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A NECESSARY AND SUFFICIENT CONDITION FOR THE NONEMPTINESS

OF THE CORES OF PARTITIONING GAMES

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by

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Abstract: Let N be a finite set of players and let π be a class of coalitions of N . We consider games with and without sidepayments such that only the coalitions in π play essential roles but not the others. For an arbitrary π , we get the class of all such games. The purpose of this note is to provide a necessary and sufficient condition with respect to π for the nonemptiness of the cores of all games in the class.

1. Introduction

In situations of n-person cooperatives games, it is not necessarily equally easy to form every coalition. For example, it might be very hard to form a large coalition because of coalition formation costs. That is, it might often happen that only some coalitions play essential roles but not the others. Myerson [7] considered such a situation from a graph theoretical viewpoint. If a situation, however, has some special structures and even if all coalitions are permitted, it happens that only some coalitions play essential roles, e.g., the assignment game and the assignment market of Shapley and Shubik [10] and Kaneko [4, 5].

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This paper concerns the nonemptiness of the cores of games of such situations. We define games with and without sidepayments called "partitioning games," which are generalizations of the assignment game of [10] and the central assignment game of [5]. The main result of this note is a necessary and sufficient condition for the nonemptiness of the cores of a class of partitioning games.

2. The Partitioning Games with and without Sidepayments

Initially we provide partitioning games with sidepayments. Let N be an arbitrary finite number of players. Let $N = \{1, 2, \ldots, n\}$. Let π be a class of nonempty coalitions satisfying $\{i\} \in \pi$ for all $i \in N$. We call S in π a $\underline{basic}$ $\underline{coalition}$. For any nonempty $S \subset N$, we call $P_S = \{T_1, \ldots, T_k\}$ a π-$\underline{partition}$ of S iff

(1) $T_t \in \pi$ for all $1, \ldots, k$ and P_S is a partition of S.

Let $P(S)$ be the set of all π-partitions of S. We call a game in characteristic function form (N,v) a $\underline{partitioning}$ $\underline{game}$ $\underline{with}$ $\underline{sidepayments}$ iff it satisfies

(2) $v(S) = \max_{P_S \in P(S)} \sum_{T \in P_S} v(T)$ for all nonempty $S \subset N$.

Note that v satisfies the super-additivity.

The basic idea is very simple. That is, only the basic coalitions can play essential roles in a partitioning game. A typical example is the assignment game of Shapley and Shubik [10].

The $\underline{core}$ of a $\underline{game}$ (N,v) $\underline{with}$ $\underline{sidepayments}$ is the set

$\{x \in R^n : \sum_{i \in N} x_i = v(N)$ & $\sum_{i \in S} x_i \geq v(S)$ for all nonempty $S \subset N\}$, where

R^n is the n-dimensional Euclidean space. The following lemma ensures that this definition does not contradict our initial motivation mentioned in Section 1.

Lemma 1. Let (N,v) be a partitioning game with sidepayments. Then the core coincides with the set $\{x \in R^n : \sum_{i \in N} x_i = v(N) \ \& \ \sum_{i \in T} x_i \geq v(T)$ for all $T \in \pi\}$.

Proof. Obvious.

The definition (2) is rewritten in terms of an integer programming as follows: for all nonempty $S \subset N$,

$$(3) \qquad v(S) = \max_{\substack{T \in \pi \\ T \subset S}} \sum X_T v(T)$$

$$\text{subject to} \quad \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \quad \text{for all } i \in S \quad \text{and} \quad X_T = 0 \quad \text{or } 1$$
$$\text{for all } T \in \pi \text{ with } T \subset S ,$$

where $(X_T)_{\substack{T \in \pi \\ T \subset S}}$ is a variable. This integer programming is usually called a set partitioning problem. See Balas and Padberg [1] and Murty [6]. This representation will play an important role in considering the nonemptiness of the core of it.

