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EXPLOITATION AND CLASS: PART II, CAPITALIST ECONOMY

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by

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1. Introduction

In the first paper of this series (Roemer (1979)), the origins of exploitation and class were studied in a pre-capitalist, subsistence economy, where every producer has access to the same technology and has the same subsistence needs, but producers differ according to the endowments of produced goods they have. The distribution of endowments gives rise to a class structure, and a hierarchy of exploitation, where a producer was said to be exploited if he has to work more time than is "socially necessary" in order to produce a bundle of goods which can be traded for his subsistence needs. An important theorem of that paper was dubbed the Class-Exploitation Correspondence Principle. Agents in the economy, facing a set of prices and wage rate, decide whether to sell labor power, hire labor power, work for themselves, or do some combination of these three activities. Thus individual optimization determines a class structure, a class being a particular way of relating to the means of production, in the Marxian

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sense. (For instance, all agents who optimize by selling some labor power and working part-time for themselves constitute one class, which might be called the class of semi-proletarians.) The Class-Exploitation Correspondence Principle (CECP) states that all producers who optimize by selling labor power are exploited at the economy's equilibrium, and all agents who optimize by hiring labor power are exploiters; furthermore, the five classes which exist in this economy always appear in a certain classical order with respect to the welfare-ordering of society's members in terms of the exploitation criterion.

In this paper, we wish to study a model which schematically represents capitalist economy, as opposed to subsistence economy. We wish to define a notion of exploitation which is faithful to the classical Marxian idea that agents are exploited when they have to work more time than is socially necessary to acquire what they get. We shall construct the model so that, as before, classes are endogenously determined by individual optimization, and then prove that the CECP is valid for capitalist economy. The validation of the CECP in the model of capitalist economy, or more precisely, the model with accumulation, will show that the association between class and exploitation is robust, even when several simple features of the subsistence model no longer attain. Alternatively, if one wishes heuristically to accept the correspondence of classes and exploitation as a Marxian premise, then the theorem indicates that the definition of exploitation which is offered in this paper is an appropriate one from the Marxian point of view, as it preserves this fundamental principle. This is a non-trivial point, since the definition of exploitation we offer is considerably more general than the classical Marxian one: in particular, we define exploitation below without reference to any subsistence concept.

In passing from a subsistence economy to a model of an economy in which agents are accumulators, it becomes necessary to define a (Marxian) concept of exploitation which is independent of any notion of subsistence. This, incidentally, is a worthwhile task in itself for Marxian economics, since most workers in advanced capitalist economies today do not receive a wage which can be called a subsistence wage under any interpretation except a tautological one (i.e., whatever workers receive is subsistence). Moreover, a Marxian notion of exploitation should be completely independent of the subjective utilities of agents, as whether an agent is exploited should not depend upon his subjective preferences. (This we take as a Marxian requirement.) Hence, the task of the paper is to propose a definition of exploitation which generalizes the Marxian idea of "surplus value," but is independent of any concept of subsistence, and makes no mention of the agents' preferences over goods.

Before becoming embroiled in the details, it is worth recalling the motivation for this study, which was provided, if inadequately, in Roemer (1979). The previous paper and this one can be regarded as preliminary work to the formulation of a general theory of exploitation which will, in particular, enable us to study the question of inequality and possible exploitation in socialist economies. The methodological approach is to induce, from the theories of exploitation of pre-capitalist and capitalist economies, the general theory which will have as special cases those two forms of exploitation, and will indicate as a third special case, a taxonomy of inequality, in socialist economies, into "exploitative" and "non-exploitative" inequality. This will facilitate a materialist approach to the formation of strata or classes under socialism, which is discussed in another paper (Roemer, 1980b).

The model used in the body of the paper is the Leontief production specification. Finally, in an Appendix, we generalize to the analysis to production sets which are cones.* (This therefore includes von Neumann technology as a special case.) Thus, the CECP is a robust phenomenon, which is true for any constant-returns technology. Moreover, a fascinating by-product of the general conical analysis is an insight gained concerning the "correct" generalization of the notion of embodied labor time to technologies more general than Leontief. This insight bears on the classical and fundamental question of whether labor values can be conceived of independently of market phenomena.

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*The ideas are sufficiently new that it was felt pedagogically less obscure to present the simpler Leontief model in the body of the paper, relegating the generalization to conical technologies to an Appendix. This does not indicate the Appendix is less important--indeed it includes the model of the paper proper as a special case.

2. A Model of Accumulation

Our schematic model of capitalist economy is that all agents are accumulators, seeking to expand the value of their endowments (capital) as rapidly as possible. All agents have access to the same technology, but they differ (as before) in their bundles of endowments. A common Leontief technology is available to all agents. An agent can engage in three types of economic activity: he can sell his labor power, he can hire the labor power of others, or he can work for himself. His constraint is that he must be able to afford to lay out the operating costs, in advance, for the activities he chooses to operate, either with his own labor or hired labor, funded by the value of his endowment.

Explicitly:

A is an $n\times n$ productive, indecomparable input matrix

L is a strictly positive $1\times n$ vector of direct labor inputs

(p,w) is a $1\times(n+1)$ price and wage vector

x^ν is an $n\times 1$ vector of activity levels

y^ν is an $n\times 1$ vector of activity levels

z^ν is a scalar

ω^ν is an $n\times 1$ vector of endowments.

Facing prices (p,w) the ν^{th} producer chooses a $(2n+1)\times 1$ activity vector (x^ν, y^ν, z^ν) to

$$
(2.1)\begin{cases}
\max\ (p-pA)x^\nu + (p-(pA+wL))y^\nu + wz^\nu \equiv \Pi^\nu(p,w) \\
\text{subject to} \\
(2.1a)\quad pA(x^\nu+y^\nu) \leqq p\omega^\nu \\
(2.1b)\quad Lx^\nu + z^\nu \leqq 1 \\
x^\nu,\ y^\nu,\ z^\nu \geqq 0
\end{cases}
$$

x^ν is the vector of activity levels agent ν chooses to operate himself with technology (A,L) ; y^ν is the vector of activity levels he hires others to operate for him; z^ν is the amount of his labor time he sells. The objective function maximizes net revenues; constraint (2.1a) is the agent's "capital constraint" and (2.1b) is his labor constraint. Let $A^\nu(p,w)$ be the set of actions (x^ν, y^ν, z^ν) which solve ν's program at prices (p,w) .

The equilibrium concept is:

Definition 2.1. A price vector (p,w) is a reproducible solution for the economy $(A, L, \omega^1, \ldots, \omega^N)$ with N producers if:

(R1) $(\forall\nu)(\exists(x^\nu, y^\nu, z^\nu) \in A^\nu(p,w))$ (profit maximization)

(R2) $A(x+y) \leqq \omega$, where $x = \sum x^\nu$, $y = \sum y^\nu$ (feasibility of production)

(R3) $Ly = z \equiv \sum z^\nu$ (labor market equilibrium)

(R4) $(I-A)(x+y) \geqq 0$ (reproducibility)

Readers not familiar with this type of equilibrium concept may refer to Roemer (1979) or Roemer (1980b). A price vector reproduces the economy if every producer can choose an individually optimal point (R1) which is globally feasible given society's total endowment of produced goods (R2) and allows trades in labor power to equilibrate (R3); moreover, we demand that society should not run down any physical stock of a produced good, which is accomplished by insisting that net output be a non-negative vector (R4).

Note the absence of any subsistence concept in this economy, or indeed any consumption demand by agents. Agents are pure-and-simple accumulators; we wish to model in as simple and stark a way, an economy whose agents are motivated by the desire to accumulate, just as the economies of the previous paper modelled stark subsistences economies. We could make this model somewhat less stark by insisting that each agent, although an accumulator, has to consume at least some minimal vector b . This would add one more constraint to the individual's program (2.1), and would change the reproducibility constraint (R4) to $(I-A)(x+y) \geqq Nb$. Such a change adds absolutely nothing to the model, however, and so we choose to keep things clean by positing, as it were, that $b = 0$. (This is, of course, just like the familiar assumption in certain growth models that the savings rate of capitalists is unity. In this model, every agent is a potential capitalist.)

Notice that time is of the essence in the model, as what constrains production decisions is the fact that inputs must be paid for today, and revenues from output accrue tomorrow. Nevertheless, we have assumed the same price vector (p,w) is used to price goods in the two periods. In the previous models of subsistence economy studied in Roemer (1979) this was also done; there it was shown the stationarity assumption was legitimate because an equilibrium in the subsistence economy is in fact a stationary state. However, an equilibrium in the accumulation economy is not a stationary state, because endowments grow. Consequently, the expectations of agents that prices will remain the same may be unfulfilled. Consequently, we must observe that the equilibrium of <u>Definition 2.1</u> is in fact descriptive of the steady state. We shall not investigate the existence question of a steady state in this paper. A steady state in the sense of <u>Definition 2.1</u>, which makes the equilibrium concept consistent with a conventional temporary equilibrium price expectation structure of the agents, exists when initial aggregate endowments $\omega = \sum\omega^{\nu}$ lie on the von Neumann ray of the economy. (For a more detailed discussion of a related model which verifies this, see the appendix of Roemer (1980a).)

We proceed to characterize the equilibrium solutions of the model, by showing first that reproducible solutions (p,w) must equalize profit rates across sectors.

Consider the dual program to (2.1): choose α , β to

$$(2.2)\begin{cases} \min \ \alpha W^{\nu} + \beta & \\ \text{subject to} & \\ (2.2a) \quad \alpha pA + \beta L \geqq p(I-A) & x^{\nu} \\ (2.2b) \quad \alpha pA \geqq p - (pA + wL) & y^{\nu} \\ (2.2c) \quad \beta \geqq w & z^{\nu} \end{cases}$$

where $W^{\nu} \equiv p\omega^{\nu}$, and the complementary primal variables are listed next to the appropriate dual constraint. Let $(\hat{\alpha},\hat{\beta})$ be an optimal solution to (2.2).

<u>Case 1</u>: $w > 0$. Suppose $\hat{\beta} > w$. Then a comparison of (2.2a) and (2.2b) shows that (2.2a) is a strict vector inequality. (This uses the assumption postulated that $L > 0$.) Hence by duality, $x^{\nu} = 0$. Also $z^{\nu} = 0$ since $\hat{\beta} > w$. Hence ν does not work at all at his optimum. But this is impossible, since $w > 0$. (ν can increase his revenues if he has some spare time by selling labor power.) Therefore $\hat{\beta} = w$. Now constraints (2.2a) and (2.2b) are identical and the program (2.2) can be simplified:

$$(2.3)\begin{cases} \text{choose } \alpha \text{ to } \min \alpha W^{\nu} + w \\ \qquad \text{s.t. } \alpha pA \geqq p - (pA + wL) \ . \end{cases}$$

Define $\pi_i = ([p-(pA+wL)]/pA)_i$ as the profit rate in sector i, and $\pi_{max} \equiv \max_i \pi_i$. Then the solution of (2.3) is

$$\hat{\alpha} = \max(0, \pi_{max}) .$$

By complementary slackness, from (2.2a) and (2.2b), we conclude:

$$x_i^\nu = 0 = y_i^\nu \quad \text{unless} \quad \pi_i = \pi_{max} \;\; \underline{\text{and}} \;\; \pi_{max} \geq 0 .$$

Only sectors generating the maximum profit rate are operated, either by hired labor or on one's own. No sector is operated if $\pi_{max} < 0$. Notice this argument is independent of ν. Hence, since reproducibility (R4) requires all sectors to operate, a necessary condition for (p,w) to be a reproducible is:

$$\forall i \quad \pi_i = \pi_{max} \quad \text{and} \quad \pi_{max} \geq 0$$

or

$$p = (1+\pi)pA + wL \tag{2.9}$$

for some number $\pi \geq 0$.

