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AN EXTENSION OF THE NASH BARGAINING PROBLEM:
INTRODUCING TIME-RELATED BARGAINING COSTS

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By Zvi A. Livne[1]

The Nash Bargaining Problem is reformulated by introducing
time-related costs into the vonNeumann-Morgenstern utility
functions of the parties. A set of mathematical requirements
on the solution is satisfied uniquely by a function called
the Extended Raiffa Solution. The properties of this func-
tion are investigated. It is shown to coincide with the
Nash Cooperative Solution (of a related Bargaining Problem)
when the parties have identical discount rates. An inter-
pretation of the solution and of the formal requirements
is detailed. In this model the 'bargaining power' of the
parties is determined by the bargaining costs related
to the duration of the negotiations.

1. INTRODUCTION

In this paper we reformulate the Nash Bargaining Problem (Nash [11]) by

introducing time-related costs into the utility functions of the parties. This

extension of the problem yields a new solution function, called here the

Extended Raiffa Solution. It also sheds some new light on the controversy

regarding the significance of the Nash Cooperative Solution (cf. Harsanyi [7],

Luce and Raiffa [10], Nydegger and Owen [13] and Roth [15].)

Nash's formulation can be viewed as a specialized model of the problem of

coordination of activities by two or more agents whose rewards depend on their

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own as well as on the others' acts. The methods of coordination available to the agents vary. They may involve face-to-face negotiations over a set of binding agreements (or, more generally, over a set of non-cooperative games.) They may involve procedures of conflict resolution like compulsory arbitration and mediation. They may involve any procedure that the parties may choose to design in order to coordinate their activities. It is not entirely clear what is the method of coordination being modeled by the original Nash Bargaining Problem, but what we wish to emphasize here is that almost all realistic methods of coordination consume a non-negligible amount of time during which costs are usually incurred. Thus the duration of the coordination effort seems to be an important factor that was abstracted out in Nash's formulation. It is the effect of the duration on the outcome that we try to capture in our new formulation of the problem.

The importance of time related costs in conflict and bargaining situations has been mentioned but rarely studied by economists and game theorists. Exceptions include the work of Bishop [1], Foldes [6], Cross [5], Coddington [4] and Clemhout et. al. [3]. This lack of interest is somewhat surprising since differences in bargaining costs and in discount factors have important effects on the 'bargaining power' of negotiators, as is well known to students of such diverse fields as collective bargaining, mergers and acquisitions, international relations, etc. It might be possible to reformulate existing models to take account of this missing factor and ours is one such attempt. (For further discussions of the role of time in negotiation see section 5).[1]

The approach taken here follows an established tradition in works on the Nash Bargaining Problem, i.e., finding a function, defined on the set of sets of outcomes in the utility plane, which satisfies uniquely a set of mathematical requirements (cf. Kalai and Smorodinsky [9], Kalai [8], Myerson [12],

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[1] Games of timing - like "duels", which were studied extensibly in the early days of game theory, address a related but still quite different problem. They deal with the timing of moves in a game rather than the costs associated with the

Roth [15] and Thomson and Myerson [16].) The requirements we chose to impose were motivated by a set of <u>arbitration principles</u> postulated in section 4. Although other interpretations are possible, this model should
be viewed as an arbitration scheme. We basically attempt to establish a solution function for an arbitrator whose perception of "fairness" in arbitration coincides with our arbitration principles.

We begin with notations, definitions and the statement of the mathematical requirements. We then state the main results and give some examples. Next we list and discuss the arbitration principles which motivated our requirements. We end with a discussion of the effects of time in negotiation. All the proofs in this paper are relegated to the Appendix.

2. THE MODEL AND DEFINITIONS

We call a subset B of the two dimensional Euclidean space a <u>bargaining</u> <u>domain</u> if it satisfies the following conditions:

1. B is convex and compact.

2. There exists a unique point $w(B) \epsilon B$ such that: ·

   (i) every $x \epsilon B$ satisfies $x \geq w(B)$, and

   (ii) for all $y \geq w(B)$, if there exists an $x \epsilon B$ such that $y \lessdot x$, then $y \epsilon B$.

   (B is comprehensive)

If there exists no point $x \epsilon B$ for which $x_1 > w_1(B)$ and $x_2 > w_2(B)$ then B is a <u>degenerate</u> bargaining domain. A degenerate domain can be <u>vertical</u> (if there exists $x \epsilon B$ such that $x_2 > w_2(B)$), horizontal (if there exists $x \epsilon B$ such that $x_1 > w_1(B)$ or trivial (if $B = \{w(B)\}$). Degenerate domains have traditionally been excluded from consideration, but they have an important role here.

For each bargaining domain B the point $w(B)$ is called the <u>conflict point</u> of B. Each point in B represents utility levels corresponding to a particular

feasible agreement. The point w(B) corresponds to the 'failure in negotiation' situation, in which the parties carry out their uncoordinated activities (sometimes called 'optimal threats').

Let the set of all bargaining domains be denoted by $\overline{B}$. It is a subset of 2^{R^2} - the set of all non-empty closed subsets of R^2, which is also a metric space with the Hausdorff metric ρ^H. Thus with the metric induced by ρ^H, $\overline{B}$ is a metric space. By definition, for all $B_1, B_2 \varepsilon \overline{B}$, $\rho^H(B_1, B_2) = \max\{\max\{d(x, \overline{B}_2) | x \varepsilon B_1\}, \max\{d(y, B_1) | y \varepsilon B_2\}\}$, where d is the usual Euclidean distance between a point and a set in R^2.

