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ADDITIVELY DECOMPOSED QUASICONVEX FUNCTIONS

Gerard Debreu and Tjalling C. Koopmans

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0. INTRODUCTION

Let $X_1,\ldots,X_m$ be m sets, and let X be their Cartesian product. We say that a real-valued function f on X is <u>additively decomposed according to the factorization</u> $\bigtimes_{i=1}^{m} X_i$ <u>of</u> X if there are m real-valued functions $f_1,\ldots,f_m$ respectively on the factor sets $X_1,\ldots,X_m$ such that for every element $x = (x_1,\ldots,x_m)$ of $X = \bigtimes_{i=1}^{m} X_i$, one has $f(x) = \sum_{i=1}^{m} f_i(x_i)$. If in addition for every $i = 1,\ldots,m$, X_i is a convex subset of a real vector space, the condition that f be quasi-convex is meaningful. This article is concerned with the conjunction of the two properties of additive decomposition and of quasiconvexity of the function f.

That conjunction has remarkably strong implications that we summarize here. Assume that the sets X_i are open convex subsets of finite-dimensional real vector spaces, that $m \geqq 2$, and that the functions f_i are not constant. Then (1) every function f_i is continuous (Theorems 1 and 9); (2) every function f_i with at most one exception is actually convex (Theorems 2 and 10); (3) if the exception arises and the function f_j is

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not convex, then all the other functions f_i $(i \neq j)$ have a strict-convexity property (Theorems 2 and 10), and the function f_j itself has several of the main properties of a convex function (Theorem 3); (4) for every i, if L is an oriented straight line intersecting X_i, and g is the restriction of f_i to L, then

(a) g is a continuous real-valued function defined on a non-empty real interval; the domain of g is partitioned into three intervals $D < C < I$ (some of which may be empty) such that g is strictly decreasing in D, constant in C, and strictly increasing in I.

In section 2 we associate with every function g satisfying (a) an extended real number $c(g)$, the convexity index of g. One of the main properties of the convexity index is stated in Theorem 6: in the case of two factors X_1, X_2 that are open real intervals and of two non-constant real-valued functions f_1, f_2 defined on X_1, X_2 respectively, the function f defined by $x = (x_1,x_2) \mapsto f(x) = f_1(x_1) + f_2(x_2)$ is quasi-convex if and only if f_1 and f_2 satisfy (a), have finite left and right-derivatives everywhere, and $c(f_1) + c(f_2) \geq 0$. In section 2 we also introduce a class of real-valued functions h_θ depending on a real parameter θ, and whose domain is an open real interval Z_θ. With the notation and the assumptions of the next to last sentence, we show (Theorem 7) that the function f is quasi-convex if and only if for every $x_1^0 \in X_1$ and $x_2^0 \in X_2$, there is a function h_θ whose graph locally separates suitable linear transforms of the graphs of f_1 and f_2. In section 3 we extend the concept of the convexity index to a real-valued function g defined on a non-empty convex subset Y of a finite-dimensional real vector space and such that for every oriented straight line L intersecting Y, the restriction g_L of g to L satisfies (a). By definition, $c(g) = \operatorname{Inf}_L c(g_L)$. In the case of two factors X_1 and X_2 , and of two C^2 real-valued functions f_1 and f_2 , Theorem 11 gives the expression of $c(f)$ in terms of $c(f_1)$ and $c(f_2)$. We conclude section 3 with a characterization, by means of convexity indices, of the quasi-convexity of a C^2 real-valued function additively decomposed into a finite number (at least equal to two) of nonconstant functions defined on finite-dimensional open convex sets.

Because of our particular backgrounds, we give examples of possible applications of these results only in economic theory, pertaining to two areas of that field. We would appreciate reader responses with regard to other fields of application of mathematics.

Utility theory. Consider the case in which the set C of the commodities of an economy is partitioned into m subsets $C_1,\ldots,C_m$. For instance the commodities in set C_i are the goods and services available in the $i^{\underline{th}}$ period. Alternatively in a study of economic uncertainty, the commodities in set C_i are the goods and services whose availability is contingent on the occurrence of the $i^{\underline{th}}$ state of the world. Still another example is the partitioning of the set C of commodities into groups of goods or services with related physical characteristics. The consumption of a given consumer in the economy is described by a vector x listing the quantities of the various commodities that he consumes. Corresponding to the partition of C into the sets $C_1,\ldots,C_m$, there is a partition $(x_1,\ldots,x_m)$ of the components of x . The vector x_i belongs to a subset X_i of the commodity space associated with the commodities in C_i and x belongs to the Cartesian product $X = \bigtimes_{i=1}^{m} X_i$. The preferences of the given consumer are represented mathematically by a complete preorder, i.e., by a complete, reflexive, transitive binary relation, $\precsim$ on X where $x \precsim x'$ is read as " x' is at least as desired as x ." If the space X and the relation $\precsim$ satisfy mild topological properties (e.g. X is connected and separable and $\{(x,x') \in X \times X \mid x \precsim x'\}$ is closed), then the preference relation $\precsim$ can be represented by a continuous real-valued utility function u in the sense that $x \precsim x'$ is equivalent to $u(x) \leq u(x')$. Denoting the set of factor spaces of X by $I = \{1,\ldots,m\}$ we consider a subset J of I and we fix the values of all the components of x for $i \notin J$ equal to $(x_i^0)_{i \notin J}$. Assume now that the preorder induced by $\precsim$ on $\bigtimes_{i \in J} X_i$ given the values $(x_i^0)_{i \notin J}$ is independent of those particular values, and denote that induced preorder by $\precsim_J$. Assume also that there are at least three factor spaces and that for every $i \in I$, the induced preorder $\precsim_i$ is not trivial in the sense that one does not have $x \precsim_i x'$ for every pair (x,x') of elements of X_i. Then (G. Debreu [1960]) there are m real-valued continuous functions u_i on X_i $(i = 1,\ldots,m)$ such that the preference preorder $\precsim$ is represented by the function u defined on X by $u(x) = \sum_{i=1}^{m} u_i(x_i)$. If in this situation the standard assumption of convexity of preferences is made, i.e., if the set X is convex (open and finite-dimensional) and if for every x' in X, the set $\{x \in X \mid x \succsim x'\}$ is convex, then the function u is quasiconcave and the results of this article apply. In particular every function u_i, with at most one exception, is concave. This observation was made for additively decomposed quasiconcave and <u>twice continuously differentiable</u> utility functions in E. E. Slutsky's classic paper of 1915, in J. Green [1961], W. Gorman [1970], and T. Rader [1972]. It was also made by K. J. Arrow and A. C. Enthoven [1961] in the context of twice continuously differentiable production functions that we discuss below. In unpublished notes of 1971, T. C. Koopmans showed that the assumptions of differentiability and even of continuity could be dispensed with. In a paper independently written at about the same time and eventually published in 1977, M. E. Yaari established the concavity of every function u_i with at most one exception dispensing with the assumption of differentiability but retaining the assumption of continuity. (An interesting sequel of this work is M. E. Yaari [1978].)

The assumption of independence for the preorder induced by $\precsim$ on $\bigtimes_{i \in J} X_i$ is of special interest in two interpretations in which it is assumed that the spaces X_i are identical with the same list of physical goods or services, and that the preorders induced on each X_i are also identical. For $m \geqq 3$, the preference relation $\precsim$ can then be represented by a utility function v of the form

$$v(x) = \sum_{i \in I} \alpha_i u(x_i) \ , \quad \alpha_i > 0 \ , \quad i \in I \ .$$

Moreover, if $\precsim$ is convex, i.e., if v is quasiconcave, then every function $\alpha_i u$, with at most one exception, is concave. Hence u is concave.

The first interpretation, which gave rise to Koopmans' initial work, is that in which the values $i = 1, 2, \ldots, m$ refer to successive periods in time, and $\alpha_i = \alpha^i$, $0 < \alpha < 1$, represent successive powers of a discount factor α.

In the second interpretation the values $i = 1, 2, \ldots, m$ refer to different states of the world, and α_i is the judgmental probability of the $i^{\underline{th}}$ state of the world for the decision maker. (K. J. Arrow [1953], J. Hirschleifer [1965], J. C. Cox [1973], C. Blackorby, R. Davidson, and D. Donaldson [1977]).

Production theory. As in K. J. Arrow and A. C. Enthoven [1961], we consider a set C of commodities used as inputs in the production of a single output and we assume that C is partitioned into m sets $C_1, \ldots, C_m$. For every $i = 1, \ldots, m$, there is an elementary production process transforming the input-vector x_i , whose components pertain to the commodities in C_i , into $f_i(x_i)$ units of output. Thus the input-vector $x = (x_1,\ldots,x_m)$ is transformed into $f(x) = \sum_{i=1}^{m} f_i(x_i)$ units of output. In an attempt to relax the standard condition of convexity of the set of feasible input-output vectors, i.e. of concavity of the function f , R. W. Shephard [1953], J. J. Laffont [1972], C. Fourgeaud, B. Lenclud, and P. Sentis [1974], E. Dierker, C. Fourgeaud and W. Neuefeind [1976], use the condition of convexity of the sets of input-vectors required to attain a certain level of output, i.e. of quasiconcavity of the function f . If the assumptions that we listed earlier are satisfied, namely if the domains of the functions f_i are open convex subsets of finite-dimensional real vector spaces, $m \geqq 2$, and the functions f_i are not constant, then all of them, with at most one exception, are actually concave. In other words every elementary production process, with at most one exception, has a convex set of feasible input-output vectors.

1. TWO FACTOR SPACES OF DIMENSION ONE

A real-valued function h on a convex subset C of a real vector space is said to be quasi-convex if for every real number k the set $\{x \in C \mid h(x) \leq k\}$ is convex. Considering a quasi-convex function f on a nonempty open real interval X, we define the intervals X^d and X^i as follows.

For every k in f(X), the set $\{x \in X \mid f(x) \leq k\}$ is a nonempty real interval with endpoints a(k) and b(k) where $a(k) \leq b(k)$. The intervals $[a(k), b(k)]$ for $k \in f(X)$ are nested since $k_2 \leq k_1$ implies $a(k_1) \leq a(k_2) \leq b(k_2) \leq b(k_1)$. We define

$$\underline{x} = \operatorname*{Sup}_{k \in f(X)} a(k) \quad \text{and} \quad \bar{x} = \operatorname*{Inf}_{k \in f(X)} b(k).$$

The function f is weakly decreasing in the interval $]\text{Inf } X, \bar{x}[$. To see this, consider x' and x" such that $\text{Inf } X < x' < x'' < \bar{x}$. Let $k = f(x')$. Then $a(k) \leq x'$ and $\bar{x} \leq b(k)$. Therefore x" belongs to the interval $]a(k), b(k)[$ hence $f(x'') \leq f(x')$. We define X^d as $]\text{Inf } X, \bar{x}]$ if f is weakly decreasing in the latter interval, and as $]\text{Inf } X, \bar{x}[$ otherwise. Similarly, f is weakly increasing in the interval $]\underline{x}, \text{Sup } X[$. We define X^i as $[\underline{x}, \text{Sup } X[$ if f is weakly increasing in the latter interval, and as $]\underline{x}, \text{Sup } X[$ otherwise.

We note that X^d and X^i together cover X, for if x in X belonged neither to X^d nor to X^i, one would necessarily have $x = \underline{x} = \bar{x}$, $f(x) > \text{Inf } f(X^d)$, and $f(x) > \text{Inf } f(X^i)$. However, the last two inequalities contradict the quasi-convexity of f.

Thus, X partitions into the three intervals $X^d \setminus X^i$, $X^d \cap X^i$, and $X^i \setminus X^d$, of which at most two may be empty. In $X^d \cap X^i$, which is a proper interval only if $\underline{x} < \bar{x}$, f takes on the constant value Inf f(X). Finally if $x \in X^d \setminus X^i$ and $x' \in X^i \setminus X^d$, then $x < x'$.

Clearly, if X^d is not empty, then f is not weakly decreasing in any interval strictly containing X^d, and if X^i is not empty, then f is not weakly increasing in any interval strictly containing X^i.

Also note that if f is nonconstant on X, it is not possible for f to take on different constant values $f(X^d)$ and $f(X^i)$ on X^d and X^i respectively. This would require X^d and X^i to be disjoint. But then $f(X^d) < f(X^i)$ would imply $X^d \subset X^i = X$ and $f(X^i) < f(X^d)$ would imply $X^i \subset X^d = X$, contradicting the disjointness.

In the remainder of this section we always make the assumptions

A.1. X and Y are open real intervals. f and g are nonconstant real-valued functions on X and Y, respectively. The function $F: X \times Y \to R$ defined by $F(x, y) = f(x) + g(y)$ is quasi-convex.

Clearly, the two functions f and g are quasi-convex. We associate with f the two intervals X^d, X^i defined earlier, and with g the corresponding intervals Y^d, Y^i.

In the sequel, we will often appeal to the following remarks. Let the function $\hat{f}$ on $\hat{X} = -X$ be defined by $\hat{f}(x) = f(-x)$, and let the function $\hat{g}$ on $\hat{Y} = -Y$ be defined by $\hat{g}(y) = g(-y)$. The three functions defined respectively on $\hat{X} \times Y$, $X \times \hat{Y}$, and $\hat{X} \times \hat{Y}$ by $(x, y) \mapsto \hat{f}(x) + g(y)$, $f(x) + \hat{g}(y)$, and $\hat{f}(x) + \hat{g}(y)$ are also quasi-convex and one has $\hat{X}^d = -X^i$, $\hat{X}^i = -X^d$, $\hat{Y}^d = -Y^i$, and $\hat{Y}^i = -Y^d$. Now consider a point (x, y) in $X \times Y$. As we noticed earlier, x belongs to X^d and/or to X^i, and y belongs to Y^d and/or to Y^i. The preceding remarks will often enable us to assume without loss of generality that x belongs to X^d or that x belongs to X^i, and that y belongs to Y^d or that y belongs to Y^i.

Theorem 1. Under assumptions A.1, the functions f and g are continuous.

Proof: Since g is nonconstant on Y^d and/or on Y^i, we can assume without loss of generality that g is not constant on Y^i.

Given any ε such that $0 < 2\varepsilon < \text{Sup } g(Y^i) - \text{Inf } g(Y^i)$, there are three points y_1, y_2, y_3 of Y^i such that

(1) $y_1 < y_2 < y_3$,

(2) $g(y_1) < g(y_2) - \varepsilon$, and

(3) $g(y_3) < g(y_2) + \varepsilon$.

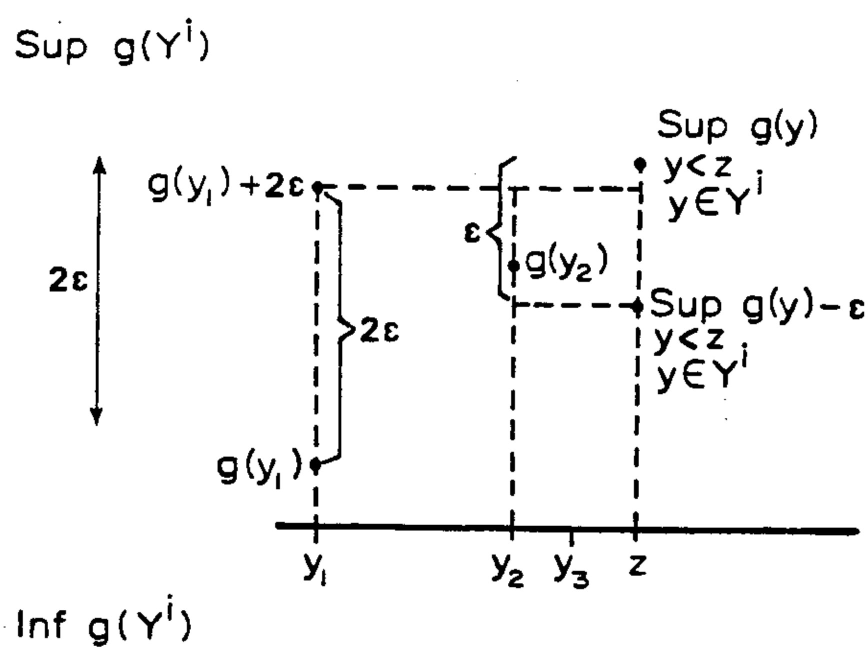

Figure 1

To show this, select first y_1 in Y^i so that

$$g(y_1) < \text{Sup } g(Y^i) - 2\varepsilon.$$

Then choose z in Y^i so that

$$\sup_{\substack{y<z \\ y\in Y^i}} g(y) > g(y_1) + 2\varepsilon.$$

Next select in Y^i a number $y_2 < z$ so that

$$g(y_2) > \sup_{\substack{y<z \\ y\in Y^i}} g(y) - \varepsilon.$$

Therefore, $y_1 < y_2$, and (2) holds. Finally, select y_3 so that $y_2 < y_3 < z$. Then (3) holds.

