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INTERGENERATIONAL POLITICAL ECONOMY

(A GAME THEORETIC MODEL OF HOW TO GLUE THE GENERATIONS TOGETHER)

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## INTERGENERATIONAL POLITICAL ECONOMY

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by

Martin Shubik

### 1. INTRODUCTION

The purpose of this paper is to show that if reasonable care is taken in well-defining the types of games of strategy which arise naturally from considerations of the multigenerational aspects of any economy; then many different plausible explanations of intergenerational links will all be consistent with a stationary economy with intergenerational transfers of wealth. Which of the models (or blends of the models) is the best appears to be more of an empirical rather than theoretical question.

This result suggests that a specific program of interdisciplinary work involving both model building based upon empirical investigation and the development of the mathematical methods of intergenerational or multistage variable person games is called for. In particular, it is argued that biology, anthropology, sociology, demography, political science,

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law and economics all are concerned with behavior with highly different time spans and different degrees of conscious behavior. The prime emphasis here is economic, but the stress is that for the successful development of an economic dynamics the correct links with the other disciplines must be forged.

## 2. MULTISTAGE VARIABLE PERSON GAMES

Consider a game with individuals who each live 3 periods. Each individual is strategically active only at age 2.

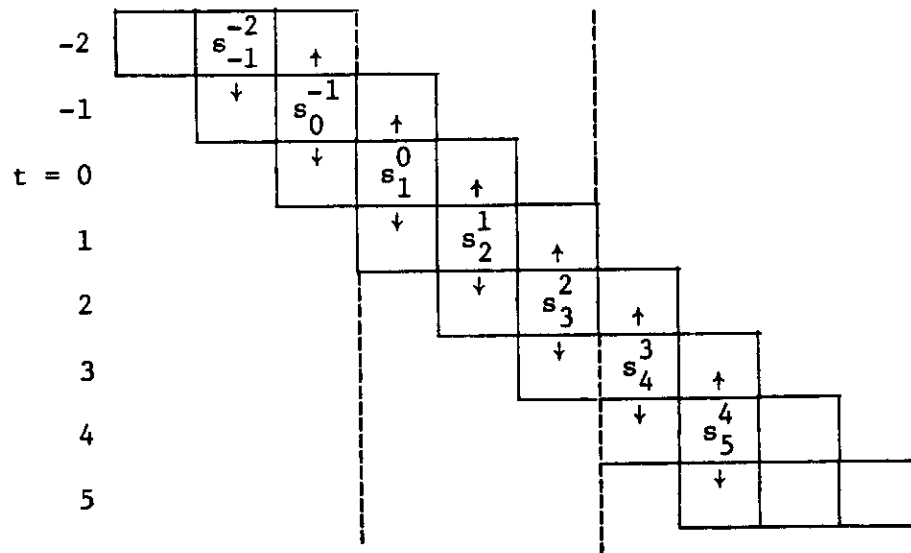


FIGURE 1

Suppose that when individual of generation  $t$  is called upon to move at time  $t+1$  he is presented with 3 units of "manna" (which cannot be stored), and a move is a selection of three numbers

$$(y_{t+1}^{t,t-1}, y_{t+1}^{t,t}, y_{t+1}^{t,t+1})$$

such that  $\sum_{j=-1}^1 y_{t+1}^{t,t+j} = 3$ , where  $y_{t+1}^{t,t+j}$  is the amount received by

$t+j$  from  $t$  at period  $t+1$ . As, at time  $t+1$  only generation  $t$  is giving we may replace  $y_{t+1}^{t,t+j}$  by  $y_{t+1}^{t+j}$ .

## 2.1. Moves and Strategies

Denote the strategy set of the player of generation  $t$  by  $S^t$  a specific strategy is  $s^t \in S^t$ .

From Figure 1 we observe that the payoff to any player  $t$  is a function of his strategy and the strategies of the preceding and succeeding generation.

Suppose that each has a payoff function of the same form. We may write it as:

$$(1) \quad f_t = f(s^{t-1}, s^t, s^{t+1}) .$$

The relationship between strategies and payoffs or utility is interlinked by the mappings

strategies  $\rightarrow$  outcomes

outcomes  $\rightarrow$  payoffs

In particular as any one generation interacts with two others it is reasonable to consider that the payoff to an individual of generation  $t$  could be a function of 9 variables  $(y_t^{t-2}, y_t^{t-1}, y_t^t; y_{t+1}^{t-1}, y_{t+1}^t, y_{t+1}^{t+1}; y_{t+2}^t, y_{t+2}^{t+1}, y_{t+2}^{t+2})$ , that is the distribution to the whole population alive while  $t$  is alive (in total five generations).

## 2.2. Strategies, Outcomes and Information

Figure 2 shows this game in extensive form beginning at time  $t = 1$  when player  $P_0$  is called upon to move. The notation  $\Gamma_{BT}$  stands for how the game was played "before time began." This is necessary as we may need to specify at least the history of those individuals who are alive at the start of the game, but who may have also been alive before the start of the game. In a similar way we may need to specify ending conditions if we try to represent this infinite game by a related finite

