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ON A LEMMA OF AMEMIYA

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by

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0. INTRODUCTION

In deriving the asymptotic properties of the maximum likelihood estimator in the general non-linear simultaneous equation model Amemiya [1] makes extensive use of the following lemma:

Lemma If $u_1,\ldots,u_n$ *are jointly normal with mean zero and covariance matrix* (σ_{ij}) *and* $h(u_1,\ldots,u_n)$ *is such that* $E(h)$ *and* $E(\partial h/\partial u_i)$ *are finite then* $E(\partial h/\partial u_i) = E(h\Sigma_{j=1}^{n}\sigma^{ij}u_j)$.

When the conclusion of this lemma holds, the limit of the mean of the gradient of the log likelihood is zero at the true values of the parameters. This property ensures that there is a consistent root of the likelihood equation, provided a number of other more usual assumptions are made concerning the existence and nature of convergence of certain summations that appear in the likelihood and its first two derivatives.

1. A COUNTEREXAMPLE TO THE LEMMA

The proof of the lemma depends on the following application of integration by parts (Amemiya's equation (3.7) on page 958):

$$(1) \quad \int_{-\infty}^{\infty}\frac{\partial h}{\partial u_i}\,\phi du_i = \Big[h\phi\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}h\sum_j\sigma^{ij}u_j\phi du_i$$

where ϕ is the joint density of $u' = (u_1,\ldots,u_n)$. Amemiya argues that "the first term on the right side of (1) is zero because E(h) is finite. Therefore, integrating both sides of (1) with respect to the remaining n-1

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components of u we get the desired result".

This argument would seem to be invalid since the existence and finiteness of E(h) is not sufficient to ensure that the first term on the right side of (1) is zero. Simple counterexamples can be constructed in the present case using one of the standard examples in analysis of convergent improper integrals whose integrands do not tend to zero at infinity (see, for example, #12 page 45 in Gelbaum and Olmsted [3]).

A complete counterexample to the lemma itself is more complicated since h is such that E(h) and $E(\partial h/\partial u_i)$ are finite. Moreover, the use of the lemma in Amemiya's paper involves setting h equal to certain derivatives of the functions that appear in the structural specification of the non linear model. Since the latter are assumed by Amemiya to be continuously differentiable, we will require the same of the function h. We can deal with the simple scalar case with n=1. A function satisfying the conditions of the lemma but not the conclusion is then

$$(2)\quad h(u) = k(u)/\phi(u)$$

with

$$(3)\quad k(u) = \left\{\frac{1}{1+(u-n)}\right\}\exp\left\{-\frac{n^4(u-n)^2}{1-4(u-n)^2}\right\}\exp\left\{-\frac{(u-n)^2}{2n^{-4}}\right\} \qquad n-\tfrac{1}{2}\le u\le n+\tfrac{1}{2} \quad (n=1,2,3,\ldots.)$$

$$0 \qquad \text{otherwise (i.e. } u<\tfrac{1}{2})$$

To verify that E(h) exists we need only check that k(u) is integrable. We have, in fact,

$$(4)\quad \int_{-\infty}^{\infty} k(u)du = \int_{\frac{1}{2}}^{\infty} k(u)du = \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k(u)du$$

provided the series converges. It is bounded by

$$(5)\quad \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |k(u)|\,du < \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} 2 \exp\left\{-\frac{(u-n)^2}{2n^{-4}}\right\} du$$

$$= 2\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{\sqrt{2\pi}n^{-2}} \exp\left\{-\frac{(u-n)^2}{2n^{-4}}\right\} du$$

$$< 2\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}n^{-2}} \exp\left\{-\frac{(u-n)^2}{2n^{-4}}\right\} du$$

$$= 2\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{2^{\frac{1}{2}}\pi^{3/2}}{3}.$$

It follows that $k(u)$ is absolutely integrable. However, $k(u)$ does not have a limit as $u \to \infty$ (the sequences $\{k(n)\}$ and $\{k(n+\frac{1}{4})\}$, for instance, tend to unity and zero respectively). This contradicts the argument in Amemiya's lemma, noted above, that the first term on the right side of (1) is zero.

