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# THE CHARACTERISTIC FUNCTION OF THE F DISTRIBUTION

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## 0. ABSTRACT

Formulae that are given in the literature for the characteristic function of the $F$ distribution are incorrect and imply that the distribution has finite moments of all orders. Correct formulae are derived and the asymptotic behavior of the characteristic function in the neighborhood of the origin is characterized.

## 1. INTRODUCTION

Major reference works[1] on distribution theory in statistics state that the characteristic function of an $F$ variate is given by the confluent hypergeometric function in the central case and an infinite series of confluent hypergeometric functions in the non-central case. Thus, if $F$ has a central $F$ distribution with $n_1$ and $n_2$ degrees of freedom the characteristic function of $F$ is stated to be

$$\phi(s) = E(e^{iFs}) = {}_1F_1\left(\frac{n_1}{2}, \frac{-n_2}{2}; \frac{-n_2}{n_1}is\right) \tag{1}$$

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[1]See, for example, Johnson and Kotz [2], p. 78; Zelen and Severo [7], p. 946.

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where ${}_1F_1( \ , \ ; \ )$ is the confluent hypergeometric function. In the non-central case, the characteristic function is said to be given by the series[2]

$$(2) \qquad \phi(s) = e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \, {}_1F_1\left(\frac{n_1}{2} + j, \frac{-n_2}{2}; \frac{-n_2}{n_1} is\right)$$

where $\lambda$ is the non-centrality parameter.

The fact that expressions (1) and (2) are incorrect seems to have past unnoticed in the literature. To see that these expressions are wrong we need only observe that, whereas the F distribution has finite moments only of order less than $n_2/2$ , (1) and (2) imply that all moments of the distribution are finite since ${}_1F_1(a,b;z)$ is an entire function of z [1]. The problem is that (1) and (2) are close to, but do not represent exactly, only one part of the relevant characteristic functions. As we see in the next section, the remaining parts do not possess continuous derivatives of all orders at the origin. These non-differentiable parts are omitted completely from (1) and (2). Moreover, as indicated above, (1) and (2) do not even give the differentiable component of the characteristic function correctly.

[2] Patnaik [5], p. 221 and Johnson and Kotz [2], p. 190.

## 2. FORM OF THE CHARACTERISTIC FUNCTION

We start with a central F variate with $n_1$ and $n_2$ degrees of freedom and density function

$$(3) \qquad \mathrm{pdf}(x) = \frac{n_1^{\frac{1}{2}n_1} n_2^{\frac{1}{2}n_2}}{B\left(\frac{1}{2}n_1, \frac{1}{2}n_2\right)} \frac{x^{\frac{1}{2}n_1 - 1}}{(n_2 + n_1 x)^{\frac{1}{2}(n_1+n_2)}}, \quad x > 0 .$$

The characteristic function of F is then given by the integral

$$(4) \qquad \mathrm{cf}(s) = \frac{1}{B\left(\frac{1}{2}n_1, \frac{1}{2}n_2\right)} \int_0^\infty e^{is\frac{n_2}{n_1}y} y^{\frac{1}{2}n_1 - 1} (1+y)^{-\frac{1}{2}(n_1+n_2)} dy .$$

For the complex variable z and complex parameters a and c such that $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(a) > 0$ we have the following integral

$$(5) \qquad \int_0^\infty e^{-zt} t^{a-1} (1+t)^{c-a-1} dt = \Gamma(a)\Psi(a,c;z)$$

which defines the confluent hypergeometric function of the second kind.[3] Now the domain of definition of the function $\Psi(a,c;z)$ can be extended beyond $\mathrm{Re}(z) > 0$ .[4] In the present case, we need only note that the integral representation continues to hold for z on the imaginary axis if $\mathrm{Re}\{(c-a-1)+(a-1)\} < -1$ , i.e. if $\mathrm{Re}(c) \leq 1-\varepsilon$ for some $\varepsilon > 0$ , since this condition ensures that the integral converges absolutely.

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[3] See [1], p. 255 and [3], pp. 267-268.

[4] [1], p. 256.