For any N and π , we denote, by $GS(N,\pi)$, the set of all partitioning games with sidepayments which have the set of players N and the set of basic coalitions π .

Next let us define partitioning games without sidepayments. Let N and π be given. A partitioning game without sidepayments (N,V) is a function from the set of all nonempty coalitions to a class of subsets of R^n such that all nonempty $S \subset N$,

(4) $V(S)$ is a closed set in R^n ,

(5)   if $x \in V(S)$ and $y \in R^n$ with $y_i \leq x_i$ for all $i \in S$,

           then $y \in V(S)$,

(6) $\text{Pro}_S[V(S) - \bigcup_{i \in S} \text{interior } V(\{i\})]$ is nonempty and bounded,[1]

(7) $V(S) = \bigcup_{P_S \in P(S)} \bigcap_{T \in P_S} V(T)$.

Assumptions (4)-(6) are the standard technical conditions. Assumption (7) means that when a coalition is formed, the players in S subdivide S into a π-partition and get payoffs guaranteed by the basic coalitions. This idea is almost the same as that of partitioning game with sidepayments. A typical example is the central assignment game of Kaneko [5].

The <u>core</u> <u>of</u> <u>a</u> <u>game</u> <u>without</u> <u>sidepayments</u> (N,V) is the set $V(N) - \bigcup_{\substack{S \subset N \\ S \neq \phi}} \text{interior } V(S)$. Parallel to Lemma 1, the following lemma holds.

<u>Lemma 2</u>. Let (N,V) be a partitioning game without sidepayments. Then the core coincides with $V(N) - \bigcup_{S \in \pi} \text{interior } V(S)$.

<u>Proof</u>. Obvious.

There is, however, a minor conceptual difference between the two above games, i.e., the partitioning game with and without sidepayments. In a partitioning game with sidepayments, it is permitted to transfer money (transferable utility) in every coalition, but in a partitioning game without sidepayments, it is only permitted to transfer something

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[1] $\text{Pro}_S X = \{(x_i)_{i \in S} : x \in X\}$ for $S \subset N$ and $X \subset R^n$.

in every basic coalition. This difference appears as follows. A game with sidepayments (N,v) can be represented as a game without sidepayments $(N,\overline{v})$ such that $\overline{v}(S) = \{x \in R^n : \sum_{i \in S} x_i \leq v(S)\}$ for all nonempty $S \subset N$. Even if (N,v) is a partitioning game with sidepayments, $(N,\overline{v})$ is not a partitioning game without sidepayments, i.e., it does not satisfy (7). But as far as we consider the core, any difficulty does not appear. Let us consider another game without sidepayments (N, V_v) such that

$$(8) \qquad V_v(S) = \bigcup_{P_S \in P(S)} \bigcap_{T \in P_S} \overline{v}(S) \quad \text{for all nonempty } S \subset N .$$

Of course, (N, V_v) is a partitioning game without sidepayments. Then the following lemma holds.

Lemma 3. Let (N,v) be a partitioning game with sidepayments. Then the core of (N,v) coincides with the core of (N, V_v) .

Proof. Obvious.

For any N and π , we denote, by $G(N,\pi)$, the set of all partitioning games without sidepayments which have the set of players N and the set of basic coalitions π . Embedding $GS(N,\pi)$ into $G(N,\pi)$ by the mapping (8): $v \mapsto V_v$, we can regard $GS(N,\pi)$ as a subset of $G(N,\pi)$.

We need another concept to state the main result of this note. Let us consider the following system of equations:

$$(9) \qquad \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \quad \text{for all } i \in N \text{ and } X_T \geq 0 \quad \text{for all } T \in \pi ,$$

where $(X_T)_{T \in \pi}$ is a variable. We say that the system of equations

(9) has the _integral_ _property_ iff every extreme solution of (9) consists

of integers. If (9) has the integral property, there exists a one-to-

one onto mapping from the set of all π-partitions of N to the set of

all extreme solutions of (9). The extreme solutions also coincide with

the feasible solutions of (3) in the case of S = N .