<u>Case 2</u>: $w = 0$. In this case, program (2.2) immediately becomes:

$$(2.4)\begin{cases} & \text{choose } \alpha,\beta \text{ to } \min \alpha W^\nu \\ & \text{subject to} \\ (2.4a) & \alpha pA + \beta L \geqq p(I-A) \\ (2.4b) & \alpha pA \geqq p(I-A) \\ & \alpha,\beta \geqq 0 . \end{cases}$$

If $\hat{\beta} > 0$, then constraints (2.4a) are all slack at the optimum and as before we see from (2.4b):

$$\hat{\alpha} = \begin{cases} \max_i \left(\frac{p(I-A)}{pA}\right)_i = \pi_{max} , & \text{if } \pi_{max} \geq 0 , \\ 0 & , \text{if } \pi_{max} < 0 . \end{cases}$$

The only activities which operate are those for which $\pi_i = \pi_{max}$, when $\pi_{max} \geq 0$, and no activities operate if $\pi_{max} < 0$. Again, reproducibility requires $\pi_i = \pi_{max} \geq 0$ for all i . If $\hat{\beta} = 0$, then (2.4a) becomes identical to (2.4b) and the same situation prevails. Summarizing:

Proposition 2.1. Let (p,w) be a non-trivial reproducible solution.

(A) Then prices must equalize profit rates in all sectors: that is, there is a non-negative number π such that

(2.9) $$p = (1+\pi)pA + wL .$$

(B) At an optimum, ν's revenues are $\pi W^{\nu} + w$.

To finish the proof of Proposition 2.1, notice that part (B) follows immediately from the duality theorem of linear programming, since the value of ν's dual program is always $\pi W^{\nu} + w$.

An important observation follows from the fact that ν's revenues are $\pi W^{\nu} + w$. Notice ν can always achieve precisely that revenue by hiring himself out for the entire day (and receiving wage w for his efforts), plus hiring others to "operate" his capital W^{ν} , bringing in revenues πW^{ν} . That is:

Lemma 2.2. At a RS, ν always possesses an individually optimal solution $(x^{\nu}, y^{\nu}, z^{\nu})$ with $x^{\nu} = 0$.

We next use Proposition 2.1 to characterize solutions of the model. The model in fact has a knife-edge property. According to Proposition 2.1, there can only be three types of reproducible solutions:

(i) (p,w) where $p = (1+\pi)pA + wL$, $\pi > 0$, $w > 0$

(ii) $(\bar{p},0)$ where $\bar{p} = (1+\bar{\pi})\bar{p}A$, $\bar{\pi} > 0$, $w = 0$

(iii) (p,w) where $p = pA + wL$, $\pi = 0$, $w > 0$.

Theorem 2.3.

(A) Let (p,w) be a non-trivial reproducible solution where $\pi > 0$ and $w > 0$. Then the knife-edge relationship $N = LA^{-1}\omega$ between the labor supply (N) and aggregate capital (ω) must hold. Conversely, if $N = LA^{-1}\omega$ then either any price vector (p,w) of type (i) supports a reproducible solution, or none does.

(B) If $N > LA^{-1}\omega$ then the only reproducible solution is $(\bar{p},0)$ where $\bar{p} = (1+\bar{\pi})\bar{p}A$.

(C) If $N < LA^{-1}\omega$ then the only reproducible solution is (p,w) where $p = pA + wL$ (and $\pi = 0$).

To prove this, consider a non-trivial reproducible solution (p,w) , where $x + y = \sum(x^\nu + y^\nu) \neq 0$. Since $(I-A)^{-1}$ exists, in fact $(I-A)(x+y) \geq 0$, and since $(I-A)^{-1} > 0$ by indecomposability of A , in fact $x + y > 0$. That is, all activities must operate at a (non-trivial) reproducible solution. By considering the objective function of program (2.1), it is seen that this can only occur if $p \geqq pA$. (If $p_i < pA_i$, no producer will operate sector i .) By productiveness of A , this implies $p \geq pA$ (since A's eigenvalue cannot be unity), and hence $p > 0$ (since $(I-A)^{-1} > 0$).

Suppose, now, that (p,w) is of the form $\pi > 0$. Then every agent at his optimum must have a binding capital constraint (from (2.1a)):

(2.6) $$pA(x^\nu + y^\nu) = p\omega^\nu ,$$

for if not, he could increase his revenues by offering to hire more workers and putting them to work on his slack capital, since $\pi > 0$. Adding (2.5) over all ν gives:

(2.6) $$pA(x+y) = p\omega .$$

In the light of (R2) and our observation that $p > 0$, (2.6) implies

(2.7) $$A(x+y) = \omega .$$

Hence $(x+y) = A^{-1}\omega$ (assuming A^{-1} exists) and

(2.8) $$L(x+y) = Lx + z = LA^{-1}\omega = N .$$

since $Ly = z$ (R3) at a reproducible solution. Hence the demand for labor is N . If $w > 0$, the supply of labor offered is also N , since any agent with slack labor could increase his revenues by selling the extra labor power. Hence (p,w) with $\pi > 0$ and $w > 0$ can only be a reproducible solution if $N = LA^{-1}\omega$.

To prove the converse statement in (A), we need only show that the aggregate activity levels $x + y = A^{-1}\omega$ can: (a) arise from individual optimization, and (b) reproduce the system. Given $N = LA^{-1}\omega$, it is easily seen that $x + y = A^{-1}\omega$ can be the outcome of individual optimization by Lemma 2.2: let $x = 0$ and $y = A^{-1}\omega$. Each agent is indifferent to which sectors he operates with hired labor, since they all enjoy the same profit rate. By Lemma 2.2 and $N = LA^{-1}\omega$, each agent can be assigned a vector $(0, y^\nu, 1) \equiv (x^\nu, y^\nu, z^\nu)$ which is individually optimal and

and which aggregate to the required activity levels. Does $x + y = A^{-1}\omega$ reproduce the system, satisfy (R4)? Only if $(I-A)(A^{-1}\omega) \geqq 0$, or $A^{-1}\omega \geqq \omega$. Thus, if $A^{-1}\omega \geqq \omega$, then all price vectors of type (i) are reproducible solutions; if $A^{-1}\omega \ngeq \omega$, then there are no reproducible solutions.

In part (B), where $N > LA^{-1}\omega$, we have a <u>capital-limited</u> economy, with excess labor. As long as $w > 0$, all labor will wish to be employed since slack labor can always increase its revenues by hiring out. Thus the only reproducible solution is $w = 0$, and $\bar{p} = (1+\bar{\pi})\bar{p}A$, where $\bar{p}$ is the Frobenius eigenvector of A. This in fact supports a reproducible solution, since agents are indifferent to offering their extra labor power for sale at $w = 0$, according to (2.1). This may be thought of as the subsistence wage case. When there is not sufficient capital in the economy to employ all labor, wages are driven down to subsistence level.

In part (C), where $N < LA^{-1}\omega$, there is a <u>labor-limited economy</u>. As long as $\pi > 0$, all agents with capital wish to gainfully employ it, which is impossible. Hence $\pi = 0$ is the only possible reproducible solution. Conversely, a reproducible solution of type (iii) can be arranged in this case. This concludes the proof of Theorem 2.3.

In summary, at the knife-edge, the profit rate is indeterminate. In the capital-limited economy, the profit rate is maximal and the wage is driven to zero, and in the labor-limited economy, the opposite occurs. This appears to be a rather uninteresting set of equilibria, and it derives from the stark specification of the model. Recall that in the subsistence economies of Roemer (1979), there was a more pleasant situation: an essentially unique profit rate π prevailed which equilibrated the demand and supply for labor, in non-pathological cases. The essential difference between the two models is that the agents in the subsistence economy were satiable: once an agent's labor time was reduced to zero, he was satiated, and was willing to allow capital to lie unused which, for other agents, had a positive shadow price. In the present model, agents are insatiable and are driven to use all their capital as long as positive profits can be made. The fixed proportions technology generates the knife-edge problem, since changes in relative prices are incapable of stimulating substitution of factors in production. We could change the equilibrium possibilities for the economy with utility functions including an accumulation-leisure trade-off. We have, however, avoided specifying such utility functions, as we wish the model to portray only the starkest kernel of the accumulation issue, so that we can examine the issues of exploitation and class in the most unfettered way. And this simple model suffices for this task.

3. Exploitation and Class

A. Definition of Exploitation

Consider a reproducible solution (p,w) for the economy $(A, L, \omega^1, \ldots, \omega^N)$ involving the total activity vector $x + y$. In classical Marxian economics, a worker is considered exploited because the bundle of goods he purchases with his wage, the subsistence bundle, embodies less labor time than he worked. In our model, agents do not purchase goods, they accumulate; nor is the subsistence notion important. We propose this generalization of the Marxian notion of exploitation. Consider all bundles of goods an agent could feasibly purchase with his revenues. An agent is exploited if there is no such bundle which emobdies as much labor as the time he worked. More precisely:

Definition 3.1. Let (p,w) support a reproducible solution $x+y$. A feasible assignment for this reproducible solution is any distribution of net output $\{f^\nu\}$, $\nu = 1, N$ such that:

$$(1)\quad \sum f^\nu = (I-A)(x+y)$$

$$(2)\quad pf^\nu = \Pi^\nu(p,w) .$$

The class of all feasible assignments is called $\mathcal{F}$.

(Recall $\Pi^\nu(p,w)$ is the optimal value for ν's program (2.1).) Let $\mathcal{F}^\nu$ be the set of bundles f^ν which ν receives under the various feasible assignments.

A feasible assignment is any distribution of society's net output to its members, distributed so each agent receives an affordable bundle. Recall the labor value of a bundle of goods v is $L(I-A)^{-1}v \equiv \Lambda v$.

Definition 3.2. An agent ν is exploited at the reproducible solution if and only if

$$\max_{\mathcal{F}^\nu} \Lambda f^\nu < Lx^\nu + z^\nu .$$

Agent ν is an exploiter iff

$$\min_{\mathcal{F}^\nu} \Lambda f^\nu > Lx^\nu + z^\nu .$$

An agent is exploited when the amount of labor embodied in any bundle of goods he could receive, in a feasible distribution of society's net product, is less than the labor he expends. Similarly, an exploiter is one whose revenues unambiguously command goods embodying more labor time than he works.

Notice that this definition is independent of subjective preferences, if we wish to endow the agents with utility functions for produced commodities. Clearly any feasible assignment can be thought of as arising from some set of preferences. (The converse, however, is not true, because an arbitrary set of utility functions will not give rise to demands which necessarily aggregate to $(I-A)(x+y)$.) The intent of the definition is to capture the notion that an agent is exploited when the goods he can command through the market embody less social labor time than he expended.

Several comments are in order. First, in general there will be agents who are neither exploiters or exploited. Under some feasible assignments they receive a bundle f^{ν} where $\Lambda f^{\nu} > Lx^{\nu} + z^{\nu}$, and under others they receive a bundle f^{ν} where $\Lambda f^{\nu} < Lx^{\nu} + z^{\nu}$. We shall be interested in analyzing this gray area of agents. The existence of the gray area is intimately related to the classical "transformation problem," of the non-proportionality of prices and labor values. Secondly, there may exist reproducible solutions with exploiters but no exploited agents, or vice versa. This may appear to be a defect of the model, but it is actually a nice point. The resolution of this issue is discussed in a later section of the paper.