Let us define another useful metric τ. For all $B \varepsilon \overline{B}$ let SP(B) denote the strong Pareto frontier of B. For all $B, B' \varepsilon \overline{B}$, let ρ^P be defined by $\rho^P(B, B') = \rho^H(SP(B), SP(B'))$, and let ρ^W be defined by $\rho^W(B, B') = d(w(B), w(B'))$. Those are semi-metrics on $\overline{B}$. Let τ be defined by $\tau(B, B') = \max\{\rho^W(B, B'), \rho^P(B, B')\}$. This is a metric on $\overline{B}$. The spaces $(\overline{B}, \rho^H)$ and $(\overline{B}, \tau)$ are not isometric. Bargaining domains B_n which converge to a bargaining domain B in ρ^H might not converge in τ. This difference is especially noticeable when degenerate bargaining domains are involved. (See for example, Figure 1. The dots denote the points $h_R(B_n)$ and $h_R(B)$ where h_R is the Raiffa Solution discussed in Section 3).
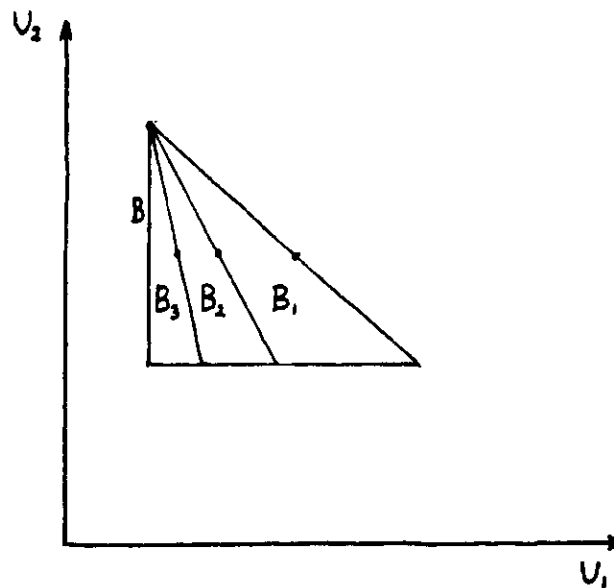

Figure 1. $\overline{B}_n$ converge to B in ρ^H but not in τ.

We now define a binary order, called <u>domination</u>, on $\bar{B}$. Let B_1 and B_2 be two bargaining domains. Then $B_1 d B_2$ (in words, B_1 dominates B_2) if : (1) The conflict points $w(B_1)$ and $w(B_2)$ satisfy $w(B_1) \geq w(B_2)$. (2) For every $y \in B_2$ there exists an $x \in B_1$ such that $x \geq y$. Alternatively, $B_1 d B_2$ if there exists a continuous function f defined on B_1 onto B_2 such that $f(w(B_1)) = w(B_2)$, and for all $x \in B_1$, $f(x) \geq x$. It's easy to check that these two definitions are equivalent and that d is a transitive, reflexive and antisymmetric relation, hence it is a partial (or proper) ordering. (For these and related definitions, and for other results concerning binary relations, see Chipman [2]).

We now turn to examining special subsets of $\bar{B}$. A subset C of $\bar{B}$ is called a <u>chain</u> if it is totally ordered by d, i.e., if for every pair B_1, $B_2 \in C$, either $B_1 d B_2$ or $B_2 d B_1$. It is called a <u>bargaining chain</u> if it is a closed set with respect to ρ^H, having both maximal and minimal elements, and if its minimal element is a trivial bargaining domain (i.e. it consists of one point).

Another condition on every bargaining chain C is that for all $B \in C$ there exist $B' \in C$, $B' \neq B$, such that $B \cap B' \neq \phi$. The reason for this condition will become clear in our discussion of the arbitration principles, in Section 4.

It can be shown that the cardinality of a bargaining chain is necessarily smaller or equal to that of the continuum. It can therefore be represented as $\{B(t) | 0 \leq t \leq T\}$ where $B(t_1) d B(t_2)$ iff $t_1 < t_2$, for some $T > 0$. (This representation is not unique.) Denote the set of all bargaining chains by $\bar{C}$. It is a metric space with the metric δ defined herein. Let $C = \{B(t) | 0 \leq t \leq T\}$ and $C' = B'(s) | 0 \leq s \leq S\}$ be two bargaining chains in $\bar{C}$. Let Q be the set of all order preserving functions ϕ from $[0,T]$ onto $[0,S]$. Then $\delta(C,C') = \underset{\phi \in Q}{\text{Inf}} \underset{t}{\text{Sup}} \ \tau(B(t), B'(\phi(t)))$.

To understand the motivation for these definitions, let us denote by $B(t)$ the bargaining domain corresponding to the set on the utility plane of

possible outcomes after negotiations of length t, and let C be the set $\{B(t) \mid 0 \le t \le \cdot \}$.

If both parties have non-negative bargaining costs, then B(t)dB(s) for every t,s

such that t≤s. The bargaining domain at the beginning of the negotiations, B(0)

is evidently a maximal element. A minimal element exists if for some reason

negotiations cannot continue past a certain duration T (T is then called a

'deadline'). Another reason for the existence of a minimal element could be

the disappearance of all duration effects from a certain time on, although

we do not think it is a realistic possibility. In both cases B(T) is mini-

mal in C and C is indeed a bargaining chain. Note that since the represent-

ation of the bargaining chain is not unique, the time T is not necessarily

the calendar time of the deadline.

Our solution function is defined only on bargaining chains, so a minimal

element must exist. However, in many negotation situations there is no clear

deadline. This problem is overcome by the continuity property of the so-

lution. By truncating the chain at various times T, computing the solutions

for the truncated problems and then sending T to ∞ we might get convergence

to a unique solution. (See Remark 2 in Section 3.)

A pictorial representation of a bargaining chain is shown in Fig. 2.

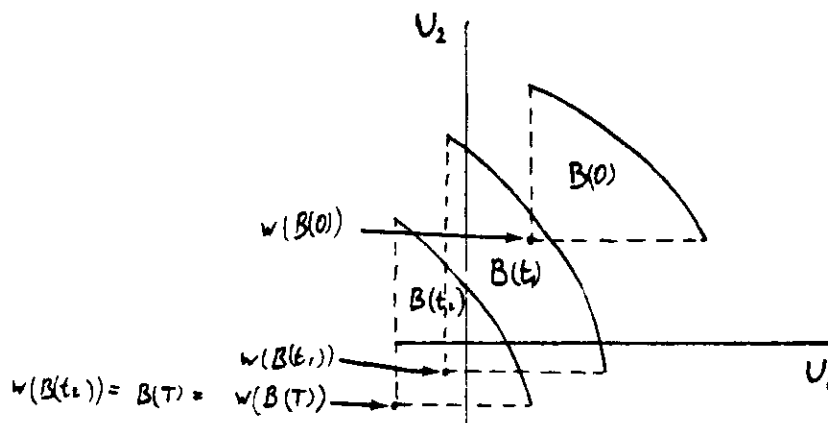Note that only several representative bargaining domains are drawn.

