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THE CENTRAL ASSIGNMENT GAME AND THE ASSIGNMENT MARKETS

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by

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ABSTRACT

Initially we prove the nonemptiness of the core of an n-person game without side payments called a central assignment game. Next, we provide a market model with indivisible goods but without the transferable utility assumption. Applying the first result to the market model, we prove the nonemptiness of the core and the existence of a competitive equilibrium. Finally, we provide a generalization of the market model and also show the nonemptiness of the core and the existence of a competitive equilibrium using the results in the previous model.

1. Introduction

The perfect divisibility of goods is assumed in most usual studies of market economies. When they consider market economies where the numbers of units of goods consumed or produced are not small, this assumption could not be inadequate even if the goods are considered to be substantially indivisible, because models with perfect divisibility could be good approximations of such markets. But when some goods have indivisible and large units and come in small number of units for some economic agents,

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The purpose of this paper is to extend the models of Shapley and Shubik [1972] and Kaneko [1976a] to cases without the transferable utility assumption. In Section 2, we provide an abstract game without side payments which we call a central assignment game, and prove the nonemptiness of the core, showing the balancedness. In Section 3, we consider the market model given by Shapley and Shubik [1972] without making the transferable utility assumption, and prove the nonemptiness of the core of the market model, using the nonemptiness of the central assignment game. Further we show the equivalence of the core and the competitive equilibria, which implies the existence of a competitive equilibrium. It is noted that no assumption on the marginal utility of money is necessary to prove these propositions. In Section 4, the model of Section 3 is extended to a case where sellers may own more than one unit of indivisible goods initially but the transferable utility assumption is imposed upon the sellers. This model is a generalization of that of Kaneko [1976a].[3] We also consider the existence of the core and the competitive equilibria, which are proved, and the relationships between them. The equivalence of them does not necessarily hold, but we provide a sufficient condition for it, though it is a less general one than that of Kaneko [1976a] in the case with the transferable utility assumption. As the sufficient condition, however, is very weak, it teaches us that the competitive equilibria can be representatives of the core in the market models, which makes our further analysis much easier.[4]

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[3]Exactly, this market model is neither any generalization of that of Section 2 nor that of Kaneko [1976a] in the usual mathematical sense.

[4]Now I am preparing a paper which applies the theory of this paper to a housing market.

2. The Central Assignment Game

We consider an m+n-person game $(M \cup N, V) \equiv (\{1, \ldots, m\} \cup \{1', \ldots, n'\}, V)$ without side payments, where V is a characteristic function from the class of all coalitions to a class of subsets of $R^{M \cup N}$, i.e., $V(S) \subset R^{M \cup N}$ for all $S \subset M \cup N$. Here $R^{M \cup N}$ is the m+n-dimensional Euclidean space whose coordinates are indexed by the members in $M \cup N$. We assume: for all $S \subset M \cup N$,

(2.1) $V(S)$ is a closed set in $R^{M \cup N}$,

(2.2) if $x \in R^{M \cup N}$ and $y \in V(S)$ with $y_i \geqq x_i$ for all $i \in S$, then $x \in V(S)$,

(2.3) $\mathrm{pro}_S[v(S) - \bigcup_{i \in S} (\text{interior of } v(\{i\}))]$ is bounded and nonempty.[5]

We say that a nonempty coalition S can improve upon a vector $x \in V(M \cup N)$ iff there is a vector $y \in V(S)$ such that

(2.4) $y_i > x_i$ for all $i \in S$.

The core is the set of all vectors in $V(M \cup N)$ which can not be improved upon by any coalition.

Let $\Phi = \{S : |S| = 1 \text{ or } (|S| = 2 \text{ and } S \cap M \neq \phi \text{ and } S \cap N \neq \phi)\}$, where $|S|$ denotes the number of members in S. We call $p_S = \{T_1, \ldots, T_k\}$ a Φ-partition of $S \subset M \cup N$ iff

(2.5) $T_t \in \Phi$ for all $t = 1, \ldots, k$,

(2.6) $\bigcup_{t=1}^{k} T_t = S$ and $T_t \cap T_{t'} = \phi$ for all t, t' with $t \neq t'$.

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[5] $\mathrm{Pro}_S X = \{(x_i)_{i \in S} : x \in X\}$ for all $S \subset M \cup N$ and $X \subset R^{M \cup N}$.

Let P(S) be the set of all Φ-partitions of S . We call $(M \cup N, V)$ a <u>central assignment game</u> iff V satisfies

$$(2.7) \qquad V(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} V(T) \quad \text{for all } S \subset M \cup N .^{6}$$

Definition (2.7) means that every coalition is subdivided into a partition which permits only coalitions consisting of one player or a couple of players in M and N . Thus the central assignment games have the same thought as the assignment games of Shapley and Shubik [1972]. It is, however, not permitted in ours that any commodity (transferable utility or other perfectly divisible commodity) is transferred freely in coalitions with more than two members. This is the main difference from Shapley and Shubik's model. We will consider a generalization of Shapley and Shubik's model, using the central assignment games in the succeeding sections.

The main result of this section is the following theorem.

<u>Theorem 1</u>. Every central assignment game has a nonempty core.

In the following we prove this theorem. So, it is necessary to prepare certain concepts. Let us call a family T of nonempty coalitions of $M \cup N$ <u>balanced</u> iff the system of equations

$$(2.8) \qquad \sum_{S: S \ni j} \delta_S = 1 \quad \text{for all } j \in M \cup N ,$$

has a nonnegative solution with $\delta_S = 0$ for all $S \notin T$. The numbers $\{\delta_S\}$ are called <u>balanced weights</u> for T . A game $(M \cup N, v)$ in

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[6]Note that V satisfies the super additivity.

characteristic function form is said to be balanced iff the following inclusion statement:

$$\bigcap_{S \in T} v(S) \subset v(M \cup N) \tag{2.9}$$

holds for all balanced families T. The fundamental theorem of Scarf [1967] states that the core of a balanced game $(M \cup N, v)$ with (2.1), (2.2) and (2.3) is nonempty. Hence, it is sufficient to show that a central assignment game is balanced. Before proving the balancedness, we need to show a lemma.

For any nonempty coalition S, if $(m+n)$ by $(m+n)$ zero-one matrix $A_S = (a_{S:ij})$, all rows and columns of which are indexed by members in $M \cup N$, satisfies

$$\begin{aligned} &\sum_{i \in M \cup N} a_{S:ij} = 1 \text{ if } j \in S \text{ and } \sum_{j \in M \cup N} a_{S:ij} = 1 \text{ if } i \in S \\ &\sum_{i \in M \cup N} a_{S:ij} = 0 \text{ if } i \notin S \text{ and } \sum_{j \in M \cup N} a_{S:ij} = 0 \text{ if } j \notin S , \end{aligned} \tag{2.10}$$

then we call A_S an S-permutation matrix. We define $D_S(b) = (d_{S:ij}(b))$ for each b in $R^{M \cup N}$ and each nonempty coalition S as follows:

$$d_{S:ij}(b) = \begin{cases} 1 \text{ if} & \begin{array}{l} (i \in S \cap M ,\ j \in N ,\ b \in V(\{i,j\})) \text{ or} \\ (i \in S \cap M ,\ j \in M ,\ b \in V(\{i\})) \text{ or} \\ (i \in S \cap N ,\ j \in N ,\ b \in V(\{j\})) \text{ or} \\ (i \in S \cap N ,\ j \in M) \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

<u>Lemma 1</u>. $V(S) = \{b \in R^{M \cup N} : D_S(b) \geqq A_S$ for some S-permutation matrix $A_S\}$ for all $S \subset M \cup N$.

<u>Proof</u>: Suppose that there is an S-permutation matrix A_S such that $D_S(b) \geqq A_S$. Let $\{(i,j) : a_{S:ij} = 1,\ i \in M \text{ and } j \in N\} = \{(i_1, i_2), \ldots, (i_k, j_k)\}$. Note that $i_t \in S$ and $j_t \in S$ for all $t = 1, \ldots, k$ by (2.10). Let $M \cap S - \{i_1, \ldots, i_k\} = \{i(1), \ldots, i(g)\}$ and $N \cap S - \{j_1, \ldots, j_k\} = \{j(1), \ldots, j(h)\}$. Of course $S = \{i_1, \ldots, i_k, i(1), \ldots, i(g), j_1, \ldots, j_k, j(1), \ldots, j(h)\}$. We define a Φ-partition $p_S = \{T_1, \ldots, T_f\}$ $(f = k+g+h)$ by

$$T_t = \{i_t, j_t\} \quad \text{for all } t = 1, \ldots, k ,$$

$$T_{k+t} = \{i(t)\} \quad \text{for all } t = 1, \ldots, g ,$$

$$T_{k+g+t} = \{j(t)\} \quad \text{for all } t = 1, \ldots, h .$$

Since $d_{S:i_tj_t}(b) = a_{S:i_tj_t} = 1$, $i_t \in M \cap S$ and $j_t \in N \cap S$, we have $b \in V(\{i_t, j_t\})$ by (2.11). Since $a_{S:i(t)i} = 1$ for some $i \in M \cap S$ and $a_{S:j(t)j} = 1$ for some $j \in N \cap S$ by (2.10) and the definition of $i(t)$ and $j(t)$, we have $d_{S:i(t)i}(b) = 1$ and $d_{S:j(t)j}(b) = 1$, which implies $b \in V(\{i(t)\})$ and $b \in V(\{j(t)\})$ by (2.11). Thus we have shown that $b \in V(T_t)$ for all $t = 1, \ldots, f$, i.e., $b \in \bigcap_{T \in p_S} V(T)$.

