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**OPTIMAL INTERTEMPORAL TAXATION AND THE PUBLIC DEBT**

**Christophe Chamley**

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\*Discussions with Dale Jorgenson and Laurence Weiss were very helpful.

## ABSTRACT

The optimal taxation problem is analyzed in a general equilibrium model of optimal growth. The private sector is represented by a single competitive household endowed with perfect foresight, and an infinite life. This household maximizes an intertemporal stationary utility function. Public consumption is financed by taxes on consumption, labor income and capital income (or wealth), or by borrowing. Different policies (first-best and second-best), are analyzed for various subsets of instruments. The problem of the optimal level of the public debt is also considered. The general conclusion supports the relative efficiency of the consumption tax with respect to the other instruments.

## Introduction

Numerous studies on the efficiency of taxation in a dynamic economy have appeared since Ricardo. In this literature two main traditions can be distinguished. The first one considers the familiar tax instruments on capital and labor income, and on consumption. It analyzes the relations between taxes and the productivity of the economy in an intertemporal framework; a typical problem is the tax incidence on savings and on the long-run capital stock. The arguments developed are often more intuitive than rigorous. For example, it is frequently argued in this context that capital income taxation is inefficient because it taxes income twice (Kaldor, 1955).<sup>1</sup>

In the second type of studies, pioneered by Ramsey (1927), the private sector's demand and supply functions are derived from utility maximization. The optimal taxation problem is then to determine a set of tax rates which maximizes the private sector's utility subject to the constraints of the private sector optimizing behavior, and of a given amount of tax revenues.<sup>2</sup> This method has widespread applications, and it has been used recently to analyze some of the propositions developed by the first approach (as for example, the relative inefficiency of the capital income tax).<sup>3</sup> However, in general, these studies use a partial equilibrium framework, and assume that gross factor prices are exogenous.

The problem of the optimal determination of taxes on capital and labor income and on consumption is analyzed here in an intertemporal general equilibrium model. We do not consider the questions of intra- or intergenerational equity, and assume that the private sector is represented by a single household who lives indefinitely.<sup>4</sup>

The household is endowed with perfect foresight, and behaves competitively: in the maximization of its intertemporal utility function, it takes future prices as given. These prices depend on the endogenous gross factor prices (wage and interest rate), which are determined by the capital accumulation and the production technology; they depend also on the future tax policy, which is announced at time zero.<sup>5</sup>

The household's intertemporal utility function is assumed to be additively separable and stationary. This is a restrictive assumption, but commonly used in the optimal growth approach that we follow here.<sup>6</sup>

At time zero, the policy maker determines the intertemporal program of tax rates (not necessarily constant over time), which maximizes the household's utility under the constraints of private optimizing behavior, and the government budget constraint.<sup>7</sup> The latter can take two forms: In the first, the government balances the present value of the flows of expenditures and revenues over the entire horizon. At a given instant these flows are not equal, and the government can accumulate capital or issue bonds, which are perfectly substitutable for capital in the household's portfolio. In the second more restrictive form, the time path of the public debt is exogenous. Without loss of generality, we can assume that there is no public debt, and that the flows of the government's expenditures and tax revenues are balanced at each instant.<sup>8</sup>

The model and the lump-sum taxation solution are described in the first section. This lump-sum solution is identical to the solution determined by a planner endowed with complete control over the private household's decisions. One possible method of decentralization is a capital levy at time zero. But this method is not unique: a combination of taxes and subsidies on the incomes of labor and capital, and on

consumption, can allow the policy maker to achieve the same allocation of resources.

In the second section we analyze various second best situations where the policy maker can rely only on a limited set of instruments. The case of the labor income tax is considered first, without and with public debt. The method developed for this problem is then used to determine the optimal tax rate on wealth or capital income. Since the use of this instrument raises some specific issues, an example is analyzed in detail in the appendix.

In the third section, the values of the tax rates are assumed to be fixed over time. These values are chosen at time zero such that the present value of government expenditures is equal to the present value of the tax revenues, and the private welfare is maximized. The determination of a solution does not appear feasible except by numerical computation, and we rely on an approximation method which is valid for small tax rates. We consider the case of a combination of taxes on capital and labor income. The optimal policy minimizes the excess burden of taxation for a given amount of tax revenues. The sensitivity of the welfare cost of taxation to the choice of policy appears to depend on the elasticity of substitution between capital and labor in the production function.

The concluding section is devoted to some brief remarks about the financing of exceptional government expenditures, and the possibility of the use of deceit by the policy maker.

## 1. First Best Solutions

### 1.1. The Model

For reasons of simplicity, we assume that there is only one produced good in the economy. This good can be either consumed, or invested to increase the capital stock. The production technology is represented by a neoclassical function which depends on the inputs of capital and labor,  $k$  and  $\ell$ , and satisfies the usual properties:

$$(1) \quad y = f(k, \ell) .$$

There are two agents in the economy, which represent respectively the private sector and the government.

In the private sector, consumers and producers are represented by a single household which grows at the rate  $n$ , and lives forever. It is endowed with perfect foresight, and behaves competitively, taking future (endogenous) prices as given. The household's intertemporal utility function is of the form:

$$(2) \quad U = \int_0^{\infty} e^{-\rho t} e^{nt} u(c_t, \ell_t) dt$$

with the following notation:

$\rho$  = rate of time preference ( $\rho > n$ )

$c_t$  = consumption per capita

$\ell_t$  = labor supply per capita.

The representative household chooses an intertemporal program  $(c_t, \ell_t)$ , which maximizes the utility function  $U$ , under the following budget constraint:

$$(3) \quad \begin{cases} \dot{a}_t = (\bar{r}_t - n)a_t + \bar{w}_t \ell_t - p_t c_t \\ (4) \quad \lim_{t \rightarrow \infty} e^{-R(t)} e^{nt} a_t = 0 \end{cases}$$

with the notation:

$a_t$  = assets per capita

$p_t$  = net price of consumption

$\bar{w}_t$  = net wage rate

$\bar{r}_t$  = net rate of return

$\bar{R}(t)$  = net rate of return between time 0 and time  $t$

$$\bar{R}(t) = \int_0^t \bar{r}_\tau d\tau .$$

The first order conditions satisfied by the household's program are given by:

$$(5) \quad \frac{u'_c(c_t, \ell_t)}{p_t} = - \frac{u'_\ell(c_t, \ell_t)}{\bar{w}_t} = q_t ,$$

where  $q_t$  is the marginal utility of assets, and

$$(6) \quad \frac{\dot{q}_t}{q_t} + \bar{r}_t = \rho .$$

The level of government expenditures per capita for consumption purpose, is measured by  $g_t$ , and is exogenous.<sup>9</sup> These expenditures can be financed by taxes and borrowing. In this risk-free economy, government bonds are perfectly substitutable with capital, and provide the same gross return. The total amount of private assets,  $a_t$ , is the



sum of capital  $k_t$ , and public debt  $b_t$  :

$$a_t = k_t + b_t .$$

At a given instant, the variation of the public debt is equal to the government deficit:

$$(7) \quad \dot{b}_t = (r_t - n)b_t + g_t - (\theta_t^r r_t a_t + \theta_t^w w_t k_t + \theta_t^c c_t) .$$

The term in brackets is equal to the amount of tax revenues;  $\theta_t^r$ ,  $\theta_t^w$  and  $\theta_t^c$  represent respectively the taxes on the return from assets, labor income and on consumption; the gross factor prices of capital services and labor are, denoted by  $r_t$  and  $w_t$ . The net household's prices are expressed as functions of the gross prices and the tax rates by:

$$p_t = 1 + \theta_t^c$$

$$\bar{w}_t = (1 - \theta_t^w) w_t$$

$$\bar{r}_t = (1 - \theta_t^r) r_t .$$

In subsequent sections, we will also consider the level of public asset  $h$ , which is the opposite of the public debt:

$$h = -b .$$

## 1.2. The Planner's Solution

The first best solution is the program  $(c_t, \ell_t)$ , chosen at time zero, by a social planner who has complete control over the private consumption and labor supply, and maximizes the utility of the household,

subject to the constraints of production, and of government consumption. This solution is identical to the program chosen by the private sector, when the government expenditures are financed by lump-sum taxation.

It is well-known that in a static framework, a tax on profits is equivalent to lump-sum taxation. The same rule is valid in the dynamic framework considered here. However the tax on pure profits cannot be implemented by a tax on the return to capital: such a tax would distort the intertemporal allocation of resources. Pure profits can be taxed by a lump-sum wealth levy at time zero. This wealth appropriation can also be obtained by different schemes which we now consider.

Assume that consumption is taxed at the rate  $\theta$ , and that labor is subsidized at the same rate. This rate is constant over time. No price distortion is introduced. Denote by  $V(x)$  the present value of a program  $(x_t)_{t \geq 0}$ , at the gross factor prices:

$$(8) \quad V(x) = \int_0^{\infty} e^{-R(t)} e^{nt} x_t dt .$$

The intertemporal household's budget constraint can be written:

$$(9) \quad (1+\theta)V(c) = (1+\theta)V(w \cdot \ell) + a_0$$

or

$$(10) \quad V(c) = V(w \cdot \ell) + \frac{a_0}{1+\theta} .$$

From this equation, we see that a wealth confiscation (reduction of  $a_0$ ), or a combination of tax and subsidy on consumption and labor achieve the same result, and reduce the second term of the RHS of (10) (which is equal to the present value of pure profits).

For example, no future taxation is necessary ( $\theta = 0$ ), if the government can confiscate an amount of wealth at time zero, such that the level of public assets is equal to the present discounted value of public expenditures.