Figure 2: A Bargaining Chain

Let us make a few more definitions. For $B \varepsilon B$ the <u>degenerate chain</u> $\bar{B} \varepsilon \bar{C}$ is the chain consisting of B and the trivial bargaining domain $\{w(B)\}$. It corresponds to negotiations in which the utility functions and the threat point are not affected by the duration but in which the parties must execute their threats before a certain deadline.

A bargaining chain C^e is the <u>extension</u> of a bargaining chain C if they can be represented by $C^e = \{B^e(t) \mid o \leq t \leq \bar{T}\}$, $C = \{B(t) \mid o \leq t \leq T\}$, with $B^e(t) = \{x \mid x \geq w(T)\}$; there is a $y \varepsilon B(t)$ such that $y \geq x\}$. We will later show that our solution function is invariant under extension.

A bargaining chain $C = \{B(t) \mid o \leq t \leq T\}$ is discrete if the set of Pareto frontiers $\{SP(B(t)) \mid o \leq t \leq T\}$ consists of isolated points in $(\bar{B}, \cdot)$. Equivalently, the extended chain C^e consists of isolated points in $(\bar{B}, \tau)$. It is continuous at t, $o \leq t \leq T$, if $B^e(t)$ is a left and right convergence point of C^e.

For all $o \leq s \leq T$, the s-tail of C, denoted by C_s, is the set $\{B(t) \mid B(t) \varepsilon C, t \geq s\}$. The s-truncation, denoted by C^s, is the set $\{B(t) \mid B(t) \varepsilon C, o \leq t \leq s\} \cup \{w(B(s))\}$. Both are bargaining chains.

Now we turn to the solution function. A function g, defined on $\bar{C}$ into R^2, is a <u>solution function</u> if it satisfies the set of requirements listed below. (We attach brief explanations to some of these requirements. They will be discussed further in section 4. Note that some of the names we use are borrowed from similar requirements found in the literature on the traditional Bargaining Problem).

1. <u>Strong Pareto Optimality</u>. The solution g(C) belongs to the Strong Pareto frontier of the maximal element in C, B(0).

2. <u>Independence of Affine Transformations</u>. For every $d, r \varepsilon R^2$ such that $d_1 > 0$, $d_2 > 0$, let $F : R^2 \rightarrow R^2$ be the affine transformation $F(v_1, v_2) = (d_1 v_1 + r_1, d_2 v_2 + r_2)$. Let $F(C) = \{F(B(t)) \mid o \leq t \leq T\}$. Then $g(F(C)) = F(g(C))$.

3. <u>Symmetry</u>. If $B(t)$ is symmetric for all t (i.e., $u=(u_1,u_2)\epsilon B(t)$ implies $(u_2,u_1)\epsilon B(t)$), then $g_1(C)=g_2(C)$.

These three requirements are generalizations of three of Nash's original requirements, commented upon in Luce and Raiffa [10] and Roth [15].

4. <u>Independence of Irrelevant Changes of Tails</u>. Let $\{B'(s)|0\leq s\leq S\}$ be a bargaining chain such that there exist $0\leq s\leq S$, $0\leq t\leq T$, for which $C'^s=C^t$ and $g(C_s')=g(C_t)$. Then $g(C)=g(C')$.

This requirement guarantees that changes in the utility functions (or in the sets of possible agreements) which occur after t=s but which do not affect the 'strategic position' of the parties in time t (in the sense that $g(C_s')=g(C_t)$) do not affect the solution.

To state the next requirement we need a new definition. For each bargaining domain $B\epsilon\overline{B}$ and for each point $y\epsilon R^2$, let the bargaining domain induced by B and y, denoted by $B[y]$, be the set $\{x|x\geq y$, and there is $r\epsilon B$ such that $r\geq x\}$ (it may be an empty set.) A bargaining chain C' is induced by a bargaining chain C and a point y (notation-$C[y]$) if C' can be represented as the union of $\{B(t)[y]|B(t)\epsilon C,0\leq t\leq T\}$ and the trivial bargaining domain $\{y\}$.

5. <u>Independence of Irrelevant Changes of Truncated Portions</u>. Let $C'=\{B'(s)|0\leq s\leq S\}$ be a bargaining chain such that there exist $0\leq s\leq S$, $0\leq t\leq T$ for which $C'_s=C_t$ and $C'^s[g(C')]=C^t[g(C_t)]$. Then $g(C')=g(C)$.

This requirement will be discussed in the context of the arbitration principles in Section 4. Briefly, it says that for every t the only relevant portion of C^t is the part which dominates $g(C_t)$.

The next requirement deals with degenerate bargaining chains. It is similar (but not identical) to Kalai and Smorodinsky's [9] Monotonicity Axiom. To state it we use the following definition. A bargaining domain B' is an <u>i-change</u> of a bargaining domain B (i may be 1 or 2) if $w(B')=w(B)$, $B'\subseteq B$, and if the levels of

highest possible utilities (denoted $\bar{x}_i(B)$ and $\bar{x}_i(B')$, i=1,2) satisfy $\bar{x}_i$ $(B') \leq \bar{x}_i(B)$ and $\bar{x}_j(B')=\bar{x}_j(B)$ (for j≠i).

6. Strong Individual Monotonicity. Let $\tilde{B}$ and $\tilde{B}'$ be two degenerate bargaining chains, derived from $\bar{B},B'\in\bar{B}$. If B' is an i-change of B, then $g_i(\tilde{B}') < g_i(\tilde{B})$.

7. Continuity. Let $\{C_n\}$ be a sequence of bargaining chains such that $\delta(C_n,C) \to 0$ as $n \to \infty$. Then $g(C_n) \to g(C)$ as $n \to \infty$.

Requirement 7 guarantees that small changes in the utility functions have little effect on the solution. We are now ready to state the main results of this paper.

3. RESULTS

The requirements listed above are satisfied by a unique function g, to be called here the Extended Raiffa Solution (Theorem 1). This function is a generalization of one solution to the Nash Bargaining Problem attributed to Raiffa (although it is only an approximation to the solution originally proposed in Raiffa [14]). This generalization should be distinguished from other extensions of the Raiffa solution, for example one that was offered by Kalai and Smorodinsky [9] and Kalai [8] under the name 'proportional solution'. It applies to bargaining domains and not to bargaining chains. Before proving the main results we remind the reader of the Raiffa Solution and we state several useful lemmas.