More important is the philosophical question, why do we choose to call this phenomenon exploitation? For the moment, this will not be defended: the justification of the normative term exploitation must await the next paper in this series (Roemer, 1980b). We are, after all, attempting to construct a generalized Marxian theory of exploitation. All that is asked of the reader at this point is that he/she evaluate the definitions in the light of the classical Marxian one in terms of surplus value, rather than against some other conception of justice or exploitation.

B. Characterization of Exploited and Exploiters

Note. For the duration of the paper, we assume the reproducible solution (p,w) has $w > 0$. According to Theorem 2.3, this excludes the capital-limited economy. In fact, the theorems proved below have immediate analogs for the case $w = 0$, and the principal results, such as the Class-Exploitation Correspondence Principle, remain true. We assume $w > 0$ only for simplicity, so that we may avoid giving two cases in every theorem. Thus:

Working Assumption. At the reproducible solution (p,w) under consideration, $w > 0$.

Notation. We denote the maximal ratio of two vectors thus:

$$\left(\frac{\Lambda}{P}\right)_{max} \equiv \max_{i}\left(\frac{\Lambda_i}{P^i}\right) .$$

We wish to characterize, at a reproducible solution (p,w), the wealths $W^\nu = p\omega^\nu$ which will be associated with exploited agents, and with exploiting agents. It is obvious from the Definition 3.2 that there are two cut-off points: if wealth is greater than a certain amount, one is an exploiter; if wealth is less than a certain amount one is exploited. To characterize simply these two cut-off points, we require:

Assumption of a Large Economy (ALE). Every agent can spend all his revenue on the purchase of any one good.

In other words, ALE guarantees that, given any agent ν and any good i, there is a feasible assignment in which ν receives a bundle consisting of only good i. This means each agent's revenue is small compared to the entire economy's net product.

Theorem 3.1. Let (p,w) be a RS with $w > 0$, and $p = (1+\pi)pA + wL$. Let ALE hold. If $\pi > 0$ then:

$$\text{agent } \nu \text{ is exploited} \iff W < \frac{1-\rho_{max}w}{\pi\rho_{max}}$$

$$\text{is an exploiter} \iff W > \frac{1-\rho_{min}w}{\pi\rho_{min}}$$

where $\rho_i = \Lambda_i/p_i$ and $\rho_{max} = \max_i \rho_i$, $\rho_{min} = \min_i \rho_i$. If $\pi = 0$ there are no exploiters and no exploited agents.

Proof. Since $w > 0$, at an optimal solution, every agent works all day: $Lx^{\nu} + z^{\nu} = 1$. From Theorem 2.1, his revenues are $\Pi^{\nu}(p,w) = \pi W^{\nu} + w$. How can ν receive goods embodying maximal labor value for his money? By spending his entire revenue on the good with the maximal labor value-price ratio, $(\Lambda/p)_{max}$. By ALE, there is, in fact, a feasible assignment which assigns this particular bundle to agent ν. ν is exploited if the labor value he thus commands through goods is less than the time he worked; that is, if:

$$(3.1) \qquad \left(\frac{\Lambda}{p}\right)_{max}(\pi W^{\nu}+w) < 1$$

(3.1) is equivalent to $W^{\nu} < (1-\rho_{max}w)/\pi\rho_{max}$.

In like manner, ν is an exploiter if, when he spends his entire revenues on the good with the minimal labor value-price ratio, ρ_{min}, he nevertheless commands more labor time in his goods than he works:

$$(3.2) \qquad \left(\frac{\Lambda}{p}\right)_{min}(\pi W^{\nu}+w) > 1 ,$$

or $W^\nu > (1-\rho_{min}w)/\pi\rho_{min}$.

Q.E.D.

(Note: When $\pi = 0$, it follows from (3.1) and (3.2) that there are no exploiters or exploited, since $p = w\Lambda$.)

Hence the gray area of agents who are neither exploited nor exploiters consists of all agents possessing wealth W such that $(1-\rho_{max}w)/\pi\rho_{max} \leqq W \leqq (1-\rho_{min}w)/\pi\rho_{min}$. Notice this interval of wealth is a single point if and only if $\rho_{min} = \rho_{max}$, which occurs if and only if prices are proportional to labor values. If $\pi > 0$, this arises if and only if organic compositions of capital are equal in all sectors. Thus, when $\pi > 0$, the existence of a non-trivial gray area is equivalent to the transformation problem.

C. Characterization of Class Membership

As in Roemer (1979), an agent's class membership is determined by whether he hires labor power, sells labor power, or works for himself when he optimizes. An agent, for instance, is said to belong to the class (+,0,+) if he possesses an individually optimal solution of the form (x^ν, y^ν, z^ν) with $x^\nu \neq 0 \neq z^\nu$ and $y^\nu = 0$. A priori, there are eight possible classes, depending on how we arrange the + and 0 symbols in the three cells. In Lemma 2.2, we have shown that every agent is a member of class (0,+,+) . We use the set-theoretic notation $\nu \in C^1\backslash C^2$ to mean " ν is a member of class C^1 but not of class C^2 ." We can now easily characterize class membership in the model.

<u>Theorem 3.2</u>. <u>At the RS</u> (p,w) :

(A) $\nu \in (+,+,0)\backslash(+,0,0) \iff \forall$ individually optimal solutions of form $(0, y^\nu, z^\nu)$, $Ly^\nu > 1$

(B) $\nu \in (+,0,0) \iff \exists$ an individually optimal solution of form $(0, y^\nu, z^\nu)$, $Ly^\nu = 1$

(C) $\nu \in (+,0,+)\backslash(+,0,0) \iff \forall$ individually optimal solutions of form $(0, y^\nu, z^\nu)$, $Ly^\nu < 1$, and $W^\nu \neq 0$

(D) $\nu \in (0,0,+) \iff W^\nu = 0$.

Let us denote these four classes as:

$$C^H = (+,+,0)\backslash(+,0,0)$$

$$C^N = (+,0,0)$$

$$C^S = (+,0,+)\backslash(+,0,0)$$

$$C^P = (0,0,+)$$

C^H is the class of agents who must hire labor power to optimize; C^S consists of those who must sell labor power to optimize, but have some capital; C^N is the "neutral" class of agents who can optimize without either hiring or selling labor power; C^P is the class of pure proletarians who have no capital.

<u>Proof of Theorem 3.2</u>

<u>Part (C)</u>. Consider any solution for ν of form $\xi^\nu = (0, y^\nu, z^\nu)$. Note $z^\nu = 1$ since $w > 0$, and $y^\nu \neq 0$ since $W^\nu \neq 0$. Define a new solution:

$$\overline{\xi}^\nu = (\overline{x}^\nu, \overline{y}^\nu, \overline{z}^\nu) \quad \text{where} \quad \begin{aligned} \overline{x}^\nu &= y^\nu \\ \overline{y}^\nu &= 0 \\ \overline{z}^\nu &= 1 - Ly^\nu . \end{aligned}$$

Check that $\overline{\xi}^\nu$ is a feasible solution for ν (program (2.1)). (Notice ν uses the same amount of capital in both solutions, and the same amount of labor.) Check also that the solution $\overline{\xi}^\nu$ generates the same revenue as ξ^ν, so that $\overline{\xi}^\nu$ is individually optimal. Hence $\nu \in (+,0,+)$, since $Ly^\nu < 1$ and $y^\nu \neq 0$.

Suppose $\nu \in (+,0,0)$: let $(x^\nu, 0, 0)$ be optimal. Then $Lx^\nu = 1$, or else ν was not optimizing: since $w > 0$, he could have sold slack labor power for increased revenue. Form the solution $\overline{\xi}^\nu = (\overline{x}^\nu, \overline{y}^\nu, \overline{z}^\nu)$ defined by:

$$\begin{aligned} \overline{x}^\nu &= 0 \\ \overline{y}^\nu &= x^\nu \\ \overline{z}^\nu &= 1 \end{aligned}$$

Check that $\overline{\xi}^\nu$ is individually optimal for ν ; this contradicts the assumption that $Ly^\nu < 1$ for all individually optimal solutions. Hence $\nu \in (+,0,+)\setminus(+,0,0)$.

The converse direction follows by making these same constructions.

Parts (A), (B), and (D) have proofs with the same format.

Q.E.D.

<u>Corollary 3.3.</u> <u>At a RS</u> (p,w) , <u>society is exhaustively partitioned into the four pairwise disjoint classes</u> C^H , C^N , C^S <u>and</u> C^P .

Proof. Both exhaustiveness and pairwise disjointedness follow immediately from Theorem 3.2.

Q.E.D.

A quick review is useful. Since every agent belongs to the class $(0,+,+)$, that class description is not interesting. The class partition C^H , C^N , C^S , C^P is much more interesting because of Corollary 3.3, and because it distinguishes unambiguously between labor-hiring and labor-selling. The possibility of a Class-Exploitation Correspondence Principle for this economy now materializes: to wit, is it true that every member of C^H is an exploiter and every member of C^S and C^P is exploited?

To approach this question, we require a characterization of the class memberships C^H , C^N , C^S and C^P according to wealths, for we have already in Theorem 3.1 a characterization of the exploitation status of an agent according to his wealth.

Theorem 3.4. Let (p,w) be a RS with $\pi > 0$. Then:

$$\nu \in C^H \iff W^\nu > \left(\frac{pA}{L}\right)_{max}$$

$$\nu \in C^N \iff \left(\frac{pA}{L}\right)_{min} \leq W^\nu \leq \left(\frac{pA}{L}\right)_{max}$$

$$\nu \in C^S \iff 0 < W^\nu < \left(\frac{pA}{L}\right)_{min}$$

$$\nu \in C^P \iff W^\nu = 0 .$$

To prove Theorem 3.4, first notice, from Theorem 3.2 and the canonical optimal solutions of the form $(0, y^\nu, z^\nu)$ of Lemma 2.2, that the four classes must fall in the order C^H , C^N , C^S , C^P as wealth decreases. Recall how the solution $(0, y^\nu, z^\nu)$ was constructed. Agent ν hires himself out all day $(z^\nu = 1)$, and then hires others to work his entire capital stock. If, in so doing, the amount of labor ν employs, Ly^ν , is always greater than the amount of labor μ employs, Ly^μ , then ν must have more capital than μ . (The reason there is indeterminacy as to the amount of labor ν employs, Ly^ν , is because organic compositions of capital differ across sectors and so ν can exhaust his capital according to $pAy^\nu = W^\nu$ using different activity vectors y^ν , and hence different labors Ly^ν --all generating the same amount of profit, πW^ν .) Hence, from Theorem 3.2, the ordering of the four classes by wealths follows.

Secondly, it is obvious that $\nu \in C^P \iff W^\nu = 0$; for if $W^\nu > 0$, then $(0,0,+)$ cannot be optimal for ν , as his revenue is only w and not $\pi W + w$. (We use $\pi > 0$ here.)

From these two remarks, to complete the proof of the theorem we need only show that:

$$\nu \in C^N \iff \left(\frac{pA}{L}\right)_{min} \leqq W^\nu \leqq \left(\frac{pA}{L}\right)_{max} .$$

This constitutes the heart of the argument.