Also, a combination of a tax and subsidy on consumption and labor at the constant rate  $\theta$ , reduces the value of private assets in terms of consumption, and assuming that the level of private assets is positive, is equivalent to a capital levy, with the following restriction: the present discounted value of the tax revenues raised by this scheme is always smaller than the level of private assets at time zero,  $a_0$ . The government budget constraint can be written as follows:

$$(11) \quad b_0 + V(g) = \frac{\theta}{1+\theta} a_0 .$$

Assume  $a_0$  to be positive; since  $\theta > -1$  (labor cannot be taxed at a rate higher than 100%), and  $a_0 = k_0 + b_0$ ,

$$b_0 + V(g) < k_0 + b_0$$

or,

$$V(g) < k_0 .$$

When the present value of public expenditures is smaller than the capital stock, these can be financed without distortion by a capital levy, or by a combination of a tax on consumption and a subsidy on labor.

Consider now the case where the present value of public expenditures is greater than the capital stock: a capital levy may be required since the level of public assets  $b_0$  must be greater than the capital stock  $k_0$ ; the level of private assets is negative ( $a_0 < 0$ ).<sup>10</sup> The value

of the tax rate  $\theta$  is given by:

$$\theta = \frac{h_0 - V(g)}{V(g) - k_0}.$$

If  $k_0 < h_0 < V(g)$ ,  $\theta < 0$ , consumption is subsidized and labor is taxed.

If  $k_0 < V(g) < h_0$ ,  $\theta > 0$ , consumption is taxed and labor is subsidized.

If the level of public assets  $h_0$  cannot be adjusted by a capital levy such that  $h_0 > k_0$ , and no other lump-sum tax is available, the first best solution cannot be decentralized.

Is the public debt (positive or negative), a necessary instrument for the decentralization of the first-best solution? In the policies considered above, the tax rates are constant over time. When the economy is not in a steady state, a temporary government deficit (or surplus), is balanced by the variation of the public debt. Only the intertemporal government budget constraint matters. However, public expenditures and revenues could also be balanced at each instant by taxes on consumption and labor and capital income:<sup>11</sup> consumption and leisure should be taxed at the same rate  $\theta_t$  which depends on time (there is no distortion between consumption and leisure in this case).

The balanced budget equation at time  $t$  can be written:

$$(12) \quad g_t = \theta_t (c_t - w_t \ell_t) + \theta_t^r r_t k_t$$

On a transition path, the rate  $\theta_t$  is not constant. Its variations have to be corrected by the interest tax, such that the ratio between the net consumption prices at two consecutive instants is always

equal to the gross rate of return:

$$\frac{(1 + \theta_t + \dot{\theta}_t dt)}{1 + \theta_t} \frac{1}{1 + r_t(1 - \theta_t^r) dt} = \frac{1}{1 + r_t dt}$$

or

$$(13) \quad \frac{\dot{\theta}_t}{1 + \theta_t} + r_t \theta_t^r = 0 .$$

It is shown in the Appendix that when the level of government consumption is not too large, and the first best solution tends to a steady state, there exists a unique value of  $\theta$  at time zero,  $\theta_0$ , such that tax program  $(\theta_t, \theta_t^r)$  defined by  $\theta_0$ , and by the equations (12) and (13), converges to a steady state.<sup>12</sup>

A decentralization policy balancing the government budget at each instant, requires continuous adjustments of the tax rates, and may be more difficult to implement than a policy which uses time invariant tax rates and bonds, in order to balance the government budget over the planning period. In this framework, bonds are not necessary but useful.<sup>13</sup>

## 2. Second Best Policies

We have seen in the previous section that when the level of government expenditures is large, and lump-sum taxation or a capital levy is not feasible, no first-best solution exists. In fact, no second best solution may exist either. Assume for example that the present value of government consumption  $V(g)$  is equal to the capital stock at time 0,  $k_0$ , and that the level of public asset  $h_0$  is equal to zero. The first best solution is approximated arbitrarily (but not attained), by large values of  $\theta$ . In general, when  $V(g)$  is greater than  $k_0$ , any policy using taxes (positive or negative), on consumption and labor is dominated by another policy with a higher tax rate on consumption, and a lower tax rate on labor income.<sup>14</sup> In particular, the consumption tax is superior to the labor income tax.

In this section, we assume that the level of public assets at time zero  $h_0$ , is exogenous and fixed, and that the consumption tax is not an available instrument.<sup>15</sup> Two types of policies are considered: in the first, the dynamic path of the public debt is exogenous. The government faces a budget constraint at each instant  $t$ ; without loss of generality, we assume that the level of the public debt is equal to zero, and that the government budget is balanced at each date. In the second class of policies, the government meets its budget constraint only over the entire horizon, and can borrow or lend at each date.

### 2.1. Balanced Budget Policies

When the government budget is balanced at each instant, by a unique tax instrument, the labor income tax, there is apparently no trade-off, and no need to determine an optimal policy. However, because of the household's optimizing behavior, the levels of private consumption and labor supply at a given instant  $t$ , do not depend only on the tax

rate on labor income at the same date, but also on the entire dynamic path of the tax policy. Therefore, all tax rates at different dates must be determined simultaneously. In order to take advantage of the property of additive separability of the model, the problem is solved by the same method as in the case of many tax instruments (examples of generalizations with more than one instrument are given below). Furthermore, this approach allows us to determine the shadow price of taxation, which can be used in the cost-benefit analysis of public consumption.

The optimal policy maximizes the household's intertemporal utility subject to the government budget constraints at each date, and to the constraint of the household's optimizing behavior.

The household's program satisfies the first order conditions (5) and (6). From (5) the levels of private consumption and labor supply, and also the level of utility at time  $t$ , depend only on the net wage rate  $\bar{w}_t$ , and on the private marginal value of assets  $q_t$ :

$$(13) \quad v(\bar{w}_t, q_t) = u(c(\bar{w}_t, q_t), \ell(\bar{w}_t, q_t)) .$$

The second condition (6), determines the time variation of  $q_t$ . Therefore, the optimal policy solves the following problem ( $P_1$ ):

$$(14) \quad \text{Maximize } V = \int_0^{\infty} e^{-(\rho-n)t} v(\bar{w}_t, q_t) dt$$

subject to the following constraints:

$$(15) \quad \dot{q}_t = q_t(\rho - r_t) \quad (\text{household's intertemporal optimization})$$

$$(16) \quad \dot{k}_t = f(k_t, \ell_t) - nk_t - c_t - g_t \quad (\text{capital accumulation})$$

$$(17) \quad 0 = -g_t + (w_t - \bar{w}_t) \ell_t \quad (\text{government budget constraint at time } t)$$

In these relations, the gross factor prices  $r_t$  and  $w_t$  are functions of capital and labor, and the private consumption and labor supply  $c_t$  and  $\ell_t$ , are functions of  $\bar{w}_t$  and  $q_t$ .

We assume that this second best solution converges in the long-run to a balanced growth path.<sup>16</sup> In this case, the discounted value at time zero of the total capital stock at time  $t$ ,  $e^{-rt} e^{nt} k_t$ , tends to zero, and the household's budget constraint is redundant in the optimization problem. The problem (P1) is solved by considering the current value Hamiltonian.<sup>17</sup>

$$(18) \quad H(q_t, k_t, \xi_t, \lambda_t, \mu_t, \bar{w}_t) \\ = v(\bar{w}_t, q_t) + \xi_t q_t (\rho - r_t) + \lambda_t (f(k_t, \ell_t) - nk_t - c_t - g_t) + \mu_t (-g_t + (w_t - \bar{w}_t) \ell_t) .$$

The variables  $\xi_t$ ,  $\lambda_t$  and  $\mu_t$  are the current values of the shadow prices of the respective constraints at time  $t$  (measured per capita). Their present value at time zero is given by the relation:

$$\hat{x}_t = x_t e^{-(\rho-n)t} , \text{ where } x \text{ represents } \xi , \lambda \text{ or } \mu .$$

The current value of the shadow price associated to the constraint of the household intertemporal optimization at time  $t$  is equal to  $\xi_t$ . The social value of the capital stock is measured by  $\lambda_t$ ; in the presence of distorting taxation, it is different from the private value of capital  $q_t$ , which is equal to the household's marginal utility. From (18), we see that if, at time  $t$ , the government increases tax revenues by one unit, and refunds this unit to the household by a lump-sum transfer, the current value of the household's utility is decreased by  $\mu_t$ . Therefore, this latter variable represents the current value of the marginal



welfare cost of taxation. The optimal program satisfies the following first order conditions,<sup>18</sup> (where the time subscripts have been omitted):

$$(19) \quad \frac{\partial H}{\partial q} = -\dot{\xi} + (\rho-n)\xi$$

$$(20) \quad \frac{\partial H}{\partial k} = -\dot{\lambda} + (\rho-n)\mu$$

$$(21) \quad \frac{\partial H}{\partial w} = 0$$

$$(22) \quad \frac{\partial H}{\partial \xi} = \dot{q}, \quad \frac{\partial H}{\partial \lambda} = \dot{k}, \quad \frac{\partial H}{\partial \mu} = 0.$$

The initial value of the state variable  $k$  is given, and equal to  $k_0$ . The initial value of the other state variable, the marginal utility of consumption  $q$ , can be chosen arbitrarily at time zero. Therefore its shadow price  $\xi$ , is initially equal to zero:  $\xi_0 = 0$ . We assume that the optimal program converges in the long-run to a steady state,<sup>19</sup> defined by the stationary equivalents of (19)-(22):

$$(19a) \quad \frac{\partial H}{\partial q} = (\rho-n)\xi$$

$$(20a) \quad \frac{\partial H}{\partial k} = (\rho-n)\mu$$

$$(21a) \quad \frac{\partial H}{\partial w} = 0$$

$$(22a) \quad \frac{\partial H}{\partial \xi} = \frac{\partial H}{\partial \lambda} = \frac{\partial H}{\partial \mu} = 0.$$