The main result of this note is the following theorem.

Theorem. The following three statements are equivalent:

    (i)   The system of equations (9) has the integral property.

    (ii) The core of (N,V) is nonempty for all (N,V) in $G(N,\pi)$.

    (iii) The core of (N,v) is nonempty for all (N,v) in $GS(N,\pi)$.

3. Remarks

3.1. Let us represent the system of equations (9) as the matrix form,

i.e., $AX = e \ \& \ X \geqq 0$, where $X = (X_T)_{T \in \pi}$ and e is the vector every

component of which equals 1. A sufficient condition for (9) to have

the integral property is the unimodular property of A , i.e., every

minor determinant of A equals 0, 1 or -1. (Hoffman and Kruskal [2,

Theorem 2].) Hoffman and Kruskal gave also several necessary and suf-

ficient conditions and more convenient sufficient conditions for the

unimodular property.

Let us consider one necessary and sufficient condition of [2].

For more details, see [2]. Let G = (N,E) be an oriented graph (i.e.,

N = {1, ..., n} is the set of vertices and E is the set of edges)

(a) which has no circular edges, (b) which has at most one edge between

any two given vertices and (c) in which each edge has an orientation.

We call (i_1, i_2) a <u>direct</u> (<u>inverse</u>) edge iff $(i_1, i_2) \epsilon E$

$((i_2, i_1) \epsilon E)$. A <u>path</u> is a sequence of distinct vertices

$i_1, i_2, \ldots, i_k$, such that for each t $(1 \leq t \leq k-1)$, (i_t, i_{t+1})

is either a direct or an inverse edge. A path is <u>directed</u> iff every

edge is oriented forward. A path is <u>alternating</u> iff successive edges

are oppositely oriented. A <u>loop</u> is a path which closes back on itself.

A graph is <u>alternating</u> iff every loop in it is alternating.

Let $\pi = \{T_1, \ldots, T_k\}$ be some set of directed path in G .
Then the incidence matrix $A = (a_{it})_{\substack{i \epsilon N \\ t=1,\ldots,k}}$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is in } T_j \\ 0 & \text{otherwise.} \end{cases}$$

Then it holds:

<u>Theorem (Hoffman and Kruskal [2, Theorem 4])</u>: For A to have a uni-
modular property, it is sufficient that G be alternating. If π
consists of the set of all directed paths of G , then for A to have
the unimodular property it is necessary and sufficient that G be
alternating.

It is easily seen that this theorem can be directly applicable
to the partitioning games. For example, let us consider the graph drawn
in Figure 1 and let π be some set of directed paths with $\{i\} \epsilon \pi$
for all $i \epsilon N$.[2] Then we can construct the classes $G(N,\pi)$ and $GS(N,\pi)$.
Since the graph is alternating, the core of every game in $G(N,\pi)$ is
nonempty by the above theorems.

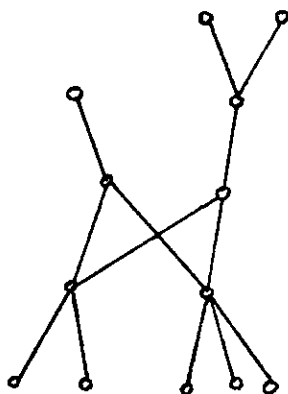[2]Every vertex is a path with length 0.

Figure 1. An Alternating Graph (arrows omitted—
all should be upward)

3.2. There are a lot of papers on the set partitioning problem. Some

of them concerned the integral property of (9) and gave preciser con-

ditions for it. See Balas and Padberg [1].

4. Proof of Theorem

We prove that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i). Since $GS(N,\pi)$

is a subset of $G(N,\pi)$, (ii) $\implies$ (iii) is trivial.