An agent ν who possesses an optimal solution to (2.1) of the form $(+,0,0)$ possesses the same optimal solution to the following restricted program :

$$(3.2)\begin{cases} & \max\ p(I-A)x^\nu \\ (3.2a) & \text{s.t. } pAx^\nu \leqq W^\nu \\ (3.2b) & \qquad Lx^\nu \leqq 1 \end{cases}$$

In the program (3.2), agent ν has no option to sell or hire labor power: he optimizes solely with respect to the activities he can operate himself, using his own resources.

The dual to the restricted program is:

$$(3.3)\begin{cases} \min\ \alpha W^{\nu} + \beta \\ (3.3a)\quad \text{s.t.}\quad \alpha pA + \beta L \geqq p(I-A)\ . \end{cases}$$

Note that $(\pi,w) = (\alpha,\beta)$ is an optimal solution to (3.3). (Feasibility of (π,w) for (3.3) follows since $p = (1+\pi)pA + wL$. Optimality follows since by hypothesis the solution of (3.2) for ν is the same as the solution of (2.1) for ν , and the solution of (2.1) is $\pi W^{\nu} + w$ by Theorem 2.1. Finally the solution of (3.2) has the same value as the solution of (3.3) by duality.)

For the solution (π,w) , (3.3a) becomes a vector equation. That is, the point (π,w) lies on every constraint line of the constraint set for program (3.3). In Figure 1, we make a picture of program (3.3) in R^2 :

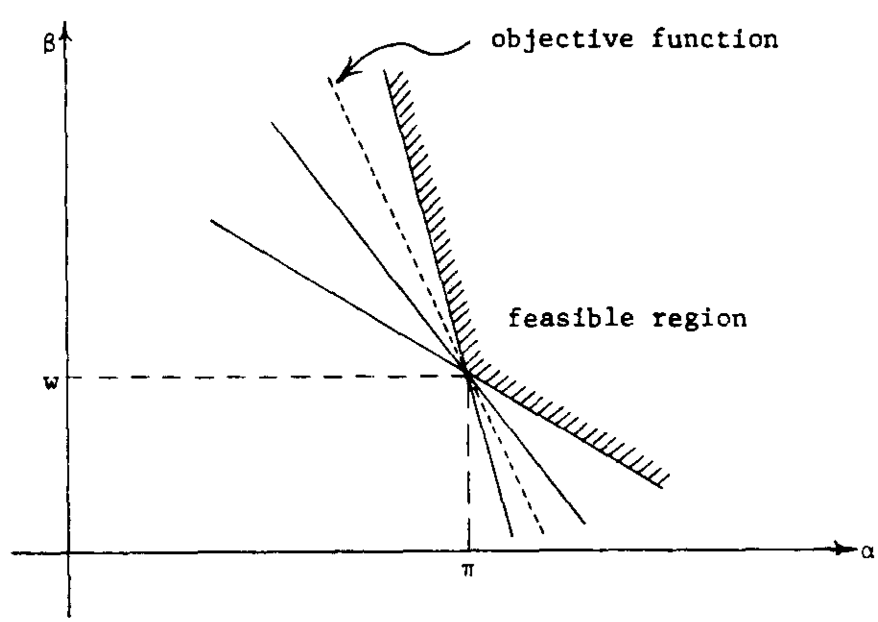

FIGURE 1. The Linear Program (3.3)

(π,w) lies on every constraint line as required. For the objective function $\alpha W^\nu + \beta$ to achieve its optimum at (π,w), it is clearly necessary and sufficient that the slope of the contour lines $\alpha W^\nu + \beta = k$ lie between the maximal and minimal slopes of the various constraints; that is:

$$\left(\frac{pA}{L}\right)_{min} \leqq W^\nu \leqq \left(\frac{pA}{L}\right)_{max} .$$

(The slope of the i^{th} constraint is $-(pA_i/L_i)$, and the slope of the contour lines is $-W^\nu$.) This completes the proof.

Q.E.D.

Note. The restriction $\pi > 0$ of Theorem 3.4 is in reality no restriction. If $\pi = 0$, the value of every agent's objective function becomes $\pi W^\nu + w = w$, and so it is clear every agent has an optimal solution of form $(0,0,+)$. This situation is one of no interest, as agents become identical for all practical purposes, since capital ceases to matter. When $\pi = 0$, there will be no exploitation. This is, in fact, a variant of the so-called fundamental Marxian theorem (FMT) of Morishima (1973), Roemer (1980a) and others. The variant of the FMT for the subsistence economy was proved in Roemer (1979). In this model it is an immediate fact, since the revenues of all agents are identical when $\pi = 0$.

4. The Class-Exploitation Correspondence Principle (CECP)

We now have two independent characterizations of class membership and exploitation according to wealth, according to Theorems 3.1 and 3.4. We proceed to compare the two characterizations to conclude a relationship between exploitation and class.

Theorem 4.1. Let (p,w) be a RS with $\pi > 0$, $w > 0$. Let the Assumption of Large Economy (ALE) hold. Then:

(A) Every member of C^H is an exploiter;

(B) Every member of $C^S \cup C^P$ is exploited.

A picture of the two real lines, along which are arranged the four crucial numbers of Theorem 3.1 and Theorem 3.4, shows the relationship we wish to prove.

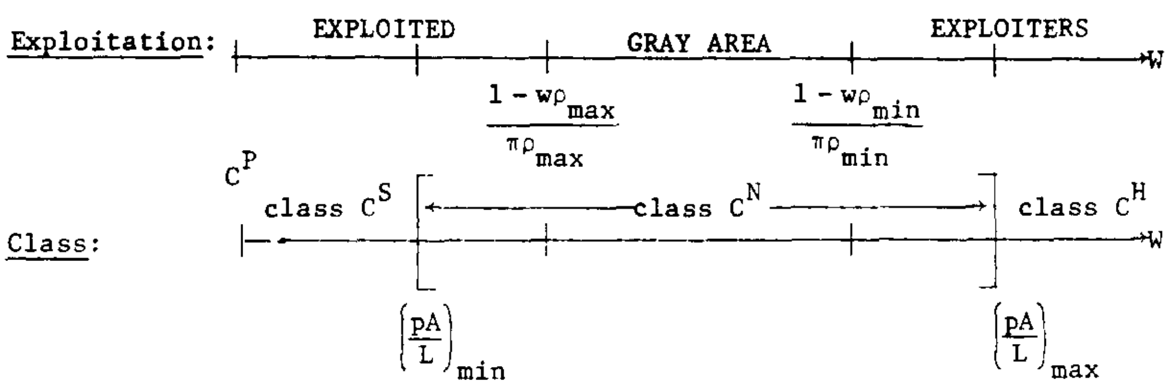

FIGURE 2. Relationship of Exploitation to Class

Thus, Theorem 4.1 is proved if we verify these two inequalities:

$$(4.1)\qquad \left(\frac{pA}{L}\right)_{max} \geqq \frac{1 - w\rho_{min}}{\pi\rho_{min}}$$

$$(4.2)\qquad \left(\frac{pA}{L}\right)_{min} \leqq \frac{1 - w\rho_{max}}{\pi\rho_{min}} .$$

We verify (4.1). Suppose it were false. Then

$$(4.3)\qquad \left(\frac{pA}{L}\right)_{max} < \frac{1 - w\rho_{min}}{\rho_{min}}/\pi .$$

Let $x \geq 0$ be an arbitrary non-zero, non-negative n vector. Notice this averaging identity holds:

$$(4.4) \qquad \frac{pAx}{Lx} \equiv \sum_{i=1}^{n} \left(\frac{pA}{L}\right)_i \left(\frac{L_i x_i}{Lx}\right) .$$

(Note $x \geq 0$ implies $Lx > 0$ since $L > 0$.) Consequently $pAx/Lx \leqq (pA/L)_{max}$ and therefore from (4.3):

$$(4.5) \qquad \frac{pAx}{Lx} < \frac{1 - w\rho_{min}}{\rho_{min}}/\pi .$$

Multiplying (4.5) by πLx and adding wLx to both sides:

$$(4.6) \qquad \pi pAx + wLx < \left(w + \frac{1 - w\rho_{min}}{\rho_{min}}\right) Lx$$

which simplifies to:

$$(4.7) \qquad \pi pAx + wLx < \frac{1}{\rho_{min}} Lx .$$

Adding pAx to both sides, and observing the form of the price vector gives:

$$(4.8) \qquad px < \frac{1}{\rho_{min}} Lx + pAx$$

or

$$(4.9) \qquad p(I-A)x < \frac{1}{\rho_{min}} Lx .$$

Inequality (4.9) holds for <u>any</u> $x \geq 0$. It follows that:

(4.10) $$p(I-A) < \frac{1}{\rho_{min}} L$$

and since $(I-A)^{-1} > 0$:

(4.11) $$p < \frac{1}{\rho_{min}} L(I-A)^{-1} \equiv \frac{1}{\rho_{min}} \Lambda .$$

But (4.11) is impossible by the definition of ρ_{min}, since $\rho_{min} p_i = \Lambda_i$ for some i. By contradiction, (4.1) is verified. A symmetrical argument verifies (4.2).

Q.E.D.

We may now discuss the precise form the transformation problem takes in this model. Suppose the organic compositions of capital are equal in the economy (A,L). (That is, $\Lambda A_i/L_i$ is the same for all sectors i.) Then $\rho_{min} = \rho_{max}$ and $(pA/L)_{max} = (pA/L)_{min}$. It follows from (4.1) and (4.2) that $(pA/L) = (1 - w\rho)/\pi\rho$ in this case. Hence: the wealths associated with class C^N constitute a single point, and the gray area reduces to that same point. The class C^H is precisely the set of agents who exploit, and the class union $C^S \cup C^P$ is precisely the set of agents who are exploited.

When, however, the organic compositions of capital are unequal then in general (4.1) and (4.2) are strict inequalities. Thus, there are in general agents in the class C^N who are exploiters or exploited. Although the gray area is always contained in the class C^N, the converse is not true. Hence class membership does not provide a completely unambiguous proxy for an agent's exploitation status.

The transformation problem renders ambiguous the exploitation status of agents in the class (+,0,0) . But Theorem 4.1 can be viewed as providing a tidy resolution to the transformation problem, if one believes such questions require resolution. For it says that although the non-proportionality of labor values and prices produces a gray area of agents whose exploitation status is ambiguous, in fact that gray area is always contained in the class (+,0,0) .

5. Exploited and Exploiting Sets

The definition of exploitation which has been studied proposes, in a sense, the strongest possible criterion for an agent to be exploited or exploiting. We do not consider an agent to be exploited if he happens to purchase a bundle of goods which embodies less labor time then he worked; he is only exploited if he could not have feasibly purchased a bundle of goods embodying as much labor time as he worked. Because of the severity of the exploitation criterion there is a large gray area of agents who are neither exploited nor exploiting. It is therefore a theorem of some power that, despite the largeness of this gray area, it is always contained entirely in the class of agents (+,0,0) . In this way, the ambiguity in the classification of agents by their exploitation status seems well resolved.

There is, however, another way of examining the agents' exploitation status which gives rise to further questions. It is possible that there can be economies where exploiters exist, but there are no exploited individuals (or vice versa). If an agent j is an exploiter then the complementary coalition to him, $N - \{j\}$, is certainly an exploited set, in the sense that in toto that coalition never receives goods under feasible assignments embodying as much labor as they collectively worked; but no individuals in the complementary coalition need be exploited. The agent(s) who receive(s) the bundle embodying less labor than he worked can change as we range over feasible assignments. In general, $N - \{j\}$ will contain some individual exploiters.