Using (22a), we see that the rate of return on capital in the steady state is fixed and equal to  $\rho$ . This value determines the capital labor ratio and the gross factor prices, which are independent of the tax policy. The combination of the equations (19)-(22), the initial conditions ( $k = k_0$  and  $\xi_0 = 0$ ), and the convergence to a steady state determines the solution to the problem (p1).<sup>20</sup>

An Example

Consider the case where the household's utility function takes the form:

$$U = \int_0^{\infty} e^{-(\rho-n)t} (\beta \log c_t + (1-\beta) \log(1 - \ell_t)) dt .$$

The levels of consumption and labor supply are given by:

$$c = \frac{\beta}{q} , \quad \ell = 1 - \frac{1-\beta}{q\bar{w}} ,$$

and the indirect current utility function can be written:

$$v(\bar{w}, q) = -\log q - (1-\beta) \log \bar{w} + \beta \log \beta + (1-\beta) \log(1-\beta) .$$

Using (18), (19a)-(22a), after some manipulations, we can derive the following relations in the steady state:

$$(23) \quad q = \lambda + \mu$$

$$(24) \quad \frac{\mu}{q} = \frac{\theta}{1-\theta} \frac{1}{1 + \frac{\beta}{1-\beta} \frac{1}{1 + \frac{\alpha}{1-\alpha} \cdot \frac{\rho}{\rho-n} \cdot \frac{1}{1-\theta}}}$$

where  $\theta$  and  $\alpha$  represent respectively the wage tax rate, and the share of capital in gross income.

From (23), the private value of capital  $q$ , is greater than its social value. Also the household's marginal utility of private consumption  $q$ , is equal to the shadow price of government consumption  $\lambda + \mu$ . This implies that in the determination of the optimal level of public consumption, the government should use only the private valuation of this public consumption, independently of the tax distortion.<sup>21</sup>

The ratio between the marginal welfare cost of taxation and the marginal value of private or public consumption is expressed by (24). Some indicative numbers are reported in the following table for the parameters:  $\alpha = 1/4$ ,  $\rho = 4\%$ ,  $n = 2\%$  (the choice of these values is not critical);  $\gamma$  is equal to the short-run elasticity of the labor supply with respect to the net wage rate (i.e., the elasticity of  $l_t$  with respect to  $\bar{w}_t$ , keeping  $q_t$  constant).<sup>22</sup>

TABLE 1

The Marginal Welfare Cost of the Labor Income Tax  
(in percentage of the private marginal utility of consumption)

$\theta \backslash \gamma$	.1	.25	.5	1.0	2.0	$\infty$
.15	2.3	4.9	7.6	10.7	13.3	17.6
.3	6.1	12.6	19.6	26.8	33.0	42.9
.4	10.2	20.8	31.7	42.9	52.2	66.7
.5	16.7	33.3	50.0	66.7	80.0	100.0
.6	27.9	54.5	80.0	104.3	123.1	150.0

## 2.2. Taxation and the Public Debt

In this section, we consider the case where the government can borrow and lend at each moment. It faces only one budget constraint: the present value of public expenditures over the planning period is equal to the sum of the present value of tax revenues and of the amount of public assets at the beginning of the planning period.<sup>23</sup> At time zero the level of public assets is exogenous (no capital levy is allowed). The variation of this level is equal to the budget surplus:

$$(25) \quad \dot{h} = (r-n)h + (w-w)z - g .$$

In the previous section, the variation of  $h$ ,  $\dot{h}$ , was exogenous (typically  $h = \dot{h} = 0$ ). Now this variation is endogenous, and provides a new additional instrument. The optimal policy can be determined by the same method as before: the balanced budget constraint at a given time (17), is replaced by the equation (25), and  $h$  is an endogenous state variable. The optimal policy satisfies the following property:

The marginal welfare cost of taxation measured in terms of a consumption loss is constant in the planning period.

This property can be proved by using a current value Hamiltonian as in (18). However, the following informal presentation may be more enlightening:

At a given date, government expenditures can be financed by taxation, or by a reduction of the level of public assets. Therefore, when the optimal policy is implemented, the social marginal value of public assets is equal to the excess-burden of taxation,  $\mu$ . Furthermore, the time variation of  $\mu$  is given by:

$$(26) \quad \frac{\dot{\mu}}{\mu} = \rho - r .$$

This relation is justified as follows: denote by  $\mu_t$  and  $\mu_{t+dt}$ , the current value of the excess-burden at time  $t$  and time  $t+dt$ , and assume for example that the LHS of (26) is greater than the RHS; for a small interval of time  $dt$ ,

$$(27) \quad \mu_{t+dt} > \mu_t(1 + (\rho-r)dt) .$$

In this case, the level of social welfare can be augmented by increasing tax revenues, and accumulating a surplus at time  $t$ , when the excess-burden of taxation is relatively low, and by decreasing taxes later: if tax revenues increase by an amount  $z$  at time  $t$  (without change in government consumption), the household incurs a utility loss which is measured (in current value) by  $z\mu_t$ . At time  $t+dt$ , the surplus of public wealth,  $z(1+rdt)$ , can be used to decrease taxes by the same amount, and increase the household's current utility by  $\mu_{t+dt}z(1+rdt)$ . Discounting this term back to time  $t$ , and using (27), we see that the net effect on the household's utility is positive. The same argument applies mutatis mutandis, when the LHS is smaller than the RHS in (26).

A similar thought experiment (or formula (6)), would show that the time variation of marginal utility of consumption is given by:

$$(6a) \quad \frac{\dot{q}}{q} = \rho - r .$$

From (26) and (6a), the ratio  $\mu/q$  is independent of time.

The ratio between the marginal welfare cost of taxation and the marginal utility of consumption, depends on the level of public assets at the beginning of the planning period. If this level is equal to the present value of government expenditures, the ratio is equal to zero (at the no-tax position, the marginal welfare cost of taxation is nil).

When the initial level of public assets decreases with respect to public consumption, the welfare cost of taxation increases. For example, if at time zero, the public debt inherited from the past is large, its service imposes a heavy burden on the economy. Some economists (Meade, 1958), suggested that in such a situation, the government should increase tax-revenues for a limited period of time; this reduces the

debt, and the welfare cost of taxation, in the future. (Meade proposes also the accumulation of capital to finance future public consumption.) This argument is not supported by the previous analysis. We have seen that because of the household's optimizing behavior over time (equation (6a)), the marginal excess-burden of taxation is constant and cannot be reduced over time. Also, even if there is no public consumption, the long-run value of the public debt is not in general equal to zero. Consider the following example:

There is no public consumption, the level of the public debt at time zero is positive, gross factor prices are exogenous, and the rate of return on capital is equal to the rate of time preference  $\rho$ . A straightforward manipulation shows that, when the only available tax is the labor income tax, the steady state policy is optimal: the level of the debt per capita  $b$  is constant over time, and equal to the exogenous value at time zero,  $b_0$ . The debt grows at the rate  $n$ , and the constant term  $(r-n)b$  is balanced by the revenues of the labor income tax (at a constant rate).<sup>24</sup>

The redemption of the public debt by a tax surplus does not bring social benefits because the welfare gain of a reduced service of the debt in the long-run is compensated by the welfare cost of additional taxes in the short-run. The only policy which could reduce the burden of a large public debt is an unexpected default on government bonds.

### 2.3. The Wealth Tax

In this section, we analyze how the optimal tax on wealth is determined simultaneously with other (non lump-sum) taxes. The wealth tax allows the policymaker to control the net interest rate available to the household. As before, we assume that the optimal policy converges in the long-run to a steady state<sup>25</sup> which we consider first:

In the long-run steady state, the tax rate on wealth is equal to zero; the gross and the net rates of return are identical.

This result can be obtained by considering the Hamiltonian as in (18). We give here an intuitive derivation: the social marginal value of capital at time  $t$ ,  $\lambda_t$  can be determined as a function of variables at time  $t+dt$  by the recurrent equation:

$$(28) \quad \lambda_t = (1 - \rho dt) \left[ \lambda_{t+dt} \left( 1 + \frac{\partial f}{\partial k}(k_{t+dt}, \ell_{t+dt}) dt \right) + \mu_{t+dt} \left( \frac{\partial f}{\partial k}(k_{t+dt}, \ell_{t+dt}) - \bar{r}_{t+dt} \right) dt \right].$$

This relation is interpreted as follows:

One additional unit of capital at time  $t$  provides the economy with  $1 + f'(k_{t+dt})dt$  units of capital at time  $t+dt$  which are valued at time  $t+dt$ , at the price  $\lambda_{t+dt}$ . Also, the revenues of the wealth tax  $t+dt$  are increased by the amount  $\Delta T = ((\partial f / \partial k)_{t+dt} - \bar{r}_{t+dt})dt$ ; therefore revenues generated by other taxes can be decreased by the same amount. This implies a social welfare gain equal to  $\mu_{t+dt} \Delta T$ . These changes occur at time  $t+dt$ , and are discounted by the rate  $\rho dt$ , to obtain the marginal social value of capital at time  $t$ ,  $\lambda_t$ , which is expressed by (28). When  $dt$  tends to zero, this relation can be rewritten:

$$(29) \quad \dot{\lambda}_t = \rho \lambda_t + \bar{r}_t \mu_t - (\lambda_t + \mu_t) r_t .$$

In the steady state, the variables  $\lambda_t$ ,  $\mu_t$ ,  $k_t$ ,  $l_t$ ,  $r_t$  are constant over time, and the net rate of return available to household  $\bar{r}$ , is equal to its discount rate  $\rho$ . From (29), we obtain:

$$(30) \quad 0 = (\lambda + \mu)(r - \bar{r}) .$$

The term  $\lambda + \mu$  is equal to the social marginal cost of government consumption (measured by the sum of value of goods consumed  $\lambda$ , and the welfare cost of the tax  $\mu$ ), It is strictly positive. Therefore the gross and the net rates of return are equal:

$$\bar{r} = r = \frac{\partial f}{\partial k} .$$

What about the optimal tax rate in the short-run, at the beginning of the planning period? At time zero, the wealth tax is equivalent to a lump-sum tax. Two cases have to be distinguished.