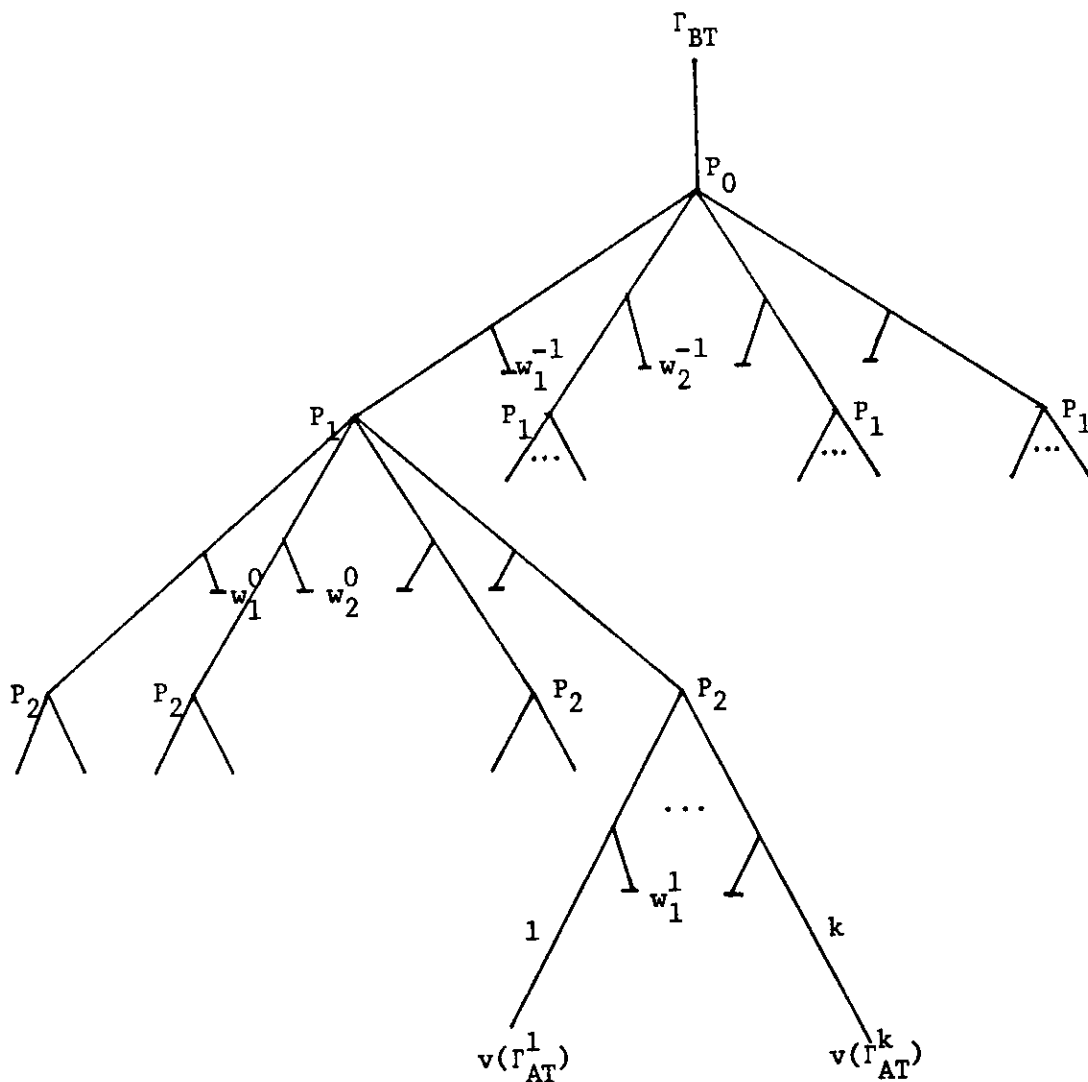


FIGURE 2

game. We return to a discussion of how to set initial and final conditions when we consider how the multigenerational model could actually be used as a finite experimental game.

Returning to Figure 2, the point marked  $P_0$  is at time  $t = 1$  and is the point where generation 0 has the move. The move involves dividing up 3 units into as many as three parts hence it is really continuous. For simplicity the set of moves have been drawn as though they were finite (for example there might be a minimum quantity that can be transferred).

After  $P_0$  moves  $P_{-1}$  dies and obtains his final payoff which is denoted by  $w_j^{-1}$  where  $j$  is the index of the move selected by  $P_0$ . The choice of move  $j$  may affect  $P_{-1}$ 's final payoff hence the subscript is needed. After  $P_0$  has moved and  $P_{-1}$  is dead the move goes to  $P_1$ . After  $P_1$  moves  $P_0$  dies and the move passes to  $P_2$ .  $P_2$  makes his move,  $P_1$  dies and the game ends with a final evaluation after move  $j$  of  $v(\Gamma_{AT}^j)$ . This indicates that there is some way of evaluating the truncated game as a function of the ending conditions including the remainder of the strategies of those alive "after the end of time."

In Figure 2 no information sets have been indicated. As it is drawn we imply perfect information. In this simple case where there is only one individual in each generation it appears to be reasonable to assume perfect information, i.e. that any generation knows what the previous one has done.

A strategy is a choice rule that is a function of available information, thus in principle it could involve contingency plans based upon what individuals did hundreds of generations previously. We may wish to limit the degree of complexity of individual strategies. But

it must be stressed that in the study of equilibrium points in the games to be considered, the justification for limiting the complexity of strategies is not logical nicety but must be based upon empirical fact or justified in some other manner.

A natural justification for selecting a special class of strategies could be simplicity. In particular we may wish to consider Markovian strategies which are based only upon the current state that the individual finds himself in. It ignores any history concerning how the state was attained.

### 2.3. Some Special Choices of Utility Functions

The formal methods of the theory of games in strategic or extensive form are of natural use in modelling intergenerational aspects of the political economy. When however, the solution concept of the noncooperative equilibrium is applied to the resultant games, unless restrictions have been placed upon strategy sets and preferences the solution set tends to be large and accordingly the solution may be regarded as weak. The noncooperative equilibrium solution is discussed in Section 3.

Three relatively natural restrictions are placed here upon the utility functions prior to examining some illustrative examples in Section 5.

#### (a) Pure Selfishness

We may specify that the utility function of the individual depends upon only that which he obtains. We may write this as

$$(2) \quad u^t = U(y_t^{t-1}, y_{t+1}^t, y_{t+2}^{t+1}) .$$

(b) Altruism or Intergenerational Concern

The utility function for the individual of the  $t^{\text{th}}$  generation may be described as:

$$(3) \quad u^t = U(y_t^{t-1}, y_{t+1}^t, y_{t+2}^{t+1}; y_{t+1}^{t-1}, y_{t+1}^{t+1})$$

this includes the resources supplied by  $t$  to generations  $t+1$  and  $t-1$ . We note that (3) shows the value  $t$  places upon these transfers and is not a statement about the preferences of  $t+1$  or  $t-1$ .

Conceivably  $t$  might be concerned for generations unborn; but he only interacts with those living. His strategy set contains no element which influences the dead and any influence he has on the unborn is only indirect. Thus strategic considerations rule out extending the utility function backwards and make it scarcely necessary to go beyond those living during the time generation  $t$  is strategically active.

(c) Instinctive or Coded Behavior

We may wish to assume that as a reasonable first order approximation individuals rear their offspring to maturity primarily as a matter of instinct or behavior that has been coded within them. this does not seem to be the case for the taking care of the old. Modifying the utility function shown in (3) to leave out conscious altruism towards the young we have:

$$(4) \quad u^t = U(y_t^{t-1}, y_{t+1}^t, y_{t+2}^{t+1}; y_{t+1}^{t-1})$$

with a condition on the strategy sets that

$$(5) \quad y_{t+1}^{t+1} = g(y_{t+1}^t) .$$



This condition states that what the child obtains is some function of what the adult obtains.

### Historical Strategies, Threats, Aggregation and Information

An important contribution of the formal theory of games to an understanding of the logical possibilities in the variation of strategic behavior is made in the treatment of information and strategy. Even for a simple matrix game played more than once, as information is made available the domain of the strategy sets expands enormously. A simple example of a 3x3 matrix game played twice serves to illustrate both the expansion of strategies and the concept of threat.

	1	2	3
1	5,5	-5,10	-10,-5
2	10,-5	0,0	-10,-5
3	-5,-10	-5,-10	-20,-20

TABLE 1

The game shown in Table 1 has three strategies for each of two players. A strategy (which here is the same as a move) for each player is to pick a number  $i = 1, 2, 3$  or  $j = 1, 2, 3$ . Suppose that the game is played twice and after the first play both players are informed of each other's choice in the play. Each player can now recognize 9 positions  $(1,1), (1,2), \dots, (3,3)$  at each of these positions he has 3 alternatives thus the number of different strategies available to each is  $3^9 = 19,583$ . A strategy will contain 9 contingent clauses; more completely it is a number (the first move) followed by a function on the 9 information sets. In total there are  $3 \cdot 3^9$  strategies but many are redundant —for example if Player 1 chooses 1 for his first move he really does

not need to plan for a position of (2,2) for his second move, as his action has ruled it out.