From a philosophical point of view, this phenomenon need not be troublesome. We view exploitation as a social phenomenon, and the existence of exploitation need not imply, in principle, the existence of individual exploiters or exploited. Nevertheless, this ambiguity requires us to analyze more carefully the set-theoretic nature of exploitation in the economy. In particular, the study can be guided by this question: Despite the non-existence, in general, of individual members of society who are exploited or exploiter, can we identify two sets of agents which we consider to be the "canonical exploited set" and the "canonical exploiting set"?

For preciseness we repeat:

Definition 5.1. Let S be a set of agents at a RS. Then S is an exploited set if, under all feasible assignments $\{f^\nu\}$ in the economy, $\sum_S \Lambda f^\nu < \sum_S (Lx^\nu + z^\nu)$; S is an exploiting set if under all feasible assignments, $\sum_S \Lambda f^\nu > \sum_S (Lx^\nu + z^\nu)$.

Note the following follows immediately from Definition 5.1:

Lemma 5.1. S *is an exploited set if and only if* S' , *its complement, is an exploiting set.*

In general, there are many exploited and exploiting sets at a RS. It would not be correct to think of an agent as in some sense unlucky if he is a member of an arbitrary exploited set. For instance, the wealthiest member of society will be a member of some exploited set, if there is a an exploiting agent other than him. (Let the other exploiting agent be j ; then the wealthiest agent is a member of the exploited set $N - \{j\}$.) This motivates us to examine minimal exploited sets: for their status as exploited sets depends on the presence of every member.

Definition 5.2. An agent is vulnerable if he is a member of some minimal exploited set, V^t . An agent is culpable if he is a member of some minimal exploiting set, C^t . $V = \cup V^t$ is the set of vulnerable agents and $C = \cup C^t$ is the set of culpable agents.

We propose to consider V as the "canonical exploited set" in the economy, and C as the "canonical exploiting set." V contains, of course, all the exploited individuals, but in general it contains much more. (C also contains all individual exploiters.) Moreover, as long as there is some exploiting set in society, then both V and C are non-empty, even if there are no individual exploiters or exploited.

For the rest of the analysis we fix a reproducible solution; let W_N be the total wealth of the N agents in society at the RS. Let $W \leqq W^N$ be any wealth level; we define the function $\phi(W)$ as the maximal amount of labor value that can be embodied in goods at the RS, purchaseable with wealth W . That is:

$$\phi(W) = \max\{\Lambda f \,|\, f \leqq (I-A)(x+y) \text{ and } pf = W\} .$$

For specificness, the function $\phi(W)$ is graphed in Figure 3 for an economy with seven agents. In Figure 3, we have named various crucial levels of wealth. W_n is defined as that wealth which would just allow its possessor to purchase goods from society's net output embodying n units of labor: that is, $\phi(W_n) = n$. The property of ϕ crucial for the analysis is

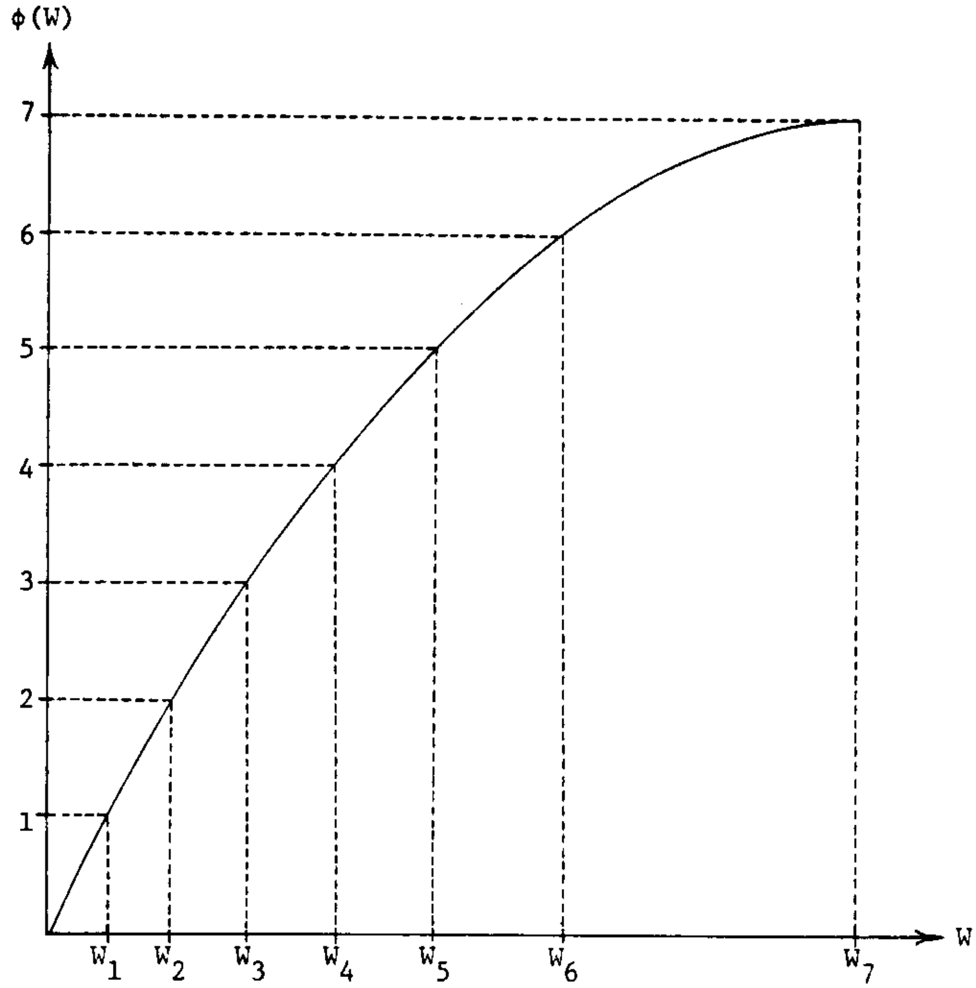

FIGURE 3. The Function $\phi(W)$

its concavity, which is obvious. For if W is small, then it can all be spent on goods which maximize the value-price ratio, Λ/p ; as W grows, goods with smaller Λ/p-ratios must be purchased, and hence dollars become less efficient in purchasing labor value. Hence the concavity of ϕ . Actually, ϕ is piece-wise linear, with as many linear segments on its graph as there are goods in the economy. It is drawn smoothly since the piece-wise linearity plays no role in the analysis.

From the concavity of $\phi(W)$, it immediately follows that:

Lemma 5.2. $(W_n - W_{n-k})$ is increasing in n , for $k \leq n$.

(This is evident from Figure 3, and its proof is immediate.)

Note the relevance of the function $\phi(W)$ for our analysis. A set S with s members is exploited if and only if the wealth of S, $W(S)$, satisfies $W(S) < \phi(s)$. Exploited individual agents are precisely those for whom $W^{\nu} < \phi(1)$.

We now prove a fact which helps put together the story for the canonicality of the sets $\mathcal{C}$ and $\mathcal{V}$:

Theorem 5.3. $\mathcal{V}$ is an exploited set and $\mathcal{C}$ is an exploiting set.

This follows from:

Lemma 5.4. Let R and S be two exploited sets whose intersection $R \cap S$ is not an exploited set. Then $R \cup S$ is an exploited set.

Proof. Label the sets so that $\#R \leqq \#S$.

If $R \subseteq S$, we are done. If $R \cap S = \phi$ we are done since the union of disjoint exploited sets is exploited (this follows from Definition 5.1). Hence we may assume $R \subsetneq S$ and $R \cap S \neq \phi$. Let $T = R \backslash S$. We have assumed $T \neq \phi$ and $T \subsetneq R$. Let $t = \#T$, $r = \#R$, $s = \#S$.

We must show $S \cup R = S \cup T$ is an exploited set. Suppose this is false. Then:

(1) $$W(S \cup T) \geqq W_{s+t}$$

where $W(S \cup T)$ means the total wealth of the coalition $S \cup T$. By definition, since S is exploited:

(2) $$W(S) < W_s .$$

Now

(3) $$W(T) = W(S \cup T) - W(S)$$

since $S \cap T = \phi$ by definition of T. It follows from (1)-(3) that:

(4) $$W(T) > W_{s+t} - W_s .$$

By choice, $s \geqq r$, so $s+t > r$. Hence, from Lemma 5.2:

(5) $$W_{s+t} - W_s \geqq W_r - W_{r-t} .$$

From (4) and (5):

(6) $$W(T) > W_r - W_{r-t} .$$

Now:

(7) $$W(R) = W(R\backslash T) + W(T) > W(R\backslash T) + W_r - W_{r-t} .$$

Notice that $R\backslash T = R\backslash(R\backslash S) \equiv R \cap S$. Thus $R\backslash T$ is not an exploited set by hypothesis and so:

(8) $$W(R\backslash T) \geqq W_{r-t} .$$

Combining (8) with (7) yields:

(9) $$W(R) > W_r$$

which says R is not an exploited set, which contradicts the original supposition that $S \cup R$ is not exploited.

Q.E.D.

Proof of Theorem 5.3

Inductive application of Lemma 5.4 to the set of vulnerable sets shows V is an exploited set. (That is, $V^1 \cup V^2$ is exploited. $(V^1 \cup V^2) \cup V^3$ is exploited, etc. This follows since a proper subset of a minimal exploited set cannot be an exploited set.)

Q.E.D.

The proof that C is an exploiting set is exactly the same. We must introduce the relevant dual function to $\phi(W)$, which we call $\psi(W)$. $\psi(W)$ is the minimal amount of labor power embodied in goods which can be purchased using wealth W:

$$\psi(W) = \min\{\Lambda f \mid f \leqq (I-A)(x+y),\ pf = W\} .$$

$\psi(W)$ is drawn in Figure 4.

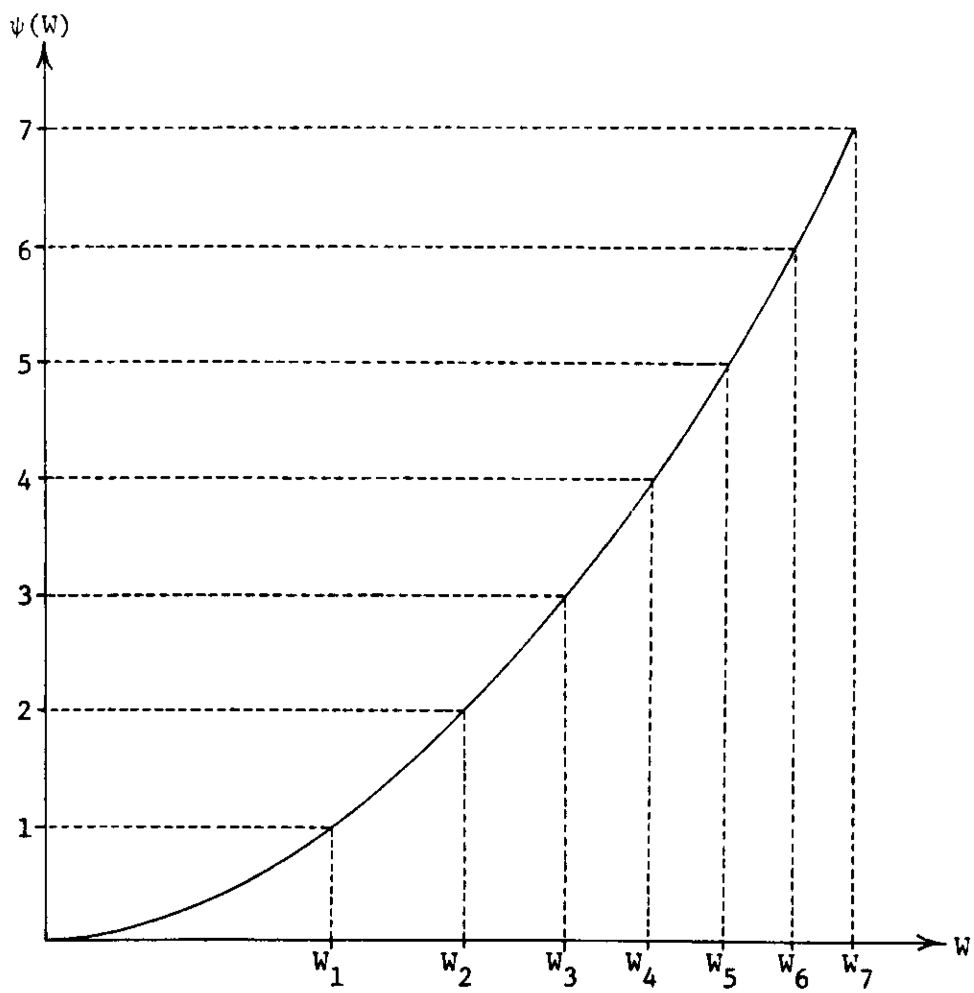

FIGURE 4. The Function $\psi(W)$

$\psi(W)$ is in fact a reflected image of $\phi(W)$, in this sense:

$$\phi(W_N - W) = N - \psi(W) .$$

That is, if a coalition with wealth W can minimally purchase n units of embodied labor time, then its complement, with wealth $W_N - W$, can maximally purchase $N-n$ units of embodied labor time.