Assume first as in Section 2.1, that there is no public debt, and that the government balances its budget at each instant. At time zero, the optimal policy is to finance all public consumption by a tax on wealth. The excess-burden of taxation  $\mu_0$  is equal to zero. However, immediately after time zero, the wealth tax introduces a wedge between the intertemporal prices, and a distortion in the private allocation of resources. A fraction of government revenues should be raised by other taxes. The welfare cost of taxation is positive, and increasing (at least shortly after time zero). As time goes on, the tax rate on wealth tends to zero, and other taxes (on labor income or consumption for example), finance entirely the government consumption.<sup>26</sup>



Assume now that the government can accumulate assets through a budget surplus or issue bonds; the optimal policy is to impose a tax rate on wealth infinitely large for an infinitesimal duration, at time zero. This provides an arbitrarily close approximation of the first best policy described in Section 1, where the level of public assets is adjusted by a capital levy to a value equal to the present discounted value of public consumption. When there is no restriction on the wealth tax rate, its optimal level is not defined.

A more interesting situation to consider is a ceiling imposed on the tax rate. For example, the wealth tax may take the form of a tax on profits: the net rate of return is constrained to be positive. This problem is analyzed in the Appendix B.<sup>27</sup> The optimal tax policy can be described as follows: up to a specific date  $t_1$  (which depends on the initial levels of capital and public debt, and on the program of public consumption),<sup>28</sup> the ceiling on the wealth tax rate is binding. Profits should be taxed as much as possible. After the date  $t_1$ , the optimal wealth tax rate is constant, and equal to zero. The other tax rates and the level of the public debt are determined by the same method as in Section 2.3. The wealth tax allows us to redeem a fraction of the public debt (or to accumulate public assets). As in Section 2.3, in general, this redemption is not complete in the long-run.

The wealth tax is a good instrument to raise revenues in the short-run, especially if the government can accumulate a surplus. Unfortunately, this is a one time bonus, and cannot be repeated in the future, unless it is never anticipated by the private household.<sup>29</sup> The announced value of the tax rate of wealth for the distant future, should be equal to zero.

### 3. The Welfare Cost of Constant Tax Rates

In the previous sections, the tax rates that we considered, could vary freely over time. However, technical constraints may restrict these variations. We analyze now briefly an example of optimal taxation, where taxes can be raised on capital and labor income respectively. These rates are constant over time, and the government budget cannot be balanced at every instant; therefore, the use of the public debt (or public assets), is required. The case of a constant tax rate on capital income is especially interesting, since we have seen in the previous section that when this rate can vary, it should be equal to infinity in the short-run, and to zero in the long-run.

When the gross factor prices are endogenous, the problem of optimal intertemporal taxation with constant tax rates does not have in general, a simple solution (the assumption of endogenous factor prices is important as we will see below). In order to obtain some information on the properties of the optimal policy, we assume that the level of the tax revenues is small, and we follow an approximation method:<sup>30</sup> the optimal values of the tax rates on capital and labor income, denoted respectively by  $\theta^r$  and  $\theta^w$  minimize the total welfare cost of taxation subject to the government budget constraint.

To the first order, this constraint can be written:

$$(31) \quad b + V(g) = \theta^r k + \frac{\theta^w w \ell}{r - n}$$

where  $b$  is the level of the public debt, and  $V(g)$  the present value of government consumption (the other notations are the same as in the previous sections), at the beginning of the planning period.

The excess-burden of taxation depends on the household's intertemporal utility function, and on the production function. We assume that the utility function takes the form:<sup>31</sup>

$$U = \int_0^{\infty} e^{-(\rho-n)t} (\beta \log c_t + (1-\beta) \log(1 - \ell_t)) dt ,$$

and we consider two examples of production functions.

Assume first that the elasticity of substitution between capital and labor is infinite, and that the production technology is represented by:<sup>32</sup>

$$y = \rho k + \ell .$$

The welfare cost of taxation is measured by the decrease of the household's utility level,  $U$ , which is due to tax distortion.<sup>33</sup> A second approximation of this cost can be derived, and expressed as a wealth equivalent  $L$ , by an application of the Harberger-Hicks-Hotelling formula<sup>34</sup> (see Appendix C):

$$(32) \quad L = \frac{1}{2} \left[ (\theta^R)^2 \left( \frac{r}{r-n} \right)^2 + (\theta^W)^2 \beta(1-\beta) \right] A ,$$

with  $A$  equal to the total private wealth:

$$(33) \quad A = k + \frac{w}{r-n} .$$

The optimal values of  $\theta^R$  and  $\theta^W$  minimize  $L$  under the budget constraint (31); therefore, they are determined by (31) and their ratio:

$$(34) \quad \frac{\theta^R}{\theta^W} = \frac{(r-n)k}{w\ell} \left( \frac{r-n}{r} \right)^2 \beta(1-\beta) .$$

Since the term  $\beta(1-\beta)$  is always smaller than  $1/4$ , this ratio is fairly small. When there is no growth and the capital income share is represented by  $\alpha$ ,  $\theta^r/\theta^w$  is smaller than  $\alpha/(4(1-\alpha))$ . When the growth rate is positive, the optimal tax rate on capital income decreases very significantly with respect to the tax on labor income.<sup>35</sup>

It is interesting to compare the excess-burden of a general income tax ( $\theta^r = \theta^w$ ),  $L_I$ , with the one of labor income tax ( $\theta^r = 0$ ),  $L_L$ , which generates the same amount of revenues. Using (31) and (32), their respective ratio can be written:

$$(35) \quad \frac{L_L}{L_I} = \left(1 + \frac{(r-n)k}{w\ell}\right)^2 \frac{\beta(1-\beta)}{\beta(1-\beta) + \left(\frac{r}{r-n}\right)^2}$$

this ratio is bounded by:<sup>36</sup>

$$\frac{L_L}{L_I} \leq \frac{1}{5} \left(1 + \frac{(r-n)k}{w\ell}\right)^2 .$$

The welfare gain obtained by a tax reform from general income taxation to labor income taxation can be quite large.

However, the excess-burden of capital income taxation depends heavily on the elasticity of substitution between capital and labor.<sup>37</sup> It is much smaller when this elasticity takes plausible values, smaller than two, than when gross factor prices are fixed. An enlightening illustration is provided by the case of an elasticity equal to zero, where the input ratio is fixed:<sup>38</sup>

The excess-burden depends only on the amount of tax revenues, and not on the combination of capital and labor income taxes.<sup>39</sup>

By continuity, we can expect that the excess-burden expressed as a function of the tax rates  $\theta^r$  and  $\theta^w$ , is fairly flat if the elasticity of substitution is small. The case of a Cobb-Douglas production function is intermediate between the two special cases considered above. Although the welfare cost of the capital income tax is then much smaller than in the case of an infinite elasticity,<sup>40</sup> we may still expect the tax rate on capital income to be smaller than the tax rate on labor income. The exact determination of the welfare gain of tax reform in this context should be analyzed in another study.

#### 4. Conclusion

The analysis presented here could be extended to a more disaggregated model including different types of consumption and capital goods without altering significantly the general character of the results.<sup>41</sup> These can be summarized briefly.

The consumption tax is more efficient than the labor income tax, because it allows a partial taxation of the pure profits generated by the existence of a capital stock at the beginning of the planning horizon. However, this difference in efficiency is rather small (when the growth rate of the economy approaches the discount rate, real capital becomes negligible with respect to human capital, and the labor income tax is asymptotically equivalent to the consumption tax).

The tax on wealth (or capital income) is much more inefficient in the long-run, than the taxes on consumption or labor income: an anticipated tax on profits in the distant future creates more distortion than an equivalent tax on consumption at the same date. In the long-run the tax on wealth should be abolished. In the short-run, the wealth tax is a good approximation of a capital levy, and is very efficient.

The main application of the wealth tax should be the reduction of the public debt. In this case, the faster, the better: when a ceiling is imposed on the tax rate, it is binding for an initial time interval during which the debt is redeemed. At the end of this interval the debt may be negative; however, at this date, the level of assets accumulated by the government is smaller than the present value of future public consumption.<sup>42</sup> After the transition period, the optimal tax rate on wealth is equal to zero. The limit case of an infinite ceiling corresponds to a capital levy, and is equivalent to lump-sum taxation.

These results allow us to formulate a theory of the public debt.<sup>43</sup>

The long-run level of the debt depends on its initial level at the beginning of the planning period, on the relative fluctuations of government expenditures and revenues, and on the possibility of raising a wealth tax. In the special case where there is no fluctuation in the economy and no wealth tax, the optimal long-run level of the debt is equal to its initial value.