Next, we prove

(a) <u>If</u> $x_0 \in X^d$, <u>then</u> $f(x_0) = f(x_0-)$,

where $f(x_0-) = \lim_{\substack{x\to x_0 \\ x<x_0}} f(x)$.

Clearly, $f(x_0) \leq f(x_0-)$. We assume that $f(x_0) < f(x_0-)$ and show that this leads to a contradiction. Select $\varepsilon > 0$ such that

$$\varepsilon < f(x_0-) - f(x_0) \text{ and } 2\varepsilon < \text{Sup } g(Y^i) - \text{Inf } g(Y^i).$$

Thus, there are three points y_1, y_2, y_3 in Y^i satisfying (1), (2), (3). There is also a number a in X such that $a < x_0$ and

(*) $\quad a \leq x < x_0$ implies $f(x_0-) \leq f(x) < f(x_0-) + \varepsilon$.

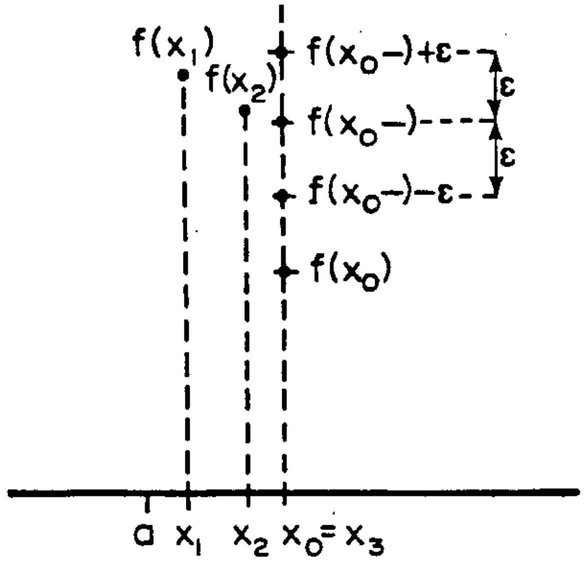

Figure 2

We can choose two real numbers $\alpha > 0$ and β such that $x_i = \alpha y_i + \beta$ $(i = 1,2,3)$ satisfy $a \leqq x_1 < x_2 < x_3 = x_0$.

Then, by (*), since $f(x_1) < f(x_0-) + \varepsilon$ and since also $f(x_0-) \leqq f(x_2)$, one has

(4) $\quad f(x_1) < f(x_2) + \varepsilon.$

Since $f(x_3) < f(x_0-) - \varepsilon$ and $f(x_0-) \leqq f(x_2)$, one has

(5) $\quad f(x_3) < f(x_2) - \varepsilon.$

Therefore, by (2) and (4),

$$F(x_1,y_1) < F(x_2,y_2),$$

while, by (3) and (5),

$$F(x_3,y_3) < F(x_2,y_2).$$

These two inequalities contradict the quasi-convexity of F on the straight line segment $[(x_1,y_1), (x_3,y_3)]$.

Having established (a), we now prove the continuity of f:

(b) <u>For every</u> $x_0 \in X$, <u>one has</u> $f(x_0-) = f(x_0+) = f(x_0)$.

Since locally f is weakly monotone on the left of x_0, as well as on the right of x_0, the two numbers $f(x_0-)$ and $f(x_0+)$ (one or both of which might be $-\infty$) are defined. Suppose that they are not equal. We can assume without loss of generality that $f(x_0-) > f(x_0+)$, which implies that $x_0 \in X^d$, and, consequently by (a), $f(x_0) = f(x_0-)$. We select $\varepsilon > 0$ such that

$$2\varepsilon < f(x_0) - f(x_0+) \text{ and } 2\varepsilon < \text{Sup } g(Y^i) - \text{Inf } g(Y^i).$$

Thus there are three points y_1, y_2, y_3 in Y^i satisfying (1), (2), and (3). As before, there is a number a in X such that $a < x_0$ and

$$a \leqq x < x_0 \text{ implies } f(x) < f(x_0) + \varepsilon.$$

There is also a number b in X such that $x_0 < b$ and

$$x_0 < x \leqq b \text{ implies } f(x) < f(x_0) - \varepsilon.$$

This last assertion is obvious if $f(x_0+) = -\infty$. If $f(x_0+)$ is finite, then there is b in X such that $x_0 < b$ and $x_0 < x \leqq b$ implies $f(x) < f(x_0+) + \varepsilon$, and consequently, $f(x) < f(x_0) - \varepsilon$.

We can now choose two real numbers $\alpha > 0$ and β such that $x_i = \alpha y_i + \beta$ $(i = 1,2,3)$ satisfy $a \leqq x_1 < x_2 = x_0 < x_3 \leqq b$.

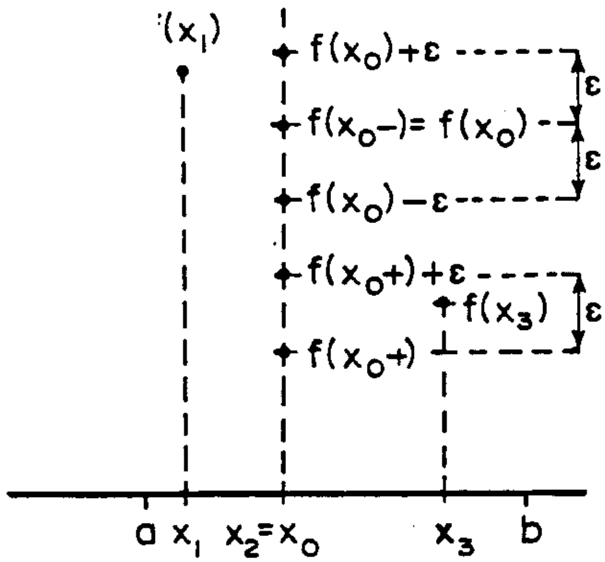

Figure 3

Since $f(x_1) < f(x_0) + \varepsilon$, one has

(4') $f(x_1) < f(x_2) + \varepsilon.$

Since $f(x_3) < f(x_0) - \varepsilon$, one has

(5') $f(x_3) < f(x_2) - \varepsilon.$

Therefore, by (2) and (4'),

$F(x_1, y_1) < F(x_2, y_2),$

while, by (3) and (5'),

$F(x_3, y_3) < F(x_2, y_2).$

We have again a contradiction of the quasi-convexity of F on the straight line segment $[(x_1, y_1), (x_3, y_3)]$, Q.E.D.

<u>Theorem 2</u>. <u>Under A.1, if f is not convex, then g has the property</u>

(P) <u>If</u> y_1, y_2 <u>are two points of</u> Y <u>such that</u> $g(y_1) \neq g(y_2)$, <u>then for every</u> t <u>in</u> $]0,1[$, <u>one has</u> $g[(1 - t)y_1 + ty_2] < (1 - t)g(y_1) + tg(y_2)$.

<u>Proof</u>: The proof rests on the same basic ideas as M. Yaari's (1977) proofs. We assume without loss of generality that f is not convex on X^i. Therefore, there are two points a_1, a_2 in X^i such that $a_1 < a_2$ and that there are points of the graph of f strictly above the chord $A = [(a_1, f(a_1)), (a_2, f(a_2))]$. Since f is weakly increasing on X^i, one cannot have $f(a_1) = f(a_2)$. Thus $f(a_1) < f(a_2)$. By continuity of f, there is a straight line L parallel to A and supporting from above the graph of f restricted to $[a_1, a_2]$. Since L is not horizontal, there is a highest point $(x^*, f(x^*))$ in the intersection of L with the graph of f restricted to $[a_1, a_2]$.

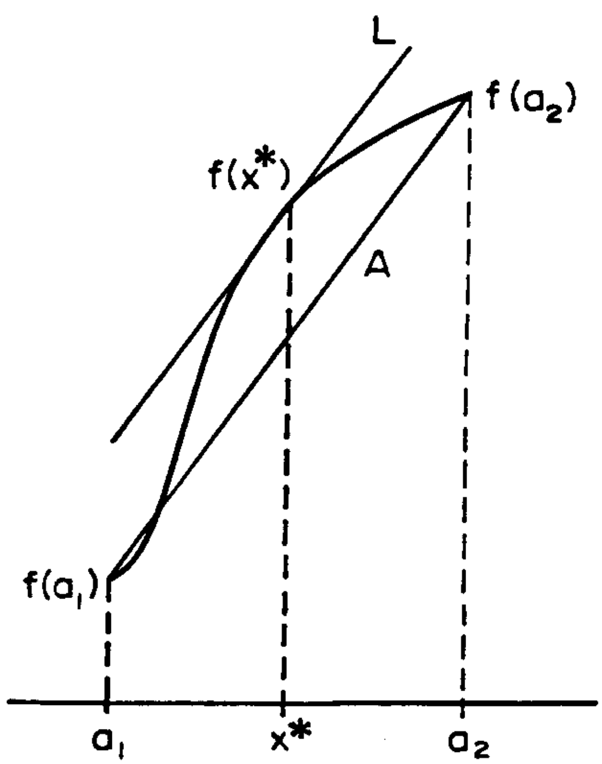

Figure 4

For every sufficiently small number $s > 0$,

(i) $\frac{1}{2}[f(x^* + s) + f(x^* - s)] < f(x^*)$.

Now consider the continuous function ϕ defined by

$$\phi(s) = f(x^* + s) - f(x^* - s).$$

One has $\phi(0) = 0$ and for every sufficiently small $s > 0$, $\phi(s) > 0$. Therefore, for every sufficiently small $\delta > 0$,

(ii) there is $s > 0$ such that $f(x^* + s) - f(x^* - s) = \delta$.

Let us then assume that there are two points b_1, b_2 in Y such that $b_1 < b_2$; $g(b_1) < g(b_2)$; and the graph of g is not strictly below the chord $B = [(b_1, g(b_1)), (b_2, g(b_2))]$. Reasoning as above, we observe that there is a straight line M parallel to B and supporting the graph of g restricted to $[b_1, b_2]$ from above. However, in the present case M may contain B. Next, we select a point $(y^*, g(y^*))$ in the intersection of M and the graph of g restricted to $[b_1, b_2]$, and different from the endpoints of B.

For every sufficiently small $t > 0$, we have this time that

(i') $\frac{1}{2}[g(y^* + t) + g(y^* - t)] \leqq g(y^*)$.

As before, we define the continuous function Ψ by

$$\Psi(t) = g(y^* + t) - g(y^* - t).$$

One has $\Psi(0) = 0$. Moreover, the inequality $g(b_1) < g(y^*)$ implies that y^* is in Y^i. Therefore, for every sufficiently small $t > 0$, $\Psi(t) > 0$. Consequently, for every sufficiently small $\delta > 0$,

(ii') there is $t > 0$ such that $g(y^* + t) - g(y^* - t) = \delta$.

We now choose $\delta > 0$ small enough so that both (ii) and (ii') are satisfied. Thus, there are $s > 0$ and $t > 0$ such that

(iii) $f(x^* + s) - f(x^* - s) = \delta = g(y^* + t) - g(y^* - t)$.

Defining $x_1 = x^* + s$, $x_2 = x^* - s$, $y_1 = y^* - t$, and $y_2 = y^* + t$, we obtain from (iii), (i), and (i'),

(iv) $F(x_1, y_1) = F(x_2, y_2) = \frac{1}{2}[F(x_1, y_1) + F(x_2, y_2)] < F(x^*, y^*)$.

Since $\frac{1}{2}[(x_1, y_1) + (x_2, y_2)] = (x^*, y^*)$, (iv) contradicts the quasi-convexity of F, Q.E.D.

(P) says that the graph of the function g is strictly below any chord $B = [(y_1, g(y_1)), (y_2, g(y_2))]$ that is not horizontal. In fact, if y_1, y_2 are two points of Y such that $y_1 < y_2$ and $g(y_1) = g(y_2) \neq \text{Inf } g(Y)$, then the graph of g is also strictly below the chord B. To see that this is implied by (P), select a point y_0 in Y such that $g(y_0) < g(y_1) = g(y_2)$. Since the graph of g is strictly below the chord with endpoints $(y_0, g(y_0))$ and $(y_1, g(y_1))$, and strictly below the chord with endpoints $(y_0, g(y_0))$ and $(y_2, g(y_2))$, it is strictly below B.

Moreover, $y_1 < y_2$ and $g(y_1) = g(y_2) = \text{Inf } g(Y)$ implies $g(y) = \text{Inf } g(Y)$ for every y in $[y_1, y_2]$ by quasi-convexity of g. Therefore, the function g is actually convex (and strictly convex on $Y^d \backslash Y^i$ and on $Y^i \backslash Y^d$).

Theorem 3. Under A.1, at every point x of X, the function f has a finite left-derivative and a finite right-derivative satisfying $f'_-(x) \leq f'_+(x)$. Both derivatives are strictly positive in $X^i \backslash X^d$ and strictly negative in $X^d \backslash X^i$. The function f'_- is continuous on the left and the function f'_+ is continuous on the right. At every point x of X outside a countable set, $f'_-(x) = f'_+(x)$ and the functions f'_- and f'_+ are continuous. At every point x of X outside a set of Lebesgue measure zero, $f'_-(x) = f'_+(x)$ and the functions f'_- and f'_+ have the same finite derivative $f''(x)$.

Proof: If f is convex, it has all the properties listed above. Therefore, we study the case in which f is not convex.

First, we assume that the open interval $X^i \setminus X^d$ is not empty and we prove that f is strictly increasing in that interval. Consider a point x_0 in $X^i \setminus X^d$. There is a point x_1 in $X^i \setminus X^d$ such that $x_1 < x_0$ and $f(x_1) < f(x_0)$, because otherwise x_0 would belong to X^d. We can without loss of generality assume that there is a point y_0 of Y such that the convex function g is strictly decreasing in a neighborhood of y_0.

We define the set

$$\Gamma = \{(x, y) \in X \times Y \mid f(x) + g(y) \leqq f(x_0) + g(y_0)\},$$

which is convex and closed relative to $X \times Y$. By continuity of g, there is a point y_1 of Y such that $y_1 < y_0$ and $g(y_1) \leqq g(y_0) + f(x_0) - f(x_1)$. Thus

(i) $x_1 < x_0$, $y_1 < y_0$, and $(x_1, y_1) \in \Gamma$.