Let $b \in \bigcap_{T \in p_S} V(T)$ for some Φ-partition p_S . Let

$$\{T : T \in p_S \text{ and } |T| = 2\} = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} ,$$

$$\{T : T \in p_S,\ |T| = 1 \text{ and } T \subset M\} = \{\{i(1)\}, \ldots, \{i(g)\}\} ,$$

$$\{T : T \in p_S,\ |T| = 1 \text{ and } T \subset N\} = \{\{j(1)\}, \ldots, \{j(h)\}\} .$$

We assume $g \leqq h$ in the following, but we can prove the following similarly when $g > h$. We define an S-permutation matrix $A_S = (a_{S:ij})$ as follows:

(2.12) for all i_t $(t = 1, \ldots, k)$,

$$a_{S:i_t j} = \begin{cases} 1 & \text{if } j = j_t \\ 0 & \text{otherwise,} \end{cases}$$

(2.13) for all $i(t)$ $(t = 1, \ldots, g)$,

$$a_{S:i(t)j} = \begin{cases} 1 & \text{if } j = j(t) \\ 0 & \text{otherwise,} \end{cases}$$

(2.14) for all $i \in M-S$,

$$a_{S:ij} = 0 \quad \text{for all } j \in M \cup N,$$

(2.15) for all $i \in N$,

$$a_{S:ij} = \begin{cases} a_{S:ji} & \text{if } i \in \{j_1, \ldots, j_k, j(1), \ldots, j(g)\} \text{ and } j \in M \\ 1 & \text{if } i \in \{j(g+1), \ldots, j(h)\} \text{ and } i = j \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that this A_S is an S-permutation matrix.

Now we show that $D_S(b) \geqq A_S$. When $a_{S:ij} = 0$, it is always true that $d_{S:ij}(b) \geqq a_{S:ij}$. So, suppose $a_{S:ij} = 1$. If $(i,j) = (i_t, j_t)$ for some $t \leqq k$, then b is in $V(\{i_t, j_t\})$ because $b \in \bigcap_{T \in p_S} V(T)$. This implies $d_{S:ij}(b) = 1$. Let $(i,j) = (i(t), j(t))$ for some $t \leqq g$. Then $b \in V(\{i(t)\}) \cap V(\{j(t)\}) \subset V(\{i(t), j(t)\})$ by the supposition, $b \in \bigcap_{T \in p_S} V(T)$, and the superadditivity of V. This implies $d_{S:ij}(b) = 1$.

When $i = j(t)$ and $j = j(t)$ $(g+1 \leqq t \leqq h)$ it is also true by the same reason that $b \in V(\{j(t)\})$, which implies $d_{S:ij}(b) = 1$. When $i \in N \cap S$ and $j \in S \cap M$, we have always $d_{S:ij}(b) = 1$ by (2.11). We have shown $D_S(b) \geqq A_S$.

Q.E.D.

Now we are in a position to prove that $(M \cup N, V)$ is a balanced game. The following proof is almost the same as that of the theorem of Shapley and Scarf [1974] that the core is nonempty in the market model where only indivisible goods are exchanged. But as our game is more complicated, we give the proof of it for mathematical completeness.

Proof of the Balancedness: Let T be an arbitrary balanced family of coalitions, and let $b \in \bigcap_{S \in T} V(S)$. Let $\{\delta_S\}$ be balancing weights for T. Then it holds that

$$D_{M \cup N}(b) = \sum_{S \in T} \delta_S D_S(b) .$$

For, if $d_{M \cup N:ij}(b) = 1$, then $d_{S:ij}(b) = 1$ if $i \in S$ and $d_{S:ij}(b) = 0$ if $i \notin S$ by (2.11), which implies $\sum_{S \in T} \delta_S d_{S:ij}(b) = \sum_{\substack{S \in T \\ S \ni i}} \delta_S = 1$, and if $d_{M \cup N:ij}(b) = 0$, then $d_{S:ij}(b) = 0$ for all $S \in T$, which implies $\sum_{S \in T} \delta_S d_{S:ij}(b) = 0$. Since $b \in V(S)$ for each $S \in T$, there is an S-permutation matrix A_S by Lemma 1 such that $D_S(b) \geqq A_S$, and so

$$\sum_{S \in T} \delta_S D_S(b) \geqq \sum_{S \in T} \delta_S A_S .$$

Call the matrix on the right B; then we have

$$D_{M\cup N}(b) \geqq B \ .$$

The crucial fact about B is that it is doubly stochastic, i.e., it is nonnegative and has all row- and column-sums equal to 1. This follows directly from the definition of balancing weights; in fact, the j^{th} column sum is

$$\sum_{i\in M\cup N} \sum_{S\in T} \delta_S a_{S:ij} = \sum_{S\in T} \delta_S \sum_{i\in M\cup N} a_{S:ij} = \sum_{S\in T} \delta_S \begin{cases} 1 & \text{if} \quad j \in S \\ 0 & \text{if} \quad j \notin S \end{cases}$$

$$= \sum_{\substack{S\in j \\ S\ni T}} \delta_S = 1 \ ,$$

and the argument for the row-sum is the same.

The next step is to change B into an $M\cup N$-permutation matrix $A_{M\cup N}$, i.e., to eliminate any fractional entries without changing the row- or column-sums and to do so without disturbing any entries which are already 0 or 1. Since all entries of $D_{M\cup N}(b)$ are 0 or 1, we will thereby ensure that $D_{M\cup N}(b) \geqq A_{M\cup N}$, which implies $b \in V(M\cup N)$.

Since a fraction can not occur alone in any row- or column, either B is already an $M\cup N$-permutation matrix or there is a closed loop of fractional entries:

$$\begin{tikzcd}
b_{i_1 j_1} \arrow[r] & b_{i_1 j_2} \arrow[d] & & \\
& b_{i_2 j_2} \arrow[r] & b_{i_2 j_3} \arrow[d] & \\
& & \cdots & \cdots \arrow[d] \\
b_{i_p j_1} \arrow[uuu] & & & b_{i_p j_p} \arrow[lll]
\end{tikzcd}$$

Alternately adding and subtracting a fixed number $\varepsilon > 0$ to the elements of this loop will clearly preserve row- and column sums. If ε is too large, the negative entries will be created, but making ε as large as possible consistent with nonnegativity will produce a new doubly stochastic matrix B' that has at least one more zero than B , and hence fewer fractional entries. If B' is not yet an $M \cup N$-permutation matrix, we repeat the same operation. Eventually we can obtain what we want--an $M \cup N$-permutation matrix $A_{M \cup N}$ such that $D_{M \cup N}(b) \geqq A_{M \cup N}$.

Q.E.D.

3. The Assignment Market

In this section we reformulate the market model of Shapley and Shubik [1972] without making the transferable utility assumption and show that there exist the nonempty core and a competitive equilibrium in the market.

We consider a market consisting of players $M = \{1, \ldots, m\}$ and $N = \{1', \ldots, n'\}$. We may also call a player in M a seller and one in N a buyer in the following. In this market s-kinds of indivisible goods are exchanged for money. A seller owns exactly one unit of indivisible good before trade. Hence M can be subdivided into $M_1 \cup M_2 \cup \ldots \cup M_s$, where $i \in M_t$ $(t = 1, \ldots, s)$ has one unit of the t^{th} good. Without loss of generality, we can assume that $M_t \neq \phi$ for all $t = 1, \ldots, s$ and $M_t = \{m_{t-1} + 1, \ldots, m_t\}$ for all $t = 2, \ldots, s$, where $0 < m_1 < m_2 < \ldots < m_s = m$. No buyer owns indivisible goods before trade. Each player $i \in M \cup N$ owns $I_i > 0$ amount of money initially, where money is perfectly divisible and should be interpreted as a composite good. Hence player i's initial endowment is

(e^t, I_i) if $i \in M_t$ and $(0, I_i)$ if $i \in N$, where e^t is the s-dimensional unit vector with $e_t^t = 1$. For simplicity, we denote the initial endowment by (a^i, I_i) $(i \in M \cup N)$.