Let B be a bargaining domain with a conflict point w(B). The Raiffa Solution, denoted here by $h_R(B)$, is the unique point on the strong Pareto frontier of B which lies on the line connecting w(B) and the point $(\bar{x}_1(B),\bar{x}_2(B))$. As noted in the literature, the use of this solution function corresponds to an ad-hoc comparison of utilities, equating the differences $\bar{x}_1(B)-w_1(B)$ and $\bar{x}_2(B)-w_2(B)$ (when B is not degenerate). The Raiffa Solution is the unique function satisfying a set of requirements listed in Lemma 1.

Lemma 1. The Raiffa Solution $h=h_R$ is the only function $h:\overline{B}\to R^2$ satisfying these

four requirements:

(i) Strong Pareto Optimality: For all $B\epsilon\overline{B}$, $h(B)$ is on the Strong

Pareto frontier of B.

(ii) Independence of Utility Scales: Let $F:R^2\to R^2$ be an affine trans-

formation. Then $h(F(B)) = F(h(B))$, for all $B\epsilon\overline{B}$.

(iii) Symmetry: If B is symmetric then $h_1(B)=h_2(B)$.

(iv) Strong Individual Monotonicity: Let $B,B'\epsilon\overline{B}$. If B' is an i-change

of B then $h_i(B') \lessgtr h_i(B)$ (i=1 or 2).

Two useful properties of the Raiffa Solution are the following:

Lemma 2. With respect to the metric τ, h_R is continuous.

As the example in Figure 1 shows, h_R is not continuous with respect to ρ.

Lemma 3. (a) For all $B\epsilon\overline{B}$ and y, $y_n\epsilon R^2$, if $B[y],B[y_n]$ are not empty and y is

not a weak Pareto point in B then if $y_n\to y$ then $h_R(B[y_n])$ $h_R(B[y])$.

(b) For all $B,B_n\epsilon\overline{B}$ and $y\epsilon R^2$, if $B[y]$, $B_n[y]$ are not empty and y is not

a weak Pareto point in B then if $B_n\xrightarrow{\tau}B$ then $B_n[y]\xrightarrow{\tau}B[y]$.

We now move to the set $\overline{C}$ of bargaining chains. We first deal with degenerate

bargaining chains, then with finite ones, and finally with continuous ones.

Lemma 4. Let $\overline{C}_d$ be the set of all degenerate bargaining chains. There exists a

unique function g_d, defined on $\overline{C}_d$ into R^2, which satisfies requirements

1-7. For all $\tilde{B}\epsilon\overline{C}_d$ (derived from $B\epsilon\overline{B}$) it is given by $g_d(\tilde{B})=h_R(B)$.

Lemma 5. Let $\overline{C}_f$ be the set of all finite bargaining chains. There exists a unique

function g_f, defined on $\overline{C}_f$ into R^2, which satisfies 1-7. For $C=\{B(0),$

$B(t_1)$, ..., $B(t_n)\}$, g_f is given recursively by:

$$g_f(C)=h_R(B(0)[g_f(C_{t_1})],$$
(1)

and

$$g_f(C_{t_k})=h_R(B(t_k)[g_f(C_{t_{k+1}})], \quad (k=1, \ldots, n-1),$$
(2)

with the boundary condition

$$g_f(C_{t_n})=B(t_n).$$
(3)

In Figure 3 we show how to compute the Extended Raiffa Solution of a finite bargaining domain.

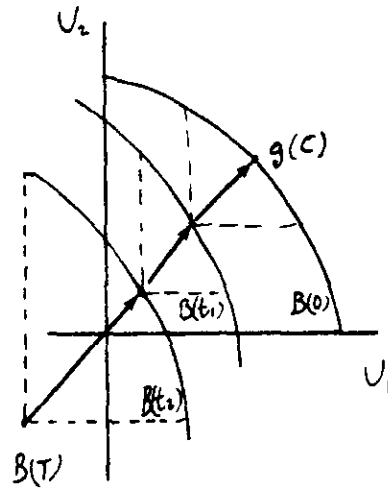Figure 3: The Extended Raiffa Solution g(C).

For all $C\epsilon\overline{C}$ let us define an ϵ-net $C(\epsilon)$ to be a finite subset of C which includes the maximal and the minimal elements of C, and which satisfies $\tau(C(\epsilon),C-C(\epsilon))\leq\epsilon$. For all $\epsilon>0$ such a set exists, since C (if viewed as a metric subspace of $\overline{B}$) is totally bounded with respect to the metric τ (it is compact with respect to the metric ρ^H). We can prove:

Theorem 1. There exist a unique function $g:\overline{C}\rightarrow R^2$ satisfying requirements 1-7. It is given by $g(C)=\lim_{\epsilon\rightarrow 0} g_f(C(\epsilon))$.

We now list a few properties of the Extended Raiffa Solution. First, note that the solution does not depend directly on the threat points $w(B(t))$ for $0\leq t<T$. More precisely; from requirement 5 it is immediate to show:

Lemma 6. For every $C\epsilon\overline{C},g(C)=g(C^e)$.

Let us turn to continuous bargaining chains C for which the Pareto frontiers of the domains $B(t)$ are differentiable. For each t, $0\leq t\leq T$ we denote $I(t)=[w_1(B(t)),\overline{x}_1(B(t))]$. For each $u\in I(t)$ we define $p(t,u)=\max\{v|(u,v)\epsilon B(t)\}$. The curve $\{(u,p(t,u))|u\epsilon I(t)\}$ is the Pareto frontier of $B(t)$. The bargaining chain C is locally-continuous at t if there exists a closed neighborhood $V=[t',t'']$ of t such that the set $\{B(s)|s\leq t'\}[g(C_{t''})]\cup\{g(C_{t''})\}$ is a continuous bargaining chain. A bargaining chain which is locally continuous for all t is continuous. We can prove:

Theorem 2 If $C = \{B(t) \mid 0\leq t\leq T\}$ is continuous, and if $p(t,u)$ is differentiable, then $g(C)$ is the end point $k(0)$ of a differentiable curve $k(t)$, $0\leq t\leq T$, whose derivative satisfies:

$$\frac{d}{dt}k_1(t) = -\frac{1}{2}\frac{\partial p}{\partial t}/\frac{\partial p}{\partial u}\bigg|_{(t,k_1(t))} , \qquad (4)$$

and

$$k_2(t) = p(t,k_1(t)),\qquad (5)$$

with the initial point $k(T)$ satisfying

$$k(T) = w(B(T)).\qquad (6)$$

Let us examine the special case in which the only effect of duration is discounting. When the discount factors are identical the Extended Raiffa Solution converges to the Nash Cooperative solution of $B(0)$ as $T \to \infty$. More precisely:

