>Proof</u>: Assumptions (i) and (ii) guarantee that $\mathscr{E}$ satisfies the conditions of Theorem (1), hence $\mathscr{E}$ has a competitive equilibrium $\langle p , g , s \rangle$. Let $q = \frac{1}{\alpha} p$, where $\alpha = p \cdot \frac{1}{m} \sum_{t\epsilon T} e(t)$. We shall show that $\langle q , g , s \rangle$ is a near standard competitive equilibrium and that $q >_\chi > 0$.

$q \cdot \frac{1}{m} \sum_{t \in T} e(t) = 1$, hence by assumption (iv) q is near

standard. "Almost all" traders have near standard allocations,

since e(t) is integrable and we can apply lemma (1) of Brown-Khan

(3) , which states: If $\mathcal{A}$ is an integrable function from T to

*R , then $(\forall \epsilon >_{\chi} 0)$ $(\exists n \in N)\{|W^n|/m \geq 1-\epsilon\}$ where $W^n = \{t \in T | |\mathcal{A}(t)| \leq n\}$.

We simply use this lemma to construct a standard sequence, which for

the given ϵ bounds the allocations, on the finite indices, of all

but $\epsilon|S|$ traders. Hence $\langle q , q , S \rangle$ is a near standard competi-

tive equilibrium.

Suppose q_i is infinitesimal for all finite i . Since

$q \cdot \frac{1}{m} \sum_{t \in T} e(t) = 1$, there is a noninfinitesimal fraction of traders

in S who have a noninfinitesimal income at prices q . In addition,

a noninfinitesimal fraction of these traders have near standard allo-

cations.

Let t_1 be one such trader, i.e., $t_1 \epsilon S$, $q \cdot e(t_1) >_{\chi} 0$, and

$g(t_1)$ is near standard. Let $y = g(t_1) + (1, 0, 0, \ldots)$, then

$y >_{t_1} g(t_1)$. Hence $\overset{v}{y}_n >_{t_1} g(t_1)$ for some finite n . This

follows from the assumptions that each preference relation in $\mathcal{P}$ is

strongly monotonic, continuous in the product topology--therefore

strongly myopic--, and that $g(t_1)$ is near standard. Hence, for

any standard preference $\succ$ in $*\mathcal{P}$, $\overset{v}{y}_n \succ g(t_1)$ for some finite

n . Since $>_{t_1}$ is near standard in the topology of closed conver-

gence, (or near standard with respect to the Hausdorff uniformity,

see appendix), the same conclusion holds for $>_t$. But $q \cdot \overset{v}{y}_n \sim 0$,

hence $\overset{v}{y}_n$ is in the budget set of agent t_1 , contradicting the

assumption that $t_1 \epsilon S$, i.e., $g(t_1) \epsilon D(q, t_1)$, the demand set of

agent t_1 .

Hence, for some finite i , say $i = 1$, $\sigma_1 \not\gtrsim 0$.

Suppose for some finite $i_0 \neq 1$, that $q_{i_0} \sim 0$. By assumption (iv) and an argument similar to that above, we can find a trader t_2 such that $g_1(t_2) >_\chi 0$, $g(t_2)$ near standard, and $t_2 \in S$. Again using strong monotonicity, strong myopia, and the fact that preferences are near standard, we can find a $v \in R_w^+$ such that $\overset{v}{\hat{v}}_n >_{t_2} g(t_2)$ for some finite n , $q \cdot \overset{v}{\hat{v}}_n \leq q \cdot e(t_2)$, a contradiction.

Therefore, $q_i \not\gtrsim 0$ for all finite i .

III. Markets with a Measure Space of Agents

Under assumptions (i), (iii), and (iv) of theorem (2), Rashid (8) has given a method for constructing a measure-theoretic exchange economy $^0\mathcal{E}$ from a hyperfinite exchange economy $\mathcal{E}$, for finite d . The measure space is simply the Loeb space over T , generated by the albegra of internal subsets of T with counting measure. The measurable assignments of traders to preferences and endowments is just the standard parts of $\succ_t$ and $e(t)$, denoted $^0\succ_t$ and $^0e(t)$. [+]

Theorem (3): If $\mathcal{E}$ satisfies the assumptions of theorem (2) and d is finite, then $^0\mathcal{E}$ has a competitive equilibrium.