Each player $i \in M \cup N$ has a preference relation R_i on the consumption set $X = I_+^s \times R_+$, where I_+ is the set of all nonnegative integers, I_+^s the direct product of s number of I_+ and R_+ is the set of all nonnegative numbers. $(x,m) \in X$ means amounts of the indivisible goods and money to be consumed. We assume that every R_i $(i \in M \cup N)$ is a weak ordering. $(x, m_1)R_i(y, m_2)$ means that player i prefers (x, m_1) to (y, m_2) or is indifferent between (x, m_1) and (y, m_2). We define the strict preference P_i be $(x, m_1)P_i(y, m_2)$ iff not $(y, m_2)R_i(x, m_1)$ and the indifference relation I_i by $(x, m_1)Q_i(y, m_2)$ iff $(x, m_1)R_i(y, m_2)$ and $(y, m_2)R_i(x, m_1)$. We assume: for all $i \in M \cup N$,

(A) <u>(Monotonicity with respect to money)</u>: if $m_1 > m_2$, then $(x, m_1)P_i(x, m_2)$ for all $x \in I_+^s$;

(B) <u>(Archimedean property)</u>: if $(x, m_1)P_i(y, m_2)$, then there is an m_3 such that $(x, m_1)Q_i(y, m_3)$.

<u>Lemma 2</u>: If $(x,m)R_i(0,0)$ for all $(x,m) \in X$, then there is a continuous utility function U^i on X, i.e., $(x, m_1)R_i(y, m_2)$ iff $U^i(x, m_1) \geqq U^i(y, m_2)$, where the relative topology of the s+1-dimensional Euclidean space R^{s+1} is introduced into $X = I_+^s \times R_+$.

This lemma is important for the representation of the market game in characteristic function form, but is not of central interest in this paper. So, the proof of it will be given in the Appendix.

We assume the two following conditions separately on the sellers and the buyers: for all sellers $i \in M_t$ $(t = 1, \ldots, s)$,

(CS) (Satiation): for each $(x,m) \in X$, if $x_t = 0$, then $(x,m)Q_i(0,m)$ and if $x_t \geqq 1$, then $(x,m)Q_i(e^t, m)P_i(0,m)$;

and for all $i \in N$,

(CB) (Satiation): for each $(x,m) \in X$, if $x_t = 1$ for some t and $(e^t, m)R_i(e^k, m)$ for all k with $x_k \geqq 1$, then $(x,m)Q_i(e^t, m)P_i(0,m)$.

The last assumption is:

(D) For all $i \in N$, $(0, I_i)P_i(x,0)$ for all $x \in I_+^s$.

It should be noted that assumptions (CS) and (CB) are stronger than the supposition of Lemma 2, which implies that Lemma 2 holds under these assumptions.

Assumption (CS) means that even if seller $i \in M_t$ has indivisible goods other than the t^{th} good or more than one unit of the t^{th} good, then his utility does not become greater than that of the initial state. It follows from this assumption that sellers never become buyers. Assumption (CB) means that even if buyer $i \in N$ consumes more than one unit of indivisible goods, then his utility does not become greater than that of the state having one unit of the most preferred one in goods to be consumed. More precisely, if the buyer purchases one unit of a more preferred good as the second unit, his utility increases, but it is indifferent from this for him to purchase the good as the first unit. This assumption implies that buyers never purchase more than one unit if all of the prices of the goods are positive. Assumption (D) means that any buyer purchases no indivisible good by paying all his income. When the marginal utility of money at $(x,0)$ is very large and the income I_i is not too small, this assumption is satisfied. This would be a natural assumption. Since assumption (CS) implies that no seller buys any indivisible good, i.e., decreases his amount of money, it is not necessary to consider the case where the amount of money is zero. This is the reason why we do not impose assumption (D) upon the sellers.

These assumptions (CS) and (CB) seem to look like strange ones at a first glance and are unfamiliar to one having the knowledge of the standard equilibrium theory. But these never make the modeling of situation which we want to consider inadequate. Now we consider an example of a market with s kinds of houses for rent. Let us consider two cases where a buyer i rents one unit of t^{th} house and where he rents two units of the house. These states are represented as (e^t, m_1) and $(2e^t, m_2)$, where m_1 and m_2 are the amounts of money after paying the rents for the house(s), respectively. It is natural to suppose that there is a nonnegative real number ε such that

$$(e^k, m_1)Q_i(2e^k, m_1 - \varepsilon) .$$

This ε is the maximal amount that i pays for the 2nd unit of the house when he has already rented one unit of it. When the t^{th} house is not small to live in for him, the real number ε could be considered to be small relative to the rent of the house. Then it is reasonable and convenient in order to formulate such a kind of market as a mathematical model that the number ε is assumed to be zero. Any way, these are assumptions for idealization. Next let us consider the case where

$(e^t, m_1)R_i(e^k, m_2)$ and $x = e^t + e^k$. Then it is natural to assume that $(x, m_1)R_i(e^t, m_1)$, and that $(x, m_1 - \varepsilon)Q_i(e^t, m_1)$ for some $\varepsilon \geqq 0$. By the same reasoning as the above, this ε could be assumed to be zero. We can also explain the meaning of (CS) in almost the same way.

The above formulation is a reformulation of Shapley and Shubik's market model in the case where the transferable utility assumption is not made. We call the above market an <u>assignment market</u>. Now we are in a position to discuss the theme of this section.

For any nonempty coalition S, we call $(x^S, m^S) = ((x^i, m^i))_{i \varepsilon S}$ an <u>S-allocation</u> iff

(3.1) $\quad (x^i, m^i) \varepsilon X$ for all $i \varepsilon S$,

(3.2) $$\sum_{i \varepsilon S} (x^i, m^i) = \sum_{i \varepsilon S} (a^i, I_i) .$$

We call an $M \cup N$-allocation simply an <u>allocation</u>. We say that a nonempty coalition S <u>can improve upon</u> an allocation $(x^{M \cup N}, m^{M \cup N})$ iff there is an S-allocation (y^S, m_1^S) such that

(3.3) $\quad (y^i, m_1^i)P_i(x^i, m^i)$ for all $i \varepsilon S$.

The <u>core</u> of the assignment market is the set of all allocations which can never be improved upon by any coalitions.

The characteristic function V of the assignment market is defined in the usual manner as follows: for all $S \subset M \cup N$,

(3.4) $$V(S) = \{b \varepsilon R^{M \cup N} : \text{for some S-allocation } (x^S, m^S),$$
$$b_i \leqq U^i(x^i, m^i) \text{ for all } i \varepsilon S\} .$$

It is easily verified that for any b in the core of $(M \cup N, V)$, there is an allocation $(x^{M \cup N}, m^{M \cup N})$ in the core of the assignment market such that $U^i(x^i, m^i) \geqq b_i$ for all $i \in M \cup N$, and conversely that $(U^i(x^i, m^i))_{i \in M \cup N}$ belongs to the core of $(M \cup N, V)$ for any allocation $(x^{M \cup N}, m^{M \cup N})$ in the core of the assignment market. Hence the nonemptiness of the core of $(M \cup N, V)$ is equivalent to that of the core of the assignment market. So, we show the nonemptiness of the core of $(M \cup N, V)$.

We define another characteristic function V_0 using V as follows: for all $S \subset M \cup N$,

$$(3.5) \qquad V_0(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} V(T) ,$$

Of course, the new game $(M \cup N, V_0)$ is a central assignment game. Hence the core of $(M \cup N, V_0)$ is nonempty by Theorem 1. Though V_0 is different from V, the following relations (3.6) hold:

$$(3.6) \qquad V_0(S) \subset V(S) \quad \text{for all} \quad S \subset M \cup N ,$$

$$V_0(S) = V(S) \quad \text{for all} \quad S \in \Phi .$$

Further we can prove:

Theorem 2: The core of $(M \cup N, V_0)$ coincides with the core of $(M \cup N, V)$.

We get the following theorem by Theorems 1 and 2.

Theorem 3: The cores of $(M \cup N, V)$ and the assignment market are nonempty.

The following lemmas are necessary to prove Theorem 2. We will give the proofs in the Appendix.

Lemma 3: If for any allocation $(x^{M\cup N}, m^{M\cup N})$, a nonempty coalition S satisfies

$$(3.7)\qquad \sum_{i\in S}(a^i, I_i) \geq \sum_{i\in S}(x^i, m^i) \text{ and } m^i > 0 \text{ for all } i \in S ,$$

then S can improve upon the allocation $(x^{M\cup N}, m^{M\cup N})$. Here $x \geq y$ means $x \geqq y$ but $x \neq y$.

Lemma 4: Let $(x^{M\cup N}, m^{M\cup N})$ belong to the core of the assignment market. Then there are partitions of M and N such that $M = \{i_1, \ldots, i_k\} \cup M_0$, $N = \{j_1, \ldots, j_k\} \cup N_0$ and

$$(3.8)\qquad (x^{i_t}, m^{i_t}) + (x^{j_t}, m^{j_t}) = (a^{i_t}, I_{i_t}) + (a^{j_t}, I_{j_t}) \text{ and}$$

$$x^{i_t} = 0 \text{ for all } t = 1, \ldots, k ,$$

$$(3.9)\qquad (x^i, m^i) = (a^i, I_i) \text{ for all } i \in M_0 \cup N_0 .$$

We note that $x^{j_t} = a^{i_t}$ for all $t = 1, \ldots, k$ by (3.8).