Theorem 3. Let the utility functions be of the multiplicative form:

$$\left. \begin{aligned} U_1(a,t) &= \alpha(t)f_1(a) \\ U_2(a,t) &= \alpha(t)f_2(a) \end{aligned} \right\} \quad 0 \le t < \infty, \ a\varepsilon[0,1], \qquad (7)$$

where $0 \lesssim \alpha(t) \le 1$ is differentiable and is non-increasing, $f_i(a)$ are differentiable $(i = 1,2)$ and $f = f_2 f_1^{-1}$ is decreasing and concave. There exists an α^*, $0 < \alpha^* < 1$ such that for all T satisfying $\alpha(T) < \alpha^*$, $g(C^T)$ is the Nash Cooperative Solution of $B(0)$.

Remark 1. Note that if the discount factors are not identical the solution generally does not converge to the Nash Cooperative Solution of $B(0)$. Theorem 3 therefore sheds a new light on the Nash Cooperative Solution. It shows that although this solution can be obtained from a set of requirements which is quite different from that of Nash, it is so only when both parties have identical discount rates. The Nash Cooperative Solution is therefore in-sensitive (in the sense of Theorem 3) to differences in discount rates and to

other duration effects. Its applicability as either a "fairness" rule or as a "normative solution" is therefore more questionable.

Remark 2. It should be noted that in general bargaining chains $g(C^T)$ does not necessarily converge as $T \to \infty$. Consider for example, the use of the "oscillating" utility functions:

$$V_1(a,t) = \begin{cases} a - \alpha(t-n) & 2n \leq t < 2n+1 \\ a - \alpha(n+1) & 2n+1 \leq t < 2n+2 \end{cases}$$

$$V_2(a,t) = \begin{cases} 1-a-\alpha(t-n) & 2n+1 \leq t \cdot 2n+2 \\ 1-a-\alpha n & 2n \leq t < 2n+1 \end{cases} \quad a \in [0,1], \quad t \in [0,T]. \tag{8}$$

It is easy to see that $g(C^T)$ oscillates between two values as $T \to \infty$. However, weak sufficient conditions for the convergence of $g(C^T)$ can be established. One such condition is that $B(t)$ be bounded from below by some bargaining domain $B(\infty)$.

Examples Let us now compute the Extended Raiffa Solution for two cases. Consider first a distributive bargaining problem on $[0,1]$, where one party (the seller) wants the agreement to be as close as possible to 1, and the other (the buyer) wants it close to 0. The parties are both risk neutral, have positive discount rates but no other bargaining costs. If money is compounded instantaneously, the utility functions are of the multiplicative form:

$$V_1(a,t) = \exp(-r_1 t) a$$
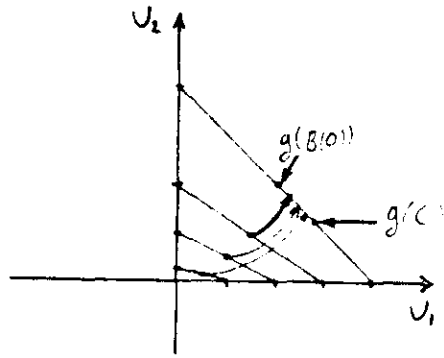$$V_2(a,t) = \exp(-r_2 t)(1-a) \quad a \in [0,1], \quad 0 \leq t < \infty \tag{9}$$

Figure 3: The Extended Raiffa Solution $g(C^T)$
for Several Truncation Points T.

Here $p(t,u) = \exp(-r_1 t) - \exp((r_2-r_1)t)$, for $u\epsilon[0,\exp(-r_1 t)]$ From (4),(5) and (6) we get

$$g(C^T) = (\frac{r_2}{r_1+r_2} + K(T), \frac{r_1}{r_1+r_2} - K(T)), \qquad (10)$$

where

$$K(T) = \exp(-\frac{r_1+r_2}{2} T)\cdot(\frac{1}{2} - \frac{r_2}{r_1+r_2}) \qquad (11)$$

Letting $T\rightarrow\infty$ we get

$$g(C) = (\frac{r_2}{r_1+r_2}, \frac{r_1}{r_1+r_2}) \qquad (12)$$

The solution of the infinite deadline case is therefore the point which divides [0,1] by the ratio of the discount rates. This solution is identical to a solution obtained by Bishop [1] and Foldes [6] using an entirely different method.

Let us now introduce additive, linear bargaining costs, with no discounting. The utility functions are of the form:

$$\left.\begin{array}{l} V_1(a,t) = a-\alpha_1 t \\ \\ V_2(a,t) = 1-a-\alpha_2 t \end{array}\right\} a\epsilon[0,1], \; 0\le t<\infty \qquad (13)$$

For $\alpha_2 \ge \alpha_1$ (the other case can be treated analogously) we get:

$$g(C^T) = \begin{cases} (\frac{1}{2} + \frac{1}{2}T \cdot (\alpha_2 - \alpha_1), \frac{1}{2} + \frac{1}{2}T (\alpha_1 - \alpha_2)) & T < \frac{1}{\alpha_2 - \alpha_1} \\ (1,0) & T \geq \frac{1}{\alpha_2 - \alpha_1} \end{cases} \quad (14)$$

The deadline is obviously of great importance in this example. When it is large enough, the party with lower bargaining costs receives the whole pie.

4. ARBITRATION PRINCIPLES

The set of requirements listed in Section 2 is motivated by several arbitration principles. Here we define an arbitrator as an intervenor whose role is to take certain information concerning the bargaining process and the bargainers and to translate it, through an "arbitration scheme", into a unique agreement which is then recommended as a "fair" solution. The word "fair" relates both to the information which is deemed relevant and to the principles underlying the arbitration scheme.