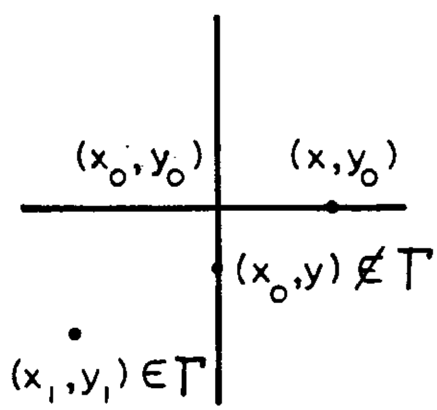

Figure 5

Moreover, for every y in Y such that $y < y_0$, one has $g(y) > g(y_0)$. Consequently,

(ii) $y \in Y$ and $y < y_0$ imply $(x_0, y) \notin \Gamma$.

(i), (ii), and the convexity of Γ imply that for every x in X such that $x > x_0$, the point (x, y_0) is not in Γ. Therefore, x in X and $x > x_0$ imply $f(x) > f(x_0)$.

Having established that f is strictly increasing on $X^i \setminus X^d$, we choose an open interval I around x_0 sufficiently small so that $I \subset X^i \setminus X^d$ and that the open interval $f(x_0) + g(y_0) - f(I)$ is contained in $g(Y^d)$. Clearly, there is an open interval J around y_0 and contained in $Y^d \setminus Y^i$ such that $g(J) = f(x_0) + g(y_0) - f(I)$.

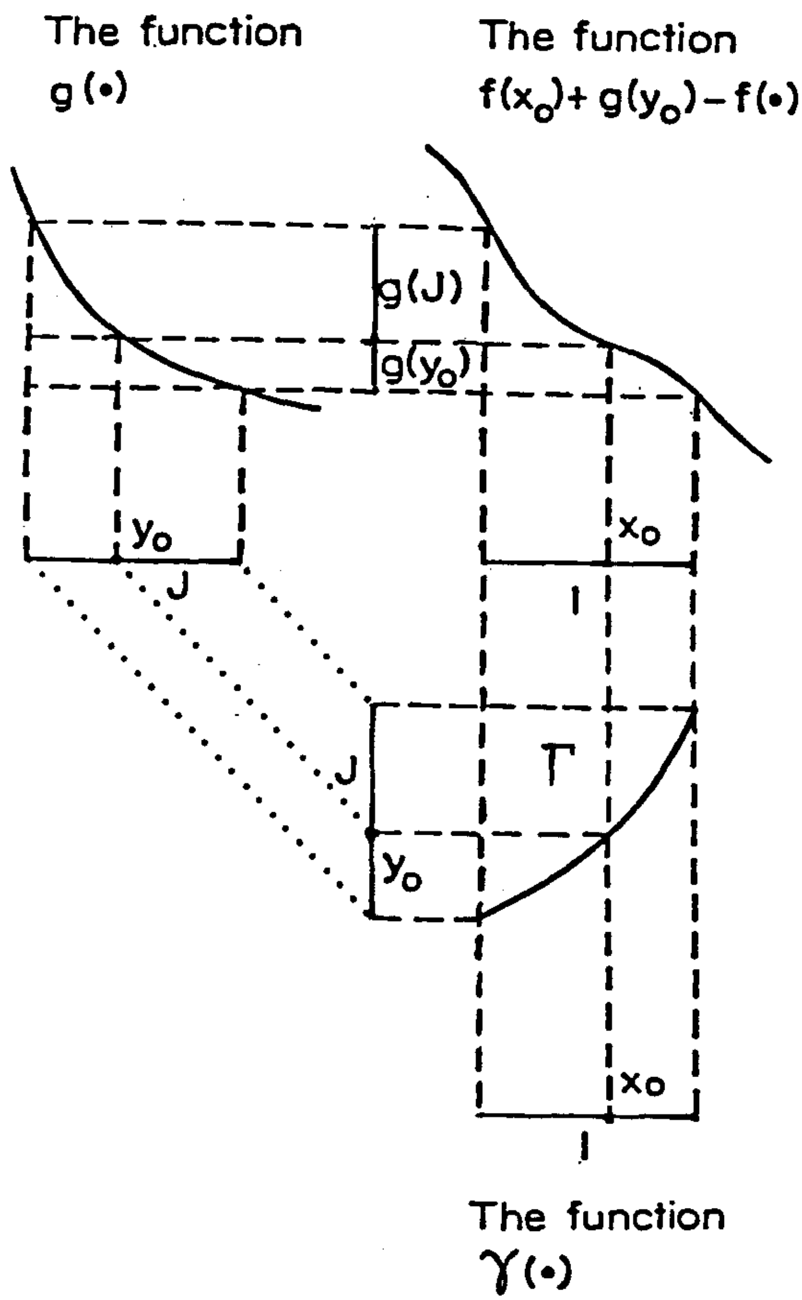

Figure 6

For every x in I, there is a unique y in J such that $g(y) = f(x_0) + g(y_0) - f(x)$. Let γ be the function from I to J defined in this manner. γ is strictly increasing on I. Moreover, for every x in I, the point $(x, \gamma(x))$ is in the lower boundary of the convex set Γ. To see this, note that obviously $(x, \gamma(x))$ belongs to Γ, while, as in (ii), for every y in Y such that $y < \gamma(x)$, one has $(x, y) \notin \Gamma$. Thus, the function γ is convex.

Let x tend to x_0 from the left. Then $\gamma(x)$ tends to $\gamma(x_0)$ from below. One also has

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{g[\gamma(x_0)] - g[\gamma(x)]}{\gamma(x) - \gamma(x_0)} \cdot \frac{\gamma(x) - \gamma(x_0)}{x - x_0}.$$

Since g is convex, the first factor of the right-hand side tends to $-g'_-[\gamma(x_0)]$. The second factor tends to $\gamma'_-(x_0)$. Therefore, $\frac{f(x) - f(x_0)}{x - x_0}$ has a limit $f'_-(x_0) = -g'_-(y_0)\gamma'_-(x_0)$, where $g'_-(y_0) < 0$ and $\gamma'_-(x_0) > 0$. Consequently, $f'_-(x_0) > 0$. Similarly, $f'_+(x_0)$ exists and equals $-g'_+(y_0)\gamma'_+(x_0)$.

We have chosen for y_0 any point of Y where g is strictly decreasing. Since the convex function g has a finite derivative at every point of Y outside a countable set, we could have chosen y_0 in Y such that $g'_-(y_0) = g'_+(y_0) < 0$. But then $\gamma'_-(x_0) \leqq \gamma'_+(x_0)$ implies $f'_-(x_0) \leqq f'_+(x_0)$.

In the preceding reasoning we have associated with every point a in $X^i \setminus X^d$, an open interval ${}_aI$ around a and contained in $X^i \setminus X^d$, and a strictly increasing convex function ${}_a\gamma$ on ${}_aI$ such that for every point x in ${}_aI$ one has

$$f'_-(x) = -g'_-({}_a\gamma(x))\,{}_a\gamma'_-(x) \text{ and } f'_+(x) = -g'_+({}_a\gamma(x))\,{}_a\gamma'_+(x).$$

Since the functions g'_- and ${}_a\gamma'_-$ are continuous on the left and the function ${}_a\gamma$ is increasing, the function f'_- is continuous on the left. Similarly f'_+ is continuous on the right.

At every point x of ${}_aI$ outside a countable set ${}_aD_1$, one has ${}_a\gamma'_-(x) = {}_a\gamma'_+(x)$ and the functions ${}_a\gamma'_-$ and ${}_a\gamma'_+$ are continuous. And at every point y of ${}_a\gamma({}_aI)$ outside a countable set ${}_aD_2$, one has $g'_-(y) = g'_+(y)$ and the functions g'_- and g'_+ are continuous. Therefore at every point x of ${}_aI$ outside the countable set ${}_aD = {}_aD_1 \cup {}_a\gamma^{-1}({}_aD_2)$, one has $f'_-(x) = f'_+(x)$ and the functions f'_- and f'_+ are continuous.

At every point x of ${}_aI$ outside a null (i.e. of Lebesgue measure zero) set ${}_aN_1$, one has ${}_a\gamma'_-(x) = {}_a\gamma'_+(x)$ and the functions ${}_a\gamma'_-$ and ${}_a\gamma'_+$ have the same finite derivative ${}_a\gamma''(x)$. And at every point y of ${}_a\gamma({}_aI)$ outside a null set ${}_aN_2$, one has $g'_-(y) = g'_+(y)$ and the functions g'_- and g'_+ have the same finite derivative $g''(y)$. We claim that ${}_a\gamma^{-1}({}_aN_2)$, the image of the null set ${}_aN_2$ by the concave function ${}_a\gamma^{-1}$, is null. Given any compact interval E contained in ${}_a\gamma({}_aI)$, the function ${}_a\gamma^{-1}$ is Lipschitzian on E and consequently the image of the null set $E \cap {}_aN_2$ by ${}_a\gamma^{-1}$ is null. However the open interval ${}_a\gamma({}_aI)$ can be covered by a countable union $\bigcup_i E_i$ of compact subintervals E_i. The equality ${}_a\gamma^{-1}({}_aN_2) = \bigcup_i {}_a\gamma^{-1}(E_i \cap {}_aN_2)$ proves our claim. Consider now a point x_0 of ${}_aI$ outside the null set ${}_aN = {}_aN_1 \cup {}_a\gamma^{-1}({}_aN_2)$. One has $f'_-(x_0) = f'_+(x_0)$. Moreover for any $x \neq x_0$ in ${}_aI$,

$$\frac{f'_-(x) - f'_-(x_0)}{x - x_0} = \frac{-g'_-({}_a\gamma(x))\,{}_a\gamma'_-(x) + g'({}_a\gamma(x_0))\,{}_a\gamma'(x_0)}{x - x_0}$$

$$= -g'_-({}_a\gamma(x)) \cdot \frac{{}_a\gamma'_-(x) - {}_a\gamma'(x_0)}{x - x_0} - {}_a\gamma'(x_0) \cdot \frac{g'_-({}_a\gamma(x)) - g'({}_a\gamma(x_0))}{{}_a\gamma(x) - {}_a\gamma(x_0)} \cdot \frac{{}_a\gamma(x) - {}_a\gamma(x_0)}{x - x_0} .$$

When x tends to x_0, $\frac{f'_-(x) - f'_-(x_0)}{x - x_0}$ tends to $-g'({}_a\gamma(x_0))\,{}_a\gamma''(x_0) - g''({}_a\gamma(x_0))[{}_a\gamma'(x_0)]^2$. Clearly $\frac{f'_+(x) - f'_+(x_0)}{x - x_0}$ tends to the same finite limit when x tends to x_0.

In summary, with every point a of $X^i \setminus X^d$ we have associated an open interval, ${}_aI$, around a and two exceptional subsets of ${}_aI$, one of them, ${}_aD$, countable, and the other, ${}_aN$, null. The intervals I_a cover $X^i \setminus X^d$. Thus, by Lindelöf's theorem (Kelley (1955, p. 49), we can cover $X^i \setminus X^d$ with a countable collection of intervals ${}_aI$. The countable unions of the corresponding exceptional sets ${}_aD$ and ${}_aN$ form the two exceptional subsets D and N of $X^i \setminus X^d$ respectively, the former being countable and latter being null.

The properties of the function f listed in Theorem 3 have been established in $X^i \setminus X^d$. In a similar manner they hold in $X^d \setminus X^i$. They also hold trivially in the interval $]\text{Inf } X^i, \text{Sup } X^d[$. Only the cases in which $x = \text{Inf } X^i$ or $x = \text{Sup } X^d$ remain to be considered. A minor modification of the preceding proof, namely choosing an interval I having x as an endpoint, establishes that in the first case f has a finite left derivative $f'_-(x) \leq 0$ continuous on the left, and in the second case f has a finite right derivative $f'_+(x) \geq 0$ continuous on the right, Q.E.D.

The function g obviously has in Y the properties that the function f has in X.

2. THE CONVEXITY INDEX OF A QUASI-CONVEX FUNCTION AND A CLASS OF SEPARATING FUNCTIONS

We have shown that under A.1, the function f is continuous and the interval X is partitioned into three intervals $X^{sd} = X^d \backslash X^i$, $X^c = X^d \cap X^i$, and $X^{si} = X^i \backslash X^d$ (some of which may be empty) such that $X^{sd} < X^c < X^{si}$, f is strictly decreasing in X^{sd}, constant in X^c, and strictly increasing in X^{si}. To define the convexity index of f on X, we first define and study the convexity index of f on X^{si}. Then, letting $\hat{f}$ be the function with domain $\hat{X} = -X$ such that $\hat{f}(x) = f(-x)$, we define the convexity index of f on X^{sd} as the convexity index of $\hat{f}$ on $-X^{sd}$. Next we set the convexity index of f on X^c to be $+\infty$. Finally the convexity index of f on X is defined as the smallest of the convexity indices of f on X^{sd}, X^c, and X^{si} that are determined, i.e., for which the corresponding interval is not empty. Thus in section 2 (with the exception of the last three paragraphs) we make the assumptions

A.2. X and Y are non-empty open real intervals. f and g are strictly increasing continuous real-valued functions on X and Y respectively. The function $F: X \times Y \to R$ is defined by $F(x,y) = f(x) + g(y)$.

The inverse $\phi = f^{-1}$ of f is defined on the non-empty open real interval $U = f(X)$, and the inverse $\chi = g^{-1}$ of g is defined on the non-empty open real interval $V = g(Y)$.

Given a real number Δ such that $0 < \Delta < \frac{1}{2}(\text{Sup } U - \text{Inf } U)$, we let

$$\bar{\lambda}(\Delta) = \operatorname*{Inf}_{u} \log \left[\frac{\phi(u) - \phi(u-\Delta)}{\phi(u+\Delta) - \phi(u)}\right]^{\frac{1}{\Delta}}$$

In the last Infimum, u is restricted to be such that $u-\Delta$, u, and $u+\Delta$ belong to U.

The <u>convexity index</u> of the function f is defined as

$$\lambda = \limsup_{\Delta \to 0} \bar{\lambda}(\Delta) .$$

Equivalently the convexity index can be defined in the following, perhaps more intuitively appealing, manner. Let

$$\beta(u\,,\,\Delta) = - \frac{\frac{1}{\Delta^2}[\varphi(u+\Delta) - \varphi(u)] - \frac{1}{\Delta^2}[\varphi(u) - \varphi(u-\Delta)]}{\frac{1}{\Delta}[\varphi(u+\Delta) - \varphi(u)]} .$$

We claim that

(6) $$\lambda = \limsup_{\Delta \to 0} \operatorname{Inf}_{u} \beta(u,\Delta)$$

<u>Proof of (6)</u> : Let

$$\alpha(u,\Delta) = \frac{1}{\Delta} \log \left[\frac{\varphi(u) - \varphi(u-\Delta)}{\varphi(u+\Delta) - \varphi(u)} \right]$$

so that $\lambda = \limsup_{\Delta \to 0} \operatorname{Inf}_{u} \alpha(u,\Delta)$. One has

$$\alpha(u,\Delta) = \frac{1}{\Delta} \log [1 + \Delta\beta(u,\Delta)] .$$

For any $\Delta > 0$, the function $z \mapsto \frac{1}{\Delta} \log [1 + \Delta z]$ is strictly increasing ; therefore

$$\operatorname{Inf}_{u} \alpha(u,\Delta) = \frac{1}{\Delta} \log [1 + \Delta \operatorname{Inf}_{u} \beta(u,\Delta)] .$$

Let now

$$\gamma(\Delta) = \operatorname*{Inf}_{u} \beta(u,\Delta)$$

Since $\frac{1}{\Delta} \log[1 + \Delta\gamma(\Delta)] \leqq \gamma(\Delta)$, one has

(i) $$\limsup_{\Delta \to 0} \frac{1}{\Delta} \log[1 + \Delta\gamma(\Delta)] \leqq \limsup_{\Delta \to 0} \gamma(\Delta) .$$

Next consider two real numbers k_0 , k_1 such that $k_0 < k_1 < \limsup_{\Delta \to 0} \gamma(\Delta)$. There is an infinite sequence $\Delta_n > 0$ such that $\Delta_n \to 0$ and for every n , $\gamma(\Delta_n) \geqq k_1$. Therefore $\frac{1}{\Delta_n} \log[1 + \Delta_n \gamma(\Delta_n)] \geqq \frac{1}{\Delta_n} \log[1 + k_1 \Delta_n]$.