The proof of Lemma 4 implies the following corollary.

Corollary 5: Under the supposition of Lemma 4, for all $i \in M_t \cap \{i_1, \ldots, i_k\}$ and $j \in \{j_1, \ldots, j_k\}$ with $x^j = e^t$, it holds that $(x^i, m^i) + (x^j, m^j) = (a^i, I_i) + (a^j, I_j)$.

Proof of Theorem 2: Let b be in the core of $(M \cup N, V)$. Since b can not be improved upon in $(M \cup N, V_0)$ by (3.6), it is sufficient to show that b is in $V_0(M \cup N)$. There is an allocation $(x^{M\cup N}, m^{M\cup N})$ such that $U^i(x^i, m^i) \geqq b_i$ for all $i \in M \cup N$. Note that this allocation is in the core of the assignment market. Let the partitions of M and N given in Lemma 4 be $M = \{i_1, \ldots, i_k\} \cup M_0$ and $N = \{j_1, \ldots, j_k\} \cup N_0$. We put $M_0 = \{i(1), \ldots, i(g)\}$ and $N_0 = \{j(1), \ldots, j(h)\}$. We define a Φ-partition $p_{M \cup N} = \{T_1, \ldots, T_f\}$ $(f = k+g+h)$ as follows:

$$T_t = \{i_t, j_t\} \quad \text{for all} \quad t = 1, \ldots, k ,$$

$$T_{k+t} = \{i(t)\} \quad \text{for all} \quad t = 1, \ldots, g ,$$

$$T_{k+g+t} = \{j(t)\} \quad \text{for all} \quad t = 1, \ldots, h .$$

By Lemma 4 it holds that $b \in V(T)$ for all $T \in p_{M \cup N}$, i.e., $b \in \bigcap_{T \in p_{M \cup N}} V(T)$. This means that b is in $V_0(M \cup N)$.

Conversely, we show that any b in the core of $(M \cup N, V_0)$ belongs to the core of $(M \cup N, V)$. Suppose b is in the core of $(M \cup N, V_0)$. By (3.6) b is in $V(M \cup N)$. We prove that if b can be improved upon by a nonempty coalition in the game $(M \cup N, V)$, then b can be also improved upon by some nonempty coalition in the game $(M \cup N, V_0)$. So, we will complete the proof.

Since b is in $V_0(M \cup N)$, there is a Φ-partition $p_{M \cup N}$ of $M \cup N$ such that $b \in \bigcap_{T \in p_{M \cup N}} V(T)$. We put the set of pairs $\{T : |T| = 2 \text{ and } T \in p_{M \cup N}\} = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\}$ and $M_0 = M - \{i_1, \ldots, i_k\}$ and $N_0 = N - \{j_1, \ldots, j_k\}$. Then it holds that

$$b \in V(\{i_t, j_t\}) \quad \text{for all} \quad t = 1, \ldots, k ,$$

$$b \in V(\{i\}) \qquad \text{for all} \quad i \in M_0 \cup N_0 .$$

Hence there is an allocation $(x^{M \cup N}, m^{M \cup N})$ such that

$$(x^{i_t}, m^{i_t}) + (x^{j_t}, m^{j_t}) = (a^{i_t}, I_{i_t}) + (a^{j_t}, I_{j_t})$$

$$\text{for all } \ t = 1, \ldots, k \ ,$$

$$(x^i, m^i) = (a^i, I_i) \ \text{ for all } \ i \in M_0 \cup N_0 \ ,$$

$$U^i(x^i, m^i) \geqq b_i \ \text{ for all } \ i \in M \cup N \ .$$

Suppose that $U^i(x^i, m^i) > b_i$ for some $i \in M \cup N$. If $i \in M_0 \cup N_0$, then $b_i < U^i(x^i, m^i) = \sup \mathrm{pro}_{\{i\}} V(\{i\}) = \sup \mathrm{pro}_{\{i\}} V_0(\{i\})$, which is a contradiction to the supposition that b is in the core of $(M \cup N, V_0)$. Let $i = j_t$ $(t \leqq k)$. If $m^{j_t} = 0$, then $U^{j_t}(x^{j_t}, m^{j_t}) < U^{j_t}(a^{j_t}, I_{j_t})$ $= \sup \mathrm{pro}_{\{j_t\}} V_0(\{j_t\})$ by assumption (D), which is a contradiction. Then we have $m^{j_t} > 0$. So, we can choose a positive real number ε by Lemma 2 and assumption (A) such that $U^{j_t}(x^{j_t}, m^{j_t} - \varepsilon) > b_{j_t}$ and $U^{i_t}(x^t, m^{i_t} + \varepsilon) > b_{i_t}$. Since $(U^{i_t}(x^{i_t}, m^{i_t} + \varepsilon), U^{j_t}(x^{j_t}, m^{j_t} - \varepsilon)) \times (b_j)_{j \neq i_t, j_t}$ belongs to $V(\{i_t, j_t\}) = V_0(\{i_t, j_t\})$, $\{i_t, j_t\}$ can improve upon b in the game $(M \cup N, V_0)$, which is a contradiction. Let $i = i_t$ $(t \leqq k)$. Then $U^{i_t}(x^{i_t}, m^{i_t}) \geqq U^{i_t}(a^{i_t}, I_{i_t})$ together with assumptions (A) and (CS) implies $m^{i_t} \geqq I_{i_t} > 0$. So, we can get a contradiction similarly with the above if $U^{i}(x^{i_t}, m^{i_t}) > b_{i_t}$. Hence it must hold that

$$U^i(x^i, m^i) = b_i \ \text{ for all } \ i \in M \cup N \ .$$

Suppose that a nonempty coalition S can improve upon this allocation $(x^{M \cup N}, m^{M \cup N})$ in the assignment market, which is equivalent to that S can improve upon B in the game $(M \cup N, V)$. This means that there is an S-allocation (y^S, m_1^S) such that

(3.10) $\quad U^i(y\ , m_1) > U^i(x^i, m^i)$ for all $i \in S$.

We choose a buyer $j \in S$ such that $y_f^j \geqq 1$ and $U^j(e^f, m_1^j) = U^j(y^j, m_1^j)$ for some $f \leqq s$, which implies that there is a seller $i \in S \cap M_f$ with $y_f^i = 0$. This choice is always possible because if not, (3.10) can not hold. Of course, it holds that $y^i + y^j \geqq a^i + a^j$. If $m_1^i + m_1^j \leqq I_i + I_j$, then the coalition $\{i,j\}$ can improve upon $(x^{M\cup N}, m^{M\cup N})$ by (3.10) and the choice of i , which is a contradiction to the supposition that b is in the core of $(M \cup N, V_0)$. Hence we have $m_1^i + m_1^j > I_i + I_j$, which implies

$$\sum_{t\in S-\{i,j\}} y^t \leqq \sum_{t\in S-\{i,j\}} a^t \quad \text{and} \quad \sum_{t\in S-\{i,j\}} m_1^t < \sum_{t\in S-\{i,j\}} I_t \ .$$

It follows that if $S-\{i,j\} \neq \phi$, $S-\{i,j\}$ can improve upon $(x^{M\cup N}, m^{M\cup N})$, i.e., there is an $S-\{i,j\}$-allocation $(z^{S-\{i,j\}}, m_2^{S-\{i,j\}})$ such that

$$U^t(z^t, m_2^t) > U^t(y^t, m_1^t) > U^t(x^t, m^t) \quad \text{for all} \quad t \in S-\{i,j\} \ .$$

This is (3.10) for the case of $S-\{i,j\}$ and $S-\{i,j\}$ is two members less than S . If $S = \{i,j\}$, then $\{i,j\}$ can improve upon b in the game $(M \cup N, V_0)$. If $S \neq \{i,j\}$, then, repeating the same argument as the above, we can reach a T-allocation (w^T, m_0^T) such that $|T| \leqq 2$ and $U^t(w^t, m_0^t) > U^t(x^t, m^t)$ for all $t \in T$. Hence b can be improved upon in the game $(M \cup N, V_0)$.

Q.E.D.

The latter part of the proof of Theorem 2 implies the following corollary.

Corollary 6. An allocation $(x^{M \cup N}, m^{M \cup N})$ is in the core of the assignment market iff $(x^{M \cup N}, m^{M \cup N})$ can be never improved upon by any coalition S in Φ.

It should be noted that since any allocation in the core of the assignment market is the sum of partial allocations as shown in Lemma 4, we do not need to consider the grand coalition $M \cup N$.