We will assume that the information the arbitrator has includes the set of feasible agreements, the parties' utility functions (including the time-related bargaining costs), their current demands (their optimal threats), the deadline and the parties' likely activities in case of a break-off in negotiations. It does not include the details of the negotiating process before or after arbitration. The earlier demands and concessions of the parties affect the arbitrated solution only through the current demands. As is customary in the literature, we allow the arbitrator to suggest agreements based on randomized procedures, so that the set of the feasible agreements at a particular moment can be represented by a bargaining domain as defined in Section 2. Here we also assume some regularity of the utility functions and of the break-off points, namely those conditions that allow the representation of the game by a bargaining chain, as defined in Section 2.

Our arbitration scheme is based on seven principles. The first three have been discussed extensively in the literature on the traditional Bargaining Problem (Luce and Raiffa [10], Roth [15]). They are:

I   The solution should not be dominated by any other feasible agreement.

II The solution should be invariant under linear transformations of utility functions.

III The solution should not depend on the identity of the bargainers.

These three principles are the basis for requirements 1, 2 and 3 of Section 2.

In the framework of the traditional Bargaining Problem, where time effects are abstracted away, the arbitrator's task is facilitated by the existence of a unique conflict point. It represents the parties' payoffs when they refuse the arbitrated solution. As such it can be viewed as an 'alternative' to the arbitrated solution. Principles I, II, III and VI, and the requirements derived from them, are then enough to generate a unique solution point (the Solution of Raiffa). When time effects do exist, the arbitrator cannot be sure of what will happen when his solution is refused. Our fourth principle, however, maintains that the solution must depend on some outcome which is viewed as the alternative to the arbitrated solution. The arbitrator must therefore determine a unique point to serve as the representative conflict point. We will call this point the alternative point.

IV The solution depends on an alternative point, determined by the arbitrator on the basis of the utility functions and the future conflict points of the bargainers.

How should this alternative point be chosen? The arbitrator cannot use any information regarding the negotiation behavior of the parties, but he can decide whether the negotiation process continues or terminates after his solution is refused. Here is the decision rule we suggest. If the conflict point at the time of arbitration ($w(B(0))$ in our notations) is preferred, by both parties, to any agreement that can be achieved in any later time t, then the negotiations terminate after the arbitrated solution is refused. Otherwise, negotiations continue. In the first case the alternative point should be ($w(B(0))$). In the second case it is chosen to be the point, on the utility plane, that would be assigned to the bargainers by the arbitrator (or by another one) if called by the parties for another round of arbitration. In our notations it is the point $g(C_{t_1})$, where t_1 is the first next time in which an agreement can be achieved (t_1 might be infinitesimal). The reason for this choice is the assumption that an arbitrated solution reflects the 'bargaining power' of the parties at the time of arbitration. This 'bargaining power' may be viewed as the alternative to the current arbitrated solution. The solution is thus recursive in nature. It reflects the current 'bargaining power' and depends on the future 'bargaining power', which in turn is reflected by the future arbitrated solution. The fifth principle states:

> V The alternative point is the future arbitration solution, unless
> both parties are better off terminating the negotiations after
> the current solution is refused.

To guarantee that the parties always prefer to continue negotiations rather than terminate them, we imposed (in Section 2) the following condition. For all bargaining chains C and for all $B \epsilon C$, there exists $B' \epsilon C$, $B' \neq B$,

such that $B \cap B \neq \emptyset$. This condition guarantees that if $C = \{B(t)\mid 0 \leq t \leq T\}$, then for all t, $0 \leq t \leq T$, there exists $s > t$ such that $w(B(t)) \in B(0)$. The condition could be removed, and another requirement on the solution function be added, to take care of bargaining chains which do not satisfy this condition. The Extended Raiffa Solution would be computed as before, except that the boundary condition will be different.

Principles IV and V lead to requirements 4 and 5. When a tail of a bargaining chain changes in such a way that the 'bargaining power' of the bargainers does not change, there should be no change in the 'bargaining power', hence in the arbitrated solution, for preceding tails (Requirement 4). Also, to guarantee that the arbitrated solution serves as a threat point in the sense of the traditional bargaining problem, we require that only agreements (on the utility plane) which dominate this point determine the solution (Requirement 5).

To state the sixth principle, let us define two new concepts. We say that a bargainer is making a concession when he changes his current demand to one that yields him strictly lower utility and yields his opponent strictly higher utility. We say that the terms of bargaining deteriorate for a particular bargainer if the terms of some of the feasible agreements which dominate his conflict point are changed in such a fashion that they yield him lower utility, although not lower than his conflict payoff. Our arbitrator would like to view concessions and deterioration of terms of bargaining as factors which contribute to the erosion of bargaining power. However, the time element in the utility functions might have a counter effect on the bargaining power. In stating the next principle, the time element is therefore assumed away.

VI Suppose that the conditions are such that if the arbitrated solution
    is refused, negotiations are terminated. By making a concession prior
    to arbitration, the bargainer's utility from the arbitrated solution
    is strictly decreased. When the terms of bargaining deteriorate for
    a bargainer prior to arbitration, his utility from the arbitrated
    solution is decreased.

This principle translates into requirement 6. Every i-change of a bar-
gaining domain can be viewed as a combination of a concession by party i and
a deterioration in the terms of bargaining for party i. Note that arbitration
principle VI gives strong incentive to bargainers not to engage in unilateral
concessions and to prevent any deterioration in the terms of bargaining
(through changes in the terms of feasible agreements) prior to arbitration. This
is the logical outcome of the assumption that current demands, not past demands,
matter in arbitration.

VII Small changes in utility functions affect the arbitrated solution
     only slightly.

This natural principle should be treated with caution. Changes in utility func-
tions may be small if one particular metric is used, and not so small if
another is used. Since the solution of each tail of a bargaining chain should
be on the Pareto frontier of the maximal element of the tail, we chose to use a
the Euclidean distance between the corresponding Pareto frontiers, a rather
strong metric, rather than the Hausdorff metric. Principle VII is then
translated into requirement 7.

5. THE ROLE OF TIME IN NEGOTIATIONS

This paper is based on the assumption that time-related costs are of great importance in conflict situations. Lengthy negotiations might incur high bargaining costs (e.g., labor-management negotiations, conducted during a strike), and   may decrease the value of the final rewards due to the time discount factor. Various events, sometimes exogenous and random in nature, might lead to an abrupt termination of lengthy negotiations (e.g., stock market developments or new government regulations during merger negotiations). The duration of the negotiations might also have political effects. One important example is negotiations between a less developed country and a multinational organization. The government may be accused of "selling out the country" if they get a fast agreement, or be termed "indecisive", and "weak", when negotiations extend too much. Lengthy negotiations also tend to reveal internal differences between the members of a bargaining party. In short, time related costs of negotiations are important, and they do affect the outcomes. Negotiators should and do take time effects into account when they design their strategies.