The last expression tends to k_1 . Hence for n large enough ,

$\frac{1}{\Delta_n} \log[1 + k_1 \Delta_n] \geqq k_0$. Therefore $\limsup_{\Delta \to 0} \frac{1}{\Delta} \log[1 + \Delta\gamma(\Delta)] \geqq k_0$,

and

(ii) $$\limsup_{\Delta \to 0} \frac{1}{\Delta} \log[1 + \Delta\gamma(\Delta)] \geqq \limsup_{\Delta \to 0} \gamma(\Delta) .$$

Q.E.D.

We first show that under A.2, a convex function f is characterized by the fact that $\lambda \geqq 0$. To this end we need two lemmata.

Lemma 1. Under A.2, let u_1 and u_0 be two points of U such that $u_1 < u_0$; let Δ be a point in the domain of $\bar{\lambda}$ such that $\Delta \leqq u_0 - u_1$; let k be a real number different from zero such that $k \leqq \bar{\lambda}(\Delta)$; and let p be an integer such that $1 \leqq p \leqq \frac{u_0 - u_1}{\Delta}$. Then

$$\phi(u_0) - \phi(u_1) \geqq [\phi(u_0) - \phi(u_0 - \Delta)] \frac{e^{kp\Delta} - 1}{e^{k\Delta} - 1} .$$

Proof: One has $u_1 \leq u_0 - p\Delta$. Hence $\phi(u_1) \leq \phi(u_0 - p\Delta)$. Therefore

$$\phi(u_0) - \phi(u_1) \geq [\phi(u_0) - \phi(u_0 - \Delta)] + [\phi(u_0 - \Delta) - \phi(u_0 - 2\Delta)] + \dots$$
$$+ [\phi(u_0 - (p-1)\Delta) - \phi(u_0 - p\Delta)]$$
$$\geq [\phi(u_0) - \phi(u_0 - \Delta)][1 + e^{k\Delta} + \dots + e^{k(p-1)\Delta}]$$
$$= [\phi(u_0) - \phi(u_0 - \Delta)] \frac{e^{kp\Delta} - 1}{e^{k\Delta} - 1} \text{ , Q.E.D.}$$

In a completely similar manner we establish

Lemma 2. Under A.2, let u_0 and u_2 be two points of U such that $u_0 < u_2$; let Δ be a point in the domain of $\bar{\lambda}$; let k be a real number different from zero such that $k \leq \bar{\lambda}(\Delta)$; and let q be an integer such that $q \geq \frac{u_2 - u_0}{\Delta}$ and $u_0 + q\Delta \in U$. Then

$$\phi(u_2) - \phi(u_0) \leq [\phi(u_0 + \Delta) - \phi(u_0)] \frac{e^{-kq\Delta} - 1}{e^{-k\Delta} - 1} \quad .$$

Theorem 4. Under A.2, the function f is convex if and only if $\lambda \geqq 0$.

Proof: Assume that f is convex. Then ϕ is concave and for every u and Δ, one has $\frac{\phi(u)-\phi(u-\Delta)}{\phi(u+\Delta)-\phi(u)} \geqq 1$. Therefore for every Δ, one has $\bar{\lambda}(\Delta) \geqq 0$. Hence $\lambda \geqq 0$.

Conversely assume that $\lambda \geqq 0$, and consider three points u_0, u_1, and u_2 of U such that $u_1 < u_0 < u_2$. Select a real number $k < 0$. There is an infinite sequence $\Delta_n > 0$ such that

$$\Delta_n \to 0 \text{ and, for every } n,\ \bar{\lambda}(\Delta_n) \geqq k .$$

Denote by p_n the greatest integer $\leqq \frac{u_0 - u_1}{\Delta_n}$, and by q_n the smallest integer $\geqq \frac{u_2 - u_0}{\Delta_n}$. By Lemmata 1 and 2, for every n,

$$\phi(u_0) - \phi(u_1) \geqq [\phi(u_0) - \phi(u_0 - \Delta_n)] \frac{e^{kp_n\Delta_n} - 1}{e^{k\Delta_n} - 1},$$

$$\phi(u_2) - \phi(u_1) \leqq [\phi(u_0 + \Delta_n) - \phi(u_0)] \frac{e^{-kq_n\Delta_n} - 1}{e^{-k\Delta_n} - 1}.$$

$$\text{Therefore}\quad \frac{\phi(u_0)-\phi(u_1)}{\phi(u_2)-\phi(u_1)} \geqq \frac{\phi(u_0)-\phi(u_0-\Delta_n)}{\phi(u_0+\Delta_n)-\phi(u_0)} \cdot \frac{1-e^{kp_n\Delta_n}}{e^{-kq_n\Delta_n}-1} \cdot \frac{e^{-k\Delta_n}-1}{1-e^{k\Delta_n}}$$

$$\geqq e^{k\Delta_n} \cdot \frac{1-e^{kp_n\Delta_n}}{e^{-kq_n\Delta_n}-1} \cdot e^{-k\Delta_n} = \frac{1-e^{kp_n\Delta_n}}{e^{-kq_n\Delta_n}-1}.$$

As $n \to +\infty$, $p_n\Delta_n \to u_0 - u_1$ and $q_n\Delta_n \to u_2 - u_0$. Hence $\frac{\phi(u_0)-\phi(u_1)}{\phi(u_2)-\phi(u_1)} \geqq \frac{1-e^{k(u_0-u_1)}}{e^{-k(u_2-u_0)}-1}$. Now let $k \to 0$. One obtains

$$\frac{\phi(u_0) - \phi(u_1)}{u_0 - u_1} \geqq \frac{\phi(u_2) - \phi(u_0)}{u_2 - u_0},$$

which implies that ϕ is concave, Q.E.D.

We note that if the function f is convex but <u>not</u> strictly convex, there is a non-empty open interval on which f is linear, hence a non-empty open interval U_0 on which ϕ is linear. Given any $u \in U_0$, for every Δ small enough, $u+\Delta$ and $u-\Delta$ belong to U_0, hence $\bar{\lambda}(\Delta)=0$. Therefore $\lambda = 0$.

In the case of a C^2 function it is possible (Theorem 8) to give a simple expression of the convexity index. To prepare for that result we remark

(7) Under A.2, if f is C^2 and at a point x_0 of X, $f'(x_0) \neq 0$, then letting $u_0 = f(x_0)$, one has $\lim_{\Delta \to 0} \log \left[\frac{\phi(u_0) - \phi(u_0-\Delta)}{\phi(u_0+\Delta) - \phi(u_0)}\right]^{\frac{1}{\Delta}} = \frac{f''(x_0)}{(f'(x_0))^2}$.

Proof of (7): Since $f'(x_0) \neq 0$, the function ϕ is locally C^2 at u_0. An application of Taylor's theorem yields

$$\lim_{\Delta \to 0} \log \left[\frac{\phi(u_0) - \phi(u_0-\Delta)}{\phi(u_0+\Delta) - \phi(u_0)}\right]^{\frac{1}{\Delta}} = -\frac{\phi''(u_0)}{\phi'(u_0)} .$$

But $\phi = f^{-1}$ yields

$$-\frac{\phi''(u_0)}{\phi'(u_0)} = \frac{f''(x_0)}{(f'(x_0))^2} ,$$

Q.E.D.

Of special interest is the class of C^2 functions h, defined on the nonempty open real interval Z, such that for every z, $h'(z) > 0$, and, denoting h^{-1} by ψ, $\log \left[\frac{\psi(w) - \psi(w-\Delta)}{\psi(w+\Delta) - \psi(w)}\right]^{\frac{1}{\Delta}}$ is independent of w and Δ, hence equal to a real number θ. According to (7), for every z, $\frac{h''(z)}{(h'(z))^2} = \theta$. Now two functions f and f^* such that $\phi^* = a\phi + b$, where $a > 0$ and b are two real numbers, obviously have the same convexity index. Therefore there is no essential loss of generality in assuming that $0 \in Z$ and in normalizing h in such a way that $h(0) = 0$ and $h'(0) = 1$. By integration one obtains

<u>if $\theta \neq 0$, $h_\theta(z) = -\frac{\log(1-\theta z)}{\theta}$ defined in $Z_\theta = \{z \in R | \theta z < 1\}$</u> ;

<u>if $\theta = 0$, $h_0(z) = z$ defined in $Z_0 = R$</u> .

Note that if $\theta_1 < \theta_2$ and $0 \neq z \in Z_{\theta_1} \cap Z_{\theta_2}$, one has $h_{\theta_1}(z) < h_{\theta_2}(z)$.

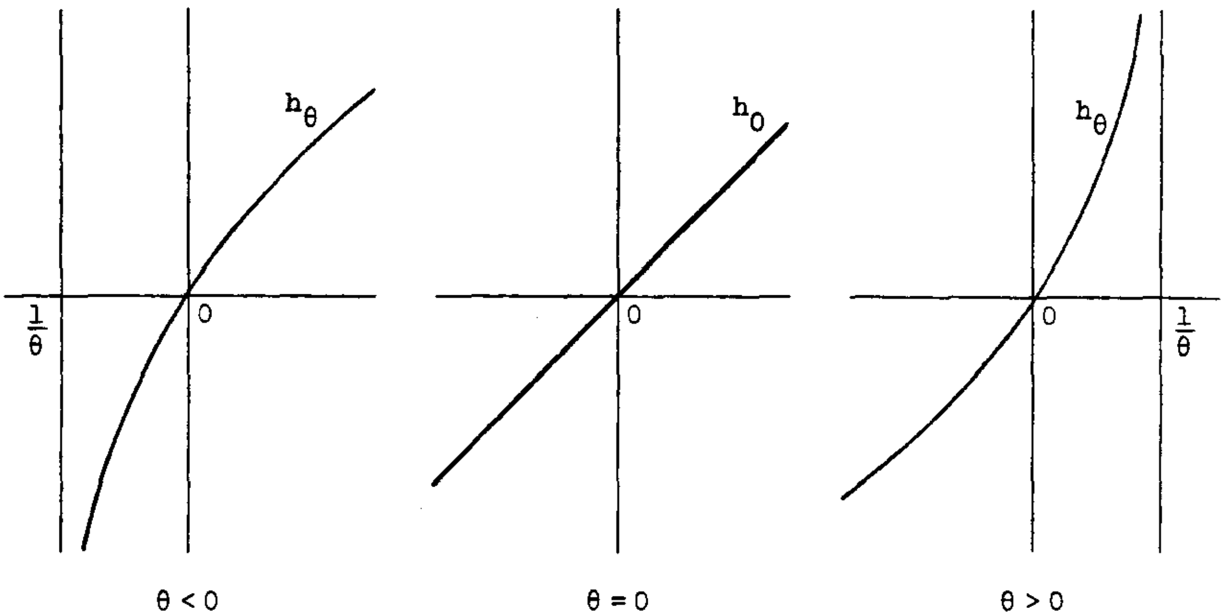

Figure 7

The inverse $\psi_\theta = h_\theta^{-1}$ is defined on R by

$$\psi_\theta(w) = \frac{1 - e^{-\theta w}}{\theta} \quad \text{if } \theta \neq 0 \; ; \quad \psi_0(w) = w \; .$$

The convexity index of h_θ is easily seen to be θ since $\left[\frac{\psi_\theta(w) - \psi_\theta(w-\Delta)}{\psi_\theta(w+\Delta) - \psi_\theta(w)}\right]^{\frac{1}{\Delta}} = e^\theta$. Some properties of ψ_θ that we will use later on are also noted here. For any w , $\psi_\theta(w)$ is continuous in θ . For $w > 0$, $\lim_{\theta \to +\infty} \psi_\theta(w) = 0$. For $w < 0$, $\lim_{\theta \to +\infty} \psi_\theta(w) = -\infty$. For $w \neq 0$, $\psi_\theta(w)$ is strictly decreasing in θ .

Theorem 5 gives conditions under which in any point x_0 of its domain, the function f is supported from below by a suitable linear transform of h_λ . Before stating and proving it we remark

(8) Under A.2, if $\lambda > -\infty$ and $f'_-(x_0)$ and $f'_+(x_0)$ exist at a point x_0 of X, then $f'_-(x_0) \leqq f'_+(x_0)$.

Proof of (8): Assume that $f'_-(x_0) > f'_+(x_0)$. Since f is increasing $f'_+(x_0) \geqq 0$. Let $u_0 = f(x_0)$. Then $0 \leqq \phi'_-(u_0) < \phi'_+(u_0)$, where $\phi'_+(u_0)$ is defined to be $+\infty$ in case $f'_+(x_0) = 0$. Moreover

$0 \leqq \lim\limits_{\Delta \to 0} \dfrac{\phi(u_0) - \phi(u_0 - \Delta)}{\phi(u_0 + \Delta) - \phi(u_0)} = \dfrac{\phi'_-(u_0)}{\phi'_+(u_0)} < 1$. Consider an arbitrary real number k. For every small enough Δ, $\log \left[\dfrac{\phi(u_0) - \phi(u_0 - \Delta)}{\phi(u_0 + \Delta) - \phi(u_0)}\right]^{\frac{1}{\Delta}} \leqq k$, hence $\bar{\lambda}(\Delta) \leqq k$. Therefore $\lambda \leqq k$. Consequently $\lambda = -\infty$, a contradiction, Q.E.D.

Theorem 5. Under A.2, the convexity index λ is strictly smaller than $+\infty$. If $\lambda > -\infty$ and $f'_-(x_0)$ exists and is finite at a point x_0 of X, then $f'_-(x_0) > 0$ and

for every $x \in X$ such that $x \leqq x_0$ and $\lambda(x - x_0) < \dfrac{1}{f'_-(x_0)}$, one has $f(x) - f(x_0) \geqq h_\lambda[f'_-(x_0)(x - x_0)]$.

If $\lambda > -\infty$ and $f'_+(x_0)$ exists and is strictly positive at a point x_0 of X, then $f'_+(x_0)$ is finite and

for every $x \in X$ such that $x \geqq x_0$, one has $f(x) - f(x_0) \geqq h_\lambda[f'_+(x_0)(x - x_0)]$.