Because a wealth tax is very efficient in the short-run, it has often been advocated for the financing of exceptional government expenditures. Keynes (1940) proposed to pay for the war by issuing bonds frozen during the war, and financed by a special capital levy after the war (Keynes, 1940 Chapter 7). The following remarks can be made about such a scheme: Keynes' purpose was to restrict private consumption without reducing the labor supply, in order to maximize the available resources for the war effort. But unless they are under liquidity constraint, households can reduce their private savings by relying on the temporarily frozen assets for future period. Also, the postponement of the capital levy until a later date is inefficient.<sup>44</sup>

A capital levy is the most efficient type of financing an exceptional government expenditure. However, this procedure should remain exceptional. It is possible to imagine that the government could mislead the private sector, and raise capital levies in the future. If the private sector can always be fooled, this situation allows to finance government expenditures by lump-sum levies, and is very efficient.<sup>45</sup> However you cannot fool one household which lives forever, all the time. It will learn to consider repeated capital levies as a wealth tax. Because of the high inefficiency of the latter in the long-run, the benefits of such a strategy of deception seem to be doubtful.<sup>46</sup>

## FOOTNOTES

1. We cannot summarize here the large literature on this subject. For a study of tax incidence in a neoclassical growth model, see Feldstein (1974a, 1974b). Examples of analysis in a life-cycle model of capital accumulation can be found in Diamond (1970), Pestieau (1974), Summers (1978). Since these models assume no bequest, the problem of inter-generational equity arises (Chamley, 1980c).
2. See Diamond and Mirrlees (1971).
3. Feldstein (1978), Helpman and Sadka (1979), Pestieau (1974).
4. For a discussion of this assumption, see Barro (1974).
5. This tax policy is known with perfect foresight by the household. One feature of an intertemporal framework where future actions cannot be known with certainty, is the possibility of deceit by the policy maker. Such a strategy can lead to a superior allocation of resources. This problem will be considered only briefly in the concluding remarks (see also Fisher, 1979).
6. This approach was pioneered by Ramsey (1927), and developed with the use of optimal control in the sixties (for a synthesis, see Arrow and Kurz, 1970). An axiomatic discussion of the assumptions of additivity and stationarity is given by Koopmans (1972). Of course, the structure of optimal taxes (as for example uniform tax rates, or no tax on wealth in the long-run), will depend on this assumption (see Sadka (1977)). It should also be noted that the additive separability is often implicitly used in studies on intertemporal optimization which rely on a more intuitive approach (see for example Bradford, 1975).
7. Since household and planner have perfect foresight, and are rational the time inconsistency problem (Prescott, 1977, Kydland and Prescott, 1978), does not arise.
8. This restriction imposes a set of new constraints which is not considered in the static optimal tax problem.
9. Once the optimal taxation problem is solved, the optimal level of  $g_t$  can be determined separately by equating the marginal benefit of  $g_t$ , with the marginal cost of raising government revenues.
10. Of course, total private wealth (human and non human), is positive, and equal to the present value of private consumption.
11. The case of an exogenous path of the public debt is technically equivalent to a balanced budget in each period.
12. In this steady state, the tax rate on consumption and the subsidy rate on labor income are identical. There is no tax on capital income.



13. This result is to be contrasted with those obtained in second best cases. In the latter, even if the available tax rates can vary freely, the existence of bonds increases private welfare.

14. Assume for simplicity that  $\theta^C$  and  $\theta^W$  are constant over time. The household's budget constraint can be written:

$$(1 + \theta^C)V(c) = (1 - \theta^W)V(w\ell) + a_0 ,$$

or

$$V(c) = ((1 - \theta^W)/(1 + \theta^C))V(w\ell) + a_0/(1 + \theta^C) .$$

The tax on consumption at the rate  $\theta^C$  allows a partial taxation of pure profits. If  $V(g) < a_0$ , the optimal policy is approximated when  $\theta^C$  tends to  $+\infty$ , and  $\theta^W$  tends to  $-\infty$ , such that the ratio  $(1 - \theta^W)/(1 + \theta^C)$  tends to  $1-z$ ;  $z$  is defined by  $V(g) = zV(w\ell) + a_0$ .

15. The analysis of the labor income tax is equivalent from a technical point of view, to that of the consumption tax. The former has been chosen to simplify the presentation.

16. It is implicitly assumed that  $g_t$  tends in the long-run to a stationary value. The convergence of the optimal path to a balanced growth path is in general, guaranteed by the assumptions of stationarity and additivity of the utility function. However, these assumptions are not always sufficient; for example, in the case of labor augmenting technological change, the elasticity of substitution between consumption and leisure has to be equal to one. The convergence to a steady state will be proved below for a specific example (Appendix B on the wealth tax).

17. A detailed exposition of the use of optimal control in optimal growth is given in Arrow-Kurz (1970).

18. As it is usual in the optimal tax literature, we analyze here only the necessary conditions satisfied by a program of optimal taxes.

19. See footnote 16.

20. For an example of complete analysis, see Appendix B.

21. The same result is true with the same utility function, in the static case where no profit exists (see for example Stiglitz and Dasgupta 1971). Of course, because of the tax distortion, the optimal levels of public and private consumption in the second best are different from their respective values in the case of lump-sum taxation.

22. This elasticity is determined by

$$\gamma = \frac{1-\beta}{\beta} \left( 1 + \frac{\rho-n}{\rho} \frac{\alpha}{1-\alpha} \right) .$$

With the parameters used here,  $\gamma$  is almost equal to  $(1-\beta)/\beta$ ,  
(  $\gamma = [(1-\beta)/\beta] \cdot (7/6)$  ).

23. As before, government bonds are perfectly substitutable with capital. At a given instant, the level of public assets can be positive or negative.
24. This steady state policy is no longer optimal when wealth can be taxed. However, even in this case, the long-run value of the debt is different from zero. An example is analyzed in Section 2.4, and in Appendix B.
25. See footnote 16.
26. The analytical solution can be determined by the same method as in Section 2.1.
27. The assumption of fixed gross factor prices, used in the appendix to obtain a closed form solution does not seem to affect the character of the optimal solution.
28. In the model described in the appendix (log utility function, no growth), the length of the period can be approximated by the formula  $t_1 = (1/\rho) \sqrt{2(\mu/q)(\rho a_0/c_0)}$ . For an annual discount rate  $\rho = 4\%$ , a welfare cost  $\mu/q = 1/4$ , and a ratio between return on wealth and consumption  $(\rho a_0/c_0) = 1/4$ , one finds:  $t_1 = 8.8$  years. This computation assumes a specific value for  $\mu/q$ ; however, this welfare cost and the time  $t_1$  should be determined simultaneously.
29. A brief discussion about the possibility of repeated capital levies over the planning period is given in the conclusion.
30. For a complete description of this approach, see Chamley (1980a).
31. The growth rate  $n$  can be considered as the sum of the rates of growth of population and labor augmenting technological change (when the latter is positive, all quantities should be measured per capita, where one caput corresponds to the product of one individual and of the efficiency index).
32. This example corresponds to a partial equilibrium assumption.
33. This definition is different from the concept used in the previous sections where  $\mu_t$  represented the marginal welfare cost of taxation during an infinitesimal period, at time  $t$  (no restrictions were imposed on the fluctuations of tax rates over time).

34. As Green and Sheshinski (1979) have pointed out, this formula gives a good approximation of the welfare gain of tax reform only for very small initial values of the tax rates. However it seems unlikely that the result obtained below (a very small  $\theta^r$ ), should be altered by a more accurate computation.
35. Consider for example the "realistic" values:  $\alpha = .2$ ,  $r = 4\%$  and  $n = 2\%$ ; we find then  $\theta^r \leq (1/128)\theta^w$ ! Of course, an infinite elasticity of substitution between capital and labor is not a good description of the U.S. technology (see the second example).
36. For the values  $\alpha = .2$ ,  $r = 4\%$  and  $n = 2\%$ , the ratio  $L_L/L_I$  is smaller than  $(1+1/8)^2/17 = .074$ . A shift from a general income tax to a labor income tax reduces the excess-burden by 92%! (See also the qualifying remark in footnote 34.)
37. See Chamley (1980a).
38. For values of the elasticity of substitution different from zero or infinity, even the second order approximation formula becomes fairly complex, and must be computed numerically. For large values of the tax rates, the excess-burden is determined by numerical dynamic simulations.
39. The derivation of this result is rather lengthy, and is available to the interested reader. An intuitive (and vague) explanation is the following: when the production technology has fixed coefficients, the burden of the capital income tax is shifted entirely to labor, (because the gross rate of return increases, and the net wage rate decreases). The capital income tax is then equivalent to a labor income tax (for the same amount of revenues).
40. See Chamley (1980a).
41. Consider for example a production function with different capital goods taxed at different rates. The welfare cost of the intersectoral misallocation is then in general, greater than the welfare cost of the intertemporal misallocation (Chamley, 1980a); this reinforces the result obtained here (no capital income tax in the long-run). The model developed in this paper can also be used to analyze the optimal inflation rate in a second-best case (Chamley, 1980b).
42. Provided that this is true at the beginning of the planning horizon (which is usually the case).
43. See also Barro (1979) for a less formal analysis.

44. In order to finance an extraordinary expenditure, when there is no wealth tax, the optimal (second-best) policy is to raise the permanent level of the tax rate on consumption or labor income by a small amount, and to pay for this temporary expenditure mainly by bonds. (After the war, the interest payments on the new debt are financed by the increment of the tax.) This procedure has been very popular historically (see also Barro, 1979).
45. For an analysis of this problem, see Fisher (1979).
46. Another problem is presented by the optimal tax policy contingent upon uncertain future public expenditures. (The contingent policy is announced and no deception occurs.) We can again suggest that unexpected expenditures should be financed by new taxes on consumption or labor income, rather than by capital levies.