It now follows from (4) and (5) that

$$(6) \qquad cf(s) = \frac{\Gamma\left(\frac{1}{2}n_1\right)}{B\left(\frac{1}{2}n_1, \frac{1}{2}n_2\right)}\Psi\left(\frac{1}{2}n_1, 1-\frac{1}{2}n_2; -\frac{n_2}{n_1}is\right) = \frac{\Gamma\left(\frac{1}{2}(n_1+n_2)\right)}{\Gamma\left(\frac{1}{2}n_2\right)}\Psi\left(\frac{1}{2}n_1, 1-\frac{1}{2}n_2; -\frac{n_2}{n_1}is\right)$$

Series representations of (6) can be obtained from the following formulae:[5]

$$(7) \qquad \Psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\, {}_1F_1(a-c+1, 2-c; z)$$

for non-integral $c$ ; and[6] for $c = 1-n$ with $n = 0, 1, 2, \ldots$

$$(8) \qquad \Psi(a, 1-n; z) = x^n\Psi(a+n, n+1; z)$$

$$= \frac{(-1)^{n-1}z^n}{n!\Gamma(a)}\left\{{}_1F_1(a+n, n+1; z)\ln z + \sum_{r=0}^{\infty} \frac{(a+n)_r}{(n+1)_r r!}[\Psi(a+n+r) - \Psi(1+r) - \Psi(1+n+r)]z^r\right\}$$

$$+ \frac{(n-1)!}{\Gamma(a+n)} \sum_{r=0}^{n-1} \frac{(a)_r}{(1-n)_r r!} z^r$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function and the final term in (8) is omitted if $n = 0$ .

Setting $a = \frac{1}{2}n_1$ , and $c = 1 - \frac{1}{2}n_2$ in (7) we deduce, for the case of $n_2$ odd, from (7) that $cf(s)$ has continuous derivatives at the origin $s = 0$ only up to order $M$ where $M$ is the largest integer $< \frac{1}{2}n_2$ . Since[7]

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[5]Equation 7 [1], p. 257.

[6]Using equation (6) on page 257 and equation (13) on page 261 of [1].

[7][1], p. 254.

$$\frac{d^n}{dz^n}\,{}_1F_1(a,c;z) = \frac{(a)_n}{(c)_n}\,{}_1F_1(a+n,\ c+n;\ z)$$

we find from (6) and (7) that

$$i^{-j}cf^{(j)}(0) = \frac{\left(\frac{1}{2}n_1\right)_j}{\left(1-\frac{1}{2}n_2\right)_j}\left(-\frac{n_2}{n_1}\right)^j$$

$$= \left(\frac{n_2}{n_1}\right)^j \frac{n_1(n_1+2)\ldots(n_1+2j-2)}{(n_2-2)(n_2-4)\ldots(n_2-2j)}\ ,\quad j = 1,\ \ldots,\ M$$

corresponding with the known $j^{th}$ moment about the origin of the F distribution.

Setting $a = \frac{1}{2}n_1$ and $c = 1 - \frac{1}{2}n_2 = 1-n$ in (8) we deduce also that, for the case of $n_2$ even, cf(s) has continuous derivatives at the origin $s = 0$ only up to order $M = n-1$ . These derivatives can be obtained from the coefficients in the final summation of (8). We find in this case

$$i^{-j}cf^{(j)}(\ ) = \frac{\Gamma\left(\frac{1}{2}(n_1+n_2)\right)}{\Gamma\left(\frac{1}{2}n_2\right)}\cdot\frac{(n-1)!\ \left(\frac{1}{2}n_1\right)_j}{\Gamma\left(\frac{1}{2}n_1+n\right)(1-n)_j}\left(-\frac{n_2}{n_1}\right)$$

$$= \left(\frac{n_2}{n_1}\right)^j \frac{n_1(n_1+2)\ldots(n_1+2j-2)}{(n_2-2)(n_2-4)\ldots(n_2-2j)}$$

once again as required.

For non-integral $c = 1 - \frac{1}{2}n_2$ we note from [7] that the first term in the representation of function $\Psi(a,c;z)$ in terms of the confluent hypergeometric function, is close to but not equivalent to the expression

(1) for the characteristic function given in the literature. Specifically, for $a = \frac{1}{2}n_1$, $c = 1 - \frac{1}{2}n_2$ ( $n_2$ odd) and $z = -(n_2/n_1)is$, we have

$$\frac{\Gamma\left(\frac{1}{2}(n_1 + n_2)\right)}{\Gamma\left(\frac{1}{2}n_2\right)} \frac{\Gamma(1-c)}{\Gamma(a-c+1)} {}_1F_1(a,c;z) = {}_1F_1\left(\frac{1}{2}n_1, 1 - \frac{1}{2}n_2; -\frac{n_2}{n_1}is\right) .$$

Note also that for $n_2$ even the conventional expression (1) for the characteristic function is undefined since the confluent hypergeometric function ${}_1F_1(a,c;z)$ has simple poles at the points $c = 0, -1, -2, \ldots$ .