Proof: By theorem (2), $\mathcal{E}$ has a near standard competitive equili-brium $\langle q , g , S \rangle$, where $q >_\chi> 0$. We shall show that $\langle {}^0q, {}^0g \rangle$,

_____

[+]Note that $^0\mathcal{E}$ satisfies the conditions of Schmeidler's existence theorem, i.e., preferences are irreflexive, transitive, strongly monotonic, and continuous. Moreover, the assignment of agents to preferences is measurable and the initial endowment is integrable.

the standard parts of q and $g(t)$ constitute an equilibrium in $^0\mathcal{E}$.

First, we show that $g(t)$ is Loeb integrable:

(i) $\dfrac{1}{m} \sum\limits_{t \varepsilon T} q(t) = \dfrac{1}{m} \sum\limits_{t \varepsilon T} e(t)$ and $\dfrac{1}{m} \sum\limits_{t \varepsilon T} e(t)$ is finite,

hence $\dfrac{1}{m} \sum\limits_{t \varepsilon T} g(t)$ is finite.

(ii) let V be an internal set of traders such that $|V|/m \underset{\sim}{} 0$ and consider $\dfrac{1}{m} \sum\limits_{t \varepsilon V} g(t)$. Then

$$q \cdot \dfrac{1}{m} \sum\limits_{t \varepsilon T} q(t) = \dfrac{1}{m} \sum\limits_{t \varepsilon V} q \cdot g(t) = \dfrac{1}{m} \sum\limits_{t \varepsilon V} q \cdot e(t)$$

$= q \cdot \dfrac{1}{m} \sum\limits_{t \varepsilon V} e(t) \underset{\sim}{} 0$. Since $e(t)$ is integrable

and q is near standard. But each component of

q is noninfinitesimal, hence $\dfrac{1}{m} \sum\limits_{t \varepsilon V} g(t) \underset{\sim}{} 0$.

Consequently, $\int {}^0g$ exists and $\int {}^0g = \int {}^0e$.

If $t \varepsilon S$, then $q \cdot g(t) = q \cdot e(t)$. Hence, $^0q \cdot {}^0g(t) = {}^0q \cdot {}^0e(t)$ on a set of Loeb measure one, i.e., $|S|/m \underset{\sim}{} 1$.

Suppose for some $t \varepsilon S$, there is a standard x such that $^0q \cdot x \leq {}^0q \cdot {}^0e(t)$ and $x ^0\succ_t {}^0g(t)$. Then by continuity and the fact that preferences in $\mathcal{E}$ are near standard in the topology of closed convergence, there is some y such that $y \succ_t g(t)$ and $q \cdot y \leq q \cdot e(t)$, a contradiction. This completes the proof.

We now drop the assumption that d is finite.

<u>Theorem (4)</u>:   If $\mathcal{E}$ satisfies the assumptions of theorem (2), then $^0\mathcal{E}$ has a competitive equilibrium.

<u>Proof</u>:   By theorem (2), $\mathcal{E}$ has a near standard competitive equilibrium $\langle q, g, S \rangle$, where $q >_{\chi} > 0$. Let q_c be the nonstandard extension of ^0q , the standard part of q , and define $q_\infty = q - q_c$

$$\text{Let } h_i(t) = g_i(t) - \frac{(q)_i[e_i(t) - g_i(t)]}{(q_c)_i} \quad \text{for all } i$$

Then for almost all t and for all finite i , $h_i(t) \underset{\sim}{\sim} g_i(t)$.

$q_c \cdot h(t) = q_c \cdot g(t) - q_\infty \cdot [e(t) - g(t)] = q \cdot g(t) - q_\infty \cdot e(t) = q \cdot e(t) - q_\infty \cdot e(t)$

$= q_c \cdot e(t)$.