4.1. Proof of (i) $\implies$ (ii): Let (N,V) be an arbitrary game in $G(N,\pi)$.

Due to Scarf's [8] fundamental theorem, it is sufficient to show that

(N,V) is a balanced game under the assumption (i).[3,4]

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[3]Let us call a family γ of nonempty coalitions of N balanced iff
the system of equations

$$\sum_{S:S \ni j} \delta_S = 1 \quad \text{for all} \quad j \in N ,$$

has a nonnegative solution with $\delta_S = 0$ for all $S \notin \gamma$. The numbers
$\{\delta_S\}$ are called balanced weights for γ. A game (N,V) without side-
payments is said to be balanced iff the following inclusion statement:

$$\bigcap_{S \in \gamma} V(S) \subset V(N) .$$

For any nonempty coalition S , we define a zero-one matrix

$$A_S = (a_{S:iT})_{\substack{i\epsilon N \\ T\epsilon\pi}} \quad \text{by}$$

(10)
$$\sum_{i\epsilon N} a_{S:iT} = \begin{cases} |T| \ \text{ or } 0 & \text{if } T \subset S \\ 0 & \text{if } T \not\subset S \end{cases}$$

$$\sum_{T\epsilon\pi} a_{S:iT} = \begin{cases} 1 & \text{if } i \epsilon S \\ 0 & \text{if } i \notin S \end{cases}$$

$$a_{S:iT} = 0 \quad \text{if } i \notin T ,$$

where $|T|$ is the number of members in T . We call A_S a _partition matrix_ of S . We define $D_S(b) = (d_{S:iT}(b))_{\substack{i\epsilon N \\ T\epsilon\pi}}$ for each $b \epsilon R^n$ and nonempty coalition S by

(11)
$$d_{S:iT}(b) = \begin{cases} 1 & \text{if } i \epsilon S , \ i \epsilon T \text{ and } b \epsilon V(T) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that p_S is a π-partition of S if and only if $P_S = \{T : a_{S:iT} = 1 \text{ for some } i \epsilon S\}$ for some partition matrix A_S of S . Hence we get the following lemma.

Lemma 4. $V(S) = \{b \epsilon R^n : D_S(b) \geq A_S \text{ for a partition matrix } A_S \text{ of } S\}$ for all nonempty $S \subset N$.

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holds for all balanced families γ . The fundamental theorem of Scarf [8] states that the core of a balanced game with (4), (5) and (6) is nonempty.

[4]This proof is a modification of the proof of the main theorem of Shapley and Scarf [9]. Regretfully, however, their game is not a partitioning game and so, our theorem does not cover their existence theorem.

Let us consider another system of inequalities:

(12)
$$\sum_{i \in N} X_{iT} \leqq |T| \quad \text{for all} \quad T \in \pi$$

$$\sum_{T \in \pi} X_{iT} = 1 \quad \text{for all} \quad i \in N$$

$$X_{iT} \geqq 0 \quad \text{for all} \quad i \in N \ \& \ T \in \pi, \quad X_{iT} = X_{jT} \quad \text{if}$$
$$i, j \in T \quad \text{and} \quad X_{iT} = 0 \quad \text{if} \quad i \notin T,$$

where $(X_{iT})_{\substack{i \in N \\ T \in \pi}}$ is a variable.

Lemma 5. There exists a one-to-one onto and linear mapping f from the set of all solutions of (9) to the set of all solutions of (12) such that to any $(X_T)_{T \in \pi}$ satisfying (9), f assigns $(X_{iT})_{\substack{i \in N \\ T \in \pi}}$ such that

(13)
$$X_{iT} = \begin{cases} X_T & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is easily verified that the mapping defined by (13) from the set of all solutions of (9) to that of all solutions of (12) is a one-to-one and linear mapping. Conversely we define the mapping which assigns to each $(X_{iT})_{\substack{i \in N \\ T \in \pi}}$ satisfying (12) $(X_T)_{T \in \pi}$ such that for all $T \in \pi$

(14)
$$X_T = X_{iT} \quad \text{for some} \quad i \in T.$$

It is easily verified that this mapping is the inverse mapping of the mapping satisfying (13).   Q.E.D.