Two examples illustrate qualitatively different strategies.

The first is the stationary or Markovian strategy. Whenever you are in the same subgame do the same thing. There are only 3 strategies of this type:

Select  $i$  ,  $i = 1, 2, 3$  then regardless of  $(i,j)$   
                   select  $k = i$  .

The second is a historical strategy which in certain contexts can be reasonably well interpreted as a threat. Consider the following strategy:

Select  $i = 1$  , if he selects  $j = 1$  select  $k = 2$   
   " " "  $j \neq 1$  "  $k = 3$

A glance at Table 1 shows that  $k = 3$  is the punishment for not cooperating by setting  $j = 1$  .

In the example above there are only two players and a threat strategy by one is clearly aimed at the other. In large societies it merits distinguishing between personal and impersonal threats. In a large society, at a personal level involving possibly high information and communication the individual recognizes family and acquaintances and "the rest of the world" where the rest of the world's behavior and identity is aggregated in some manner. It is possible to distinguish strategies where whole sets of individuals are willing to "punish" any individual, not necessarily identified by name, but only by action if that individual "steps out of line."

### 3. THE NONCOOPERATIVE EQUILIBRIUM

An n-person game has an equilibrium point, denoted by a set of n strategies  $(s_1^*, s_2^*, \dots, s_n^*)$  if the following holds true. Let the strategy set for player i be denoted by  $S_i$  and an individual strategy by  $s_i \in S_i$  then for all i

$$(6) \quad \max_{s_i \in S_i} f_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*) \implies s_i = s_i^* .$$

In the game shown in Table 1 it should be clear that the strategies for each player "play i = 2 each time, regardless" form an equilibrium point, with payoffs of 0 to each. Less obvious is the fact that the threat strategies, if used by both players also form an equilibrium point. That is "play i = 1, if competitor replies with 1 then use 2; otherwise 3. This yields 5 for each as the payoffs.

Although the second equilibrium yields a higher payoff than the first it is more complicated. The first has the nice property that it is perfect<sup>1</sup> in the sense that not only is it in equilibrium in the game as a whole, but at any stage of the game the remaining components of the strategies are in equilibrium.

In searching for equilibria in a society, there is an appeal in considering ones which are robust<sup>2</sup> in various ways such as those which maintain existence even though information is changed. The Markovian equilibria belong to this class.

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<sup>1</sup>Selten (1975).

<sup>2</sup>Shubik (1980).

#### 4. PLAYING AND BRIEFING FOR AN EXPERIMENTAL GAME

Suppose that we tried to play the game shown in Figures 1 and 2 as a three period game starting at time  $t = 1$  with three strategically active generations  $P_0$ ,  $P_1$  and  $P_2$ , and two strategically given, i.e. player  $P_{-1}$  who is alive at time  $t = 1$  but was strategically active earlier, and player  $P_3$  who will be alive at  $t = 3$ , but will not become strategically active until the game is over.

If we limit the strategies of the players to contingent plans involving only those individuals whose life spans overlap then we have a way of experimentally testing for the existence of a stationary state in a 3 period 5 player game as follows:

We inform player  $P_{-1}$  that he employed his strategy  $s_0^{-1}$  at time  $t = 0$  and the outcomes relevant to his payoffs have been  $(\alpha_{-1}^{-1}, \alpha_0^{-1}, -, \alpha_0^0, \alpha_0^{-2})$ . This tells him what his plan was and how it has fared so far. He knows what he received when young, from his parent, how he took care of his child and parent when middleaged, but the value of  $y_2^{-1}$  must be determined in the game to be played.

We inform player  $P_0$  that as a child he obtained  $\alpha_0^0$ .

Player  $P_1$  needs no initial or ending conditions as his life is completely spanned by the active periods of the game.

Player  $P_2$  will live out his old age after the experimental game is over hence he must be told the strategy of  $P_3$  which will help to determine  $\alpha_4^2$ .

Player  $P_3$  as an actual player will not select his move until after the game is over, but his strategy is relevant and has to be given parametrically. Thus  $s_0^3$  must be specified.

It is important to stress that in the initial and ending conditions

it has been necessary to specify strategies which are essentially functions hence considerable specification has been called for.

One easy way to comprehend the role of a player whose life period stretches into time before or after the game is that each player could be played by three agents one for youth, middle age and old age. In the actual game the agent for  $P_{-1}$  will have nothing to do at  $t = 1$  but his previous strategy was relevant. Similarly  $P_3$  has nothing to do at  $t = 3$  but his strategy controlled by  $P_3$ 's second agent is relevant.

## 5. GLUEING THE POLITICO-SOCIO-ECONOMIC SYSTEM TOGETHER

In this section several simple models are presented to illustrate the different hypotheses which can lead to the same outcome.

As the first multigenerational economic model considered was "the pure consumption loan" model of Samuelson<sup>1</sup> the examples considered here are closely related to his. However we vary our assumptions concerning utility functions and are explicit concerning the strategic structure.

### 5.1. Not Enough Glue (Model 1)

As our first example we assume that all generations are composed of pure individualists with no constraints on them whatsoever.

Each individual  $P_t$  has a utility function of the form

$$u^t = U(y_t^{t-1}, y_{t+1}^t, y_{t+2}^{t+1})$$

which we specialize to

$$(7) \quad u^t = \sqrt{y_t^{t-1}} + \sqrt{y_{t+1}^t} + \sqrt{y_{t+2}^{t+1}} .$$

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<sup>1</sup>Samuelson (1958).

We limit the strategy sets to simple Markovian strategies; i.e. when  $P_t$  is called upon to move he looks at the state that he is in and selects his move using that information and nothing else. This amounts to saying that his strategy and moves coincide. A strategy is really nothing more than selecting 3 numbers which add to 3 at time  $t+1$ .

Suppose that each individual  $t$  at time  $t+1$  merely gave everything to himself, i.e. set  $y_{t+1}^t = 3$ , then it is easy to see that there is a stationary state equilibrium with

$$s^t = (0, 3, 0) \text{ for } t = -1, 0, 1, 2, 3$$

and the payoff being:

$$u^t = \sqrt{3} \text{ for } t = -1, 0, 1, 2, 3 .$$

Now the paradox is upon us. Suppose all were to select the strategy  $s^t = (1, 1, 1)$ , i.e. feed young and old like yourself. This should give  $u^t = 3$  for all individuals and this is both feasible and considerably better than  $\sqrt{3}$ , but is in equilibrium? Setting up the game formally postulating the appropriate initial and ending conditions we have the following:

Player  $P_{-1}$  is assumed to have employed strategy  $s^{-1} = (1, 1, 1)$  at time  $t = 0$  and the outcomes have been  $(1, 1, -)$  i.e. he knows what his plan was, how he fared as a child, and in middle age.