Application of the argument given above to the function ψ proves, in like manner, that C is a culpable set.

We now complicate the argument by introducing another important set. Let $\{\mathcal{W}^\tau\}$ be the maximal exploited sets. We shall propose another candidate for the canonical exploited set: $\mathcal{W} = \cap \mathcal{W}^\tau$. First notice:

Lemma 5.4. The sets $\mathcal{W}^\tau$ are precisely the complements of the sets C^t .

Proof. Let $\mathcal{W}^\tau$ be a maximal exploited set. Its complement $(\mathcal{W}^\tau)'$ is exploiting, by Lemma 5.1. If $(\mathcal{W}^\tau)'$ is not a minimal exploiting set, there is an agent $j \in (\mathcal{W}^\tau)'$ who can be removed, and $(\mathcal{W}^\tau)' - \{j\}$ is still exploiting. Thus by Lemma 5.1, $\mathcal{W}^\tau \cup \{j\}$ is exploited, which means $\mathcal{W}^\tau$ is not a maximal exploited set.

Q.E.D.

Corollary 5.5. $\mathcal{W}$ is an exploited set.

Proof. $\mathcal{W} = \cap \mathcal{W}^\tau$ and so $\mathcal{W}' = \cup (\mathcal{W}^\tau)' = \cup C^t = C$ by Lemma 5.4. By Theorem 5.3, C is exploiting, and so $\mathcal{W}$ is exploited by Lemma 5.1.

Q.E.D.

Despite a plethora of sets, we must introduce the one missing notion. Let $\{\mathcal{D}^\tau\}$ be the maximal exploiting sets and $\mathcal{D} = \cap \mathcal{D}^\tau$. $\mathcal{D}$ is another candidate for a canonical exploiting set. The reasoning of Corollary 5.5 has also proved that $\mathcal{D}$ is an exploiting set, since $\mathcal{D}' = \cup \mathcal{V}^t = \mathcal{V}$.

We study next the relationship between $\mathcal{V}$ and $\mathcal{W}$. Notice $\mathcal{V}$ is a "max min" and $\mathcal{W}$ is a "min max." This motivates us to carry through this analogy, and ask whether we always have "max min $\leq$ min max," as is true of very general functions.

<u>Theorem 5.6</u>. $\mathcal{V} \subseteq \mathcal{W}$ <u>and</u> $\mathcal{C} \subseteq \mathcal{D}$.

(I.e., max min $\leq$ min max.)

<u>Proof</u>. (We need only prove $\mathcal{V} \subseteq \mathcal{W}$. Then the second statement follows by de Morgan's laws.)

We need show

$$\cup \mathcal{V}^t \subseteq \cap \mathcal{W}^\tau$$

for which it is necessary and sufficient to show:

$$\forall t, \tau \quad \mathcal{V}^t \subseteq \mathcal{W}^\tau .$$

Suppose for some t and τ , $\mathcal{V}^t \not\subseteq \mathcal{W}^\tau$. If $\mathcal{V}^t$ and $\mathcal{W}^\tau$ were disjoint, then $\mathcal{W}^\tau \cup \mathcal{V}^t$ would be an exploited set, which contradicts the maximality of $\mathcal{W}^\tau$. Hence

$$\mathcal{V}^t \cap \mathcal{W}^\tau = Q \neq \phi .$$

Since Q is a proper subset of $\mathcal{V}^t$ it is not an exploited set by the minimality of $\mathcal{V}^t$. But now Lemma 5.4 applies, and asserts that $\mathcal{V}^t \cup \mathcal{W}^\tau$

is an exploited set, which contradicts the maximality of $\mathcal{W}^{\tau}$.

Q.E.D.

Corollary 5.7. $\mathcal{V} \cap \mathcal{C} = \phi$.

Proof. Since $\mathcal{V} \subseteq \mathcal{W}$, $\mathcal{V} \cap \mathcal{W}' = \phi$. But $\mathcal{W}' = \mathcal{C}$.

Q.E.D.

This corollary contributes more to our story that $\mathcal{V}$ and $\mathcal{C}$ are canonical exploited and exploiting sets. No agent can be both vulnerable and culpable.

Definition 5.3. An agent j is strongly neutral if he is neither vulnerable nor culpable. The set of strongly neutral agents is called $\mathcal{S}$: $\mathcal{S} = N\setminus(\mathcal{V} \cup \mathcal{C})$.

We now ask under what conditions we have a duality theorem, that is, when is "min max $\leqq$ max min" as well?

Theorem 5.8. (Duality Theorem).

(a) $\mathcal{W}\setminus\mathcal{V} = \mathcal{S}$ and $\mathcal{D}\setminus\mathcal{C} = \mathcal{S}$

(b) $\mathcal{W} = \mathcal{V}$ and $\mathcal{D} = \mathcal{C}$ if and only if $\mathcal{S} = \phi$.

Proof. From Corollary 5.7, we can write the set of all agents as a pairwise disjoint union of three sets:

$$N = \mathcal{C} \cup \mathcal{V} \cup \mathcal{S} .$$

By de Morgan's laws, $\mathcal{C}' = \mathcal{W}$, and hence $\mathcal{W} = \mathcal{V} \cup \mathcal{S}$. Likewise $\mathcal{V}' = \mathcal{D}$, and so $\mathcal{D} = \mathcal{C} \cup \mathcal{S}$. This proves part (a). Part (b) follows.

Q.E.D.

Hence the "duality theorem max min = min max" holds precisely when all agents are either vulnerable or culpable. In this case, there is no ambiguity as to which of the sets $\mathcal{V}$ or $\mathcal{W}$ should be considered the canonical exploited set, as the two sets are identical, and the same for $\mathcal{C}$ and $\mathcal{D}$. If, however, there are some strongly neutral agents at the reproducible solution, then we may wish to identify the canonical exploited set as $\mathcal{V}$ and the canonical exploiting set as $\mathcal{C}$.

We summarize this classification of society by:

Theorem 5.9.

(A) If there is any exploited set or exploiting set of agents, then $\mathcal{V} \neq \phi \neq \mathcal{C}$.

(B) $\mathcal{V} \cup \mathcal{C} \cup \mathcal{S} = N$ is a pairwise disjoint representation of society.

(C) $\mathcal{V}$ is an exploited set and $\mathcal{C}$ is an exploited set.

(D) $\mathcal{V}$ consists of all agents poorer than some wealth $\underline{W}$; $\mathcal{C}$ consists of all agents wealthier than $\overline{W}$; $\mathcal{S}$ consists of all agents with wealth in the interval $(\underline{W}, \overline{W})$.

Proof. (A) is a simple derivation: if there is an exploited set, then there is an exploiting set, its complement. Some subset of the first set is a minimal exploited set and some subset of the complement is a minimal exploiting set.

(B) and (C) have been proved above.

Proof of (D): Let $\underline{W}$ be the wealth of the wealthiest agent, μ , in $\mathcal{V}$ and suppose the claim is false; there is an agent $\nu \notin \mathcal{V}$ such that $W^{\nu} < \underline{W}$. μ is a member of some minimal exploited set $\mathcal{V}^t$. Form the set

$$Q = V^t \backslash \{\mu\} \cup \{\nu\} .$$

A fortiori Q is an exploited set, since

$$W(V^t) < W_q , \quad \text{where} \quad q = \#Q = \#V^t ,$$

and

$$W(Q) < W(V^t) .$$

Either Q is a minimal exploited set or it contains one. If the former, then $\nu \in V$. If not, Q contains a proper minimal exploited set $\overline{Q}$: but $\overline{Q}$ must contain ν , since all other proper subsets of Q are proper subsets of V^t , and hence are not exploited sets. Hence in any case ν is vulnerable.

Q.E.D.

We have shown there is some inherent ambiguity in the exploitation status of members of society in the sense that there may be strongly neutral agents. It is natural to ask if there is any sense in which the set S can be ignored? For instance, we might be justified in ignoring the set S if we could show that as economies become large, the set of strongly neutral agents become small. Unfortunately (or not), this is not so. The following example, due to Roger Howe, shows that we can construct an economy with N members, $N-2$ of whom are strongly neutral.

Clearly, an economy for these purposes is specified by the function $\phi(W)$. (Any increasing, concave, piece-wise linear function with $\phi(0) = 0$ and $\phi(W_N) = N$ can be the ϕ-function associated with some reproducible solution of some economy.) The example is best comprehended by referring to Figure 3. We have an economy with seven agents. We given them wealths:

$$W^1 = W_1$$

$$W^2 = W_2 - W_1$$

$$W^3 = W_3 - W_2$$

$$\vdots$$

$$W^7 = W_7 - W_6 \ .$$

Now make the first agent a little poorer, and the last a little richer:

$$W^1 = W_1 - \varepsilon$$

$$W^\nu = W_\nu - W_{\nu-1} \quad \text{for} \quad \nu = 2, \ldots, 6$$

$$W^7 = W_7 - W_6 + \varepsilon \ .$$

Observe all exploited sets contain agent 1. In fact the exploited sets are simply: $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, ..., $\{1,2,3,4,5,6\}$, since the wealth of the j^{th} set in that sequence is $W_j - \varepsilon$. It is easily verified that any other set is too rich to be exploited. Consequently there is only one minimal exploited set: $\{1\}$. Then 1 is the only vulnerable agent. A similar argument shows 7 is the only culpable agent. Thus 2, ..., 6 are strongly neutral. Clearly this construction works for an economy with N agents. Furthermore, there is an open neighborhood of this distribution in wealth space which preserves the result of $N-2$ strongly neutral agents.