This corollary has an important implication. It is often said that the core-theory neglects costs of coalition-formations and that as costs of making large coalitions are very large, it would be implausible to assume to be able to bargain freely. I readily agree with this criticism in general. But this corollary says that this criticism has no persuasive power at least in the assignment market, because the same result is derived even when only the coalitions in Φ are permitted and the coalitions in Φ are very small. I think that this criticism has a little persuasive power even in the usual market games like perfectly competitive markets, because only small coalitions play important roles substantially in such market games. In games in which large coalitions play substantial roles, it is a weak point of the theory to ignore costs of coalition-formations.

Now we consider the relationship between the core and the competitive equilibria in the assignment market. We call a pair $(p, (x^{M \cup N}, m^{M \cup N}))$ of a price vector $p = (p_1, \ldots, p_s) \in R^s_+$ and an allocation $(x^{M \cup N}, m^{M \cup N})$ a competitive equilibrium iff

(3.11) for all $i \in M \cup N$, $(x^i, m^i)R_i(y^i, m_1^i)$ for all

$$(y^i, m_1^i) \in X \text{ such that } py^i + m_1^i \leqq I_i + pa^i ,$$

(3.12) for all $i \in M \cup N$, $px^i + m^i = I_i + pa^i$.

We call $(x^{M \cup N}, m^{M \cup N})$ a <u>competitive allocation</u> iff there is a price vector such that $(p, (x^{M \cup N}, m^{M \cup N}))$ is a competitive equilibrium, and p a <u>competitive price vector</u>.

Shapley and Shubik [1972] show that the core always coincides with the set of all competitive allocations in the assignment market with the transferable utility assumption. This theorem is true even in the assignment market without the transferable utility assumption.

<u>Theorem 4</u>: The core coincides with the set of all competitive allocations in the assignment market.

From this theorem and Theorem 3, we get

<u>Theorem 5</u>: There exists a competitive equilibrium in the assignment market.

<u>Proof of Theorem 4</u>: It can be shown in the well-known manner that the competitive allocations is included by the core. Hence we need to show the converse inclusion.

Let $(x^{M \cup N}, m^{M \cup N})$ be in the core. Let us consider the partition given by Lemma 4. If there are a seller $i \in \{i_1, \ldots, i_k\} \cap M_f$ and a buyer $j \in \{j_1, \ldots, j_k\}$ such that $x^j = e^f$, $m^j = I_j - r$, $m^i = I_i + r'$ and $r \neq r'$, then we have $m^i + m^j \neq I_i + I_j$, which is a contradiction to Corollary 5. Hence it holds that if there is a buyer $j \in \{j_1, \ldots, j_k\}$

with $x^j = e^f$, there is a real number r_f such that

$$m^i = I_i + r_f \quad \text{for all} \quad i \in M_f \cap \{i_1, \ldots, i_k\}$$

$$m^j = I_j - r_f \quad \text{for all} \quad j \in \{j_1, \ldots, j_k\} \quad \text{with} \quad x^j = e^f .$$

If $r_f \leqq 0$, then $U^i(e^f, I_i) > U^i(0, I_i + r_f)$ for all $i \in M_f \cap \{i_1, \ldots, i_k\}$ by assumption (CS), which is a contradiction. Hence we have $r_f > 0$.

$$\text{(3.13)} \qquad p_f = \begin{cases} r_f & \text{if there is a buyer } j \in \{j_1, \ldots, j_k\} \text{ with } x^j = e^f \\ \min\limits_{i \in M_f} q_f(i) & \text{otherwise.} \end{cases}$$

where we define $q_f(i)$ by $U^i(e^f, I_i) = U^i(0, I_i + q_f(i))$ for all $i \in M_f$ $(f = 1, \ldots, s)$. The existence and uniqueness of $q_f(i)$ are ensured by assumptions (A) and (B). It is clear by assumption (CS) that $q_f(i) > 0$ for all $i \in M_f$. Hence $p_f > 0$ for all $f \leqq s$. It can easily be verified that the budget constraint (3.12) follows from (3.13) and Corollary 5. Hence we need to show (3.11).

Suppose that there is a $(y^i, m_1^i) \in X$ for some $i \in M \cup N$ such that $U^i(y^i, m_1^i) > U^i(x^i, m^i)$ and $py^i + m_1^i \leqq I_i + pa^i$. First, let $i \in M_f$. In this case it is sufficient to assume $y^i = 0$ by assumption (CS) and the individual rationality $U^i(x^i, m^i) \geqq U^i(a^i, I_i)$. Then we have $p_f > \min\limits_{i \in M_f} q_f(i)$, because otherwise, $U^i(0, m_1^i) = U^i(0, I_i + p_f)$ $\leqq U^i(a^i, I_i) \leqq U^i(x^i, m^i)$, which is a contradiction. If $i \in M_f \cap \{i_1, \ldots, i_k\}$, then the existence of such a (y^i, m_1^i) is impossible by assumption (CS), (3.13) and Corollary 5. So, $i \in M_0 \cap M_f$. Since $p_f > \min\limits_{i \in M_f} q_f(i)$, there is a buyer $j \in \{j_1, \ldots, j_k\}$ with

$x^j = e^f$. For a sufficiently small real number $\varepsilon > 0$, it holds by assumption (A) that

$$U^i(0,\ I_i + p_f - \varepsilon) > U^i(x^i,\ m^i) \quad \text{and} \quad U^j(x^j,\ I_j - p_f + \varepsilon) > U^j(x^j,\ m^j) \ .$$

That is, $\{i,j\}$ can improve upon $(x^{M \cup N},\ m^{M \cup N})$, which is a contradiction. Second, let $i \in N$. Then we can assume $(y^i,\ m_1^i) = (e^f,\ I_i - p_f)$ for some $f \leqq s$. Since $U^i(e^f,\ I_i - p_f) > U^i(x^i,\ m^i) \geqq U^i(a^i,\ I_i)$, we have $I_i - p_f > 0$ by assumption (D). Hence if $M_f \cap \{i_1, \ldots, i_k\} \neq \phi$, then for any $j \in M_f \cap \{i_1, \ldots, i_k\}$, it holds that for some $\varepsilon > 0$

$$U^i(e^f,\ I_i - p_f - \varepsilon) > U^i(x^i,\ m^i)$$

and

$$U^j(0,\ I_j + p_f + \varepsilon) > U^j(0,\ I_j + p_f) = U^j(x^j,\ m^j) \ ,$$

which is a contradiction. Next, if $M_f \cap \{i_1, \ldots, i_k\} = \phi$, then for a seller j with $q_f(j) = \min_{t \varepsilon M_f} q_f(t)$,

$$U^i(e^f,\ I_i - p_f - \varepsilon) > U^i(x^i,\ m^i)$$

$$U^i(0,\ I_j + p_j + \varepsilon) > U^j(0,\ I_j + q_f(j)) = U^j(a^j,\ I_j) = U^j(x^j,\ m^j) \ ,$$

because $p_f = q_f(t)$ and $(a^j,\ I_j) = (x^j,\ m^j)$ by the assumption that $M_f \cap \{i_1, \ldots, i_k\} = \phi$, where ε is a sufficiently small positive number. This is a contradiction.

Q.E.D.

The Core of $(M \cup N, V_0) \neq \phi$ — Theorem 1

Theorem 2

The Core of $(M \cup N, V) \neq \phi$ — Theorem 3

Definition

The Core of Assignment Market $\neq \phi$ — Theorem 3

Theorem 4

The Competitive Equilibria of the Assignment Market $\neq \phi$ — Theorem 5

Definition

The Competitive Equilibria of the Agent Assignment Market $\neq \phi$

Theorem 6

The Competitive Equilibria of the Generalized Assignment Market $\neq \phi$ — Theorem 7

Under the Assumption of Theorem 10 — Well Known

The Core of the Generalized Assignment Market $\neq \phi$ — Theorem 8

4. The Generalized Assignment Market

In this Section we consider a generalization of the assignment market, in which we permit each seller $i \in M_k$ $(k = 1, \ldots, s)$ to own more than one unit of the k^{th} individual good initially. That is, his initial endowment of the individual goods is $w_i e^k$ in the model of this section, where w_i may be any positive integer. We assume: for all $i \in M_k$ $(k = 1, \ldots, s)$,

(CS') (Satiation): $(x,m)Q_i(x_k e^k, m)$ for all $(x,m) \in X$ and if $x_k \geqq w_i$ then $(x,m)Q_i(w_i e^k, m)P_i((w_i - 1)e^k, m)P_i \ldots P_i(e^k, m)P_i(0,m)$

We can justify this assumption by the similar reasoning to that of assumption (CB).

We impose the transferable utility assumption on the sellers $i \in M$:

(E) (Constant marginal utility of money): if $(x, m_1)Q_i(y, m_2)$, then $(x, m_1 + \delta)Q_i(y, m_2 + \delta)$ for all $\delta > 0$.