Player  $P_0$  is informed that as a child he obtained 1 thus he wishes to select  $s^0$  to maximize  $\sqrt{1} + \sqrt{y_1^0} + \sqrt{y_2^0}$  or

$$(8) \quad \max_{s^0} \sqrt{1} + \sqrt{y_1^0} + \sqrt{y_2^1}$$

where  $s^0 = (y_1^{-1}, y_1^0, y_1^1)$  and  $y_1^{-1} + y_1^0 + y_1^1 = 3$ , all  $y_t^{t+j} \geq 0$ .

For player  $P_1$

$$(9) \quad \max_{s^1} \sqrt{y_1^0} + \sqrt{y_2^1} + \sqrt{y_3^2}$$

where  $s^1 = (y_2^0, y_2^1, y_2^2)$  and  $y_2^0 + y_2^1 + y_2^2 = 3$  with  $y_t^{t+j} \geq 0$ .

Player  $P_2$  is informed of  $P_3$ 's strategy, which in this case because we are limiting the domain of strategies to those of the form  $(y_t^{t-1}, y_{t+1}^t, y_{t+2}^{t+1})$  he can see that no matter what he does he will obtain  $y_4^3$  from  $P_3$ . We set  $P_3$ 's strategy at  $s^3 = (1,1,1)$ , hence the maximization for  $P_2$  is:

$$(10) \quad \max_{s^2} \sqrt{y_2^1} + \sqrt{y_3^2} + \sqrt{y_4^3}$$

where  $s^2 = (y_3^1, y_3^2, y_3^3)$ ,  $y_3^1 + y_3^2 + y_3^3 = 3$  all  $y_t^{t+j} \geq 0$  and  $y_4^3 = 1$ .

A glance at equations (8), (9) and (10) shows that the optimal strategy of the type we are considering for the players is  $s^0 = (0,3,0)$ ,  $s^1 = (0,3,0)$  and  $s^2 = (0,3,0)$ . In spite of having constrained  $P_{-1}$  and  $P_3$  to  $s^{-1} = (1,1,1)$  and  $s^3 = (1,1,1)$  these initial and ending conditions were not enough to maintain a stationary equilibrium at  $s^t = (1,1,1)$ .

In this solution both the young and the old obtain 0. If 0 were to be interpreted literally then one might ask do the young survive to the second period if they obtain no resources when young? This raises some basic questions of fact and modelling concerning population. Here

the birth of the next generation is exogenous, but can parents starve offspring to death?, or kill them otherwise. In most societies today the killing or starving to death of children are not in the rules of the game. This was not always so, and is not the case with some other animals during times of food scarcity. We return to this problem in Section 6.

### Selfishness and the End of the Game (Model 1a)

If zero support of the young implies their death then the game tree sketched in Figure 2 must be modified to include a complete "end of the game" which amounts to an extinction of the species in a finite time. This is shown in Figure 3, which is essentially the same as Figure 2 with the strategic options to end the game included.

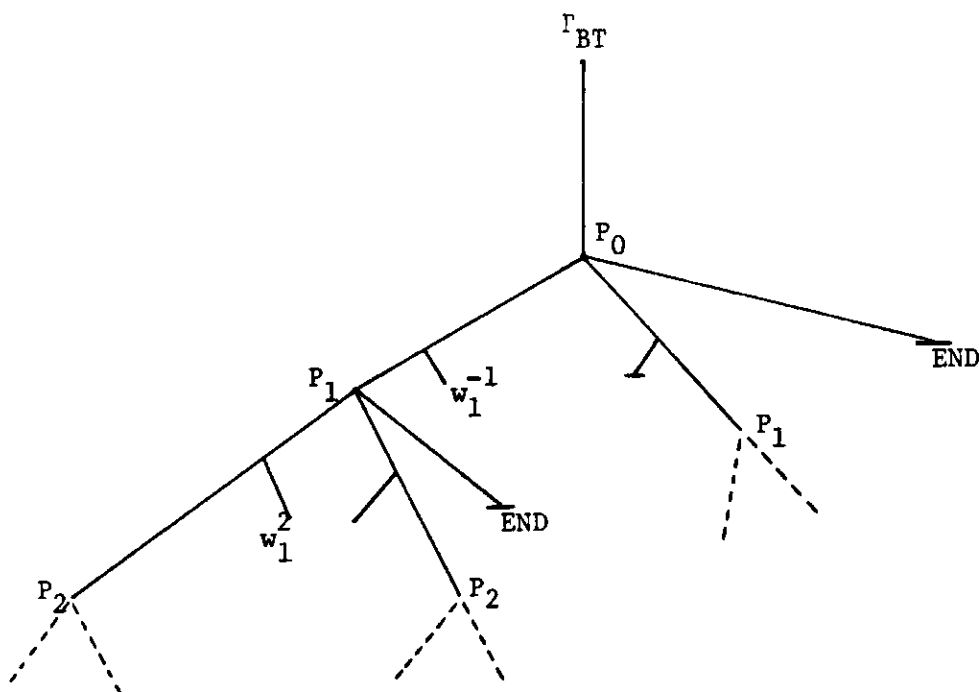


FIGURE 3



If model 1 is modified to include the feature that whenever  $y_t^t = 0$   $t$  starves and all generations from  $t+1$  onwards never exist then a glance at optimization condition (8) shows that it is not influenced by this change hence  $s_1^0 = (0, 3, 0)$ , the payoff to  $P_0 = 1 + \sqrt{3}$  and the game is over. This outcome is individually rational, but as with many noncooperative equilibria, it is not Pareto optimal.

### 5.2. Linkage by Love: The Altruism Finesse (Model 2)

Referring to equation (3) in 2.3 we may specialize the form of the utility function as follows:

$$(11) \quad u^t = \sqrt{y_t^{t-1}} + \sqrt{y_{t+1}^t} + \sqrt{y_{t+2}^{t+1}} + \theta_{t-1}^t \sqrt{y_{t+1}^{t-1}} + \theta_{t+1}^t \sqrt{y_{t+1}^{t+1}}$$

where in equation (11):

$\theta_{t-1}^t$  is the coefficient of concern<sup>1</sup> of generation  $t$  for its parent generation  $t-1$ .

$\theta_{t+1}^t$  is the coefficient of concern of generation  $t$  for its successor generation  $t+1$ .

There is no logical necessity to restrict  $\theta_{t-1}^t$ ,  $\theta_{t+1}^t$  to the range  $[0, 1]$ , although  $\theta = 0$  has the interpretation of isolated, "selfish" or orthogonal preferences and  $\theta = 1$  may be interpreted as regarding others as oneself.

Suppose that we set up our previous example with the modification  $\theta_{t-1}^t = \theta_{t+1}^t = 1$  then (11) is specialized to

$$u^t = \sqrt{y_t^{t-1}} + \sqrt{y_{t+1}^t} + \sqrt{y_{t+2}^{t+1}} + \sqrt{y_{t+1}^{t-1}} + \sqrt{y_{t+1}^{t+1}}.$$

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<sup>1</sup>The idea of a coefficient of concern was essentially suggested by Edgeworth (1887).