There is a second way in which one might hope strongly neutral could be ignored. If the addition (or subtraction) of strongly neutral agents to (or from) a set of agents never changed the exploitation status of that set, then we would be justified in treating the strongly neutral agents as irrelevant from the point of view of exploitation. In fact, such is the case for the example just given. Nevertheless, the contention is false, in general. To see this consider an economy with four agents, where wealths are assigned thus:

$$W^1 = W_1 - \varepsilon$$

$$W^2 = (W_2 - W_1) + \varepsilon$$

$$W^3 = (W_3 - W_2) - \varepsilon$$

$$W^4 = (W_4 - W_3) + \varepsilon \ .$$

The exploited sets are $\{1\}$ and $\{1,2,3\}$. Hence $V = \{1\}$. The exploiting sets are $\{2,3,4\}$ and $\{4\}$. Hence $C = \{4\}$. Thus $S = \{2,3\}$. But $\{1,2\}$ is a neutral set. Thus the augmentation of the exploited set $\{1\}$ with the strongly neutral agent 2 changes the exploitation status of that set.

Consequently, it does not appear there is any sense in which the strongly neutral agents can be ignored.

What the analysis of this section has shown is that in general the gray area can be whittled down a good deal. We can first of all discard the exploiting and exploited agents who will, in general, be in the gray area because of the transformation problem; we can then distinguish vulnerable agents and culpable agents in the gray area--agents who are individually neutral, but belong to minimal exploited or exploiting sets. After taking them out, the grayest of the gray area remains, the strongly neutral agents.

Note the key result in developing the theory of vulnerable sets is Lemma 5.4, which depended on the concavity of the function $\phi(W)$.

6. Conclusion

We have studied a generalization of the classical Marxian definition of exploitation, as part of a program for proposing a general definition of exploitation. The definition of exploitation in a capitalist economy proposed here has two noteworthy features: (1) it is independent of the subsistence concept, and thus more applicable to modern capitalist economies than Marx's concept is, and yet (2) it is also independent of subjective preferences which the agents might hold. The second feature is important because the exploitation status of an agent should be an objective feature of the economy, I believe, and not turn on what the individual subjectively chooses to do.

Classes arise in the capitalist economy, and an agent's class membership is determined at an equilibrium by his wealth. There are various ways of classifying agents into classes: the most useful way classifies them according to whether they must hire labor, must sell labor, or do neither, to optimize. Class membership of agents comes in an unambiguous order with respect to their wealths. This is a theorem which was encountered before in Roemer (1979).

The notions of exploitation and class give two classifications of society, and the key theorem which allows us to compare the two taxonomies is the Class-Exploitation Correspondence Principle, which remains true in the capitalist economy. The CECP states that the entire group of agents who are neither exploited nor exploiters (the gray area) is contained in the neutral class of petty bourgeoisie, $(+,0,0)$. That is, every compulsory labor-hirer is an exploiter and every compulsory labor-seller is exploited. This theorem can be taken to vindicate the definition of exploitation which is provided. It can also be taken to provide a new resolution to the transformation problem. The gray area exists precisely because of the transformation problem; the CECP says despite its existence, one can always infer that capitalists are exploiters and workers are exploited.

Our criterion for exploitation is sufficiently strong that we can construct economies in which exploiters exist, but there are no exploited individuals. This motivates the analysis of a two-level hierarchy of the exploitation phenomenon: vulnerability and culpability are the next level in the hierarchy above individual exploitation. By introducing these concepts, we can always provide a decomposition of society into vulnerable, culpable, and strongly neutral agents. It was shown with several examples that the strongly neutral agents cannot be ignored: they are the grayest of the gray area, and their non-vanishing significance is the most precise statement offered concerning the strength of the exploitation criterion. Even at the hierarchical level of vulnerability/culpability, there is no way of preventing significant numbers of agents from slipping through the exploitation net.

Some readers may ask whether the results of this paper depend on the absence of a credit market. The answer is no. In Roemer (1979), it was shown that a credit market and a labor market provide precisely the same functions with regard to exploitation and class in the subsistence economy, and the same is true here. That is, if we introduce a credit market instead of a labor market in the capitalist economy, a class structure emerges where labor sellers are transformed into borrowers and labor hirers become lenders. With regard to class and exploitation, the economy with labor market is isomorphic to the economy with credit market. If both markets operate in the capitalist economy, then one of them is redundant at any reproducible solution. (That is, the solution can always be arranged using only one of the markets.)

It is appropriate in this conclusion to comment on the results in the Appendix. There it is shown that the CECP remains valid when the production set is any closed convex cone, with a no-free-lunch assumption on production, providing an appropriate generalization of the concept of socially necessary labor time is taken for the conical environment. (It is well-known that in general an unambiguous vector Λ of individual-commodity labor times does not exist in the general model of production.) The obvious significance of this result is that the CECP becomes a robust principal, and therefore achieves status, I think, as a deep and important aspect of Marxian economics, as it relates two key concepts of Marxism in a formally precise way, in a general setting.

There is, as well, a more subtle insight gained from the proof of the generalized CECP. If we adopt the generalized definition of socially necessary labor time (SNLT) of Morishima (1976) and Roemer (1980a), in which SNLT is independent of the market, then the generalized CECP is false. If, however, we adopt another plausible generalization of SNLT to the conical production environment in which SNLT depends on the market then the CECP is true. *A priori*, we have no reason to formally prefer either of these two definitions of SNLT over the other. (In the Leontief model, the two become the same.) If we take a teleological position that the CECP should be a basic premise of Marxism, then we are directed to prefer the general definition of SNLT which depends upon the market. In my opinion, the dependence of the CECP on this definition does justify this choice. Now the implication of this reasoning for the primacy of labor values in Marxian theory is powerful: for it says, in general, labor values cannot be determined independently of prices! For, as is shown in the Appendix, it is the particular price equilibrium which defines the subset of production activities to which our scrutiny must be limited in defining SNLT. I think this is the most powerful evidence thusfar demonstrated that (labor) value is a concept which does not exist logically prior to the market. This constitutes a critical blow to the "fundamentalist" position within Marxism that labor values exist "prior to" and independently of prices.

Finally, a critical comment should be offered which suggests a comparison between exploitation in the capitalist economy with exploitation in the pre-capitalist, subsistence economy. In the pre-capitalist economy, a definite utility function involving labor could be associated with the producer's optimization problem: to minimize labor expended subject to survival and reproduction. Thus, a producer's exploitation status had some meaning in terms of his "welfare." In the capitalist economy, this is not the case. Utility for the agents here is linked only to profits. In what sense is "exploitation" a relevant concept for the economy? This question suggests that a more general and appealing definition of exploitation would be useful, perhaps implying the definition of exploitation under capitalism provided here. This approach to exploitation will be pursued in the next paper in this study.

APPENDIX

The CECP for General Conical Production Sets

A1. Introduction

In the body of the paper, we have assumed a Leontief technology (A,L) . It is important to ask whether the Class Exploitation Correspondence Principle (CECP) holds for more general technologies. In this Appendix, we show the CECP is valid when the production set is a cone. In particular, this includes the case of a von Neumann technology, with joint products and many alternative processes. Consequently, the CECP is robust when extended to environments including fixed capital, differential turnover times, continuous substitution possibilities among inputs, and all the aspects of production which can be modeled with general conical production sets.

A conceptual question will arise immediately: How should exploitation be defined in a technology whose individual labor values are not well-defined, as in the Leontief model? Morishima (1976) has provided an answer for the von Neumann model and Roemer (1980a) has generalized Morishima's definition to production sets which are cones (and, more generally, convex sets). Yet according to that definition, the CECP is in general false! However, another equally plausible--perhaps more plausible--definition of exploitation for a conical production environment is proposed below, which renders the CECP true. Indeed, a by-product of the argument is that it points to the generalization of the concept of labor value which should be considered the "correct" one. This has implications which will be discussed below.

A2. The Model of Production and Accumulation

Every producer faces a production set P whose elements have the form $\alpha = \langle -\alpha_0, -\underline{\alpha}, \overline{\alpha} \rangle$ where $\alpha_0 \in R_+$, $\underline{\alpha} \in R^n_+$, $\overline{\alpha} \in R^n_+$. α_0 is the labor input, $\underline{\alpha}$ is the input of non-labor commodities and $\overline{\alpha}$ is the output vector. We denote the net output vector arising from α as $\hat{\alpha} \equiv \overline{\alpha} - \underline{\alpha}$. We assume P is a closed convex cone containing the origin in R^{2n+1}. Another assumption is needed later.

Marxian value theory where the production set is a general convex set is developed in Roemer (1980a). Some reference will be made to results in that paper.

We specify the program for the individual accumulator, which is the generalization of program (2.1) of this paper. The price vector is $(p,1)$. (We may avoid the case where the wage is zero without problem: so we normalize setting the wage equal to unity.) The producer ν chooses a vector $\alpha = \langle -\alpha_0, -\underline{\alpha}, \overline{\alpha} \rangle$ which he operates himself, a vector $\beta = \langle -\beta_0, -\underline{\beta}, \overline{\beta} \rangle$ which he hires others to operate using his capital, and a scalar of labor time sold, γ_0. (We omit the superscripts ν from these three vectors for clarity.) His program is:

$$(\text{A.1}) \quad \begin{cases} \max\ p(\overline{\alpha} - \underline{\alpha}) + p(\overline{\beta} - \underline{\beta}) - \beta_0 + \gamma_0 \\ \text{s.t. } p\underline{\alpha} + p\underline{\beta} \leqq p \cdot \omega^\nu \equiv W^\nu \\ \qquad \gamma_0^\nu + \alpha_0^\nu \leqq 1 \\ \qquad \alpha, \beta \in P; \quad \gamma_0 \geqq 0 \end{cases}$$

That is, ν maximizes revenues subject to his capital constraint and labor constraint.

The definition of a reproducible solution is the straightforward analog to Definition 2.1, and shall not be repeated here.

A3. Class Division of Society

At a given price vector $(p,1)$, an agent optimizes by choosing an action $(\alpha; \beta; \gamma_0)$, which can be displayed as a vector in R^{4n+3}. As before, we consider an agent as enjoying a certain class position, according as he possesses an optimal solution with α, β, or γ_0 zero or non-zero vectors. For example, if he possesses a solution of the form $(\alpha; 0; \gamma_0)$ with $\alpha \neq 0 \neq \gamma_0$, then we say he belongs to the class $(+;0;+)$.

The class characterization of society is developed in the same manner as in Lemma 2.2 and Section 3C.

Lemma A.1. Every agent has an optimal solution to his program (A.1) of form $(0; \beta; \gamma_0)$, if he has any optimal solution at prices $(p,1)$.

(Note: This is the analog of Lemma 2.2.)

Proof. Let $(\alpha, \beta, \gamma_0)$ be a solution to (A.1) for our agent. Define a new solution by:

$$\alpha' = 0$$

$$\beta' = \beta + \alpha$$

$$\gamma' = \gamma_0 + \alpha_0 .$$

It is easily checked that $(\alpha'; \beta'; \gamma')$ is a feasible solution to (A.1), generating the same revenue as $(\alpha, \beta, \gamma_0)$, and so it is optimal.

Q.E.D.

Definition A.1. Let $\alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P$, and let prices $(p,1)$ rule. Then the profit rate at α is defined as

$$\pi(\alpha,p) = \frac{p\hat{\alpha} - \alpha_0}{p\underline{\alpha}} .$$

It is clear that if $(p,1)$ supports a reproducible solution, the profit rates of points in P must be bounded--or else producers would not be able to choose optimal solutions. By closedness of P, a maximal profit rate is attained in P, which we call π^{max}.

Corollary A.2. At an optimal solution to (A.1), agent ν's revenue is $\pi^{max}W^\nu + 1$.