Kaneko [1976b] shows that assumptions (A), (B) and (E) imply

(4.1) there is a real-valued function $u^i(x)$ on I^s such that

$$(x, m_1)R_i(y, m_2) \text{ iff } u^i(x) + m_1 \geqq u^i(y) + m_2 .$$

Assumption (E) would not necessarily be strong for the sellers. Each seller's amount of money can be only increased by selling his initial endowment. It is not inadequate to assume that the marginal utility of money is constant in the domain where the amount of money consumed is not small. That is, we could neglect the income effect if it is small. If I_i is not too small, then we need to consider only the domain where the marginal utility of money could be assumed to be a constant. It is natural in our model to assume that each I_i $(i \in M)$ is not small. Hence assumption (E) is not inadequate, though it should not be imposed upon the buyers because each buyer's amount of money is decreased by purchasing an indivisible good and the proportion of the payment to the initial income is not negligible.

In Kaneko [1976a] it is permitted that a seller owns more than one kind of goods, i.e., $a_f^i > 0$ for $i \in M_k$ $(f \neq k)$, but it is also assumed that $u^i(x)$ satisfies $u^i(x) = \sum_{f=1}^{s} u^i(x_f e^f)$. When each $u^i(x_f e^f)$ is not constant return to scale, i.e., $u^i(x_f e^f) = x_f u^i(e^f)$, this assumption is inadequate, but rather it is plausible to assume that $u^i(x) = u^i(\sum_{f=1}^{s} x_f e^f)$. Because the indivisible goods are permitted to be different but they are never substantially different goods, which is an implication of (BS') or the reasoning of (CB). By this reason we do not generalize the assignment market to such a form. But we note that the following results can be gained without any essential change in this case.

We put $a_i(g) = u^i(ge^k) - u^i((g-1)e^k)$ for all $i \in M_k$ $(k = 1, \ldots, s)$ and $g \leqq w_i$. We can put $u^i(0) = 0$ without loss of generality. For convenience sake, we put $a_i(g) = 0$ for all $g > w_i$ $(i \in M)$. Then we have

$$(4.2) \qquad u^i(x) = \sum_{g=1}^{x_k} a_i(g) \quad \text{for all } x \in I_+^s .$$

Assumption (CS') implies that $a_i(g) > 0$ for all $g \leqq w_i$ $(i \in M)$. Further we assume that the marginal utility of the k^{th} indivisible good is not increasing for all $i \in M_k$ $(k = 1, \ldots, s)$, that is

(F) (<u>Nonincreasing marginal utility of the indivisible good</u>):

$$a_i(g) \geqq a_i(g+1) \quad \text{for all} \quad g .$$

We call this market model (M,N) a <u>generalized assignment market</u>. The market models of Shapley and Shubik [1972] and Kaneko [1976a] are special cases of the generalized assignment market.[7]

In order to characterize this market model, we shall define another market model (M^*, N) , which we call the <u>agent assignment market</u> of a generalized assignment market (M,N) .[8] The buyers N are the same as the buyers N of the generalized assignment market (M,N) . The sellers M^* consist of $M_1^*, \ldots, M_s^*$, i.e., $M^* = M_1^* \cup M_2^* \cup \ldots \cup M_s^*$ such that

$$(4.3) \qquad M_k^* = \bigcup_{i \in M_k} \{i(1), \ldots, i(w_i)\} \quad \text{for all} \quad k = 1, \ldots, s .$$

We assume that each seller $i(g) \in M_k^*$ $(k = 1, \ldots, s)$ owns one unit of the k^{th} indivisible good and $I_{i(g)} > 0$ amount of money initially, i.e., $(a^{i(g)}, I_{i(g)}) = (e^f, I_{i(g)})$. Seller $i(g)$'s utility function is given as

$$(4.4) \qquad U^{i(g)}(x,m) = u^{i(g)}(x) + m ,$$

$$(4.5) \qquad u^{i(g)}(x) = \begin{cases} a_i(g) & \text{if} \quad x_k \geqq 1 \\ 0 & \text{otherwise.} \end{cases}$$

[7] Exactly, it is slightly different from a generalization of that of Kaneko [1976a] as remarked above.

[8] The same procedure was employed in Kaneko [1976a].

Each seller i attaches the label $i(g)$ showing the name of proprietor i and the number of unit g to each unit and employs an agent acting as a seller of the unit who has the valuation $a_i(g)$ of the unit given by the proprietor i. Of course, the agent assignment market is an assignment market given in the previous section.

We can show that there is a one-to-one mapping from the set of all competitive allocations in the generalized assignment market to that in the agent assignment market.

<u>Theorem 6</u>: If $(p, (x^{M \cup N}, m^{M \cup N}))$ is a competitive equilibrium in the generalized assignment market (M,N), then $(p, (x^{M* \cup N}, m^{M* \cup N}))$ defined by

$$(4.6) \qquad x^{i(g)} = \begin{cases} e^k & \text{if } x_k^i \geqq g \text{ and } i(g) \in M_k^* \\ 0 & \text{otherwise,} \end{cases}$$

$$(4.7) \qquad m^{i(g)} = \begin{cases} p_k + I_{i(g)} & \text{if } x^{i(g)} = 0 \text{ and } i(g) \in M_k^* \\ I_{i(g)} & \text{otherwise,} \end{cases}$$

is a competitive equilibrium in the agent-assignment market (M^*, N).[9] Conversely if $(p, (x^{M* \cup N}, m^{M* \cup N}))$ is a competitive equilibrium in the agent assignment market, then $(p, (x^{M \cup N}, m^{M \cup N}))$ defined by

$$(4.8) \qquad x^i = \sum_{g=1}^{w_i} x^{i(g)} \quad \text{for all } i \in M,$$

$$(4.9) \qquad m^i = I_i + \sum_{g=1}^{w_i} (m^{i(g)} - I_{i(g)}) \quad \text{for all } i \in M$$

---

[9] p, x^j and m^j $(j \in N)$ in $(p, (x^{M* \cup N}, m^{M* \cup N}))$ are the same as those in $(p, (x^{M \cup N}, m^{M \cup N}))$.

is a competitive equilibrium in the generalized assignment market (M,N) .

From Theorems 5 and 6, we get

Theorem 7: There exists a competitive equilibrium in the generalized assignment market.

Since the core includes the competitive allocations, we get

Theorem 8: The core of the generalized assignment market is nonempty.

Proof of Theorem 6: Let $(p, (x^{M\cup N}, m^{M\cup N}))$ be a competitive equilibrium in the generalized assignment market. Then it is easily verified that $x^i = x_k^i e^k \leqq w_k^i e^k$ for all $i \in M_k$ $(k = 1, \ldots, s)$ and $x^j = 0$ or e^k for all $j \in N$. Since $(x^{M\cup N}, m^{M\cup N})$ is an allocation, $\sum_{i\in M\cup N} (x^i, m^i) = \sum_{i\in M\cup N} (a^i, I_i)$.

Let $(p, (x^{M*\cup N}, m^{M*\cup N}))$ be given by (4.6) and (4.7). Clearly the budget constraint (3.12) holds for all $i \in M \cup N$. First, we show that $(x^{M*\cup N}, m^{M*\cup N})$ is an allocation. By (4.6) we have

$$\sum_{i\in M} x_k^i = \sum_{i\in M_k} x_k^i = \sum_{i\in M_k} \sum_{g=1}^{x_k^i} x_k^{i(g)} = \sum_{i(g)\in M_k^*} x_k^{i(g)} = \sum_{i(g)\in M^*} x_k^{i(g)} ,$$

which implies $\sum_{i\in M^*\cup N} x^i = \sum_{i\in M^*\cup N} a^i$. It is clear that $\sum_{i\in M} m^i = \sum_{i\in M} I_i + \sum_{k=1}^{s} \sum_{i\in M_k} (w_i - x_k^i)p_k$, which implies $\sum_{j\in N} m^j = \sum_{j\in N} I_j - \sum_{k=1}^{s} \sum_{i\in M_k} (w_i - x_k^i)p_k$.

Since by (4.7)

$$\sum_{i(g)\in M^*} m^{i(g)} = \sum_{i(g)\in M} I_{i(g)} + \sum_{k=1}^{s} \sum_{i\in M_k} (w_i - x_k^i)p_k ,$$

it holds that

$$\sum_{i(g)\varepsilon M^*} m^{i(g)} + \sum_{j\varepsilon N} m^j = \sum_{i(g)\varepsilon M^*} I_{i(g)} + \sum_{j\varepsilon N} I_j .$$

Clearly the utility maximization (3.11) is true for all $i \varepsilon N$. Hence we need to show that (3.11) is true for all $i \varepsilon M^*$. Suppose that there is an $i(g) \varepsilon M_k^*$ for whom (3.11) is not true. This means that if $x^{i(g)} = e^k$, then $p_k + I_{i(g)} > a_i(g) + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)}$, i.e., $p_k > a_i(g)$ and if $x^{i(g)} = 0$, then $a_i(g) + I_{i(g)} > p_k + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)}$, i.e., $a_i(g) > p_k$. Since $(p, (x^{M\cup N}, m^{M\cup N}))$ is a competitive equilibrium, it holds that

$$\sum_{g=1}^{x_k^i} a_i(g) + p_k(w_i - x_k^i) + I_i \geqq \sum_{g=1}^{h} a_i(g) + p_k(w_i - h) + I_i \quad \text{for all } h .$$

By this inequality and assumption (F) we have $p_k \leqq a_i(g)$ for all $g \leqq x_k^i$ and $p_k \geqq a_i(g)$ for all $g > x_k^i$. This contradicts the above fact, i.e., that if $x^{i(g)} = e^k$ i.e., $x_k^i \geqq g$ by (4.6), than $p_k > a_i(g)$ and if $x^{i(g)} = 0$ i.e., $x_k^i < g$, then $p_k < a_i(g)$. Hence (3.11) is true for all $i(g) \varepsilon M^*$.