## APPENDIX A

The Decentralization of the First BestSolution without Government Debt

The social planner relies on the tax rates on consumption, labor income and capital income, respectively  $\theta^c$ ,  $\theta^l$ , and  $\theta^r$  to maximize the private welfare represented by the function

$$U = \int_0^{\infty} e^{-(\rho-n)t} u(c_t, l_t) dt ,$$

and is subjected to the following constraints:

$$(1) \quad u'_c = p_t (1 + \theta^c)$$

$$(2) \quad u'_l = -p_t w_t (1 - \theta^w)$$

$$(3) \quad \frac{\dot{p}_t}{p_t} + r_t (1 - \theta^r_t) = \rho$$

$$(4) \quad \dot{k}_t = f(k_t, l_t) - c_t - g_t$$

$$r_t = \frac{\partial f}{\partial k}(k_t, l_t) , \quad w_t = \frac{\partial f}{\partial l}(k_t, l_t) .$$

The first three relations describe the behavior of the private sector.<sup>1</sup>

We assume that the optimal path tends to a stationary point. When this constraint and the capital accumulation relation (4) are satisfied, the budget constraint of the private and of the public sectors are also

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<sup>1</sup>For the sake of clarity, whenever it is convenient, the time subscripts will be omitted.

verified. If the private rate of substitution between consumption and labor is equal to the social rate,  $\theta^C = -\theta^W = \theta$ . Call  $\lambda$  the marginal utility of consumption,  $\lambda = p(1+\theta)$ . On the optimal path,

$$(5) \quad \frac{\dot{\lambda}_t}{\lambda_t} + r_t = \rho .$$

From (3) and (5), we deduce that

$$(6) \quad r_t \theta_t^r + \frac{\theta_t}{1+\theta_t} = 0 .$$

Using the government budget constraint to substitute  $\theta^r$  for  $\theta$ ,

$$r_t k_t \theta_t^r + \theta_t (c_t - w_t \ell_t) = g_t ,$$

we find

$$(7) \quad \frac{\dot{x}_t}{x_t} = x_t \left( \frac{c_t - w_t \ell_t}{k_t} \right) - \frac{g_t + c_t - w_t \ell_t}{k_t} ,$$

with  $x_t = 1 + \theta_t$ .

We now show that this differential equation, together with the terminal condition

$$\lim_{t \rightarrow \infty} x_t = 1 + \theta^* = \frac{g^* + c^* - w^* \ell^*}{c^* - w^* \ell^*}$$

(where a star denotes a stationary value), determines completely the values of  $x_t$  (and therefore of the different tax rates), at each instant.

The relation (7) can be rewritten as follows:

$$(8) \quad \dot{z}_t = e^{z_t} X_t - Y_t$$

where  $z_t = \log(x_t)$ ,  $X_t = (c_t - w_t l_t)/k_t$ ,  $Y_t = (g_t + c_t - w_t l_t)/k_t$

$$\lim_{t \rightarrow \infty} X_t = X^* , \quad \lim_{t \rightarrow \infty} Y_t = Y^* .$$

We assume that  $X^*$  is positive, which is equivalent to

$$g^* < (r^* - n)k^* .$$

If this condition is not satisfied, the first best policy cannot be decentralized by using only tax instruments (see main text). Also, if  $X^*$  is positive, the same property is verified for  $Y^*$  (which is greater than  $X^*$ ). If  $z_t$  converges to a stationary value, this limit is given by:

$$\lim_{t \rightarrow \infty} z_t = \log(Y^*/X^*) .$$

Define by  $z(t, z_0)$  the solution of (8) such that  $z(0, z_0) = z_0$ ; we now show that there is a unique value of  $z_0$  such that

$$\lim_{t \rightarrow \infty} z(t, z_0) = \log\left(\frac{Y^*}{X^*}\right) .$$

This value and the integration of the differential equation (8) will define  $z_t$  for all values of  $t$ .

Assume first that  $X_t$  and  $Y_t$  are always strictly positive. Since  $X_t$  and  $Y_t$  tend to the limits  $X^*$  and  $Y^*$  when  $t$  tends to infinity, we can introduce the following parameters depending on the integer  $N$ :

$$(9) \quad \begin{aligned} \bar{X}_N &= \sup_{t \geq N} \{X_t\}; & \underline{X}_N &= \inf_{t \geq N} \{X_t\} \\ \bar{Y}_N &= \sup_{t \geq N} \{Y_t\}; & \underline{Y}_N &= \inf_{t \geq N} \{Y_t\}. \end{aligned}$$

Take a value of  $N$ , and define the number  $a$  such that

$$e^{a \underline{X}_N} = \bar{Y}_N > \frac{1}{N}.$$

By integration of (8) backwards from time  $N$  to the origin, one finds the value  $\beta_N$  such that

$$z(N, \beta_N) = a$$

we have therefore:

$$(10) \quad \dot{z}(N, \beta_N) = e^{z(N, \beta_N) X_N} - Y_N \geq e^{z(N, \beta_N) \underline{X}_N} - \bar{Y}_N > \frac{1}{N}$$

Consider now the time derivative of the function

$$\begin{aligned} y_t &= e^{z(t, \beta_N) X_N} - \bar{Y}_N, \\ \dot{y}_t &= \dot{z} e^{z X_N} = (e^{z X_t} - Y_t) e^{z X_N} \geq y_t e^{z(t, \beta_N) \underline{X}_N}, \quad \text{for } t \geq N; \end{aligned}$$

therefore if  $y_t$  is positive at time  $t$  it is increasing (and positive) in the interval  $(t, +\infty)$ .

We deduce then from (10) that  $z(t, \beta_N)$  tends to the infinity when  $t$  becomes large. Applying this construction of  $\beta_N$  for each  $N$  we find a sequence of  $\beta_N$  such that:



$$(11) \quad e^{z(t, \beta_N)} X_N - Y_N \geq e^{z(t, \beta_N)} \bar{X}_N - \bar{Y}_N = \frac{1}{N}$$

Furthermore,  $\lim_{t \rightarrow \infty} z(t, \beta_N) = +\infty$ .

Symmetrically we can construct the sequence of number  $\alpha_N$  such that :

$$(12) \quad e^{z(t, \alpha_N)} X_N - Y_N \leq e^{z(t, \alpha_N)} \bar{X}_N - \bar{Y}_N = -\frac{1}{N},$$

and

$$\lim_{t \rightarrow \infty} z(t, \alpha_N) = -\infty.$$

Let us consider the differences  $\beta_N - \alpha_N$ ; it is clear that they are positive. When  $t$  increases, the difference  $z(t, \beta) - z(t, \alpha)$  varies according to the sign of  $(e^{z(t, \beta)} - e^{z(t, \alpha)})X_t$ , which is positive.

Therefore we have:

$$\begin{aligned} 0 < \beta_N - \alpha_N &= z(0, \beta_N) - z(0, \alpha_N) < z(t, \beta_N) - z(t, \alpha_N) \\ &= \text{Log} \left( \frac{\bar{Y}_N + \frac{1}{N}}{\bar{X}_N} \right) - \text{Log} \left( \frac{Y_N + \frac{1}{N}}{X_N} \right). \end{aligned}$$

When  $t$  becomes large, the right hand side of this expression tends to zero. Therefore the sequences  $\alpha_N$  and  $\beta_N$  have a common limit  $z_0$ . Since  $z(t, \alpha)$  is an increasing function of  $\alpha$ ,

$$z(t, \alpha_N) \leq z(t, z_0) \leq z(t, \beta_N).$$

Using (9), (11) and (12), we find:

$$\lim_{t \rightarrow \infty} z(t, \alpha_N) = \lim_{t \rightarrow \infty} z(t, \beta_N) = \text{Log}(Y^*/X^*),$$

which proves that

$$\lim_{t \rightarrow \infty} z(t, z_0) = \text{Log}(Y^*/X^*) .$$

If  $X$  and  $Y$  are not strictly positive, it is possible to find a time  $T$ , after which they are positive (since their limit is positive).

We can apply the previous argument at time  $T$  to find the value of  $z_t$  at time  $T$ , and then integrate backwards (8) to find the value of  $z$  at time 0.

## APPENDIX B

The Wealth Tax

We make the following assumptions to simplify the argument. The gross factor prices are fixed, and the production function takes the form:

$$(1) \quad y = \rho k + \ell .$$

The population is constant, and the private sector's intertemporal utility function is given by:

$$(2) \quad U = \int_0^{\infty} e^{-\rho t} (\beta \log c_t + (1-\beta) \log(1 - \ell_t)) dt .$$

At time zero, the capital stock and the public assets are respectively equal to  $k_0$  and  $h_0$ ;  $h_0 < 0$  (the level of the public debt is positive).

There is no public consumption; the service of the debt is financed by taxes on labor income or on wealth. The tax on wealth is subject to the arbitrary constraint that only the return on wealth can be taxed; therefore the net rate of return  $\bar{r}$  is always at least equal to zero.