<u>Proof of the Balancedness</u>: Let γ be an arbitrary balanced family of coalitions and let $b \in \bigcap_{S \in \gamma} V(S)$. Let $\{\delta_S\}$ be balanced weights for γ . Then it holds that

$$D_N(b) = \sum_{S \in \gamma} \delta_S D_S(b) \ .$$

For, if $d_{N:iT}(b) = 1$, then $d_{S:iT}(b) = 1$ if $i \in S$ and $d_{S:iT}(b) = 0$ if $i \notin S$ by (11), which implies $\sum_{S \in \gamma} \delta_S d_{S:iT}(b) = \sum_{\substack{S \in \gamma \\ S \ni i}} \delta_S d_{S:iT}(b) = 1$,

and if $d_{N:iT}(b) = 0$, then $d_{S:iT}(b) = 0$ for all $S \in \gamma$, which implies $\sum_{S \in \gamma} \delta_S d_{S:iT}(b) = 0$.

By Lemma 4, there is a partition matrix A_S such that $D_S(b) \geqq A_S$ for all $S \in \gamma$, and so we have

$$\sum_{S \in \gamma} \delta_S D_S(b) \geqq \sum_{S \in \gamma} \delta_S A_S \ .$$

Call the matrix on the right B ; then we have $D_N(b) \geqq B$. The crucial fact about B is to satisfy (12). In fact, we have by (10),

$$\sum_{i \in N} \sum_{S \in \gamma} \delta_S a_{S:iT} = \sum_{S \in \gamma} \delta_S \sum_{i \in N} a_{S:iT} = \sum_{S \in \gamma} \delta_S \begin{cases} |T| & \text{or } 0 \text{ if } T \subset S \\ 0 & \text{otherwise} \end{cases}$$

$$\leqq |T| \sum_{\substack{S \in \gamma \\ S \supset T}} \delta_S \leqq |T| \ ;$$

$$\sum_{T \in \pi} \sum_{S \in \gamma} \delta_S a_{S:iT} = \sum_{S \in \gamma} \delta_S \sum_{T \in \pi} a_{S:iT} = \sum_{S \in \gamma} \delta_S \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{\substack{S \in \gamma \\ S \ni i}} \delta_S = 1 \ .$$

Since $a_{S:iT} = 0$ if $i \notin T$ and it follows from (10) that $a_{S:iT} = a_{S:jT}$ if $i, j \in T$, we get

$$\sum_{S \in \gamma} \delta_S a_{S:iT} = 0 \quad \text{if} \quad i \notin T \quad \text{and}$$

$$\sum_{S \in \gamma} \delta_S a_{S:iT} = \sum_{S \in \gamma} \delta_S a_{S:jT} \quad \text{if} \quad i, j \in T .$$

So, B satisfies (12). If B consists of integers, then B is already a parrition matrix of N, which means $b \in V(N)$. Let us assume that B does not consist of integers. Lemma 4 says that the integral property of (9) is equivalent to that of (12). Since (9) has the integral property by the supposition of proof, (12) has the integral property. Therefore, there are integral extreme solutions of (12) as a convex combination of which B is represented. Let $B^1, \ldots, B^k$ be such integral extreme solutions. Then $D_N(b) \geq B^t$ for all $t = 1, \ldots, k$ because $D_N(b)$ has only zero or one entories. These B^t are partition matrices. Therefore b belongs to $V(N)$.   Q.E.D.

4.2. <u>Proof of (iii) $\Longrightarrow$ (i)</u>: We assume the negation of (i) and prove that there is a game (N,v) in $GS(N,\pi)$ with an empty core.