Let $(p, (x^{M^*\cup N}, m^{M^*\cup N}))$ be a competitive equilibrium in the agent assignment market, and let $(p, (x^{M\cup N}, m^{M\cup N}))$ be given by (4.8) and (4.9). Similarly with the above, we can show the budget constraint (3.12) for all $i \varepsilon M$ and that $(x^{M\cup N}, m^{M\cup N})$ is an allocation. We show the utility maximization (3.12) for all $i \varepsilon M$. Suppose that there is a seller $i \varepsilon M_k$ such that for some y_k^i

$$u^i(x^i) + m^i = \sum_{g=1}^{x_k^i} a_i(g) + p_k(w_i - x_k^i) + I_i$$

$$< \sum_{g=1}^{y_k^i} a_i(g) + p_k(w_i - y_k^i) + I_i = u^i(y_k^i e^k) + m_1^i .$$

If $x_k^i > y_k^i$, then we have $\sum_{g=y_k^i+1}^{x_k^i} a_i(g) < p\ (x^i - y^i)$, which implies that there is an $i(g) \varepsilon M_k^*$ such that $p_k > a_i(g)$ and $g \leqq x_k^i$. If $x^{i(g)} = 0$, then by (4.8) $x^{i(g')} = e^k$ for some $g' > g$. For this g' , we have $p_k > a_i(g) \geqq a_i(g')$ by assumption (F). Hence we can assume that $p_k > a_i(g)$ and $x^{i(g)} = e^k$. Thus we have $p_k + I_{i(g)} > a_i(g) + I_{i(g)}$ $= u^{i(g)}(x^{i(g)}) + m^{i(g)}$, which is a contradiction to the supposition that $(p, (x^{M* \cup N}, m^{M* \cup N}))$ is a competitive equilibrium. If $x_k^i < y_k^i$, then we have $\sum_{g=x_k^i+1}^{y_k^i} a_i(g) > p_k(y_k^i - x_k^i)$, which implies that there is an $i(g) \varepsilon M_k^*$ such that $a_i(g) > p_k$ and $x^{i(g)} = 0$. Hence we have $a_i(g) + I_{i(g)} = u^{i(g)}(a^{i(g)}) + I_{i(g)} > p_k + I_{i(g)} = u^{i(g)}(x^{i(g)}) + m^{i(g)}$, which is a contradiction.

Q.E.D.

In the assignment market, the core always coincides with the set of all competitive allocations. It is, however, not necessarily true in the generalized assignment market, but we can give a weak condition for the equivalence.

In Kaneko [1976a] the following theorems are given as more general versions in the generalized assignment market with the transferable utility assumption. As the proofs of them are almost the same as Lemma 3 and

Theorem II in Kaneko [1976a], we do not give the proofs in this paper.

If two sellers i_1 and i_2 in M have the same preference orderings, i.e., the same utility functions and the same initial endowment, the sellers are called the <u>same type</u>.

<u>Theorem 9</u>. If for each $i \varepsilon M_k$, there is at least one seller $i' \varepsilon M_k$ $(i \neq i')$ who is the same type as i, then the k^{th} indivisible good has a common price in each allocation in the core, i.e., there is a p_k such that

$$(4.10) \qquad m^i = I_i + p_k(w_i - x_k^i) \text{ for all } i \varepsilon M_k ,$$

$$(4.11) \qquad m^i = I_i - p_k \text{ for all } i \varepsilon N \text{ with } x^i = e^k .$$

<u>Theorem 10</u>. If for each $i \varepsilon M$, there is at least one seller i' $(i \neq i')$ who is the same type as i, then the core coincides with the set of all competitive allocations.

In Kaneko [1976a], the core is considered in the case where the supposition of Theorem 10 is not true, i.e., a seller becomes a monopolist in a certain sense. It says that the core permits price discrimination. This result is also true in the generalized assignment market without the transferable utility assumption, but as it can be gained in almost the same way, we give an explanation in a diagram.

Let $s = 1$. Then the supply curve and the demand curve are drawn as Figure 1. Let us consider the case where the supposition of Theorem 9 is not true. Let $a_M = \min\{a_i(g) : i \varepsilon M - \{1\}, g = 1, \ldots, w_i\}$. If a_M is greater than the intersection of the supply and demand curves, e.g., a_M in Figure 1, then the core permits price discrimination in any allocation

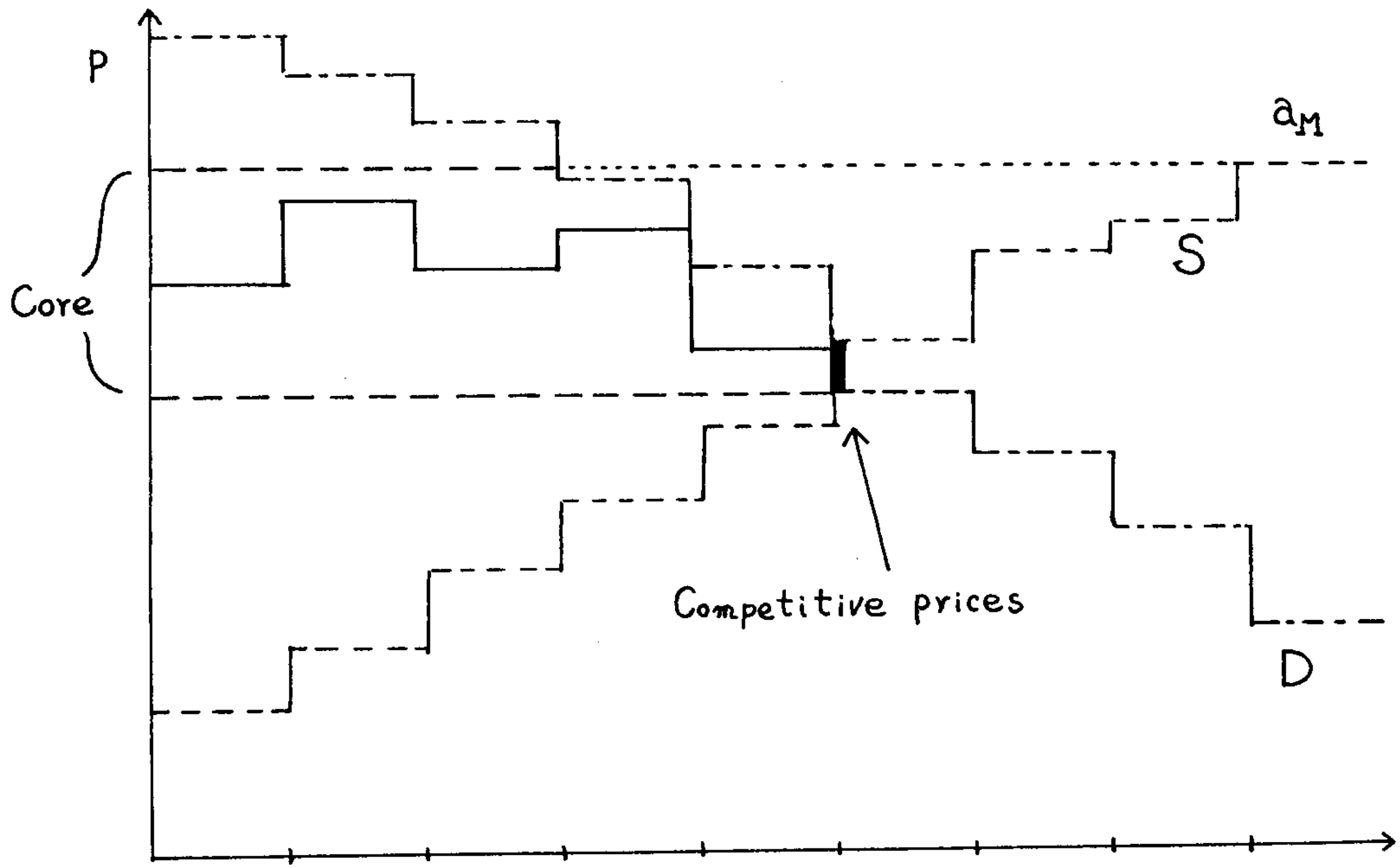

Figure 1

in the core, i.e., seller 1 trades the good at different prices not more than a_M with buyers. See Figure 1. Seller 1 is a monopolist in this sense. If a_M is not greater than the intersection, the core coincides with the set of all competitive equilibria, because one seller becomes a competitor with seller 1. Of course, the good is traded at a common price in the intersection. This is precisely a more sufficient condition than Theorem 10 for the equivalence of the core and the set of all competitive allocations, which is corresponding to the supposition of Theorem II of Kaneko [1976a].