We may now modify the optimization conditions (8), (9), (10) to become:

$$(12) \quad \max_{s_1^0} \sqrt{1} + \sqrt{y_1^0} + \sqrt{y_2^1} + \sqrt{y_1^{-1}} + \sqrt{y_1^1}$$

where  $s_1^0 = (y_1^{-1}, y_1^0, y_1^1)$  and  $y_1^{-1} + y_1^0 + y_1^1 = 3$ , all  $y_t^{t+j} \geq 0$ .

$$(13) \quad \max_{s_2^1} \sqrt{y_1^0} + \sqrt{y_2^1} + \sqrt{y_3^2} + \sqrt{y_2^0} + \sqrt{y_2^2}$$

where  $y_2^0 + y_2^1 + y_2^2 = 3$  with all  $y_t^{t+j} \geq 0$ .

$$(14) \quad \max_{s_3^2} \sqrt{y_2^1} + \sqrt{y_3^2} + \sqrt{1} + \sqrt{y_3^1} + \sqrt{y_3^3}$$

where  $y_3^1 + y_3^2 + y_3^3 = 3$  with all  $y_t^{t+j} \geq 0$ .

A glance at these three conditions and it is straightforward to verify that  $s_1^0 = s_2^1 = s_3^2 = (1,1,1)$  and there is indeed an optimal stationary state which exists as a noncooperative equilibrium where the links between the generations are "love" or altruism. This particular solution here depended upon selecting the  $\theta_{t-1}^t, \theta_{t+1}^t = 1$  which if we adopt a Dawkin's viewpoint<sup>1</sup> in a unisexual world makes sense for  $\theta_{t+1}^t = 1$ , but there seems to be little evidence, biological, sociological or otherwise that  $\theta_{t-1}^t = 1$ , i.e. that the mature adult is as concerned for his parents as himself.

It is an empirical question as to what leads to the support of

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<sup>1</sup>Dawkins (1976).

the elderly in different societies. My guess is that although  $\theta_{t+1}^t = 1$  for children might be consistent as an explanation of why children are supported  $\theta_{t-1}^t = 1$  is not an adequate explanation for the support of the elderly.

For those who wish to ignore institutions, social and political behavior and the possibility of coded or instinctive behavior the approach of plugging altruism factors into the utility function offers a logically consistent, but empirically unverified view of behavior.

### 5.3. Linkage by Coding or Instinctive Behavior (Model 2a)

Rather than explicitly put the driving mechanism for the support of children into the utility function and then using a purely conscious decisionmaking justification for the behavior towards children we can model "limited rationality" by requiring that

$$(5) \quad y_{t+1}^{t+1} = g(y_{t+1}^t)$$

as noted previously in Section 2.3. It appears that the raising of the young in all mammals is highly instinctive. In humans there may be some conscious economic problems concerning the value of child labor, or the quality of some resource allocations, but the evidence concerning the split between conscious utilitarian optimizing behavior and instinctive species determined behavior in the raising of the young goes in favor of the latter. There is nothing scientifically or even aesthetically wrong in making analogies between children and refrigerators or other consumer durables. The question that must be considered is "is a theory based upon such analogies better able to answer major socioeconomic, demographic and politico-economic questions than other theories?"

It is my belief that economic analyses per se may be of some help in providing an explanation for why some demographic phenomena may be inhibited, but not why they are motivated.

If we go back to Model 1 where

$$u^t = U(y_t^{t-1}, y_{t+1}^t, y_{t+2}^{t+1}) \quad \text{and} \quad s_{t+1}^t = (y_{t+1}^{t-1}, y_{t+1}^t, y_{t+1}^{t+1})$$

then if we impose the extra constraints

$$(15) \quad y_{t+1}^{t-1} = y_{t+1}^t, \quad y_{t+1}^{t+1} = y_{t+1}^t$$

we obtain the stationary state  $s^t = (1,1,1)$  by constraints on the strategy sets.

It might be argued that in this simple example the use of equations (15) completely forces the answer. In general this does not have to be the case. The principal still remains that the desire to have children and to support them to maturity can be modelled as constraints on economic behavior, still leaving for the economist the decisions concerning which baby bonnet the rational consumer should choose.

A further glance at equations (15) is enough to make us suspect that they are empirically wrong inasmuch as the support of elderly parents by no means appears to be instinctive hence the constraint

$$y_{t+1}^{t-1} = y_{t+1}^t$$

is implausible.

#### 5.4. Historical Strategies: A Sociological Leakage (Model 3)

Returning the the same data as in Model 1 where

$$(7) \quad u^t = \sqrt{y_t^{t-1}} + \sqrt{y_{t+1}^t} + \sqrt{y_{t+2}^{t+1}}$$

we modify the strategy sets. A strategy  $s_{t+1}^t$  is no longer merely a triad of numbers  $(y_{t+1}^{t-1}, y_{t+1}^t, y_{t+1}^{t+1})$  but a function of previous information available at the time of any move. In particular as initial and ending conditions suppose that we ascribe to  $P_{-1}$  and to  $P_3$  the following strategy written generally for generation  $t$

$$(16) \quad \begin{aligned} & \text{"I select } y_{t+1}^{t+1} = 1, \quad y_{t+1}^t = 1, \quad y_{t+1}^{t-1} = 1 \\ & \text{if } y_t^t = 1 \text{ and } y_t^{t-2} = 1 \text{ otherwise I select} \\ & y_{t+1}^{t+1} = y_{t+1}^{t-1} = 0, \quad y_{t+1}^t = 3 \text{ ."} \end{aligned}$$

In words, (16) states "I will maintain my children when young and parents when old at a level like my own provided that I see that my parents did the same. If they fail then I will support neither my children, nor my parents when they are old."

It is straightforward to check that if all use the strategy  $s^t$  described in (16) a stationary equilibrium with  $y_t^{t-1} = y_{t+1}^t = y_{t+2}^{t+1} = 1$  is formed.

Suppose that this were not true, then any individual  $t$  could break the chain and increase the value of  $y_{t+1}^t$  from 1 to 3, but the price of failing to provide for either parent or child is that the child will not provide for him in his old age. Thus a deviation from the strategy yields at most  $\sqrt{1} + \sqrt{3} + 0$ . This is less than  $\sqrt{1} + \sqrt{1} + \sqrt{1}$ .

The stability here is essentially historical and sociological.

The threat strategies can be regarded as a cultural norm or tribal custom. The model here has only one aggregate individual in each generation. A more careful and general model would explicitly represent the cohort size as this might be important to stability.

One might regard this type of solution as prelegal and pre or early institutional. Mathematically the strategies can be well defined, but societies in general evolve enforcement mechanisms more concrete, formal and institutional than verbal threats.

#### 5.5. Money: A Legal, Financial Linkage (Model 4)

A view different from all of the above, but logically consistent and plausible was basically sketched, but not fully developed by Paul Samuelson in his article on the pure consumption loan. A somewhat different version is presented here. As we wish to concentrate on economic or financial explanations of why the old are sustained the problem is simplified by following Model 2a for the support of the young, i.e.