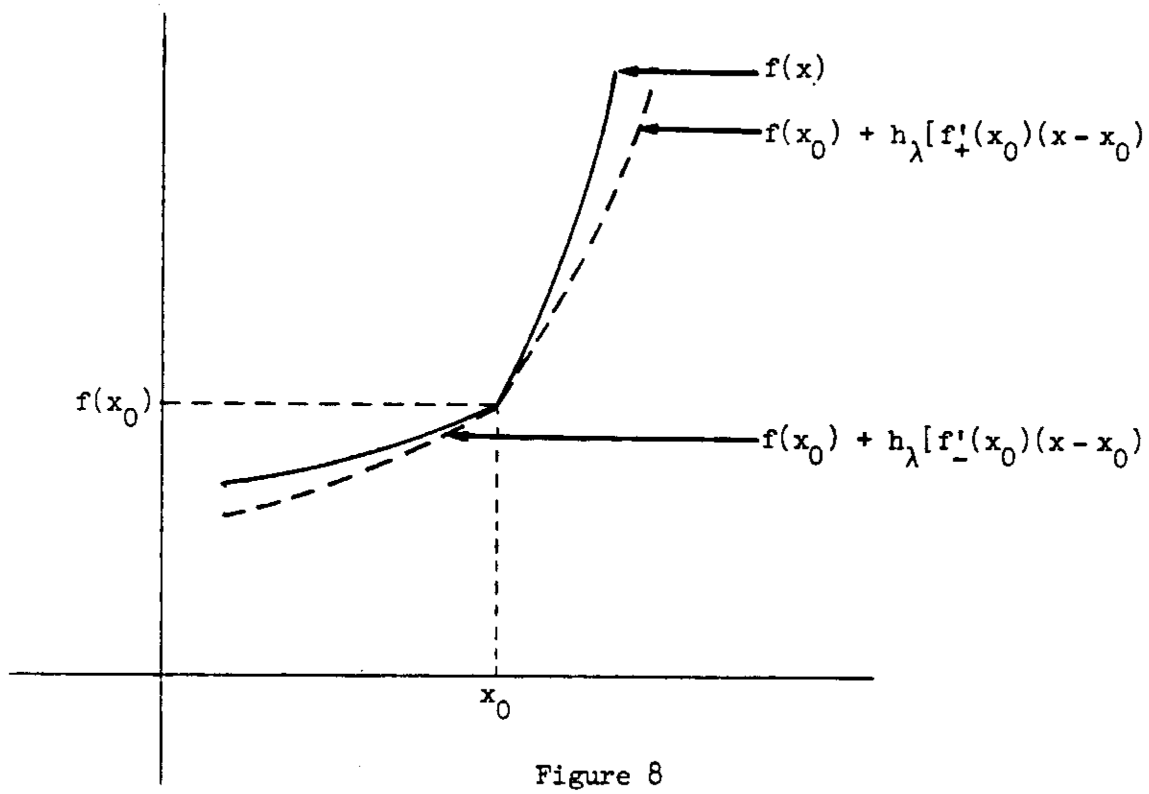

Figure 8

Proof: Assume that $\lambda > -\infty$ and $f'_-(x_0)$ exists and is finite at x_0. Choose any $x \in X$ such that $x < x_0$ and let $u = f(x)$ and $u_0 = f(x_0)$. Select a real number k such that $0 \neq k < \lambda$. There is an infinite sequence $\Delta_n > 0$ such that

$$\Delta_n \to 0 \text{ and, for every } n, \ \bar{\lambda}(\Delta_n) \geq k .$$

Denote by p_n the greatest integer smaller than or equal to $\dfrac{u_0 - u}{\Delta_n}$.

By Lemma 1,

$$\phi(u) - \phi(u_0) \leq \frac{\phi(u_0-\Delta_n) - \phi(u_0)}{-\Delta_n} \cdot \frac{1 - e^{kp_n\Delta_n}}{\frac{1}{\Delta_n}(e^{k\Delta_n} - 1)} .$$

As $n \to +\infty$, one has $\Delta_n \to 0$; $\dfrac{\phi(u_0-\Delta_n) - \phi(u_0)}{-\Delta_n} \to \phi'_-(u_0) = \dfrac{1}{f'_-(x_0)}$, which may be $+\infty$; $p_n\Delta_n \to u_0-u$; $\dfrac{1}{\Delta_n}\left(e^{k\Delta_n}-1\right) \to k$. Therefore

$$\phi(u) - \phi(u_0) \leq \phi'(u_0). \frac{1 - e^{k(u_0-u)}}{k} = \phi'_-(u_0)\ \psi_k(u-u_0).$$

Since $\psi_k(u-u_0) < 0$, if $\phi'_-(u_0) = +\infty$, one obtains $\phi(u) = -\infty$, a contradiction of the fact that $\phi(u)$ is finite for every $u \in U$. Thus $0 < f'_-(x_0)$.

Next let $k \to \lambda$. If $\lambda = +\infty$, $\psi_k(u-u_0) \to -\infty$, and again $\phi(u) = -\infty$. Thus $\lambda < +\infty$, and, in the limit,

(i) for every $u \in U$ such that $u \leqq u_0$, $\phi(u) - \phi(u_0) \leqq \phi'_-(u_0)\ \psi_\lambda(u-u_0)$.

From (i), for every $x \in X$ such that $x \leqq x_0$, one has $x - x_0 \leqq \dfrac{1}{f'_-(x_0)}\ \psi_\lambda[f(x) - f(x_0)]$. Hence

(i') $\psi_\lambda[f(x) - f(x_0)] \geqq f'_-(x_0)(x-x_0)$.

Now $f'_-(x_0)(x-x_0)$ belongs to Z_λ if and only if $\lambda f'_-(x_0)(x-x_0) < 1$, i.e., if and only if

$$\lambda(x-x_0) < \frac{1}{f'_-(x_0)}\ .$$

Therefore from (i'), for every $x \in X$ such that $x \leqq x_0$ and $\lambda(x-x_0) < \dfrac{1}{f'_-(x_0)}$, one has $f(x) - f(x_0) \geqq h_\lambda[f'_-(x_0)(x-x_0)]$.

We have shown that if $\lambda > -\infty$ and there is a point x_0 of X at which $f'_-(x_0)$ exists and is finite, then $\lambda < +\infty$. However, according to Lebesgue's theorem (F. Riesz and B. Sz.-Nagy [1955], pp. 5-9), the increasing function f has a finite derivative almost everywhere. Therefore $\lambda > -\infty$ implies $\lambda < +\infty$. So, obviously, does $\lambda = -\infty$.

Assume now that $\lambda > -\infty$ and $f'_+(x_0)$ exists and is strictly positive at x_0. In a completely similar manner one can show that for $u \in U$ such that $u > u_0$ and for $0 \neq k < \lambda$, one has

$$\phi(u) - \phi(u_0) \leqq \phi'_+(u_0)\psi_k(u-u_0) .$$

Since $\psi_k(u-u_0) > 0$, if $\phi'_+(u_0) = 0$, one obtains $\phi(u) \leqq \phi(u_0)$, a contradiction of the fact that ϕ is strictly increasing on U. Thus $f'_+(x_0)$ is finite. Letting $k \to \lambda$, one obtains

(ii) for every $u \in U$ such that $u \geqq u_0$, $\phi(u) - \phi(u_0) \leqq \phi'_+(u_0)\psi_\lambda(u-u_0)$.

From (ii), for every $x \in X$ such that $x \geqq x_0$, one has $x - x_0 \leqq \frac{1}{f'_+(x_0)} \psi_\lambda[f(x) - f(x_0)]$. Hence

(ii') $\psi_\lambda[f(x) - f(x_0)] \geqq f'_+(x_0)(x-x_0)$.

Now $f'_+(x_0)(x-x_0)$ belongs to Z_λ if and only if $\lambda f'_+(x_0)(x-x_0) < 1$. This inequality is obviously satisfied if $\lambda \leqq 0$. It is also satisfied if $\lambda > 0$, for in this case (ii') implies $\frac{1 - e^{-\lambda(u-u_0)}}{\lambda} \geqq f'_+(x_0)(x-x_0)$, hence $\lambda f'_+(x_0)(x-x_0) \leqq 1 - e^{-\lambda(u-u_0)} < 1$. Therefore from (ii'), for every $x \in X$ such that $x \geqq x_0$, $f(x)-f(x_0) \geqq h_\lambda[f'_+(x_0)(x-x_0)]$. Q.E.D.

The assumptions made in Theorem 5 lead one to ask whether $\lambda > -\infty$ implies the existence of left and right derivatives of f everywhere in X. The following example gives a negative answer to that question.

Consider the family of functions $\phi_{\alpha,\beta}$ from R to R defined by $u \mapsto \phi_{\alpha,\beta}(u) = \alpha(e^u - 1) + \beta$, depending on the two real parameters $\alpha > 0$ and β. We construct a strictly increasing continuous real-valued function ϕ such that $\phi(0) = 0$, ϕ has no right-derivative at 0, and the convexity index of ϕ^{-1} is finite. Let L be the straight line in R^2 through 0 with slope 1, and M be the straight line through 0 with slope 2. Let also P be the point of M with abscissa $p > 0$. There is exactly one function $\phi_{\alpha,\beta}$ such that the graph of $\phi_{\alpha,\beta}$ goes through P, and is tangent to L at a point to the left of P. The corresponding parameter α is the smallest of the two roots of the equation

$$\alpha e^p - \log \alpha - 2p - 1 = 0 ,$$

and the corresponding β is given by

$$\beta = \alpha - \log \alpha - 1 .$$

Moreover letting Q be the point different from P where the graph of $\phi_{\alpha,\beta}$ so determined intersects M, one can easily prove that if $p \leq 1$, then the abscissa q of Q satisfies $0 < q < \frac{p}{5}$.

Take $p_0 = 1$, obtain α_0, β_0 and the abscissa p_1 of the point of intersection of the graph of ϕ_{α_0,β_0} and M. Repeat the construction starting with p_1, obtaining α_1, β_1 and $p_2, \ldots$ Since $p_n < \frac{1}{5^n}$, the sequence p_n tends to zero.

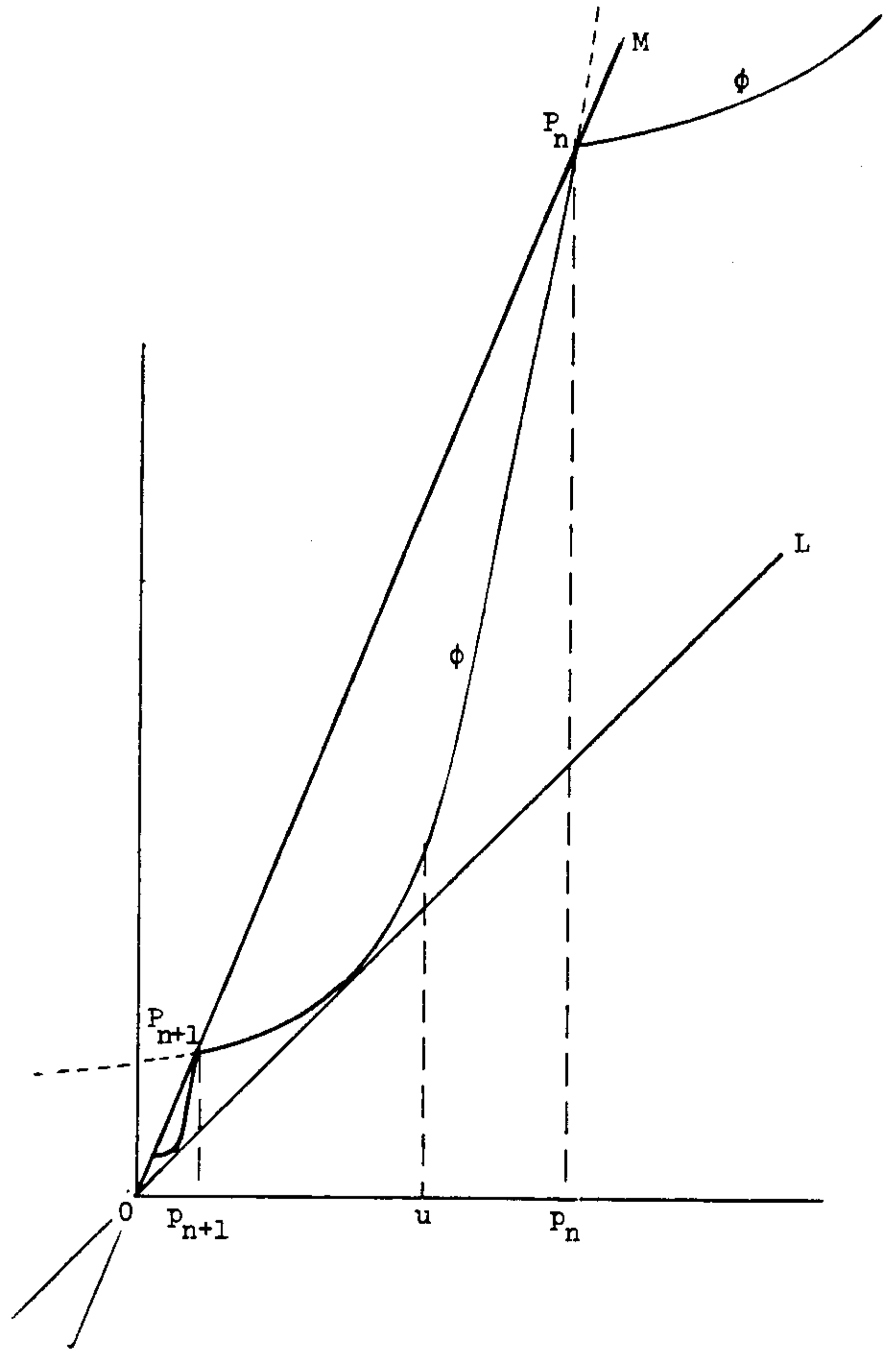

Figure 9

To define the function ϕ, consider first u such that $0<u<1$ and u does not coincide with any of the p_n. Then u belongs to exactly one of the intervals $[p_{n+1},p_n]$. Define $\phi(u)$ as $\phi_{\alpha_n,\beta_n}(u)$. For all other values of u, namely $u\leq 0$, or $u=p_n$ for some n, or $1<u$, define $\phi(u)$ as $2u$.

Clearly ϕ is increasing and continuous, $\phi(0)=0$, and ϕ has no right-derivative at 0. There remains to prove that the convexity index of ϕ^{-1} is finite. If $u\leq 0$, or $u=p_n$ for some n, or $1<u$, then obviously for every $\Delta>0$, $\dfrac{\phi(u)-\phi(u-\Delta)}{\phi(u+\Delta)-\phi(u)}\geq 1$. If $u\in\,]0,1[$ and u does not coincide with any of the p_n, then u belongs to exactly one of the intervals $[p_{n+1},p_n]$. Denote ϕ_{α_n,β_n} by ϕ_n, and consider an arbitrary $\Delta>0$. One has $\dfrac{\phi(u)-\phi(u-\Delta)}{\phi(u+\Delta)-\phi(u)}\geq\dfrac{\phi_n(u)-\phi_n(u-\Delta)}{\phi_n(u+\Delta)-\phi_n(u)}=e^{-\Delta}$. Therefore for every u, and every $\Delta>0$, $\log\left[\dfrac{\phi(u)-\phi(u-\Delta)}{\phi(u+\Delta)-\phi(u)}\right]^{1/\Delta}\geq -1$. Consequently the convexity index of ϕ^{-1} is at least equal to -1.

In an entirely simlilar manner, one could construct a function having a finite convexity index and no left-derivative at 0.

We now establish two lemmata needed in the proofs of Theorems 6-8.

Lemma 3. Assume that A.2 holds, and that f and g are C^2. The function F is quasi-convex if and only if

(i) for every $(x,y)\in X\times Y$, one has $f'(x)\neq 0$, $g'(y)\neq 0$,

$$\frac{f''(x)}{(f'(x))^2}+\frac{g''(y)}{(g'(y))^2}\geq 0 .$$

Proof: Consider $(x_0, y_0) \in X \times Y$ and let

$$\Gamma = \{(x,y) \in X \times Y \mid f(x) + g(y) \leq f(x_0) + g(y_0)\} .$$

One has, for the boundary $\partial\Gamma$ of Γ,

$$\partial\Gamma = \{(x,y) \in X \times Y \mid f(x) + g(y) = f(x_0) + g(y_0)\} .$$

Denote by proj_1 the projection from $X \times Y$ into X. Clearly $\text{proj}_1\, \partial\Gamma$ is a non-empty open real interval, and for every $x \in \text{proj}_1\, \partial\Gamma$, there is a unique point $\gamma(x)$ of Y such that $(x, \gamma(x)) \in \partial\Gamma$.