When $s \geqq 2$, the similar price discrimination occurs in the case where the supposition of Theorem 9 is not true. But since the indivisible goods are not substantially different, any seller could regard the sellers owning the other goods even beside the ones owning the same good as competitors with him. This fact makes the price discrimination in the core not so wide.

Clearly the sufficient condition for the equivalence given by Theorem 10 is weak. Further even if the equivalence does not hold, the price discrimination in the core is not too wide. Then we get the conclusion that in most classes of generalized assignment markets, the competitive equilibria could be representatives of the core.

Finally, we note that permissible coalitions can be constrained to a subclass of that of all the coalitions in the generalized assignment market for the equivalence of the core and the competitive equilibria similarly to Corollary 6, which is shown in the case with the transferable utility assumption in Lemma 2 of Kaneko [1976a].

APPENDIX

Proof of Lemma 2: For any $(x,m) \varepsilon X$ there is a unique real number d by assumptions (A) and (B) such that $(x,m)Q_i(0,d)$. We define $U^i(x,m)$ by:

(A.1) $$U^i(x,m) = d .$$

It can be easily verified that this U^i satisfies $(x, m_1)R_i(y, m_2)$ iff $U^i(x, m_1) \geqq U^i(y, m_2)$. We show that U^i is a continuous function. Let $\{(x^\alpha, m^\alpha)\}$ be a sequence in X which converges to $(x,m) \varepsilon X$. Since I_+^s has the discrete topology, there is an integer k such that $x^\alpha = x$ for all $\alpha \geqq k$. Since $\{(x^\alpha, m^\alpha)\}$ is a converging sequence, there is a m_1 such that $0 \leqq m^\alpha \leqq m_1$ for all α , which implies $0 \leqq U^i(x, m^\alpha) \leqq U^i(x, m_1)$ for all $\alpha \geqq k$. Hence the sequence $\{U^i(x, m^\alpha)\}$ has at least one limit point in R_+ . Let the sequence $\{U^i(x, m^\alpha)\}$ have a limit point not equal to $U^i(x,m)$ and let $\{U^i(x, m^{\alpha^\beta})\}$ be a converging subsequence such that $\lim U^i(x, m^{\alpha^\beta}) = u \neq U^i(x,m)$. Let ε be a sufficiently small positive number. Let $u > U^i(x,m)$. Then we have $(0, U^i(x,m) + \varepsilon)P_i(x,m)$ by assumption (A) and (A.1). By assumptions (A) and (B) there is a $\delta > 0$ such that $(0, U^i(x,m) + \varepsilon)Q_i(x, m+\delta)$, i.e., $U^i(x,m) + \varepsilon = U^i(x, m+\delta)$. But it holds that $m^{\alpha^\beta} < m+\delta$ for all $\beta \geqq$ some β_0 , which implies $U^i(x, m^{\alpha^\beta}) \leqq U^i(x, m+\delta) = U^i(x,m) + \varepsilon \leqq u$. This is a contradiction. Next let $u < U^i(x,m)$. Since $U^i(x, m^s) \geqq U^i(x,0)$ by assumption (A), we have $u \geqq U^i(x,0)$. Hence there is an m_0 such that $(0, u+\varepsilon)Q_i(x, m_0)$, i.e., $u+\varepsilon = U^i(x, m_0)$. Since ε is sufficiently small, we have $m_0 < m$. But since there is a β_0 such that

$U^i(x, m^{\alpha^\beta}) < u+\varepsilon = U^i(x, m_0)$ for all $\beta \geqq \beta_0$ by $u = \lim U^i(x, m^{\alpha^\beta})$, we have $m_0 > m^{\alpha^\beta}$ by assumption (A). This contradicts that $\lim m^{\alpha^\beta} = m$. Hence we have shown that any limit point coincides with $U^i(x,m)$.

Q.E.D.

Proof of Lemma 3: If $\sum_{i\varepsilon S} I_i > \sum_{i\varepsilon S} m^i$, then S can improve upon $(x^{M\cup N}, m^{M\cup N})$ by the S-allocation (y^S, m_1^S) such that $y^i = x^i$ for all $i \varepsilon S$ and $m_1^i = m^i + (\sum_{i\varepsilon S}(I_i - m^i))/|S|$ for all $i \varepsilon S$. Let $\sum_{i\varepsilon S} I_i = \sum_{i\varepsilon S} m^i$. Then there is an $f \leqq s$ such that $\sum_{i\varepsilon S} x_j^i < \sum_{i\varepsilon S} a_f^i$, which implies that $x_f^{i_0} = 0$ for some $i_0 \varepsilon S \cap M_f$. By assumption (CB) we have $U^{i_0}(e^f, m^{i_0}) > U^{i_0}(x^{i_0}, m^{i_0}) = U^{i_0}(0, m^{i_0})$. If ε is a sufficiently small positive number, then $U^{i_0}(e^f, m^{i_0} - \varepsilon) > U^{i_0}(x^{i_0}, m^{i_0})$ by Lemma 2. Hence it holds that $U^i(x^i, m^i + \varepsilon/(|S| - 1)) > U^i(x^i, m^i)$ for all $i \varepsilon S - \{i_0\}$, which implies that S can improve upon $(x^{M\cup N}, m^{M\cup N})$.

Q.E.D.

Proof of Lemma 4: If $m^i = 0$ for some $i \varepsilon N$, then $U^i(x^i, m^i) < U^i(a^i, I_i)$ by assumption (D), which contradicts that $(x^{M\cup N}, m^{M\cup N})$ is in the core. If $m^i = 0$ for some $i \varepsilon M_f$, then $U^i(0, m^i) < U^i(e^f, 0) < U^i(e^f, I_i)$, which is a contradiction. Hence we have $m^i > 0$ for all $i \varepsilon M \cup N$.

Suppose that there is a buyer j with $x^j \neq 0$. Then by assumption (CB) there is an $f \leqq s$ such that $x_f^j > 0$ and $U^j(e^f, m^j) = U^j(x^j, m^j)$. Then there is a seller $i \varepsilon M_f$ with $x_f^j = 0$. We show that $m^i + m^j = I_i + I_j$, $x^i = 0$ and $x^j = e^f$. If $m^i + m^j > I_i + I_j$, then the coalition $(M - \{i\}) \cup (N - \{j\})$ can improve upon $(x^{M\cup N}, m^{M\cup N})$, because $\sum_{t\neq i,j} a^t \geqq \sum_{t\neq i,j} x^t$ and $\sum_{t\neq i,j} I_t > \sum_{t\neq i,j} m_t$. If $m^i + m^j < I_i + I_j$, then the coalition $\{i,j\}$ can improve upon $(x^{M\cup N}, m^{M\cup N})$ by the allocation $((y^t, m_1^t))_{t=i,j}$ such that

$$y^i = 0 \ , \quad m_1^i = m^i + (I_i + I_j - m^i - m^j)/2$$

$$y^j = e^f, \quad m_1^j = m^j + (I_i + I_j - m^i - m^j)/2 \ .$$

Hence it holds that $m^i + m^j = I_i + I_j$. If $x_{f'}^i = 1$ for some $f' \neq f$ or $x^j \geq e^f$, then the coalition $(M - \{i\}) \cup (N - \{j\})$ can improve upon $(x^{M \cup N}, m^{M \cup N})$ by Lemma 3 because $\sum_{t \neq i,j} a^t \geq \sum_{t \neq i,j} x^t$ and $\sum_{t \neq i,j} m^t = \sum_{t \neq i,j} I_t$. Hence we have $x^i = 0$ and $x^j = e^f$.

Repeating this argument, we choose all such pairs and denote the set of them by $\{(i_1, j_1), \ldots, (i_k, j_k)\}$. We put $M_0 = M - \{i_1, \ldots, i_k\}$ and $N_0 = N - \{j_1, \ldots, j_k\}$. Clearly (3.8) holds for these $\{(i_1, j_1), \ldots, (i_k, j_k)\}$. Of course, no buyer $j \in N_0$ has any indivisible good, i.e., $x^j = 0$ for all $i \in N_0$. For, otherwise, we can find another pair (i_{k+1}, j_{k+1}) for which (3.8) holds by the above argument. Hence any sellers in M_0 do not trade indivisible goods with any buyers in N_0 . Furhter even when two sellers in M_0 trade indivisible goods with each other, they have no interest, or decrease their utilities. So, all sellers in M_0 do not trade at all. The similar argument is valid for buyers in N_0 . By this reason (3.9) holds for all sellers in M_0 and buyers in N_0 .

Q.E.D.

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