$y_{t+1}^{t+1} = y_{t+1}^t$ , it is assumed as a matter of instinct that parents treat the young as themselves.

The following noncooperative game is constructed. All trade is monetized and takes place through an organized spot market.<sup>1</sup> A bank is available which will lend any amount of money at a zero rate of interest to anyone with assets or ownership claims to assets other than money. Ruling out borrowing by children this leaves only the middleaged in a position to borrow on the 3 units of manna. In order to prevent unbounded some type of default or bankruptcy law is required. Its effect must be reflected in the utility function of the individual. The punishment for

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<sup>1</sup>A variety of different ways for specifying market structures with trade in money are given elsewhere, Shubik (1972), Dubey and Shubik (1980), Dubey, Mas-Collel and Shubik (1980).

failure to repay is not necessarily economic. It may involve prison, the loss of citizenship, physical punishment, deportation or other legal and societal retributions.

In this model we are introducing the concepts of money, markets and a price system. We must modify both the strategy sets and the utility functions.

In particular as money is assumed to be durable the strategy set of the old is enlarged as it now becomes possible for them to have saved and thereby have purchasing power in their old age. We must distinguish the variables  $y_{t+1}^{t-1}$  and  $\hat{y}_{t+1}^{t-1}$  where the first stands for what generation  $t$  gives to  $t-1$  at time  $t+1$  and the second stands for what generation  $t-1$  buys at period  $t+1$ .

We introduce the new utility function

$$(17) \quad u^t = \sqrt{y_t^{t-1}} + \sqrt{y_{t+1}^t} + \sqrt{y_{t+2}^{t+1} + \hat{y}_{t+2}^t} \\ + \mu \min[0, \{p_{t+1}(3 - y_{t+1}^{t+1} - y_{t+1}^t - y_{t+1}^{t-1}) - p_{t+2}\hat{y}_{t+2}^t\}] .$$

The third term includes gifts to and purchases by the old. The fourth term takes into account the fundamental need for a penalty against default (this is not noted by Samuelson). The form that this penalty should take is quite general, as has been noted elsewhere.<sup>1</sup> But it must exist if strategic default is to be avoided. The special form utilized here is merely for convenience and simplicity in composing different models with steady states giving the same levels of consumption.

The fourth term begins with  $\mu$  which is a one parameter index

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<sup>1</sup>Dubey and Shubik (1979).

of the severity of default punishment. The value of net positive monetary assets is assumed to be zero, but if individuals are in default the consequences are unpleasant. An individual's total earned income is  $3p_{t+1}$  his expenditures are caused by his instinctive support of the young as himself, his own support and any gift made to the old, or  $(y_{t+1}^{t+1} + y_{t+1}^t + y_{t+1}^{t-1})p_{t+1} + \hat{y}_{t+2}^t p_{t+2}$ .

When we are dealing with markets for price formation, unless we are particularly interested in oligopolistic or oligopsonistic competition it may be reasonable to assume that there is a multitude of individuals of each generation. If we do this it can be shown<sup>1</sup> that one can check for the noncooperative equilibria of strategic market games with a continuum of traders as though price were a given parameter whenever the size of any trader is of measure zero, as his behavior alone will not influence the market.

In order to fully define the game we must specify what is a strategy by a trader and fix the default penalty. We do so as follows:

$$(18) \quad s^t = (m_{t+1}^t, b_{t+1}^{t+1}, b_{t+1}^t, b_{t+1}^{t-1}, b_{t+2}^t)$$

subject to  $m_{t+1}^t \geq 0$ ;  $b_{t+k}^{t+j} \geq 0$ ;  $b_{t+1}^{t+1} = b_{t+1}^t$  and

$$(19) \quad b_{t+1}^{t+1} + b_{t+2}^t + b_{t+1}^{t-1} + b_{t+2}^t \leq m_{t+1}^t .$$

This states that at the start of period  $t+1$  an individual of generation  $t$  borrows  $m_{t+1}^t$  from the bank, he then spends  $b_{t+1}^{t+1}$  on buying manna for the young;  $b_{t+1}^t$  on buying for himself and  $b_{t+1}^{t-1}$  on

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<sup>1</sup>Dubey and Shapley (1980).



buying for the old. We could model the last act slightly differently and both the old and new alternatives have empirical meaning. In particular instead of having the middleaged buy for the old, they could give them money. In period  $t+2$  if generation  $t$  still has money it can spend it; this is indicated by  $b_{t+2}^t$ . Equation (19) indicates that the markets do not grant credit. Goods must be paid for in money.

We may check immediately that the strategies

$s_{t+1}^t = (2,1,1,0,1)$  form a stationary equilibrium for all  $t$ ;  $p_t = 1$  for all  $t$  and  $y_{t+1}^{t+1} + y_{t+1}^t + \hat{y}_{t+2}^t + 1$ ,  $y_{t+2}^t = 0$ .

In words, each individual  $t$  borrows 2 from the bank at time  $t+1$  and spends 1 on himself and 1 on his child immediately. At the end of period  $t+1$  he receives 3 from the market which sold all his manna; he repays the bank 2 (at 0 interest) and spends the 1 left over in his old age at  $t+2$ .

#### 5.6. Capital Goods, Ownership Paper and Money: A Legal, Social, Financial Linkage (Model 5)

Suppose that instead of manna coming down for generation  $t$  at time  $t+1$  we consider a production function of the form

$$(20) \quad y_t = f(w_t, k_t) = 3\sqrt{w_t k_t},$$

where  $y_t$  = total output at time  $t$ ,

$w_t$  = input of labor at time  $t$ ,

$k_t$  = input of land or capital stock at time  $t$ .

Our initial ownership conditions are that at time  $t = 1$ , the young own nothing, the middleaged (generation 0) have 1 units of labor

to supply and the old own the land hence they can lease it, sell it or give it away. Suppose that land is indestructible, does not depreciate and needs no maintenance. As there is no "natural discount" in this model; at least to the race as a whole the land is of unbounded worth, yet strategically the old have its use only for one period. Suppose further that under the inheritance laws and customs of this society the next generation inherits the land from the generation just deceased. In this model with no love between the generations there is no motive for the old to leave the middleaged an inheritance, but they cannot eat the land. There is a fallacy of composition at work, the small landowner may think that he should be able to sell rather than rent his land, but when viewed as a whole in equilibrium the middleaged will rent the land they need from their parents in one period and inherit it in the next because their parents in this game cannot take it with them.

The initial conditions required are, at time  $t = 1$  :

Generation -1    (0,1)

Generation 0    (1,0)

Generation 1    (0,0)

I.e. the old own the land, the middleaged labor and the young nothing.