We shall show that $\langle ^0q_c , ^0h(t) \rangle$ is a competitive equilibrium for $^0\mathcal{E}$.

R_w with the product topology is a metric space and Loeb (6) has shown that the standard part of an internal near standard function from an internal measure space into the nonstandard extension of a metric space is measurable. Hence, $^0e(t)$ is measurable.

Anderson (1) has extended this theorem of Loeb to uniform spaces. Hence, $^0\succ_t$ is measurable.

If $f:T \to R_w$, where $f_i(t)$ is integrable for every i , then we define the integral of f as $\int f_i$ for $i = 1, 2, \ldots$, and denote it as $\int f$.

Now applying Rashid's construction to $\mathcal{E}$, we see that $^0\mathcal{E}$ is a standard measure-theoretic exchange economy.

^0e is integrable by assumption (iii). Suppose for some
internal set of traders V that $|V|/m \sim 0$. Consider
$\frac{1}{m} \sum\limits_{t \epsilon V} h(t)$. Then $q_c \cdot \frac{1}{m} \sum\limits_{t \epsilon V} h(t) = \frac{1}{m} \sum\limits_{t \epsilon V} q_c \cdot h(t) = \frac{1}{m} \sum\limits_{t \epsilon V} q_c \cdot e(t)$

$= q_c \cdot \frac{1}{m} \sum\limits_{t \epsilon V} e(t) \sim 0$, since $e(t)$ is integrable and q_c is

standard. But $(q_c)_i$ is noninfinitesimal for each finite i ,
hence $(\frac{1}{m} \sum\limits_{t \epsilon V} h(t))_i \sim 0$ for all finite i . Therefore, ^0h is
integrable and $\int ^0h = \int ^0e$.

We have also shown that $q_c \cdot h(t)$ and $q_c \cdot t(t)$ are integrable, hence for almost all t , $q_c \cdot h(t)$ and $q_c \cdot e(t)$ are
near standard. It is clear that if $^0(q_c \cdot h(t))$ exists, then
$^0q_c \cdot ^0h(t) \leq ^0(q_c \cdot h(t))$. If $^0q_c \cdot ^0h(t) < ^0(q_c \cdot h(t))$, then using
strong monotonicity, strong myopia, and the fact that preferences
are near standard, we can find a $y \epsilon ^*R+_w$ such that $q \cdot y \leq q \cdot e(t)$
and $y \succ_t g(t)$ for some $t \epsilon S$, a contradiction. Hence, for almost
all t , $^0q_c \cdot ^0h(t) = ^0(q_c \cdot h(t))$.

Let S be an internal set of full measure, where for all
$t \epsilon S$:

      (a) $g(t)$ is near standard

      (b) $q \cdot g(t)$ is near standard

      (c) $g(t)$ is in the demand set of trader t .

Suppose $t \epsilon S$ and for some infinite n_0 and n_1 , that

$\alpha_t = \sum\limits_{j=n_0+1}^{n_1} q_j \cdot g_j(t) \not\lesssim 0$. Let $y = \check{y}_{n_0}(t) + \frac{\alpha_t}{q_1} (1, 0, 0, \ldots)$.

Then $y \succ_t \check{y}_{n_0}(t)$ and $\check{y}_{n_0}(t) \sim g(t)$, hence $y \succ_t g(t)$; but

$q \cdot y \leq q \cdot g(t) = q \cdot e(t)$ which contradicts (c). Therefore,

$$\frac{1}{m} \sum_{t \in S} \alpha_t = \frac{1}{m} \sum_{t \in S} \sum_{j=n_0+1}^{n_1} q_j \cdot g_j(t) = \sum_{j=n_0+1}^{n_1} \frac{1}{m} \sum_{t \in S} q_j \cdot g_j(t)$$

$$= \sum_{j=n_0+1}^{n_1} q_j \cdot \frac{1}{m} \sum_{t \in S} g_j(t) \underset{\sim}{\sim} 0 . \quad \text{But} \quad \sum_{j=1}^{n_0} q_j (\frac{1}{m} \sum_{t \in T} g_j(t))$$