The optimal policy is determined by considering the current value Hamiltonian (with the notations of the text):

$$(3) \quad \begin{aligned} H = & v(\bar{w}, q) + \xi q(\rho - \bar{r}) \\ & + \lambda(\rho k + \ell - c) \\ & + \mu(\bar{r}h + \rho k + \ell - \bar{r}k - \bar{w}\ell) \\ & + v\bar{r} . \end{aligned}$$

The variables  $c$ ,  $\ell$  and  $v$  are determined by:

$$(4) \quad c = \frac{\beta}{q}, \quad \ell = 1 - \frac{1-\beta}{q\bar{w}}, \quad v = -\frac{1}{q} - \frac{1-\beta}{q}.$$

The first order conditions are written as follows:

$$(5) \quad \frac{\partial H}{\partial q} = -\dot{\xi} + \rho q$$

$$(6) \quad \frac{\partial H}{\partial k} = -\dot{\lambda} + \rho \lambda$$

$$(7) \quad \frac{\partial H}{\partial h} = -\dot{\mu} + \rho \mu$$

$$(8) \quad \frac{\partial H}{\partial \bar{w}} = 0$$

$$(9) \quad \frac{\partial H}{\partial \bar{r}} = 0$$

$$(10) \quad \left\{ \begin{array}{l} \frac{\partial H}{\partial \xi} = \frac{\partial H}{\partial \lambda} = \frac{\partial H}{\partial \mu} = 0 \\ \frac{\partial H}{\partial v} \geq 0; \text{ if } v > 0, \quad \bar{r} = 0 \text{ (the constraint } \bar{r} \geq 0 \text{ is binding).} \end{array} \right.$$

The relation (9) determines  $v$  :

$$(11) \quad v = q\xi + \mu a, \quad \text{with } a = k - h = k + b.$$

From (7) and (10), we find:

$$(12) \quad \frac{\dot{\mu}}{\mu} = \frac{\dot{q}}{q} = (\rho - \bar{r})$$

Therefore, the ratio  $\mu/q$  is constant:

$$\mu = \phi q,$$

where  $\phi$  is independent of time. At time zero,  $\xi$  is equal to zero (see Section 2.1 in the text); also the level of taxes financing the

service of the debt, and their excess-burden are strictly positive. Therefore  $\phi$  is positive, and  $v$  is positive. From (11), the constraint  $\bar{r} \geq 0$ , is binding at time zero.

After some manipulations, the system of dynamic equations (5)-(10) can be transformed into the equivalent set:

$$(13) \quad z = q\xi + \mu a$$

$$(14) \quad \dot{z} = \rho z + (q - (\lambda_0 + \mu_0))c$$

$$(15) \quad \lambda + \mu = \lambda_0 + \mu_0, \text{ which is independent of time}$$

$$(16) \quad \mu = \phi q$$

$$(17) \quad \frac{\partial H}{\partial \bar{w}} = 0$$

$$(18) \quad \begin{cases} \text{if } z > 0, & v = z \text{ and } \bar{r} = 0 \\ \text{if } z < 0, & \bar{r} = +\infty \end{cases}$$

$$(19) \quad \dot{q} = q(\rho - \bar{r})$$

$$(20) \quad \dot{k} = \rho k + \ell - c$$

$$(21) \quad \begin{cases} \dot{a} = \bar{r}k + \bar{w}\ell - c, & (a = k-h) \\ \text{or} \\ \dot{h} = \bar{r}h + \rho k + \ell - \bar{r}k - \bar{w}\ell. \end{cases}$$

The variable  $z$ , positive at time zero, cannot remain so indefinitely (the government would accumulate all the capital stock with no purpose). Call  $t_1$  the smallest value of  $t$ , for which  $z$  is equal to zero.

Using (14), we can show that at time  $t_1$ , the value of  $q$  (call it  $q_1$ ), must be equal to  $\lambda_0 + \mu_0$ : if  $q_1 > \lambda_0 + \mu_0$ ,  $\dot{z}_1 > 0$ ,

and  $t_1$  cannot be the smallest value of time for which  $z = 0$ ; if  $q_1 < \lambda_0 + \mu_0$ , after time  $t_1$ ,  $z$  is negative, and  $\bar{r}$  is infinite (an infinite subsidy on the rate of return is absurd).

The same type of argument (using (14), (18), and (19)), shows that the only meaningful dynamic solution after time  $t_1$ , is the stationary solution characterized by the following equations:

$$(13a) \quad 0 = q_1 \xi_1 + \mu_1 a$$

$$(14a) \quad q_1 = \lambda_0 + \mu_0$$

$$(15a) \quad \lambda_1 + \mu_1 = \lambda_0 + \mu_0$$

$$(16a) \quad \mu_1 = \phi q_1$$

$$(17a) \quad \mu_1 = \frac{1-\beta}{w_1} (1 - \bar{w}_1)$$

$$(18a) \quad \bar{r} = \rho$$

$$(20a) \quad 0 = \rho k_1 + \ell_1 - c_1$$

$$(21a) \quad 0 = \rho h_1 + (1 - \bar{w}_1) \ell_1 .$$

These 8 equations define the 8 values  $a_1$ ,  $k_1$ ,  $h_1$ ,  $\xi_1$ ,  $\lambda_1$ ,  $\mu_1$ ,  $\bar{w}_1$ ,  $\bar{r}_1$  ( $c_1$  and  $\ell_1$  are determined by (4)).

A necessary condition for the optimal solution is to satisfy the dynamic equations (13)-(21) in an interval of time  $(0, t_1)$ , and to converge at time  $t_1$  to the steady state defined by the equations (13a)-(21a). Because the constraint  $\bar{r} \geq 0$  is binding in the interval  $(0, t_1)$ , and  $\bar{r}$  is equal to zero, the dynamic solutions satisfying (13)-(21) can be written under a closed form. Using (13), (14), (18), (19), and (14a),  $z$  can be expressed as a function of time:

$$(22) \quad z = \frac{\beta}{\rho}(-1 + \text{ch}(\rho(t - t_1))) \quad , \quad \text{with} \quad \text{ch}(x) = \frac{e^x + e^{-x}}{2} .$$

Taking  $t = 0$  in (22), and  $\xi_0 = 0$  in (13), we find the value of  $t_1$  by:

$$(23) \quad \text{ch}(\rho t_1) = \mu_0 \frac{\rho a_0}{\beta} + 1$$

This value increases with the excess-burden  $\mu_0$ .<sup>1</sup>

The existence of a solution which satisfies the dynamic system (13)-(21), and converges at time  $t_1$  to the stationary solution given by (13a)-(21a) can be proven as follows: (the proof provides also a sketch for a method of numerical computation):

Choose a value for  $q_1$  and  $\bar{w}_1$  (in the steady state after time  $t_1$ );  $\mu_1$  is then given by (17a), and  $t_1$  is determined by (23) (where  $\mu_0 = \mu_1 e^{-\rho t_1}$ ). Using  $\bar{r} = 0$  (by (18)), and successively, (19), (16), (14a), (15), (14), (13), (17), with integrations over time, we can compute the time path of the variables  $q$ ,  $\mu$ ,  $\lambda$ ,  $\xi$ ,  $w$  and also of  $c$  and  $\ell$  (by (4)), in the interval  $(0, t_1)$ .

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<sup>1</sup>Around  $t_1 = 0$ , the function  $\text{ch}x - 1$  is very well approximated by  $x^2/2$ . Using (23), we find

$$(\rho t_1)^2 = 2 \frac{\mu_0}{\beta} \rho a_0 = 2 \frac{\mu_0}{q_0} \frac{\rho a_0}{\beta} q_0, \quad \rho t_1 = \sqrt{2\phi \frac{\rho a_0}{c_0}}, \quad (\text{since } c_0 = \frac{\beta}{q_0}).$$

Taking a ratio between gross capital income and consumption at time zero ( $\rho a_0/c_0$ ), equal to 1/4, and an annual rate of return  $\rho$  equal to 4% the value of  $t_1$  is equal to 8.8 years. Of course, this computation is only indicative; the value of  $\phi$ ,  $c_0$ , and  $t_1$  should be determined simultaneously.

Integrate (20) and (21), to calculate the value of  $k$  and  $h$  at time  $t_1$ , respectively  $k_1$  and  $h_1$ .

The RHS of (20a) and (21a) can now be determined; they depend only on the initial guess  $(q_1, \bar{w}_1)$ :

$$(24) \quad E(q_1, \bar{w}_1) = \rho k_1 + \ell_1 - c_1 = \rho k_1 + 1 - \frac{1-\beta}{q_1 \bar{w}_1} - \frac{\beta}{q_1}$$

$$(25) \quad F(q_1, \bar{w}_1) = \rho h_1 + (1 - \bar{w}_1) \ell_1 = \rho h_1 + (1 - \bar{w}_1) \left( 1 - \frac{1-\beta}{q_1 \bar{w}_1} \right).$$

By construction the variables  $q$ ,  $\xi$ ,  $\lambda$ ,  $\mu$ , converge at time  $t_1$ , to values which satisfy the stationary relations (13a), (14a), (15a), (16a), (17a); (18) determines  $\bar{r}$  after  $t_1$ , the remaining equations are:

$$(20a) \quad 0 = E(q_1, \bar{w}_1)$$

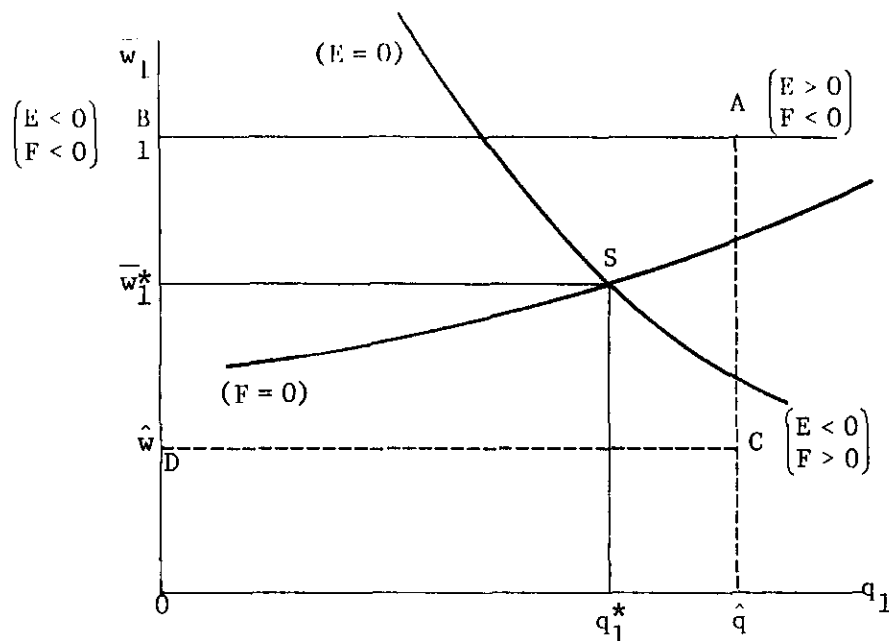
$$(21a) \quad 0 = F(q_1, \bar{w}_1).$$

The initial guess  $(q_1, \bar{w}_1)$  can be represented by a point on the following diagram;  $\hat{q}$  and  $\hat{w}$  represent the values of  $q$  and  $\bar{w}$  when instead of following an optimal path after time zero, the economy stays in the steady state with no tax on wealth, and which is defined by:

$$(26) \quad 0 = \rho k_0 + 1 - \frac{1-\beta}{\hat{q} \hat{w}} - \frac{\beta}{\hat{q}}$$

$$(27) \quad 0 = \rho h_0 + (1 - \hat{w}) \left( 1 - \frac{1-\beta}{\hat{q} \hat{w}} \right).$$