If $f'(x) \neq 0$ and $g'(y) \neq 0$ everywhere, the function γ is C^2 and one has for every $(x,y) \in \partial\Gamma$,

$$\frac{f''(x)}{(f'(x))^2} + \frac{g''(y)}{(g'(y))^2} = -\gamma''(x)\, \frac{g'(y)}{(f'(x))^2} .$$

Assume now that F is quasi-convex, then the set Γ is convex and the function γ is concave. By Theorem 3, $f'(x_0) > 0$ and $g'(y_0) > 0$. The concavity of γ implies $\gamma''(x_0) \leq 0$. Therefore (x_0, y_0) satisfies condition (i).

Conversely if (i) is satisfied, one has $\gamma''(x) \leq 0$ for every $x \in \text{proj}_1\, \partial\Gamma$. Therefore the function γ is concave, and the set Γ is convex. Consequently F is quasi-convex, Q.E.D.

Lemma 4. Assume that A.2 holds, and that for every $(x_0, y_0) \in X \times Y$, there are two open intervals I and J containing x_0 and y_0 respectively and two strictly increasing functions f^* and g^* defined on I and J

<u>respectively such that (i) $f^*(x_0) = f(x_0)$ and for every $x \in I$, $f^*(x) \leqq f(x)$, (ii) $g^*(y_0) = g(y_0)$ and for every $y \in J$, $g^*(y) \leqq g(y)$, (iii) the function from $I \times J$ to R defined by $(x,y) \mapsto f^*(x) + g^*(y)$ is quasi-convex. Then F is quasi-convex.</u>

<u>Proof</u>: Consider $(x_0,y_0) \in X \times Y$ and the sets $\Gamma_1 = \{(x,y) \in I \times J | F(x,y) \leqq F(x_0,y_0)\}$ and $\Gamma_2 = \{(x,y) \in I \times J | f^*(x) + g^*(y) \leqq F(x_0,y_0)\}$. Since $f^*(x) + g^*(y) \leqq f(x) + g(y)$, one has $\Gamma_1 \subset \Gamma_2$. Since f, g, f^*, g^* are strictly increasing, $(x_0,y_0) \in \partial\Gamma_1 \cap \partial\Gamma_2$. Moreover Γ_2 is convex. Consequently there is through (x_0,y_0) a straight-line supporting Γ_2, hence Γ_1, from above. This implies that the set $\Gamma = \{(x,y) \in X \times Y | F(x,y) \leqq F(x_0,y_0)\}$ is convex. Hence F is quasi-convex, Q.E.D.

In Theorem 6 and in its proof, the function $\bar{\mu}$ and the convexity index μ are associated with g as $\bar{\lambda}$ and λ are associated with f.

<u>Theorem 6</u>. <u>Under A.2, the function F is quasi-convex if and only if $\lambda + \mu \geqq 0$ and at every $(x,y) \in X \times Y$, $f'_-(x)$, $f'_+(x)$, $g'_-(y)$, and $g'_+(y)$ exist and are finite.</u>

There is clearly some redundancy in the assumptions following "if and only if" since $\lambda + \mu \geqq 0$ implies that at least one of the two convexity indices is $\geqq 0$. However, by Theorem 4, under A.2, a function whose convexity index is $\geqq 0$ is convex, and therefore has finite left and right-derivatives everywhere.

Proof of Theorem 6. Assume first that F is quasi-convex. Consider x, x', x'' in X and y, y', y'' in Y such that $x' < x < x''$, $y'' < y < y'$ and $F(x',y') = F(x,y) = F(x'',y'')$. Since f and g are strictly increasing, the point (x,y) is not strictly below the diagonal of the rectangle in Figure 10.

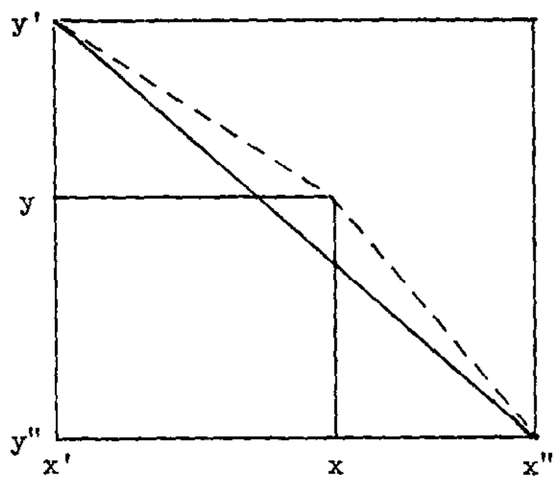

Figure 10

Therefore the absolute values of the slopes of the dashed segments satisfy $\frac{y'-y}{x-x'} \leqq \frac{y-y''}{x''-x}$. Hence

$$\frac{x-x'}{x''-x} \cdot \frac{y-y''}{y'-y} \geqq 1 .$$

Let now u, v and $\Delta > 0$ be such that $u-\Delta$, u, and $u+\Delta$ are in U; $v-\Delta$, v, and $v+\Delta$ are in V. One has

$$\frac{\phi(u) - \phi(u-\Delta)}{\phi(u+\Delta) - \phi(u)} \cdot \frac{\chi(v) - \chi(v-\Delta)}{\chi(v+\Delta) - \chi(v)} \geqq 1 .$$

Therefore

(9) $\underline{\text{for every } \Delta \in (\text{domain } \bar{\lambda}) \cap (\text{domain } \bar{\mu}), \text{ one has } \bar{\lambda}(\Delta) + \bar{\mu}(\Delta) \geqq 0}$.

Since, by Theorem 5, $\lambda < +\infty$ and $\mu < +\infty$, the sum $\lambda + \mu$ is well defined. If one had $\lambda + \mu < 0$, there would be λ' and μ' such that $\lambda < \lambda'$, $\mu < \mu'$ and $\lambda' + \mu' < 0$. For every Δ small enough, $\bar{\lambda}(\Delta) < \lambda'$ and $\bar{\mu}(\Delta) < \mu'$. Hence $\bar{\lambda}(\Delta) + \bar{\mu}(\Delta) < 0$, a contradiction of (8). Therefore $\lambda + \mu \geqq 0$. The existence of the finite left and right derivatives of f and g is asserted by Theorem 3.

Assume now that f and g have finite left and right derivatives everywhere and $\lambda + \mu \geqq 0$. Since $\lambda < +\infty$ and $\mu < +\infty$, one has $\lambda > -\infty$ and $\mu > -\infty$. Consider $(x_0, y_0) \in X \times Y$.

According to Theorem 5 and (8), $0 < f'_-(x_0) \leqq f'_+(x_0)$ and $0 < g'_-(y_0) \leqq g'_+(y_0)$. Select two numbers α, β such that

$$f'_-(x_0) \leqq \alpha \leqq f'_+(x_0) \quad \text{and} \quad g'_-(y_0) \leqq \beta \leqq g'_+(y_0).$$

By Theorem 5, since h_λ is a strictly increasing function, for every $x \in X$ such that $x \geqq x_0$, or such that $x \leqq x_0$ and $\lambda(x-x_0) < \frac{1}{\alpha}$, one has $f(x) - f(x_0) \geqq h_\lambda\,[\alpha(x-x_0)]$.

This leads us to define the function $\bar{f}$ and its domain $\bar{X}$ as follows. If $\lambda \geqq 0$, then $\bar{X} = X$. If $\lambda < 0$, then $\bar{X} = \left]\mathrm{Max}\{x_0 + \frac{1}{\alpha\lambda}, \mathrm{Inf}\, X\}, \mathrm{Sup}\, X\right[$. In either case, for every $x \in \bar{X}$, $\bar{f}(x) = h_\lambda\,[\alpha(x-x_0)]$. Therefore

$$\text{for every } x \in \bar{X}, \quad f(x) \geqq f(x_0) + \bar{f}(x).$$

Note that $\bar{f}(x_0) = 0$, and for every $x \in \bar{X}$, $\bar{f}'(x) > 0$ and $\frac{\bar{f}''(x)}{(\bar{f}'(x))^2} = \lambda$.

Similarly define the function $\bar{g}$ and its domain $\bar{Y}$ as follows. If $\mu \geq 0$, then $\bar{Y} = Y$. If $\mu < 0$, then $\bar{Y} =]\text{Max}\{y_0 + \frac{1}{\beta\mu}, \text{Inf } Y\}, \text{Sup } Y[$. In either case, for every $y \in \bar{Y}$, $\bar{g}(y) = h_\mu [\beta(y-y_0)]$. Therefore

$$\text{for every } y \in \bar{Y}, \quad g(y) \geq g(y_0) + \bar{g}(y).$$

As before, $\bar{g}(y_0) = 0$, and for every $y \in \bar{Y}$, $\bar{g}'(y) > 0$ and $\frac{\bar{g}''(y)}{(\bar{g}'(y))^2} = \mu$.

According to Lemma 3, the function from $\bar{X} \times \bar{Y}$ to R defined by $(x,y) \mapsto f(x_0) + g(y_0) + \bar{f}(x) + \bar{g}(y)$ is quasi-convex. According to Lemma 4, the function F is quasi-convex, Q.E.D.

Note that if F is quasi-convex and f is not convex, then $\lambda < 0$ and, by Theorem 6, $\lambda + \mu \geq 0$. Hence $\mu > 0$. Consequently, according to the remark we made after proving Theorem 4, g is strictly convex, a restatement of Theorem 2.

As a consequence of the preceding results, the function F is quasi-convex if and only if the two functions f and g can be separated by a function of the class h_θ in the following sense.

<u>Theorem 7</u>. <u>Under A.2, the function F is quasi-convex if and only if for every $(x_0,y_0) \in X \times Y$, there are three real numbers $\alpha > 0$, $\beta > 0$, and θ and an open interval I containing x_0 such that</u>

$$\text{for every } x \in I, \quad g(y_0) - g[y_0 - \frac{\alpha}{\beta}(x-x_0)] \leq h_\theta[\alpha(x-x_0)] \leq f(x) - f(x_0).$$

Proof. Assume first that F is quasi-convex, and consider $(x_0,y_0) \in X \times Y$. By Theorem 3, $0 < f'_-(x_0) \leqq f'_+(x_0)$ and $0 < g'_-(y_0) \leqq g'_+(y_0)$. By Theorems 5 and 6, $\lambda + \mu \geqq 0$, $\lambda > -\infty$, $\mu > -\infty$.

Select two numbers α and β such that

$$f'_-(x_0) \leqq \alpha \leqq f'_+(x_0) \quad \text{and} \quad g'_-(y_0) \leqq \beta \leqq g'_+(y_0).$$

For every x such that $y_o - \frac{\alpha}{\beta}(x-x_0) \in Y$, i.e., $x \in x_0 - \frac{\beta}{\alpha}(Y-y_0)$, we define

$$\tilde{g}(x) = g(y_0) - g[y_0 - \frac{\alpha}{\beta}(x-x_0)].$$

Note that $\tilde{g}(x_0) = 0$; $\tilde{g}'_-(x_0) = \frac{\alpha}{\beta} g'_+(y_0) \geqq \alpha \geqq f'_-(x_0)$; $g'_+(x_0) = \frac{\alpha}{\beta} g'_-(y_0) \leqq \alpha \leqq f'_+(x_0)$.

The sets $\bar{X},\bar{Y}$ and the functions $\bar{f},\bar{g}$ are defined as in the proof of Theorem 6. For every $y_o - \frac{\alpha}{\beta}(x-x_0) \in \bar{Y}$, one has $g[y_0 - \frac{\alpha}{\beta}(x-x_0)] - g(y_0) \geqq \bar{g}[y_0 - \frac{\alpha}{\beta}(x-x_0)] = h_\mu[-\alpha(x-x_0)]$. However $z \in Z_\mu$ implies $-z \in Z_{-\mu}$ and $h_{-\mu}(-z) = -h_\mu(z)$. Consequently for every $x \in x_0 - \frac{\beta}{\alpha}(\bar{Y}-y_0)$,

(i) $\tilde{g}(x) \leqq h_{-\mu}[\alpha(x-x_0)]$.

Moreover $-\mu \leqq \lambda$ and $h_\theta(z)$ is increasing in θ. Therefore if $\alpha(x-x_0) \in Z_\lambda$, i.e., $x \in x_0 + \frac{1}{\alpha} Z_\lambda$, and $-\mu \leqq \theta \leqq \lambda$, one has

(ii) $h_{-\mu}[\alpha(x-x_0)] \leqq h_\theta[\alpha(x-x_0)] \leqq h_\lambda[\alpha(x-x_0)]$.

Finally for every $x \in \bar{X}$,

(iii) $h_\lambda[\alpha(x-x_0)] \leqq f(x) - f(x_0)$.

Let $I = [x_0 - \frac{\beta}{\alpha}(\bar{Y}-y_0)] \cap [x_0 + \frac{1}{\alpha} Z_\lambda] \cap \bar{X}$, and note that I is an open interval containing x_0. Summing up (i), (ii) and (iii), one has

for every $\theta \in [-\mu,\lambda]$, for every $x \in I$, $\tilde{g}(x) \leq h_\theta[\alpha(x-x_0)] \leq f(x) - f(x_0)$.

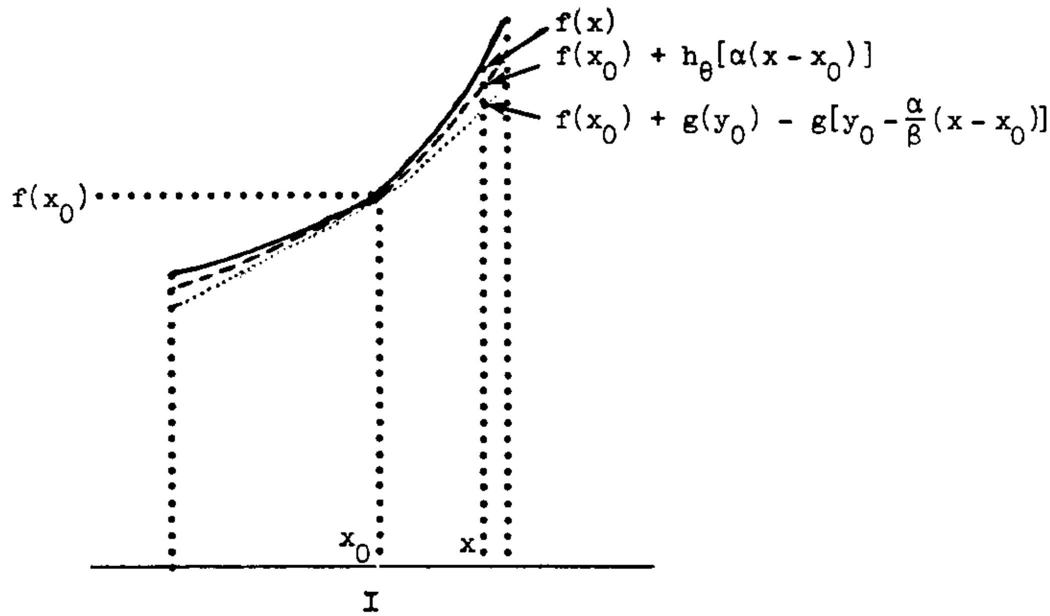

Fig. 11

Conversely assume that for every $(x_0,y_0) \in X \times Y$, there are three real numbers $\alpha > 0$, $\beta > 0$, and θ, and an open interval I containing x_0 such that for every $x \in I$,

$$g(y_0) - g[y_0 - \frac{\alpha}{\beta}(x-x_0)] \leq h_\theta[\alpha(x-x_0)] \leq f(x) - f(x_0).$$

Let $J = y_0 - \frac{\alpha}{\beta}(I-x_0)$. For every $y \in J$, one has $g(y_0) - g(y) \leq h_\theta[-\beta(y-y_0)] = -h_{-\theta}[\beta(y-y_0)]$. Hence

for every $y \in J$, $g(y) - g(y_0) \geq h_{-\theta}[\beta(y-y_0)]$.

Moreover,

for every $x \in I$, $f(x) - f(x_0) \geqq h_\theta[\alpha(x-x_0)]$.