$$= \frac{1}{m} \sum_{t \in T} (\sum_{j=1}^{n_0} q_j \, g_j(t)) \underset{\sim}{\sim} \frac{1}{m} \sum_{t \in S} (\sum_{j=1}^{n_0} q_j \, g_j(t)) \underset{\sim}{\sim} \frac{1}{m} \sum_{t \in S} \sum_{j=1}^{n_0} q_j \, g_j(t))$$

$$+ \frac{1}{m} \sum_{t \in S} \sum_{j=n_0+1}^{d} q_j \, g_j(t)) = \frac{1}{m} \sum_{t \in S} q \cdot g(t) = \frac{1}{m} \sum_{t \in S} q \cdot e(t)$$

$$\underset{\sim}{\sim} \frac{1}{m} \sum_{t \in T} q \cdot e(t) = \frac{1}{m} \sum_{t \in T} (\sum_{j=1}^{n_0} q_j \, e_j(t)) + \frac{1}{m} \sum_{t \in T} (\sum_{j=n_0+1}^{d} q_j e_j(t))$$

$$= \sum_{j=1}^{n_0} q_j (\frac{1}{m} \sum_{t \in T} e_j(t)) + \sum_{j=n_0+1}^{d} q_j (\frac{1}{m} \sum_{t \in T} e_j(t)) .$$

Hence $\sum_{j=n_0+1}^{d} q_j (\frac{1}{m} \sum_{t \in T} e_j(t)) \underset{\sim}{\sim} 0$, since $\frac{1}{m} \sum_{t \in T} q_j(t) = \frac{1}{m} \sum_{t \in T} e_j(t) \; \forall_j$.

Therefore, $\frac{1}{m} \sum_{t \in T} (\sum_{j=n_0+1}^{d} q_j \, e_j(t)) \underset{\sim}{\sim} 0$, which implies for almost

all $t \in S$, $\sum_{j=n_0+1}^{d} q_j \, e_j(t) \underset{\sim}{\sim} 0$.

Consequently $q_c \cdot e(t) \underset{\sim}{\sim} q \cdot e(t)$ for almost all $t \in S$. Hence,

$^0q_c \cdot {}^0e(t) = {}^0(q_c \cdot e(t))$, and therefore $^0q_c \cdot {}^0h(t) = {}^0q_c \cdot {}^0e(t)$ for

almost all $t \in S$.

The argument that $h(t)$ is maximal in the budget set $\{y \epsilon R_w^+ \mid {}^0q_c \cdot y \leq {}^0q_c \cdot {}^0e(t)\}$ is now the familiar one given above, employing strong myopia and the fact that preferences are near standard.

This completes the proof.

Corollary 4.1: If (i) $\mathcal{E}$ satisfies the assumptions of theorem (4)

(ii) there exists finite noninfinitesimal positive c_0 and c_1 such that for all $t \epsilon T$,

$$c_0 u \leq e(t) \leq c_1 u \quad \text{where}$$

$$u = (1, 1, \ldots, 1) \epsilon {}^*R_d^+ .$$

Then $^0\mathcal{E}$ has a competitive equilibrium, with prices in ℓ_1 .

Proof: By assumption (i), $^0\mathcal{E}$ has a competitive equilibrium $\langle {}^0q_c , {}^0h(t) \rangle$.

By assumption (ii), $\int {}^0e(t)$ is uniformly bounded away from zero. Hence ^0q_c is in ℓ_1

If one wishes to consider ℓ_∞^+ as the commodity space with the Mackey topology for the pairing $\langle \ell_\infty , \ell_1 \rangle$; assumes that preferences are Mackey continuous; and assumes that consumption sets are norm bounded. Then corollary 4.1 implies Bewley's (unpublished) theorem on the existence of a competitive equilibrium in an economy with a measure space of agents and a countable number of commodities, where we have dropped his assumptions that preferences are complete or convex.

This result follows from the observation that the Mackey topology reduces to the product topology on norm bounded subsets of ℓ_∞ .