Finally, in order to verify that all the conditions of the stated lemma hold we need to show that $E(\partial h/\partial u)$ is finite. Now,

$$\frac{\partial h}{\partial u} = \frac{k'(u)}{\phi(u)} + \frac{u}{\sigma}\frac{k(u)}{\phi(u)}$$

so it will suffice to show that k' is integrable. We have

$$(6)\quad k'(u) = -\left(\frac{1}{1+(u-n)}\right)^2 \exp\left\{-\frac{n^4(u-n)^2}{1-4(u-n)^2} - \frac{(u-n)^2}{2n^{-4}}\right\}$$

$$-\left(\frac{1}{1+(u-n)}\right)\left(\frac{2n^4(u-n)}{(1-4(u-n)^2)^2}\right) \exp\left\{-\frac{n^4(u-n)^2}{1-4(u-n)^2} - \frac{(u-n)^2}{2n^{-4}}\right\}$$

$$- \left(\frac{1}{1+(u-n)}\right) \left(\frac{u-n}{n^{-4}}\right) \exp \left\{ - \frac{n^4 (u-n)^2}{1-4(u-n)^2} - \frac{(u-n)^2}{2n^{-4}} \right\}$$

for $n-\frac{1}{2}<u\leq n+\frac{1}{2}$ (n=1,2,3....) and zero elsewhere. Using the same argument as in (4) and (5), we can establish that the first term on the right side of (6) is absolutely integrable. For the second term we have a series whose n'th term is given by

$$(7) \quad - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{1+z}\right) \left(\frac{2n^4 z}{(1-4z^2)^2}\right) \exp \left\{ -n^4 \left[\frac{z^2}{1-4z^2} + \frac{z^2}{2} \right] \right\} dz$$

As $n\to\infty$ the region in which the exponential factor in the integrand is significantly different from zero becomes a smaller and smaller neighborhood of the origin. At this point the integrand itself is zero and the function $\left[z^2/(1-4z^2) + z^2/2\right]$ in the exponent has a minimum. The latter function has a non-zero second derivative at z=0 so that, using Laplace's method to represent (7) as $n\to\infty$ (see, for instance, equation (5-1.21) on page 185 of Bleistein and Handelsmon [2]) we find that it is of $0\left(n^4 (n^4)^{-3/2}\right) = 0(n^{-2})$. The series involving (7) as its n'th term therefore converges by comparison. A similar argument verifies that the third term on the right side of (6) is integrable. This proves that the function h defined by (2) and (3) satisfies the conditions of the lemma. However, as noted above, $h(u)\phi(u) = k(u)$ does not approach a limit of zero at infinity and the lemma is false.

2. COMMENTS

One apparent way of tightening up the Lemma is to restrict the class of allowable h functions so that the argument following (1) is valid. It is too much to require that $h(u)$ tends to zero at the limits of its domain since we want to set $h(u)$ equal to the functions that appear in the structural specification of the model and their derivatives. It will often be unrealistic to require these functions to vanish at infinity. On the other hand, we may suitably bound the growth of these functions to ensure that $h(u)\phi(u)$ tends to zero as $u \to \infty$. This requirement then allows the structural functions that are present in the model to be unbounded over the whole space of realizations of the random elements in the model (just as they are in the linear simultaneous equations framework or the simple regression model). But the requirement will also prevent the structural functions from taking on values which become too large for certain realizations of the random elements relative to the probability that the random elements actually assume these realizations.[1]

An alternative approach to the development of a theory for non-linear simultaneous equations would be to work with a more limited class of non-linear functions as structural specifications in the first place and attempt to characterize the associated class of error distributions for which the quasi-maximum likelihood (QML) estimator is consistent. This approach would have the advantage of allowing the linear simultaneous equations model results (namely, that the class of error distributions for which the QML is consistent is very wide) to fall out naturally, rather than appear as pathological cases of a general theory where normality of the errors appears as a crucial assumption in the consistency proof.

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1. This mild requirement is certainly satisfied in linear models for a wide class of error distributions.

REFERENCES

1 Amemiya, T., "The Maximum Likelihood and Non Linear Three Stage Least Squares Estimator in the General Non-Linear Simultaneous Equations Model," Econometrica, 45, 1977, pp. 955-968.

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3 Gelbaum, B. R. and J. R. Olmsted, Counterexamples in Analysis, Holden Day, 1964.