Now let $f^*(x) = h_\theta[\alpha(x-x_0)]$ and $g^*(y) = h_{-\theta}[\beta(y-y_0)]$. By Lemma 3, the function from $I \times J$ to R defined by $(x,y) \mapsto f^*(x) + g^*(y)$ is quasi-convex. By Lemma 4, the function F is quasi-convex, Q.E.D.

Theorem 8 gives an explicit expression of the convexity index of a C^2 function.

<u>Theorem 8</u>. <u>Under A.2, if the function f is C^2 and for every $x \in X$, $f'(x) \neq 0$, then $\lambda = \operatorname{Inf}_x \frac{f''(x)}{(f'(x))^2}$</u>.

<u>Proof</u>. Assume that $\lambda > -\infty$ and select a real number $k < \lambda$. There is an infinite sequence $\Delta_n > 0$ such that

$\Delta_n \to 0$ and for every n, $\bar{\lambda}(\Delta_n) \geqq k$.

Therefore for every n, for every $u \in U$ such that $u - \Delta_n$ and $u + \Delta_n$ are in U, $\log \left[\frac{\phi(u) - \phi(u-\Delta_n)}{\phi(u+\Delta_n) - \phi(u)}\right]^{\frac{1}{\Delta_n}} \geqq k$. Hence, by (7), for every $x \in X$, $\frac{f''(x)}{(f'(x))^2} \geqq k$. Consequently

$$\operatorname{Inf}_x \frac{f''(x)}{(f'(x))^2} \geqq \lambda.$$

This inequality also obviously holds if $\lambda = -\infty$.

Conversely assume that $\operatorname*{Inf}_x \frac{f''(x)}{(f'(x))^2} > -\infty$

and let $k = \operatorname*{Inf}_x \frac{f''(x)}{(f'(x))^2}$. We have defined earlier the function h_{-k} from $Z_{-k} = \{y \in R \mid -ky < 1\}$ to R and noted that its convexity index is $-k$. Moreover for every $y \in Z_{-k}$, $h'_{-k}(y) > 0$ and

$\frac{h''_{-k}(y)}{(h'_{-k}(y))^2} = -k$. Therefore for every $(x,y) \in X \times Z_{-k}$, $\frac{f''(x)}{(f'(x))^2} + \frac{h''_{-k}(y)}{(h'_{-k}(y))^2} \geqq 0$. According to Lemma 3, the function defined on $X \times Z_{-k}$ by $(x,y) \mapsto f(x) + h_{-k}(y)$ is quasi-convex. Therefore by Theorem 6, $\lambda - k \geqq 0$. Hence

$$\lambda \geqq \operatorname*{Inf}_x \frac{f''(x)}{(f'(x))^2} .$$

This inequality also obviously holds if $\operatorname*{Inf}_x \frac{f''(x)}{(f'(x))^2} = -\infty$, Q.E.D.

The assumption that the intervals X and Y are open was made only to simplify the exposition. All the definitions and results of this section can be extended to the case in which X and Y are non-degenerate, i.e., contain at least two distinct points. It suffices to observe that (1) the convexity index of f on X is equal to the convexity index of the restriction of f to <u>int</u> X, (2) if the restriction of F to <u>int</u> $X \times$ <u>int</u> Y is quasi-convex, then F is quasi-convex. To prove the last assertion we show that if K is a convex set with a non-empty interior, G is a continuous real-valued function on K such that the restriction of G to int K is quasi-convex, then G is quasi-convex. Let x, y be two points of K, and z be a point of the straight line segment $[x,y]$.

Subjecting K to a translation if need be, we assume without loss of generality that the origin of the space belongs to int K . Define $x_n = (1-\frac{1}{n})x$, $y_n = (1-\frac{1}{n})y$, $z_n = (1-\frac{1}{n})z$, where n is a positive integer. x_n, y_n and z_n are in int K . The point z_n belongs to $[x_n, y_n]$. By quasi-convexity of G on int K , one has $G(z_n) \leqq \text{Max}\,\{G(x_n), G(y_n)\}$. As $n \to +\infty$, $G(z) \leqq \text{Max}\,\{G(x), G(y)\}$.

The convexity index can readily be defined for a broader class of functions than those we have considered so far in this section. If f is a strictly decreasing continuous real-valued function on a non-degenerate real interval X , let $\hat{f}$ be the function defined on $\hat{X} = -X$ by $\hat{f}(x) = f(-x)$. The convexity index c(f) of f is defined as $c(\hat{f})$, the convexity index of $\hat{f}$. If f is a constant real-valued function on a non-empty real interval X , the convexity index of f is defined to be $+\infty$.

Consider now the class $\mathscr{C}$ of continuous real-valued functions f such that (i) the domain of f is a non-empty real interval partitioned into three intervals $X^{sd} < X^{c} < X^{si}$ (some of which may be empty), and (ii) f is strictly decreasing on X^{sd} , constant on X^{c}, and strictly increasing on X^{si} . According to the results of section 1, if f and g are non-constant real-valued functions defined on the non-empty open real intervals X and Y repectively, and the function from $X \times Y$ to R defined by $(x,y) \mapsto f(x) + g(y)$ is quasi-convex, then f and g belong to $\mathscr{C}$. Let f be a function in $\mathscr{C}$ and denote by f^{sd} , f^{c} , and f^{si} the restrictions of f to X^{sd} , X^{c} and X^{si} respectively. The convexity indices $c(f^{sd})$, $c(f^{c})$, and $c(f^{si})$ are determined if and only if the corresponding intervals X^{sd} , X^{c} , and X^{si} are non-empty. The convexity index c(f) is defined as the smallest of the convexity indices $c(f^{sd})$, $c(f^{c})$, and $c(f^{si})$ that are determined. With this definition, the extension of the results of this section to the class $\mathscr{C}$ is straightforward.

3. EXTENSIONS TO HIGHER DIMENSIONS AND TO MORE THAN TWO FACTORS

In this section we make the assumptions

A.3. S and T are finite-dimensional real vector spaces; X and Y are open convex subsets of S and T, respectively; f and g are non-constant real-valued functions on X and Y, respectively; the function $F: X \times Y \to R$ defined by $F(x, y) = f(x) + g(y)$ is quasi-convex.

We first note

Theorem 9. Under A.3, the functions f and g are continuous.

Proof: Since g is non-constant on Y, there are two points y_1, y_2 of Y such that $g(y_1) \neq g(y_2)$. Let M be the straight line through y_1 and y_2 and let g_M be the restriction of g to $Y \cap M$, which is a convex subset of M open relative to M.

Consider now an arbitrary straight line L in S and let f_L be the restriction of f to $X \cap L$, which is a convex subset of L open relative to L. The function from $(X \cap L) \times (Y \cap M)$ to R defined by $(x, y) \mapsto f_L(x) + g_M(y)$ is quasi-convex. Therefore, by Theorem 1, if f_L is non-constant, it is continuous, while if f_L is constant, it is trivially continuous. Thus for every L, the function f_L is continuous. We shall show that, according to Proposition 4 of Crouzeix [1977], this implies that the quasi-convex function f is continuous. Following Crouzeix,

(i) f is lower semi-continuous.

For this one has to show that for every real number k, the set

$$V_k = \{x \in X \mid f(x) \leq k\}$$

is closed relative to X. By quasi-convexity of f, the set V_k is convex. Moreover, for every straight line L in S, the set $V_k \cap L$ is closed relative to X by continuity of f_L. These two properties together imply that V_k is closed relative to X. To see this, consider a point x_0 of X adherent to V_k.

Select a point x_1 in the relative interior of V_k. If $x_1 = x_0$, then x_0 is in V_k. If $x_1 \neq x_0$, then every point of the straight line segment $[x_1, x_0[$ is in V_k (Eggleston (1969, pp. 9-11)). Since the intersection of the straight line through x_1 and x_0 with V_k is closed relative to X, the point x_0 is in V_k.

(ii) f is upper semi-continous.

For this one has to show that for every real number k, the set

$$U_k = \{x \in X \mid f(x) < k\}$$

is open. By quasi-convexity of f, the set U_k is convex. Moreover, for every straight line L in S, the set $U_k \cap L$ is open relative to L by continuity of f_L. These two properties together imply that U_k is open. To see this, assume that U_k is not empty and consider a point x_0 of U_k. Let $\{e_1, \ldots, e_m\}$ be a basis of S. For every $i = 1, \ldots, m$, there is $t_i > 0$ such that $x_0 - t_i e_i$ and $x_0 + t_i e_i$ belong to U_k. The convex hull of the 2m points obtained in this manner is a neighborhood of x_0, and is contained in the convex set U_k.

Hence f is continuous, Q.E.D.

We also note

Theorem 10. Under A.3, if f is not convex, then g has property (P) of Theorem 2.

Proof: If f is not convex, there is a straight line L in S such that the restriction f_L of f to $X \cap L$ is not convex. Let then y_1, y_2 be two points of Y such that $g(y_1) \neq g(y_2)$, and denote by M the straight line through y_1 and y_2, and by g_M the restriction of g to $Y \cap M$. By Theorem 2, the function g_M has property (P), Q.E.D.

For every $i = 1, \ldots, m \geq 2$, let now S_i be a finite-dimensional real vector space, X_i be an open convex subset of S_i, f_i be a non-constant real-valued function on X_i, and assume that the function $f: \bigtimes_{i=1}^{m} X_i \to R$ defined by $x \mapsto f(x) = \sum_{i=1}^{m} f_i(x_i)$ is quasi-convex. By Theorem 9, every function f_i is continuous, and by Theorem 10, if a certain function f_j is not convex, then for every $i \neq j$, the function f_i has property (P) of Theorem 2.

Our next purpose is to extend the definition of the convexity index given in Section 2. Let K be a non-empty convex subset of a finite-dimensional real vector space, and G be a real-valued function on K such that for every straight line D intersecting K, the restriction G_D of G to D belongs to the class $\mathscr{C}$ defined at the end of Section 2. By definition, the convexity index $c(G)$ of G is

$$c(G) = \operatorname*{Inf}_{D} c(G_D) ,$$

where the Inf is defined over the set of straight lines intersecting K.

We note that f and g satisfy all the conditions imposed on G.

Let L be a straight line in S intersecting X and such that f_L is not constant. Let M be a straight line in T intersecting Y and such that g_M is not constant. According to Theorem 6, $c(f_L) + c(g_M) \geq 0$. Therefore $c(f)$ and $c(g)$ are both finite and satisfy

$$c(f) + c(g) \geq 0 .$$

<u>Theorem 11</u>. <u>Assume that A.3 holds, and that f and g are C^2. If $c(f) + c(g) > 0$, then $\frac{1}{c(F)} = \frac{1}{c(f)} + \frac{1}{c(g)}$. If $c(f) + c(g) = 0$ and $c(f) \cdot c(g) \neq 0$, then $c(F) = -\infty$. If $c(f) = 0 = c(g)$, then $c(F) = 0$.</u>

In the case covered by the second sentence, if one of the two convexity indices, say $c(f)$, vanishes, the other is strictly positive. Thus $\frac{1}{c(f)}$

is infinite while $\frac{1}{c(g)}$ is finite. Therefore $\frac{1}{c(F)}$ is infinite and $c(F) = 0$.

Proof of Theorem 11. We assume at first that X and Y are open real intervals and f and g are strictly increasing. Consider an arbitrary point $(x_0,y_0) \in X \times Y$ and an arbitrary straight line D through (x_0,y_0) with slope σ . If σ is finite, the restriction F_D of F to D satifies $F_D(x,y) = f(x) + g[y_0 + \sigma(x-x_0)]$ which we denote by $t(x;\ x_0,y_0,\sigma)$. If σ is infinite, the restriction F_D of F to D satisfies $F_D(x,y) = f(x_0) + g(y)$. Since t is C^2, by Theorem 8 we have if σ is finite and (x_0,y_0,σ) are such that $t'(x_0;\ x_0,y_0,\sigma) \neq 0$,

$$c(F) = \operatorname*{Inf}_{x_0,y_0,\sigma} \frac{t''(x_0;\ x_0,y_0,\sigma)}{(t'(x_0;\ x_0,y_0,\sigma))^2} .$$

To cover the case in which σ is infinite, it is sufficient to check that

$$\operatorname*{Inf}_{\sigma \text{ finite}} \frac{t''(x_0;\ x_0,y_0,\sigma)}{(t'(x_0;\ x_0,y_0,\sigma))^2} \leq \frac{g''(y_0)}{(g'(y_0))^2}$$

which we will do below in every case.

Now

$$t'(x_0;\ x_0,y_0,\sigma) = f'(x_0) + \sigma g'(y_0) \quad \text{and}$$

$$t''(x_0;\ x_0,y_0,\sigma) = f''(x_0) + \sigma^2 g''(y_0) .$$

To lighten notation we let

$$\gamma(x_0,y_0,\sigma) = \frac{t''(x_0;\ x_0,y_0,\sigma)}{(t'(x_0;\ x_0,y_0,\sigma))^2} , \quad a' = f'(x_0) , \quad a'' = f''(x_0) ,$$

$$b' = g'(y_0) , \quad b'' = g''(y_0) .$$

Thus

(10) if $b'\sigma + a' \neq 0$, $\gamma(x_0,y_0,\sigma) = \dfrac{b''\sigma^2 + a''}{(b'\sigma + a')^2}$.

For a given point (x_0,y_0), we seek the infimum of this ratio as σ varies, recalling that by Theorem 3, $a'>0$ and $b'>0$, and by Lemma 3, $\frac{a''}{(a')^2}+\frac{b''}{(b')^2} \geq 0$.

<u>The case in which</u> $\frac{a''}{(a')^2}+\frac{b''}{(b')^2} > 0$.

A routine study of the function $\sigma \mapsto \frac{b''\sigma^2+a''}{(b'\sigma+a')^2}$ shows that its minimum is $\dfrac{\frac{a''}{(a')^2}\cdot\frac{b''}{(b')^2}}{\frac{a''}{(a')^2}+\frac{b''}{(b')^2}}$. Moreover this number is indeed $\leq \frac{b''}{(b')^2}$.

<u>The case in which</u> $\frac{a''}{(a')^2}+\frac{b''}{(b')^2} = 0$.

If $a'' \neq 0 \neq b''$, then $\frac{b''\sigma^2+a''}{(b'\sigma+a')^2} = \frac{b''}{(b')^2}\cdot\frac{b'\sigma-a'}{b'\sigma+a'}$.

Letting σ tend to $-\frac{a'}{b'}$ on the right if $b''>0$, on the left if $b''<0$, one obtains $\operatorname*{Inf}_{\sigma} \frac{b''\sigma^2+a''}{(b'\sigma+a')^2} = -\infty$.

If $a''=0=b''$, then for every σ, $\frac{b''\sigma^2+a''}{(b'\sigma+a')^2}=0$. Therefore $\operatorname*{Min}_{\sigma} \frac{b''\sigma^2+a''}{(b'\sigma+a')^2} = 0$, which is indeed $\leq \frac{b''}{(b')^2} = 0$.

In the following we appeal to the properties of the function

$$(u,v) \mapsto e(u,v) = \frac{uv}{u+v}$$

<u>defined on</u> $H = \{(u,v) \in R^2 \mid u+v>0\}$.

The function e is continuous and strictly increasing in u and in v. Moreover let $(u_0,v_0) \in \partial H$. If $u_0 \neq 0 \neq v_0$, then as (u,v) in H tends to (u_0,v_0), $e(u,v) \to -\infty$. If $u_0 = 0 = v_0$, then as u tends to 0 on the right and v tends to 0 on the right in such a way that $(u,v) \in H$, one has $e(u,v) \to 0$ because $e(u,v) < u+v$.

We now distinguish two cases.

$\underline{c(f) + c(g) > 0}$.