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APPENDIX

The Compact Topology on the Space of Preferences

In "Neighboring Economic Agents," Debreu remarks that an appropriate way to topologize the space of preferences over a commodity space, whose topology is metrizable, is to define a uniform structure over the space of preferences. This uniformity is called the Hausdorff uniformity by Bourbaki and is derived from the uniformity, generated by the metric, on the underlying commodity space.

For the case at hand, the metric space is $R_w \times R_w$ with the product topology and metric d. Let U be the uniformity generated by the bounded metric $d_1 = d/1 + d$. As is well known, d and d_1 generate the same uniformity on $R_w \times R_w$. Since $R_w \times R_w$ is a bounded metric space under d_1, the Hausdorff metric on the closed subsets of $R_w \times R_w$, denoted $2^{R_w \times R_w}$, generates the Hausdorff uniformity on $2^{R_w \times R_w}$ (see E. Michael, "Topologies on Spaces of Subsets").

Narens, in "Topologies of Closed Subsets," defines a topology of seminal importance in the nonstandard analysis of exchange economies. This is his so-called compact topology. The properties of the compact topology necessary for this paper are given in the next three propositions of Narens.

<u>Proposition (1)</u> Let (X, Γ) be a topological space. $B\epsilon^*\Gamma$ can

be considered in two ways: as a closed subset of $^*X$ and as a

point in the space $(^*\Gamma, ^*C)$ where $C = \{\Psi | \Psi \subset \Gamma, A\epsilon^*\Psi \Rightarrow {}^{0}A\epsilon\Psi\}$.

Consider B as a closed subset of $^*X$ and let $A = {}^{0}B$. Then in

the space $(^*\Gamma, ^*C)$, B is near standard to $^*A$, and ${}^{0}(^*A) = A$.

<u>Proposition (2)</u> (Γ, C) is a compact space.

<u>Proposition (3)</u> If (X, Γ) is a bounded metric space, C is

the compact topology on Γ , and $\mathcal{H}$ the Hausdorff topology on Γ ,

then $C \subset \mathcal{H}$.

From (2) and (3) we see that a subset of Γ which is compact

with respect to $\mathcal{H}$ is also compact with respect to C . For our

purposes, $X = R_w \times R_w$ with the bounded metric d_1 .

If $\succ$ is a binary relation on R_w^+ which has open graph,

i.e., continuous in the product topology, then $\not\succ$, the complement of

$\succ$ in $R_w^+ \times R_w^+$ is relatively closed. We assume that our prefer-

ences, i.e., $\not\succ$, belong to a set P which is compact with respect

to the topology induced by the Hausdorff uniformity, hence compact

in Narens' compact topology. If $\not\succ \epsilon^*P$, then ${}^{0}\not\succ$ denotes its

standard part with respect to $\mathcal{H}$. Since $\mathcal{H}$ is finer than C ,

the standard part of $\not\succ$ with respect to $\mathcal{H}$ equals the standard

part of $\not\succ$ with respect to C . The following theorem is

essential for the analysis of this paper.

<u>Theorem</u>   If x and y are near standard, then ${}^{0}x \; {}^{0}\succ \; {}^{0}y \iff \mu(x) \succ \mu(y)$

, where $\mu(x)$ denotes the monad of x with respect to the product

topology on R_w .

Proof: Immediate consequence of proposition (1).

Note that if the underlying space X is compact then Narens shows that $C = \mathcal{H}$. Hence, when the commodity space is a finite dimensional locally convex (linear) topological vector space, i.e., R_n , the topology of closed convergence on the space of closed subsets is identical to the compact topology of Narens, i.e., we compactify R_n by taking its Alexandroff compactification.

If X is an arbitrary uniform topological space, then we do now know the relationship between the compact topology and the topology induced by the Hausdorff uniformity on the space of closed subsets of X . To extend the analysis of this paper to commodity spaces which are nonmetrizable infinite dimensional locally convex (linear) topological vector spaces--hence uniform topological spaces--we would have to show that the topology induced by the Hausdorff uniformity is finer than the compact topology of Narens.