A strategy in this game is as follows for generation  $t$  :

At time $t$	when young	no strategic decisions are available
" "	$t+1$	" middleaged
		borrow money from bank
		rent or buy land
		sell or use labor
		sell or use food
		buy good for some subset of self, child and parent
" "	$t+2$	" old
		lease or sell land
		borrow money from bank
		buy food

We assume that neither land nor labor enters directly into anyone's utility function. They are both intermediate goods needed to produce food. Then we may reinterpret (17) the utility function in Section 5.5, as applying to the model here.

Mathematically a strategy is:

$$(21) \quad s^t = (m_{t+1}^t, b_{t+1}^{t+1}, b_{t+1}^t, b_{t+1}^{t-1}; v_{t+1}^t, w_{t+1}^t, r_{t+1}^t, \hat{r}_{t+1}^t, z_{t+1}^t; m_{t+2}^t, \hat{b}_{t+2}^t, k_{t+2}^t, \hat{k}_{t+2}^t)$$

$m_{t+1}^t, m_{t+2}^t$  are the amounts borrowed by  $t$  at  $t+1$  and  $t+2$ ,  
 $b_{t+1}^{t+1}, b_{t+1}^t, b_{t+1}^{t-1}$  are the bids by  $t$  during period  $t+1$  to buy food for child, self and parent,  
 $v_{t+1}^t$  is the amount of money bid for labor by  $t$  at period  $t+1$ ,  
 $w_{t+1}^t$  is the amount of labor offered for sale by  $t$  at  $t+1$ ,  
 $r_{t+1}^t$  is the amount bid for land for one period rental bid by  $t$ ,  
 $\hat{r}_{t+1}^t$  is the amount of money bid for land for outright purchase bid for by  $t$ ,

- $\hat{b}_{t+2}^t$  is the amount of money spent by  $t$  during  $t+1$  ,  
 $k_{t+2}^t$  is the amount of land offered for rent by  $t$  at  $t+2$  ,  
 $\hat{k}_{t+2}^t$  is the amount of land for sale by  $t$  at  $t+2$  ,  
 $z_{t+1}^t$  is the amount of food offered for sale at  $t$  at  $t+1$  .

### Cottage Industry (Model 5a)

For specificity especially in taking care to describe the need for working capital, the organization of industry for production must be noted. In this model, the simplest assumption is made. We imagine that the middle-aged conduct their own cottage industry, i.e. they supply their own labor and rent or buy land from their elders.

There are two choices we must face concerning the availability of labor, either the individuals supply labor directly to themselves and avoid trading in the market, or they are required to offer their labor in a labor market and buy it back in a cash transaction. The accounting is easiest under the second and cash flow needs are higher. We choose the second

$$(22) \quad \text{The price of labor during } t+1 \text{ is } p_{1,t+1} = v_{t+1}^t / w_{t+1}^t$$

$$(23) \quad \text{" " " land for rent during } t+1 \text{ is } p_{2,t+1} = r_{t+1}^t / k_{t+1}^t$$

$$(24) \quad \text{" " " food during } t+1 \text{ is } p_{3,t+1} = (b_{t+1}^{t+1} + b_{t+1}^t + b_{t+1}^{t-1} + \hat{b}_{t+1}^{t-1}) / z_{t+1}^t$$

The amount of food offered for sale  $z_{t+1}^t$  must be less than or equal to  $y_{t+1}^t$  which is the amount produced by  $t$  during  $t+1$  .

We may check that the following strategies are in equilibrium:

$$(25) \quad s^t = (5, 1, 1, 0; 3/2, 3/2, 0, 3; 1/2, 1, 1, 0)$$

$t$  begins by borrowing 5, he spends 1 on his child, 1 on himself 0 on his parent; he pays  $3/2$  for labor and also offers 1 units for sale; he pays  $3/2$  to rent land and 0 to buy land; he also offers 3 units of food for sale; when he becomes old at period  $t+2$  he will have received an income of  $4-1/2$  at the end of  $t+1$ , 3 from the sale of food and  $3/2$  from the sale of labor.

$$P_{1,t+1} = 3/2/1 = 3/2 \quad \text{price of labor}$$

$$P_{2,t+1} = 3/2/1 = 3/2 \quad \text{price of land rented}$$

$$P_{3,t+1} = (1+1+0+1)/3 = 1 \quad \text{price of food.}$$

At age  $t+2$ ,  $t$  borrows  $1/2$  more, spends 1 on buying food and offers 1 unit of land for rent. He receives an income of  $3/2$  from his land which at the end of  $t+2$  pays back the bank debt.

The equilibrium noted in (25) is not unique, but we do not discuss the others here.

#### The Joint Stock Firm (Model 5b)

As an alternative to the cottage industry model, we can imagine that all production is via firms of indefinite duration. The accounting is made possibly easier and more realistic by assuming the independent existence of a class of small profit maximizing firms whose shares are held by the middleaged and/or the old. Why self-seeking managers should wish to maximize the profits of their firms requires proof. This is discussed elsewhere.<sup>1</sup>

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<sup>1</sup>Dubey and Shubik (1980).

In this example, at equilibrium, because the production technology has been chosen to homogeneous of order one, the profits will be zero hence ownership at that point will apparently not matter.

If we postulate the existence of firms, then we may as well give them an infinite life and we assume that at time  $t$ , representative firm  $j$  bids to buy labor and rent or buy land; it also offers its produce for sale. We may wish to include explicitly the feature that production takes time, thus instead of

$$(20) \quad y_t = f(w_t, k_t)$$

we might use

$$(26) \quad y_{t+1} = f(w_t, k_t)$$

Following this latter alternative we now state the strategy set of the representative firm, its initial conditions and we restate the strategy set for the individual and the initial conditions for the individual. In this model all trade, production and consumption is monetized.

A strategy by a firm  $j$  starting at time  $t = 1$  is as follows

$$(27) \quad \hat{s}^j = (m_1^j, z_1^j, r_1^j, \hat{r}_1^j, x_1^j; m_2^j, z_2^j, \dots),$$

where  $m_t^j$  = the amount borrowed (paid back) at time  $t$  by  $j$ ,

$z_t^j$  = the amount of food offered for sale by  $j$ ,

$r_t^j$  = the amount of money offered for land for rent for one period bid for by  $j$ ,

$\hat{r}_t^j$  = the amount of money offered for land for purchase bid for by  $j$ ,

$x_t^j$  = the amount of money offered for labor bid for in time  $t$  by  $j$ .

The initial conditions for the firm at the start of time  $t = 1$  are  $(3,0,0,0,0)$ . It begins with 3 units of food, no money, no labor and no land rented or bought.