The points  $A$ ,  $B$  and  $C$  are defined by their respective coordinates  $A(\hat{q}, 1)$ ,  $B(0, 1)$  and  $C(\hat{q}, \hat{w})$ . The respective signs of the functions  $E(q_1, \bar{w}_1)$  and  $F(q_1, \bar{w}_1)$  at these points are analyzed in a subsequent paragraph, and reported on the figure. Since  $E$  and  $F$  are continuous, the curves  $(E = 0)$  and  $(F = 0)$  must separate the points  $A$ ,  $B$ ,  $C$  as indicated. Therefore they must intersect (point  $S$  on the figure), and there exists at least one solution  $(q_1^*, \bar{w}_1^*)$  such that:

$$0 = E(q_1^*, \bar{w}_1^*)$$

$$0 = F(q_1^*, \bar{w}_1^*) .$$

This method does not prove the unicity of the solution.<sup>2</sup> However, common sense dictates that the solution must be in the quadrangle  $ABDC$  (the net wage rate and the level of consumption are higher in the long-run when the debt is partially redeemed, than at time zero). It seems then unlikely that more than one solution exists (the number of solutions must be odd). The numerical values of  $q_1$  and  $\bar{w}_1$  can be computed by

<sup>2</sup>It is easy to show that  $\partial E/\partial q_1 > 0$  and  $\partial E/\partial \bar{w}_1 > 0$ , therefore the slope of  $(E = 0)$  is negative. Also  $\partial F/\partial q_1 > 0$ , but the sign of  $\partial F/\partial \bar{w}_1$  is a priori undecided.

by the following method: find ( $E = 0$ ) by an iterative procedure (if  $E > 0$ , reduce  $q_1$  and  $\bar{w}_1$ , etc....); then search along  $E = 0$  in the quadrangle ABDC for  $F = 0$ .

The function  $F$  is negative on the segment AB; therefore the long-run wage rate is smaller than 1:  $\bar{w}_1^* < 1$ . In the long-run, the tax rate on labor income is positive; by the government budget constraint, we find that the level of the public debt is constant and positive.

We conclude by the determination of the signs of the functions  $E$  and  $F$  at the points A, B, C:

Consider first the point A which corresponds to a guess ( $\bar{w}_1 = 1$ ,  $\bar{q}_1 = \hat{q}$ ). Since  $\bar{w}_1 = 1$ , by (17a) and (23),  $\mu = 0$ , and  $t_1 = 0$ . Therefore, there is no transition period, and  $k_1 = k_0$ ,  $h_1 = 0$ . But because  $\hat{w}_1 < 1$ , it follows from (24) and (26) that  $E(\hat{q}, 1) > 0$ ; also, use (25) and (27) to find that  $F(\hat{q}, 1) < 0$ .

Following the same method, at the point B:

$$E(0, 1) < 0 \quad \text{and} \quad F(0, 1) < 0.$$

Consider now the point C:  $\hat{w}$  is smaller than 1, and  $t_1$  is positive. From (14a), (15), and (17), we have:

$$\frac{1-\beta}{q\bar{w}} \left( -1 + \frac{q_1}{q\bar{w}} \right) = \phi.$$

The product  $q\bar{w}$  is constant on the transition path and equal to  $q_1 \bar{w}_1 = \hat{q}\hat{w}$ . Therefore in the interval  $(0, t_1)$ , the labor supply is equal to its value before time zero. However  $q_0$  is lower than  $q_1$  (since  $\dot{q} = \rho q$ ), and because of the higher consumption, the capital stock is smaller at  $t_1$  than at  $t_0$ , and cannot provide the same level of consumption. Therefore  $E(\hat{q}, \hat{w}) < 0$ .

by the following method: find ( $E = 0$ ) by an iterative procedure (if  $E > 0$ , reduce  $q_1$  and  $\bar{w}_1$ , etc....); then search along  $E = 0$  in the quadrangle ABDC for  $F = 0$ .

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$$\frac{1-\beta}{q\bar{w}} \left( -1 + \frac{q_1}{q\bar{w}} \right) = \phi .$$

The product  $q\bar{w}$  is constant on the transition path and equal to  $q_1\bar{w}_1 = \hat{q}\hat{w}$ . Therefore in the interval  $(0, t_1)$ , the labor supply is equal to its value before time zero. However  $q_0$  is lower than  $q_1$  (since  $\dot{q} = \rho q$ ), and because of the higher consumption, the capital stock is smaller at  $t_1$  than at  $t_0$ , and cannot provide the same level of consumption. Therefore  $E(\hat{q}, \hat{w}) < 0$ .

In the interval of time  $(0, t_1)$ ,  $h$  is increasing ( $\dot{h} = \rho k + (1-w)\ell$  in (21)), and  $h_1 > h_0$ . From (25) and (27),  $F(\hat{q}, \hat{w}) > 0$ .

## APPENDIX C

The Welfare Cost of the Capital and Labor Income Tax

The level of private utility is represented by the function

$$U = \int_0^{\infty} e^{-rt} e^{nt} (\beta \log c_t + (1-\beta) \log(1 - \ell_t)) dt ,$$

$n$  represents the population growth rate. Labor augmenting technological change takes place, at the constant rate  $\mu$ . The gross factor prices  $r$  and  $w$  are fixed, and  $r$  is equal to the sum of the rate of time preference and of  $\mu$ .<sup>1</sup>

At the origin of time, the private sector is endowed with an amount of assets equal to the sum of the capital stock  $k_0$ , and of the public debt  $b_0$ . The amount of the debt is assumed to be small, and we consider the welfare cost induced by the taxes raised on capital and labor income to equilibrate the government budget.

The tax rates  $\theta^r$  and  $\theta^w$  on the interest and the wage rate are constant through time. Up to the first order, the government budget constraint can be written:

$$V(g) + b = \theta^r k_0 + \theta^w \frac{w\ell}{r-n-\mu} .$$

The excess-burden  $L$  of the debt is determined by the well-known Harberger-Hicks-Hotelling formula. It is applied here in a continuous formulation: Call  $z_t$  the leisure  $1 - \ell_t$ , and  $p_t$  and  $q_t$  the

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<sup>1</sup>When capital and labor are not perfect substitutes in the production process, the gross rate of return will tend to this stationary value.

prices at time zero of  $c_t$  and  $z_t$  :

$$p_t = e^{-r(1-\theta^r)t} ; \quad q_t = (1-\theta^w)we^{-r(1-\theta^r)t}$$

the (small) values of the tax rates  $\theta^r$  and  $\theta^w$  induce the following variations of  $p_t$  and  $q_t$  :

$$\Delta p_t = rt\theta^r p_t ; \quad \Delta q_t = (rt\theta^r - \theta^w)q_t .$$

With these notations, the excess-burden  $L$ , the public debt is given by the formula:

$$(1) \quad -2L = \int \Delta p_t dt \int \left( \Delta p_t, \left( \frac{\partial c_t}{\partial p_t} \right)^U + \Delta q_t, \left( \frac{\partial c_t}{\partial q_t} \right)^U \right) dt' \\ + \int \Delta q_t dt \int \left( \Delta p_t, \left( \frac{\partial z_t}{\partial p_t} \right)^U + \Delta q_t, \left( \frac{\partial z_t}{\partial q_t} \right)^U \right) dt' .$$

Call  $\Delta c_t$  and  $\Delta z_t$  the uncompensated changes of  $c_t$  and  $z_t$  induced by the tax rates; for example:

$$\Delta c_t = \int \left( \Delta p_t, \frac{\partial c_t}{\partial p_t} + \Delta q_t, \frac{\partial c_t}{\partial q_t} \right) dt' .$$

An application of Slutsky's equation transforms (1) into:

$$(2) \quad -2L = \int (\Delta p_t \Delta c_t + \Delta q_t \Delta z_t) dt + \frac{1}{A} (\int (\Delta p_t c_t + \Delta q_t z_t) dt)^2 ,$$

where  $A$  represents the total private net wealth (including human capital):

$$\Lambda = k_0 + \frac{w}{r^* - n - \mu} .$$

The paths of  $c_t$  and  $z_t$  are characterized by the relations:

$$c_t = \beta(r - n - \mu) A e^{-r\theta^r t} ,$$

and

$$z_t = \frac{(1-\beta)(r - n - \mu)}{w(1 - \theta^w)} A e^{-r\theta^w t} .$$

A lengthy but straightforward manipulation of (2) gives the following result:

$$L = \frac{1}{2} \left[ (\theta^r)^2 \left( \frac{r}{r - n - \mu} \right)^2 + (\theta^w)^2 \beta(1-\beta) \right] A .$$

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