Since we assume the negation of (i), there is an extreme point $(X_T^*)_{T \in \pi}$ of the set of all solutions of (9) which is not an integral point. Let X be the set of all solutions of (9). Since X is a convex polyhedron, there is a vector $(u_T)_{T \in \pi}$ such that

(15) $$\sum_{T \in \pi} u_T X_T^* > \sum_{T \in \pi} u_T X_T \quad \text{for all} \quad (X_T)_{T \in \pi} \in X \quad \text{with} \quad (X_T)_{T \in \pi} \neq (X_T^*)_{T \in \pi} .$$

Using this $(u_T)_{T \in \pi}$, we define a game (N,v) with sidepayments as follows:

(16)
$$v(S) = \max \sum_{\substack{T \in \pi \\ T \subset S}} X_T u_T$$

subject to $\sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1$ for all $i \in S$ and $X_T = 0$ or 1

for all $T \in \pi$ with $T \subset S$.

By Lemma 1, a necessary and sufficient condition for the nonemptiness

of the core of (N,v) is that $L \leqq v(N)$, where

$$L = \min \sum_{i \in N} x_i \quad \text{subject to} \quad \sum_{i \in T} x_i \geqq v(T) \quad \text{for all} \quad T \in \pi .$$

The dual of the above linear programming is

$$M = \max \sum_{T \in \pi} v(T) X_T$$

subject to $\sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1$ for all $i \in N$ and $X_T \geqq 0$ for all $T \in \pi$.[5]

By the duality theorem, it holds that L = M . Since $u_T \leqq v(T)$ for

all $T \in \pi$, it holds that $M' \leqq M$, where

$$M' = \max \sum_{T \in \pi} u_T X_T$$

subject to $\sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1$ for all $i \in N$ and $X_T \geqq 0$ for all $T \in \pi$.

The constraint of the above problem coincides with (9). Hence

$M' = \sum_{T \in \pi} u_T X_T^*$. Since $(X_T^*)_{T \in \pi}$ is not a feasible solution of (16) in

the case of S = N because it is not an integral point, v(N) is smaller

than M' by (15). Hence $v(N) < M' \leqq M = L$. This implies that the

core of (N,v) is empty.   Q.E.D.

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[5]Note that this is a linear programming with mixed constraints. See
Goldman and Tucker [3].

REFERENCES

[1]   Balas, E. and Padberg, M. W. (1976). Set Partitioning: A Survey.
        Management Science Research Report No. 386, Carnegie-Mellon
        University.

[2]   Hoffman, A. J. and Kruskal, J. B. (1956). Integral Boundary
        Points of Convex Polyhedra. H. W. Kuhn and A. W. Tucker, eds.,
        Linear Inequalities and Related Systems, 223-246. Annals of
        Mathematical Studies, No. 38, Princeton University Press.

[3]   Goldman, A. J. and Tucker, A. W. (1956). Theory of Linear Pro-
        gramming. H. W. Kuhn and A. W. Tucker, eds., Linear Inequalities
        and Related Systems. Annals of Mathematical Studies, No. 38,
        Princeton University Press.

[4]   Kaneko, M. (1976). On the Core and Competitive Equilibria of a
        Market with Indivisible Goods, Nav. Res. Log. Q., 23, 321-337.

[5]   _____ (1980). The Central Assignment Game and the Assignment
        Markets, Cowles Foundation Discussion Paper No. 563.

[6]   Murty, K. (1976). Linear and Combinatorial Programming. John
        Wiley and Sons, Inc.

[7]   Myerson, R. B. (1977). Graphs and Cooperation in Games, Mathematics
        of Operations Research 2, 225-229.

[8]   Scarf, H. E. (1967). The Core of an n-Person Game. Econometrica
        35, 50-69.

[9]   Shapley, L. S. and Scarf, H. E. (1974). On Cores and Indivisibility,
        J. Math. Economics 1, 23-37.

[10]   Shapley, L. S. and Shubik, M. (1972). The Assignment Game I:
        The Core. Inter. J. Game Theory 1, 111-130.