If this model were actually played as an experimental game which ended after  $t = 3$ , then the player playing the representative firm would need to be supplied with the firm's strategy for after  $t = 3$ . We set this at

$$\left( \underbrace{\dots}_{t=1,3}; 0, 3, \frac{3}{2}, 0, \frac{3}{2}; 0, 3, \frac{3}{2}, 0, \frac{3}{2}; \dots \right).$$

The only modification we need to make to (21) is that if all production is through a factory then individuals will not rent or buy land, except for speculation. For simplicity then in this example we modify (21) by leaving out  $r_{t+1}^t$  and  $\hat{r}_{t+1}^t$ , also  $v_{t+1}^t$  and  $z_{t+1}^t$ , etc.

$$(28) \quad s^t = (m_{t+1}^t, b_{t+1}^{t+1}, b_{t+1}^t, b_{t+1}^{t-1} w_{t+1}^t, m_{t+2}^t, b_{t+2}^t, k_{t+2}^t, \hat{k}_{t+2}^t).$$

The same type of equilibrium as displayed in (25) for model 5a will still be in equilibrium here. In particular

$$(29) \quad s^t = (2, 1, 1, 0, 1; 1, 1, 1, 0).$$

Generation  $t$  at time  $t+1$  borrows 2, spends 1 on himself and 1 on his child, nothing on the parent, offers 1 unit of labor, he borrows 1 more at period  $t+1$ , spends it on himself, offers 1 unit of land for rent, and 0 for sale. At equilibrium prices of  $p_{3,t} = 1$ ,  $p_{1,t} = p_{2,t} = 3/2$  for all  $t$  the income during  $t+1$  from sale of labor will be  $1-1/2$  and during  $t+2$  income from rent of land will be  $1-1/2$ :

$$(30) \quad \hat{s}^j = (3, 3, 3/2, 0, 3/2; 0, 3, 3/2, 0, 3/2, 0, \dots) .$$

The firm  $j$  starts by borrowing 3 units of working capital, offers 3 units of food for sale, bids  $3/2$  to rent land, 0 to buy and  $3/2$  for labor.

#### Equilibria with the Sale of Land

In this model the land is of unbounded worth, but the only actors with unbounded potential life are the corporations. If borrowing at 0% were unbounded then we would have a "bid the highest number" game with no equilibrium.

The possibility for an equilibrium depends upon the definition of extra (and possibly quite realistic) rules. In particular there may be a bound upon the amount that an individual or a firm is permitted to borrow. Furthermore there may be constraints upon the payment or nonpayment of dividends which depend upon the definition of short term and long term profits.

One way of modeling the infinite horizon which we have adopted elsewhere is to consider a finite horizon of length  $T$  and a salvage value to be paid at the end of  $T$  for land left over, then study what happens as  $T$  becomes arbitrarily large. We do not develop this model here.

#### 5.7. Subsidies, Taxes and Voting

##### The Subsidy Solution (Model 6)

We may consider that the land were owned by the government which in turn rents it to the factory and uses the income to subsidize the old by paying them  $3/2$  per period. It is straightforward to check that this gives the same stationary state as does Model 3.



### The Tax Solution (Model 7)

Suppose that labor belongs to the middleaged and land to the old, as in Model 5. A combination of an income tax and a property tax can be applied to adjust the income of the middleaged and the old to any levels whatsoever when combined with the government use of subsidies in disposing of the tax monies taken in.

### A Political Solution (Model 8)

The tax and subsidy models suggested in Models 6 and 7 were introduced exogenously; the decision to tax and subsidize came from some outside undefined government imposing these features as rules of the game. If we go back to a model such as Model 1 where, without the existence of land, no Markovian stationary strategy exists, we can enlarge the model by considering a voting game grafted onto the economic game. Suppose for example, every two periods all the middleaged and old (here as before the young are treated as dummies) are given the opportunity to vote in a referendum on taxes and subsidies for the following two periods. Consider the following referendum: "For the next two periods, the middleaged will be taxed one third of their income and this will be paid as a subsidy to the old." It is straightforward to check that in equilibrium both the middleaged and old will vote in approval of this taxation and subsidy scheme. In order to completely well define the new game however it is necessary to introduce one extra feature into the model in order to fully establish the stability of the new equilibrium. That is a penalty on those who fail to pay taxes which have been voted into the law of the society. As a matter of simple observable fact societies, ancient and modern have had many ways to discourage their members from tax evasion.

## 6. POLITICAL ECONOMY, COMPARATIVE ECONOMICS AND DEVELOPMENT

### 6.1. The Market Model Is Misleading

The market model and price system is undoubtedly central to a good part of the distribution and production system of any modern economy. It gives no basic insights into the driving forces of population and investment. The development of modern price theory has been a masterful abstraction away from the institutional structure that links the economy to the society and polity. This abstraction has been so complete that it was possible to invent a highly specialized solution concept--the competitive equilibrium of a price system--which is only adequately defined over an extremely limited class of models. This concept however is conceptually nothing more than a highly specialized form of a far more general solution--the noncooperative equilibrium of a game in strategic or extensive form.

The appeal and simplicity of the perfect economic market structure and price system is misleading. One can easily confuse generality with highly limiting simplification. The simplifications of general equilibrium theory cut out the study of distribution and production in a nonstrategic setting and totally separated it from the study of investment, political-economy and demography.

If we give up the simplification of the competitive equilibrium and model the political-economy as a game of strategy then the noncooperative equilibrium solution provides a more general solution concept which is consistent with the ideas of mass markets and a price system where they apply, but which enables us to analyze for more general structures involving strategic optimization in a politico-economic or socio-economic context.

In particular some of the models suggested here indicate the variety of socio-economic and politico-economic structures that are consistent with obtaining similar economic results in socio-economic settings which are comparatively different. Especially in the study of developing economies or in the study of economies with markedly different political and other institutional structures it is important to be able to generalize usefully beyond the bounds and constraints of the pure market models.

## 6.2. Mathematical Institutional Economics

The institutions of society, the polity and the economy provide the structure which carries the dynamics and serves as the interchange or exchange points among biological, sociological, political and economic motivations. The study of individual, family or group optimization, conflict and cooperation goes far beyond the market even if our view of individual motivation were naively utilitarian.

The methods of modeling and analysis of the theory of games are naturally directed towards the inclusion of institutions at their most abstract level. In essence the institutions of a society and political-economy emerge as the rules of the game. The approach of modeling games in extensive or strategic form is fundamentally oriented towards the study of process and institutions are the carriers of process.

The concept of a mathematical institutional economics may seem to be almost a contradiction in terms. Earlier attempts at the introduction of institutional considerations into economic analysis have tended to be descriptive and nonmathematical. Here the suggestion is that the very act of attempting to model process mathematically raises the questions which lead to the formal description of institutions.

### 6.3. Afterthoughts

The major purposes of this article were to indicate the broad spectrum of alternatives which are available in the construction of multigenerational models and to show that the study of games in extensive and strategic form provides a natural way for studying mixed socio-political-economic models. For simplicity our analysis was confined to the study of models where the individual lives for three time periods.

Two complications of considerable importance were omitted. They are the role of endogenous birth processes and the role of exogenous uncertainty. Leaving aside at this point the considerable empirical difficulties in providing a good description of either, each introduce basic new problems in economic theorizing. In particular if M and F must jointly choose to decide whether to give birth to C the definition of the strategy feasible for all players requires care in specification. And this in turn influences the structure of the Pareto optimal set.

The presence of exogenous uncertainty, paradoxically provides extra glue to help cement the generations inasmuch as the need for insurance calls for the carrying of extra capital stock.

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