These results can readily be extended to the case of a non-central $F$ variate and we only state the final formula here. If $F$ has a non-central $F$ distribution with degrees of freedom $n_1$ and $n_2$ and non-centrality parameter $\lambda$, then the characteristic function of $F$ is given by

$$\text{(9)} \qquad cf(s) = \frac{e^{-\frac{1}{2}\lambda}}{\Gamma\left(\frac{1}{2}n_2\right)} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\lambda\right)^j \Gamma\left(\frac{n_1 + n_2}{2} + j\right)}{j!} \Psi\left(\frac{1}{2}n_1 + j, 1 - \frac{1}{2}n_2; -\frac{n_2}{n_1}is\right)$$

where, as before, the series representations (7) and (8) for the $\Psi( , ; )$ function can be used in the respective cases of $n_2$ odd and $n_2$ even.

## 3. BEHAVIOR AS $s \to 0$

Considering first the case where $n_2$ is odd we have from (6) and (7)

$$cf(s) = {}_1F_1\left(\tfrac{1}{2}n_1,\ 1-\tfrac{1}{2}n_2;\ -\frac{n_2}{n_1}is\right)$$

$$+\frac{\Gamma\left(-\frac{1}{2}n_2\right)}{B\left(\frac{1}{2}n_1,\ \frac{1}{2}n_2\right)}|s|^{\frac{1}{2}n_2} e^{\frac{in_2\pi}{4}\operatorname{sgn}(s)}\ {}_1F_1\left(\tfrac{1}{2}n_1+\tfrac{1}{2}n_2,\ 1+\tfrac{1}{2}n_2,\ -\frac{n_2}{n_1}is\right)$$

(10)
$$= \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}n_1\right)_k\left(-\frac{n_2}{n_1}\right)^k}{\left(1-\frac{1}{2}n_2\right)_k k!}(is)^k + \frac{\Gamma\left(-\frac{1}{2}n_2\right)}{B\left(\frac{1}{2}n_1,\ \frac{1}{2}n_2\right)}|s|^{\frac{1}{2}n_2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}$$

$$\cdot\frac{\left(\frac{1}{2}n_1+\frac{1}{2}n_2\right)_m\left(\frac{in_2\pi}{4}\right)^{\ell}\left(-\frac{n_2}{n_1}\right)^m}{\left(1+\frac{1}{2}n_2\right)_m \ell!m!}(\operatorname{sgn}(s))^{\ell}(is)^m .$$

When $n_2$ is even and $\frac{1}{2}n_2 = n$ we have from (6) and (8)

(11)
$$cf(s) = \sum_{k=0}^{n-1}\frac{\left(\frac{n_1}{2}\right)_k\left(-\frac{n_2}{n_1}\right)^k}{\left(1-\frac{1}{2}n_2\right)_k k!}(is)^k - \frac{(is)^n}{B\left(\frac{1}{2}n_1,\ \frac{1}{2}n_2\right)}$$

$$\cdot\left[{}_1F_1\left(\tfrac{1}{2}n_1+\tfrac{1}{2}n_2,\ 1+\tfrac{1}{2}n_2;\ -\frac{n_2}{n_1}is\right)(\ln(n_2/n_1)+\ln|s|-\tfrac{1}{2}i\pi\ \operatorname{sgn}(s))\right.$$

$$\left.+\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}n_1+\frac{1}{2}n_2\right)_m\left(-\frac{n_2}{n_1}\right)^m}{\left(1+\frac{1}{2}n_2\right)_m m!}\left\{\Psi\left(\tfrac{1}{2}n_1+\tfrac{1}{2}n_2+m\right)-\Psi(1+m)-\Psi\left(1+\tfrac{1}{2}n_2+m\right)\right\}(is)^m\right]$$

which can be written as an asymptotic series as $s \to 0$ by expanding the ${}_1F_1(\ ,\ ;\ )$ function in the usual way.

Both (10) and (11) belong to the class of characteristic functions whose behavior as $s \to 0$ is given by the following asymptotic series

$$\text{(12)} \qquad cf(s) \sim e^{i\eta s}\left\{\sum_{m=0}^{M-1} p_m (is)^m + |s|^\mu \sum_{j=0}^{\infty} \sum_{k=0}^{K(j)} \sum_{\ell=0}^{L(j)} q_{jk\ell} |s|^j (\text{sgn}(s))^k (\ln|s|)^\ell \right\}$$

where $p_m$ and $q_{jk\ell}$ are coefficients, $\mu \geq M$ and $L(j)$ is usually either 0 or 1 for all $j$. The first summation in (12) is analytic and ensures that integral moments of the distribution exist to order $M-1$ if this is an even integer and $M-2$ if $M-1$ is odd. The second summation in (12) is instrumental in determining the form of the tails of the distribution corresponding to $cf(s)$. In cases where the density function $pdf(x)$ exists, (12) can be used directly to extract an asymptotic expansion of $pdf(x)$ as $x \to \pm\infty$. This work is being reported elsewhere [6].

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