In many conflict situations there are no credible threats that bargainers can make. If there exists no time effect, there is no reason for negotiations to end. When time effects are present, the existence of a deadline and the emergence of a new type of credible threats (stalling negotiations), might affect the 'bargaining power' significantly and yield fast and perhaps previously unexpected agreements. The force that really moves negotiations and makes them all terminate in a finite time is indeed those time-related costs.

When time-related effects are of great importance, and our point is that this is almost always the case, they should not be abstracted away, but rather be modeled explicitly. Time-related strategies have to be formalized and be taken into consideration in the search for equilibrium strategies, and "bounded-rationality" strategies.

APPENDIX

Proof of Lemma 1. It is easy to check that h_R satisfies (i) - (iv). Kalai and Smorodinsky [9] showed that h_R is the only function satisfying (i') Weak Pareto Optimality, (ii), (iii) and (iv') Weak Individual Monotonicity, i.e., if B' is an i - change of B then $h_i(B) \leq h_i(B)$ (i=1 or 2). Obviously, any solution function which staisfies (i) and (iv) satisfies also (i') and (iv'), therefore it must be h_R.

Proof of Lemma 2. Let B, B' $\in$ $\overline{B}$. If both bargaining domains are degenerate, then $d(h_R(B), h_R(B')) = \rho^P(B,B') \leq \tau(B,B')$. If B is degenerate and B' is not, then $d(h_R(B), h_R(B)) = d(SP(B), h_R(B') \leq \rho^P(B,B') \leq \tau(B,B')$. On the set of non-degenerate bargaining domains, h_R is continuous with respect to τ because it is continuous with respect to ρ^H. (It is not uniformly continuous, though.)

Proof of Lemma 3 (a): Based on Lemma 2, it is enough to show that $y_n \to y$ implies $B[y_n] \overset{\tau}{\to} B[y]$. Clearly $\tau(B[y], B[y_n]) = \text{Max} \{ d(y,y_n), d[(y_1,\overline{x}_2(B[y])), (y_{n1}, \overline{x}_2(B[y_n]))], d[(\overline{x}_1(B[y]), y_2), (\overline{x}_1(B[y_n]), y_{n2})]$. As $y_n \to y$ the first term converges to zero. Due to the continuity of the Pareto frontiers of B, the last two terms converge to zero too.

(b): From $B_n \overset{\tau}{\to} B$ we get $\rho^H(SP(B), SP(B_n)) \to 0$. From here it is easy to see that $\rho^H(SP(B[y]), SP(B_n[y])) \to 0$. Since $\rho^W(B[y]), B_n[y]) = 0$, we get $B_n[y] \overset{\tau}{\to} B[y]$.

<u>Proof of Lemma 4</u>. The spaces $(\overline{C}_d, \cdot)$ and $(\overline{B}, \cdot)$ are isometric under the mapping $\overline{B} \cdot B$. The requirements (i), (ii), (iii), (iv) (in $\overline{B}$) are identical to requirements 1,2,3,6, (in $\overline{C}_d$) respectively. Requirement 7 is staisfied by g_d (from Lemma 2), and so are 4 and 5 (trivial to show). Hence g_d satisfies 1-7 uniquely.

<u>Proof of Lemma 5</u>. The proof is by induction on the number of bargaininn domains in the bargaining claim. Let $\overline{C}_n$ be the set of all bargaining chains with n bargaining domains ($n \geq 2$). For $n = 2$, $\overline{C}_2 = \overline{C}_d$ and $g_f = g_d$, so the claim is proved exactly as in Lemma 4. Suppose the claim holds for $\overline{C}_3, \ldots, \overline{C}_k$. Let $\overline{C}_{k+1} \ni C = \{B(d), \ldots, B(t_k)\}$. By its definition $g_1(C) = h_R (B(0) [g_f(C_{t_1})])$. From Lemma 1 and the induction assumption, requirements 1, 2 and 3 are satisfied. From the definition of g_f it is clear that 4 and 5 are satisfied. Requirement 6 is not applicable for $n>2$. To prove that requirement 7 is satisfied, denote (without loss of generality) $C_n = \{B_n(0), \ldots B_n(t_k)\}$. If $C_n \overset{\delta}{\to} C$, then the t_1-tail of C_n converges to the t_1-tail of C. From the induction assumption $g_f((C_n)_{t_1}) \to g_f(C_{t_1})$. We also have $B_n(0) \overset{I}{\to} B(0)$. From Lemma 3 we get $g_f(C_n) \to g_f(C)$. It is left to prove that g_f satisfies 1-7 uniquely. Suppose g_f' satisfies 1-7. From the induction assumption $g_f'(C_{t_1}) = g_f(C_{t_1})$. From requirement 4 and the induction assumption $g_f'(C) = h_R(B(0)[g_f'(C_{t_1})]) = h_R (B(0)[g_f(C_{t_1})]) = g_f(C)$.

<u>Proof of Theorem 1</u>. First, g is well defined. The sequence $\{g_f(C(\varepsilon)) | \varepsilon > 0\}$ is a Cauchy-sequence in the closed set $SP(B(0))$ and therefore converges to a unique point there. Any other such sequence converges to the same point. It is immediate to show that the function g satisfies 1, 2, and 3. Let $C = \{B(t) | 0 \leq t \leq T\}$, $C' = \{B'(s) | 0 \leq s \leq S\}$ such that there are s_0, t_0 for which $C'^{s_0} = C^{t_0}$, $g(C_{s_0}) = g(C_{t_0})$. To simplify notations, and without loss of generality, assume $T=S$, $s_0 = t_0$. Let $C(\varepsilon)$ and $C'(\varepsilon)$ be ε-nets which coincide on $C^{t_0}(\varepsilon) \cup C'_{t_0}(\varepsilon)$, where $C^{t_0}(\varepsilon)$, $C_{t_0}(\varepsilon)$ are the parts of the ε-net which belong to the

t_0-truncation of C and to the t_0-tail of C, respectively (same for C$'$).