For any $(x,y) \in X \times Y$, one has $\frac{f''(x)}{(f'(x))^2} \geqq c(f)$ and $\frac{g''(y)}{(g'(y))^2} \geqq c(g)$, hence $\frac{f''(x)}{(f'(x))^2} + \frac{g''(y)}{(g'(y))^2} > 0$. According to the above $\operatorname*{Inf}_{\sigma} \gamma(x,y,\sigma) = \dfrac{\frac{f''(x)}{(f'(x))^2} \cdot \frac{g''(y)}{(g'(y))^2}}{\frac{f''(x)}{(f'(x))^2} + \frac{g''(y)}{(g'(y))^2}}$. By continuity and monotony of the function e , $c(F) = \operatorname*{Inf}_{x,y} \operatorname*{Inf}_{\sigma} \gamma(x,y,\sigma)$

$$= \frac{\operatorname*{Inf}_{x} \frac{f''(x)}{(f'(x))^2} \cdot \operatorname*{Inf}_{y} \frac{g''(y)}{(g'(y))^2}}{\operatorname*{Inf}_{x} \frac{f''(x)}{(f'(x))^2} + \operatorname*{Inf}_{y} \frac{g''(y)}{(g'(y))^2}} = \frac{c(f) \cdot c(g)}{c(f) + c(g)} .$$

$\underline{c(f) + c(g) = 0}$.

Two subcases have to be considered.

(i) $\underline{c(f) \neq 0 \neq c(g)}$. There is a sequence $(x_n,y_n) \in X \times Y$ such that $\frac{f''(x_n)}{(f'(x_n))^2}$ tends to $c(f)$, and $\frac{g''(y_n)}{(g'(y_n))^2}$ tends to $c(g)$. If for some n ,

$$\frac{f''(x_n)}{(f'(x_n))^2} + \frac{g''(y_n)}{(g'(y_n))^2} = 0 \text{ , then } \frac{f''(x_n)}{(f'(x_n))^2} = c(f) \text{ and } \frac{g''(y_n)}{(g'(y_n))^2} = c(g) .$$

According to the above, $\operatorname*{Inf}_{\sigma} \gamma(x_n,y_n,\sigma) = -\infty$. Hence $c(F) = -\infty$. If for every n , $\frac{f''(x_n)}{(f'(x_n))^2} + \frac{g''(y_n)}{(g'(y_n))^2} > 0$, then according to the above,

$$\operatorname*{Inf}_{\sigma} \gamma(x_n,y_n,\sigma) = \frac{\frac{f''(x_n)}{(f'(x_n))^2} \cdot \frac{g''(y_n)}{(g'(y_n))^2}}{\frac{f''(x_n)}{(f'(x_n))^2} + \frac{g''(y_n)}{(g'(y_n))^2}}$$

. Moreover as $n \to +\infty$, the

last ratio tends to $-\infty$. Hence again $c(F) = -\infty$. In summary, in subcase (i), $c(F) = -\infty$.

(ii) <u>$c(f) = 0 = c(g)$</u>. Consider an arbitrary $(x,y) \in X \times Y$. One has $f''(x) \geqq 0$ and $g''(y) \geqq 0$. According to (10), for every σ, one has $\gamma(x,y,\sigma) \geqq 0$. Therefore $c(F) \geqq 0$. Moreover there is a sequence $(x_n,y_n) \in X \times Y$ such that $\frac{f''(x_n)}{(f'(x_n))^2}$ tends to 0, and $\frac{g''(y_n)}{(g'(y_n))^2}$ tends to 0. If for some n, $\frac{f''(x_n)}{(f'(x_n))^2} + \frac{g''(y_n)}{(g'(y_n))^2} = 0$, then $f''(x_n) = 0 = g''(y_n)$, and by (9), $\gamma(x_n,y_n,\sigma) = 0$ for every σ. Hence $c(F) = 0$. If for every n,

$$\frac{f''(x_n)}{(f'(x_n))^2} + \frac{g''(y_n)}{(g'(y_n))^2} > 0 \text{ , then } \operatorname*{Inf}_{\sigma} \gamma(x_n,y_n,\sigma) = \frac{\frac{f''(x_n)}{(f'(x_n))^2} \cdot \frac{g''(y_n)}{(g'(y_n))^2}}{\frac{f''(x_n)}{(f'(x_n))^2} + \frac{g''(y_n)}{(g'(y_n))^2}} .$$

When $n \to +\infty$, the last ratio tends to 0. Hence again $c(F) = 0$. In summary, in subcase (ii), $c(F) = 0$.

This concludes the proof in the case in which X and Y are open real intervals and f and g are strictly increasing. The preceding results hold as well if f and g are no longer restricted to be strictly increasing. In that case f and g belong to the class $\mathscr{C}$ defined at the end of Section 2 and the definition of the convexity indices $c(f)$ and $c(g)$ makes the extension immediate.

Finally we consider the case in which X and Y are open convex subsets of finite-dimensional real vector spaces. In the remainder of this proof, L denotes a straight line in S intersecting X; M denotes a straight line in T intersecting Y; K denotes the product $L \times M$;

F_K denotes the restriction of F to K ; L_n denotes a sequence such that $c(f_{L_n})$ tends to $c(f)$; M_n denotes a sequence such that $c(g_{M_n})$ tends to $c(g)$.

As we noted before stating Theorem 11, one has $c(f) + c(g) \geq 0$.

The case in which $c(f) + c(g) > 0$.

Consider a straight line D in $S \times T$ intersecting $X \times Y$. There are L and M such that $D \subset K$. One has $c(f_L) + c(g_M) > 0$, hence $c(F_D) \geq c(F_K) = \frac{c(f_L) \cdot c(g_M)}{c(f_L) + c(g_M)} \geq \frac{c(f)\,c(g)}{c(f) + c(g)}$. Consequently $c(F) \geq \frac{c(f) \cdot c(g)}{c(f) + c(g)}$.

Consider now the sequences L_n , M_n , and K_n . One has $c(F_{K_n}) = \frac{c(f_{L_n}) \cdot c(g_{M_n})}{c(f_{L_n}) + c(g_{M_n})}$. As $n \to +\infty$, $c(F_{K_n})$ tends to $\frac{c(f) \cdot c(g)}{c(f) + c(g)}$. However $c(F) \leq c(F_{K_n})$. Hence $c(F) = \frac{c(f) \cdot c(g)}{c(f) + c(g)}$.

The case in which $c(f) + c(g) = 0$.

Two subcases have to be considered.

(i) *$c(f) \neq 0 \neq c(g)$* . Let L_n , M_n , K_n be as above. If for some n , $c(f_{L_n}) + c(g_{M_n}) = 0$, then $c(f_{L_n}) = c(f)$ and $c(g_{M_n}) = c(g)$. Consequently $c(F_{K_n}) = -\infty$. Hence $c(F) = -\infty$. If for every n , $c(f_{L_n}) + c(g_{M_n}) > 0$, then $c(F_{K_n}) = \frac{c(f_{L_n}) \cdot c(g_{M_n})}{c(f_{L_n}) + c(g_{M_n})}$. As $n \to +\infty$, $c(F_{K_n})$ tends to $-\infty$. Hence again $c(F) = -\infty$.

(ii) *$c(f) = 0 = c(g)$* . Let D be a straight line in $S \times T$ intersecting $X \times Y$. There are L and M such that $D \subset K$. If $c(f_L) + c(g_M) > 0$, then $c(F_K) = \dfrac{c(f_L)\cdot c(g_M)}{c(f_L)+c(g_M)} \geqq 0$. If $c(f_L) + c(g_M) = 0$, then $c(f_L) = 0 = c(g_M)$ and $c(F_K) = 0$. In either case, $c(F_D) \geqq c(F_K) \geqq 0$. Consequently $c(F) \geqq 0$.

Let now L_n , M_n , K_n be as above. If for some n , $c(f_{L_n}) + c(g_{M_n}) = 0$, then $c(f_{L_n}) = 0 = c(g_{M_n})$. Consequently $c(F_{K_n}) = 0$. Hence $c(F) = 0$. If for every n , $c(f_{L_n}) + c(g_{M_n}) > 0$, then $c(F_{K_n}) = \dfrac{c(f_{L_n})\cdot c(g_{M_n})}{c(f_{L_n})+c(g_{M_n})}$. As $n \to +\infty$, $c(F_{K_n})$ tends to 0 . Hence again $c(F) = 0$, Q.E.D.

For every $i = 1,\ldots,m \geqq 2$, let S_i be a finite-dimensional real vector space, X_i be an open convex subset of S_i , f_i be a C^2 non-constant, real-valued function on X_i such that the restriction of f_i to every straight line intersecting X_i belongs to the class $\mathscr{C}$. Define the function $f\colon \bigtimes_{i=1}^{m} X_i \to R$ by $x \mapsto f(x) = \sum_{i=1}^{m} f_i(x_i)$. Theorem 11 permits a complete characterization of the functions f that are quasi-convex.

Consider first the case of two factors S_1 and S_2 . As we noted before stating Theorem 11, if the function f is quasi-convex, one has $c(f_1) + c(f_2) \geqq 0$. Conversely assume that $c(f_1) + c(f_2) \geqq 0$. For $i = 1,2$, let L_i be a straight line in S_i intersecting X_i . Denote by $f_{i\cdot L_i}$ the restriction of f_i to L_i . One has $c(f_{1\cdot L_1}) + c(f_{2\cdot L_2}) \geqq 0$, and by Theorem 6, the restriction of f to $L_1 \times L_2$ is quasi-convex. Let D be a straight line in $S_1 \times S_2$ intersecting $X_1 \times X_2$. Choose L_1 and L_2 such that $D \subset L_1 \times L_2$. The restriction of f to D is quasi-convex. Therefore f is quasi-convex. Summing up, f is quasi-convex if and only if $c(f_1) + c(f_2) \geqq 0$.

Consider now an arbitrary number $m \geqq 2$ of factors. Two cases must be distinguished.

(a) If f is convex (equivalently if $c(f) \geqq 0$), then for every i, f_i is convex, and $c(f_i) \geqq 0$. Conversely if for every i, $c(f_i) \geqq 0$, then every f_i is convex, and f is convex.

(b) If f is quasi-convex but not convex (hence $c(f) < 0$), then there is j for which f_j is not convex, hence $c(f_j) < 0$. For every $i \neq j$, f_i is convex, hence $c(f_i) \geqq 0$. Let $I = \{1, \ldots, m\} \setminus \{j\}$, and f_I be the function $\bigtimes_{i \in I} X_i \to R$ defined by $x \mapsto f_I(x) = \sum_{i \in I} f_i(x_i)$. According to the preceding paragraph $c(f_j) + c(f_I) \geqq 0$. Therefore $c(f_I) > 0$. Consequently by Theorem 11, $c(f_i) > 0$ for every $i \in I$, and $\frac{1}{c(f_I)} = \sum_{i \in I} \frac{1}{c(f_i)}$. However $c(f_I) \geqq -c(f_j)$ implies $\frac{1}{c(f_I)} \leqq -\frac{1}{c(f_j)}$. Hence $\frac{1}{c(f)} = \sum_{i=1}^{m} \frac{1}{c(f_i)} \leqq 0$.

Conversely assume that for some j, $c(f_j) < 0$; for every $i \neq j$, $c(f_i) > 0$; and $\sum_{i=1}^{m} \frac{1}{c(f_i)} \leqq 0$. Then $\frac{1}{c(f_I)} = \sum_{i \in I} \frac{1}{c(f_i)}$, and $\frac{1}{c(f_I)} + \frac{1}{c(f_j)} \leqq 0$. Hence $c(f_I) + c(f_j) \geqq 0$, and f is quasi-convex.

Summing up, f is quasi-convex if and only if either (1) for every i, $c(f_i) \geqq 0$, or (2) for some j, $c(f_j) < 0$; for every $i \neq j$, $c(f_i) > 0$; and $\sum_{i=1}^{m} \frac{1}{c(f_i)} \leqq 0$.

REFERENCES

Arrow, K. J., 1953, "Le rôle des valeurs boursières pour la répartition la meilleure des risques," Econométrie, Paris, Centre National de la Recherche Scientifique, 41-48. Translated as "The Role of Securities in the Optimal Allocation of Risk-Bearing," Review of Economic Studies, 31, 1964, 91-96.

Arrow, K. J., and Enthoven, A. C., 1961, "Quasi-Concave Programming," Econometrica, 29, 779-800.

Blackorby, C., Davidson, R., and Donaldson, D., 1977, "A Homiletic Exposition of the Expected Utility Hypothesis," Economica, 44, 351-358.

Cox, J. C., 1973, "A Theorem on Additively-Separable Quasi-Concave Functions," Journal of Economic Theory, 6, 210-212.

Crouzeix, J. P., 1977, "Conjugacy in Quasi-Convex Analysis," in Convex Analysis and Its Applications, A. Auslender, ed., Springer-Verlag, Berlin, 66-99.

Debreu, G., 1960, "Topological Methods in Cardinal Utility Theory," in Mathematical Methods in the Social Sciences, 1959, K.J. Arrow, S. Karlin, and P. Suppes, eds., Stanford University Press, 16-26.

Dierker, E., Fourgeaud, C., and Neuefeind, W., 1976, "Increasing Returns to Scale and Productive Systems," Journal of Economic Theory, 13, 428-438.

Eggleston, H. G., 1969, Convexity, Cambridge University Press.

Fenchel, W., 1953, "Convex Cones, Sets, and Functions," Lecture Notes, Department of Mathematics, Princeton University.

Fourgeaud, C., Lenclud, B., and Sentis, P., 1974, "Equilibre, Optimum et Décentralisation dans un Cas de Rendement Croissant," Cahiers du Séminaire d'Econométrie, Centre National de la Recherche Scientifique, Paris, 15, 29-46.

Gorman, W. M., 1970, "The Concavity of Additive Utility Functions," Research Memorandum, The University of North Carolina.

Green, J., 1961, "Direct Additivity and Consumers' Behaviour," Oxford Economic Papers, 13, 132-136.

Greenberg, H. J., and Pierskalla, W. P., 1971, "A Review of Quasi-Convex Functions," Operations Research, 19, 1553-1570.

Hirschleifer, J., 1965, "Investment Decision under Uncertainty: Choice-Theoretic Approaches," Quarterly Journal of Economics, 79, 509-536.

Kelley, J. L., 1955, General Topology, Van Nostrand Reinhold, New York.

Koopmans, T. C., 1971, "If $f(x) + g(y)$ is Quasi-Convex, at Least One of $f(x), g(y)$ is Convex," Mimeographed, Yale University.

Laffont, J. J., 1972, "Une Note sur la Compatibilité entre Rendements Croissants et Concurrence Parfaite," Revue d'Economie Politique, 82, 1188-1193.

Rader, T., 1972, Theory of Microeconomics, Academic Press, New York.

Riesz, F., and Sz.-Nagy, B., 1955, Functional Analysis, Ungar, New York.

Roberts, A. W., and Varberg, D. E., 1973, Convex Functions, Academic Press, New York.

Shephard, R. W., 1953, Cost and Production Functions, Princeton University Press.

Slutsky, E. E., 1915, Sulla teoria del bilancio del consumatore, Giornale degli economisti, 51, 1-26. Translated as "On the Theory of the Budget of the Consumer," in Readings in Price Theory, American Economic Association, 1952, Richard Irwin, 27-56.

Thorlund-Petersen, L., 1980, "Concave, Additive Utility Representations and Fair Division of a Random Income," Institute of Economics, University of Copenhagen.

Yaari, M. E., 1977, "A Note on Separability and Quasi-concavity," Econometrica, 45, 1183-1186.

Yaari, M. E., 1978, "Separably Concave Utilities or the Principle of Diminishing Eagerness to Trade," Journal of Economic Theory, 18, 102-118.