Then g(c) = $\lim_{\varepsilon \to 0} \lim_{j \to 0} g_f(C^t o(\varepsilon) \cup C_{t_c} (n))$. By Lemma 5 this term is equal to

$\lim_{r \to 0} g_f(C^t o(\varepsilon)[g(C_{t_c})])$. Similarly g$'$ (C) is equal to the same term. There-

fore 4. is satisfied. The proof of 5. is similar. Now requirement 7 is a

direct result of its equivalent requirement in the finite chains.

Proof of Theorem 2. Equation (5) is an immediate result of requirement

1. To prove (4) denote by C(t,ε) the bargaining chain $\{B'(0)\,|\,0 \leq s < T\}$ with

B$'$(s) = B(s) for o$\leq$s$\leq$t-ε and t$\leq$s$\leq$T, and B$'$(s) = B(t) for t-ε<s<t. For every

sequence $\varepsilon_n \downarrow 0$ we have $g(C(t,\varepsilon_n)_t) = g(C_t)$ (for all n), and $g(C(t,\varepsilon_n)_{t-\varepsilon_n}) \xrightarrow{n}$

$g(C_t)$. From the properties of the Raiffa Solution we get

$$\frac{g_2 (C(t,\varepsilon_n)_{t-\varepsilon n}) - g_2(C_t)}{g_1(C(t,\varepsilon_n)_{t-\varepsilon n}) - g_1(C_t)} = \frac{p(t-\varepsilon_n, g_1(C_t) - p(t,g_1(C_t))}{\Delta u} \quad ,$$

where Δu satisfies $p(t-\varepsilon_n, g_1(C_t) + \Delta u) = p(t, g_1(C_t))$. From these equalities

we get

$$\frac{\frac{d}{dt} g_2(C_t)}{\frac{d}{dt} g_1(C_t)} = - \frac{\partial p}{\partial u}\Big|(t,g_1(C_t)).$$

Since $g_2(C_t) = p(t,g_1(C_t))$, we have

$$\frac{d}{d_t} g_2 (C_t) = \frac{\partial p}{\partial t}\Big| (t,g_1(C_t)) + \frac{dg_1(C_t)}{dt} \cdot \frac{\partial p}{\partial u}\Big| (t,g_1(C_t))$$

Denote k(t) = g(C$_t$). From the last two equalities (4) is obtained immediately.

Proof of Theorem 3. It is easy to check that (7) (with 0$\leq$t$\leq$T instead of

0$\leq$t<∞) defines a bargaining chain. Here $p(t,u) = \alpha(t) \cdot f\left(\frac{u}{\alpha(t)}\right)$, for $u \in [0, f_1(1)\cdot\alpha(t)]$.

Using the notations of Theorem 2, and denoting $\frac{d_1(t)}{dt}$ by $\dot{}\;(t)$ and $\frac{k_1(t)}{\alpha(t)}$ by x_t, we get from (4):

$$\dot{\alpha}(t)\,(x_t\,f'(x_t) + f(x_t)) + 2x_t\alpha(t)\,f'(x_t) = 0 \ . \qquad (A-1)$$

For each $B\,\bar{B}$ let $h_N(B)$ denote its Nash Cooperative Solution. Suppose first that there exists t_0, $0\leq t_0\leq T$, and that $k(t_0) = h_N(B(t_0)$. We show that this implies $k(s) = h_N B(s)$ for all $0\leq s\leq t_0$, and especially $g(C) = k(0) = h_N(B(0))$. If $k(t_0) = h_N(B(t_0))$ then $x_{t_0}\,f'(x_{t_0}) + f(x_{t_0}) = 0$, and from equation (A-1) $\dot{x}_{t_0} = 0$. Equation (A-1) with the boundary condition $\dot{x}_{t_0} = 0$, is then solved for all $0\leq s\leq t_0$ by $\dot{x}_s=0$, or $k(s) = \alpha(0)x_s = \alpha(s)x_{t_0} = h_{N1}(B(s))$.

We now have to show that if $\alpha(t)$ is small enough then there exists t_0 for which $\dot{x}_{t_0} = 0$. From (A-1) we get

$$\dot{x}_t = -\,\frac{\dot{\alpha}(t)}{2\alpha(t)}\,\frac{(f(x_t)\cdot(x_t))'}{f'(x_t)} \qquad (A-2)$$

Since $\dot{\alpha}(t) < 0$, $f'(x_t)<0$ (the case $\dot{\alpha}(t) = 0$ can be excluded since then $k(t)$ "freezes" until $\dot{\alpha}(t)<0$; $f'(x_t) = 0$ would violate the strong Pareto optiamality) we get

$$\dot{x}_t<0 \text{ iff } (f(x_t)\cdot x_t)' <0 \text{ iff } x_t>x^*$$

and

$$\dot{x}_t<0 \text{ iff } (f(x_t)\cdot x_t)' > 0 \text{ iff } x_t<x^* \ , \text{ where } x^* = h_{N1}(B(0)).$$

So if $x_t<x^*$ then $x_t\uparrow x^*$, and if $x_t>x^*$ then $x_t\downarrow x^*$. We have to prove that convergence occurs in a finite time. Assume $x_T = h_R(B(0))<x^*$. Since $x_T\leq x_t\leq x^*$ for all t, $0\leq t\leq T$, we have

$$\dot{x}_t = -\frac{\dot{\alpha}(t)}{2\alpha(t)}\,(x_t + \frac{f(x_t)}{f'(x_t)}) \leq -\frac{\dot{\alpha}(t)}{2\;(t)}\,(x_T + \frac{f(x_T)}{f'(x_T)}),$$

therefore

$$x_0 = x_T - \int_0^T \dot{x}_t \, dt \ge x_T + \frac{1}{2} (x_T + \frac{f(x_T)}{f'(x_T)}) \cdot \int_0^T \frac{\dot{\alpha}(t)}{\alpha(t)} \, dt =$$

$$x_T - \frac{1}{2} \varepsilon \ln \alpha(T), \quad \text{where} \quad \varepsilon = \left| x_T + \frac{f(x_T)}{f'(x_T)} \right| .$$

It is therefore enough that $x_T - \frac{1}{2} \varepsilon \ln \alpha(T) \ge x^*$, or $\alpha(T) \le \exp(2(x_T - x^*)/\varepsilon)$

for the Nash Cooperative Solution of $B(0)$ to coincide with the Extended Raiffa

Solution of C.

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