Proof. By Lemma A.1, ν has an optimal solution of the form $(0, \beta, \gamma_0)$. Since the wage is positive, $\gamma_0 = 1$. (The agent must work all day to maximize revenue.) The revenues from hiring others, received by ν, are maximally $\pi^{max}W^\nu$, by definition of $\pi(\alpha,p)$. Hence his revenue at $(0, \beta, \gamma_0)$ must be $\pi^{max}W^\nu + 1$.

Q.E.D.

Theorem A.3. The class division of society is precisely as in Theorem 3.2.

Proof. Same as Theorem 3.2; employ Lemma A.1.

Q.E.D.

We proceed to characterize class membership in terms of agent's wealth.

<u>Theorem A.4</u>. <u>In this model</u>, <u>Theorem 3.4 holds</u>. <u>In the present notation</u>, <u>the important statement is</u>:

$$\nu \in (+;0;0) \iff \min_{\beta\in\overline{P}}\left(\frac{p\underline{\beta}}{\beta_0}\right) \leqq W^\nu \leqq \max_{\beta\in\overline{P}}\left(\frac{p\underline{\beta}}{\beta_0}\right)$$

<u>where</u> $\overline{P} = \{\beta \in P \mid \pi(\beta,p) = \pi^{max}\}$.

<u>Note</u> the qualification of Theorem 3.4. Here, the minimization and maximization of the "capital-wages ratio" is taken over only those activities in P which achieve the maximal profit rate. In the Leontief model, <u>all</u> activities achieve the maximal profit rate at a reproducible solution, and so this distinction was not evident.

<u>Proof of Theorem A.4</u>. We wish to characterize the wealths associated with class membership $(+;0;0)$. Note, according to the construction of Lemma A.1, that ν possesses an optimal solution of form $(+;0;0)$ if and only if he has a solution of form $(0;\beta;1)$ with $\beta_0 = 1$. Compute $(p\underline{\beta}/\beta_0)_{min}$ and $(p\underline{\beta}/\beta_0)_{max}$ where β ranges over $\overline{P}$. (It does no harm if the max is infinite.) If

$$\text{(1)} \qquad \min_{\beta\in\overline{P}}\left(\frac{p\underline{\beta}}{\beta_0}\right) \leqq W^\nu \leqq \max_{\beta\in\overline{P}}\left(\frac{p\underline{\beta}}{\beta_0}\right)$$

then by convexity of $\overline{P}$, there is a process in $\overline{P}$, call it β , such that $p\underline{\beta}/\beta_0 = W^\nu$. The ray through β belongs to $\overline{P}$, since P and $\overline{P}$ are cones. Hence ν can sink all his wealth into some process along the β ray, and he will thereby hire one unit of labor. (That is, ν chooses a multiple of β , call it $t\beta$, in which he sinks all his wealth:

$$tp\underline{\beta} = W^{\nu} .$$

It follows that $t\beta_0 = 1$.)

But with choice of activity $t\beta$, ν will make revenues $\pi^{max}W^{\nu} + 1$ at action $(0; t ; 1)$, since $t\beta \in \overline{P}$. It follows by the second sentence of this proof that $\nu \in (+;0;0)$. The converse--that $\nu \in (+;0;0)$ implies W^{ν} lies in the appropriate interval--follows in like manner by consideration of Lemma A.1.

Q.E.D.

A4. Exploitation

As mentioned in the Introduction A1, a definition of exploitation for a convex technology is provided in Roemer (1980a). We provide here a slightly different definition, and discuss the distinction below.

To define exploitation in terms of surplus value, we must first define the labor value of a vector of commodities. Our definition of labor value shall depend, in part, on the particular equilibrium the economy is in. Let $c \in R^n_+$ be a vector of produced commodities. Let

$$\phi(c;p) = \{\alpha \in \overline{P} | \hat{\alpha} \geqq c\} .$$

$\phi(c;p)$ is those profit-rate-maximizing actions which produce, as net output vectors, at least c . The dependence of ϕ on p is through the profit-maximizing set $\overline{P}$.

Definition A.2. The labor value of c is:

$$l.v.(c) \equiv \min\{\alpha_0 | (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in \phi(c;p)\} .$$

The labor value of c , or "socially necessary labor time to produce c ," is the smallest amount of direct labor which must be employed to produce at least c as net output, using only profit-maximizing activities. By convexity and closedness of $\overline{P}$, and the assumption that $0 \in P$, it can be shown that $\text{l.v.}(c)$ is well-defined if $\phi(c;p)$ is not empty (see Roemer (1980a) for the argument).

What is special about this definition is that in computing socially necessary labor time, we restrict our search for labor-cheap ways of producing c to processes which are earning the maximal profit rate--that is, to processes which capitalists would in fact operate at the equilibrium prices $(p,1)$. It is a consequence of Lemma A.2 that no point in P achieving a profit rate less than π^{max} will be chosen by any agent, either to operate on his own or to operate with hired labor. Thus, in fact, the only process we will observe at the reproducible solution are processes in $\overline{P}$, and we define these observable processes as the universe for the computation of socially necessary labor time. In the definition of labor value in Morishima (1976) and Roemer (1980a), no such restriction was put on the universe: rather, $\phi(c)$ was defined over all processes $\alpha \in P$ such that $\hat{\alpha} \geqq c$.

This distinction involves the conception of what constitutes "socially necessary" labor. The labor value of a bundle will in general be greater defined over, the restricted set $\phi(c;p)$ than over $\phi(c)$. Moreover, in general labor value will change with different reproducible price vectors. There are two formal justifications for choosing the restricted set $\phi(c;p)$ as an appropriate universe for defining labor value. First, both the definition using $\phi(c;p)$ and the one using $\phi(c)$ are generalizations of labor value in the Leontief model, because all processes enjoy the maximal profit rate at a reproducible solution in the indecomposable Leontief model. Secondly, the "Fundamental Marxian Theorem" continues to hold using the $\phi(c;p)$ definition. (This can be checked by reading through the proof of Theorem 3.1 in Roemer (1980a), substituting the new definition. An assumption of independence of production, like Assumption 7 of that paper, must be made concerning the restricted production set $\overline{P}$.) There therefore appears to be no formal reason to prefer the $\phi(c)$ generalization of labor value for the cone model, to the $\phi(c;p)$ generalization.

In fact, one might argue that the $\phi(c;p)$ definition is preferable. For "socially necessary" labor time embodied in a bundle c , under Definition A.2, is the amount of labor time actually embodied in c , if c were to be produced by capitalists choosing processes guided by today's prices. The socially necessary labor time of the $\phi(c)$ definition cannot be realized at today's prices, for in general, capitalists would have to operate some inferior processes to achieve it; for this reason, the latter definition is somewhat idealistic or counterfactual.

In addition, under the $\phi(c;p)$ definition, labor values are themselves a function of prices, as the set $\overline{P}$ is a function of prices. Thus, exploitation is itself mediated through the market. This adds further fuel to recent interpretations of Marxian value theory, in which labor values are not viewed as existing at a level "more basic" than prices. (See Steedman (1978) and Roemer (1980c, Chapter 7).)

The reader should perhaps be reminded that the CECP shall be true under the definition of labor values via $\phi(c;p)$, but false under the $\phi(c)$ definition. Thus, if we take the CECP as a basic Marxian principle, then we are directed to adopt the $\phi(c;p)$ definition as the preferred or "correct" one. This is perhaps the strongest argument that has been made for the dependence of labor-value phenomena on the market.

It must be remarked that under Definition A.2, l.v.(c) is not in general defined for all bundles c : it is defined only for those bundles which are producible at prices (p,1) , that is, for which $\phi(c;p)$ is non-empty. Conceptually, this is appropriate: goods which are not commodities, that is, which capitalists will not produce, have no labor value. Labor value is not the labor time embodied in the production of goods in general, but in the production of goods for the market. This also fits nicely with classical Marxian ideas, that value is a market phenomenon, not a technological phenomenon.*

We next assume:

Assumption of Large Economy (A.L.E.): Same as A.L.E. of Section 3B.

We now characterize the wealth levels of exploiters and exploited. Agent ν has revenue $(\pi^{\max}W^{\nu} + 1)$. To ascertain whether he is an exploiter, we shall assign him a bundle of goods, under a feasible assignment, which minimizes the labor value he can buy with his revenue. If that labor value exceeds one, then he is an exploiter. Consider the set of all commodity bundles of labor value unity:

*A very similar, perhaps equivalent, definition of labor value in the context of the von Neumann model has been independently developed by Flaschel (1979). He examines the square matrices of input and output coefficients defined by the processes in the von Neumann technology which are maximally profitable at a particular equilibrium, and defines additive labor values by inverting this system.

$$\Theta = \{c \in R^n_+ | \ \text{l.v.}(c) = 1 \}.$$

Let

$$\Theta^* = \{\alpha \in \overline{P} | (\exists\, c \in \Theta)(\hat{\alpha} = c \text{ and } \alpha_o = 1 \}.$$

An additional assumption we now require on P is:

<u>Assumption NFL</u> (no free lunch). Any semi-positive net output vector from activities in P requires labor input:

$$\alpha \in P , \ \ \hat{\alpha} \geq 0 \Longrightarrow \alpha_0 > 0 .$$

Under the NFL assumption and the convexity of $\overline{P}$, a standard argument shows that Θ is bounded; it is also closed, and hence compact. Thus $p \cdot c$ achieves a maximum and minimum value on Θ.

Let $\tilde{c} \in \Theta$ be such that $p \cdot \tilde{c}$ is maximal over Θ:

$$p \cdot \tilde{c} \geqq p \cdot c \quad \forall c \in \Theta$$

and $\underset{\sim}{c} \in \Theta$ such that:

$$p \cdot \underset{\sim}{c} \leqq p \cdot c \quad \forall c \in \Theta .$$

Since Θ is the set of commodity bundles of labor value unity, $\tilde{c}$ is that bundle which maximizes price per unit labor value. Purchasing $\tilde{c}$ is the "least efficient" way of commanding labor value through the goods one buys. Hence, if our agent ν spends all his revenue on a bundle proportional to $\tilde{c}$, and still commands more than one unit of labor, he is an exploiter.

Thus, ν is an exploiter if and only if:

$$(2) \qquad \pi^{max} W^{\nu} + 1 > p \cdot \tilde{c}$$

This says he can purchase more than one unit of bundle $\tilde{c}$, and hence command more than one unit of labor, even when he commands embodied labor in the least efficient way. Note the use of A.L.E.: we need to know there is enough of every good produced that ν can spend his money on goods in the required proportions.

In like manner, ν is exploited if and only if:

$$(3) \qquad \pi^{max} W^{\nu} + 1 < p \cdot \underset{\sim}{c} .$$

Thus:

$$(2') \qquad \nu \text{ is an exploiter} \iff W^{\nu} > \frac{p\tilde{c} - 1}{\pi^{max}}$$

$$(3') \qquad \nu \text{ is exploited} \iff W^{\nu} < \frac{p\underset{\sim}{c} - 1}{\pi^{max}} .$$

A5. <u>The CECP</u>

<u>Theorem A.5 (CECP)</u>. <u>Every agent who must hire labor power to optimize is an exploiter, and every agent who must sell labor power to optimize is exploited</u>.

<u>Note</u>: This is the precise analog to Theorem 4.1. We state it with less jargon for brevity.

Proof. The proof follows the same format as Theorem 4.1. We must simply compare the wealth characterization of exploitation (inequalities (2') and (3')), with the wealth characterization of class membership (from inequality (1)).

We show the first statement of the theorem, the proof of the second is identical. To verify the first statement, it is necessary and sufficient to show that

$$W^{\nu} > \max_{\beta \varepsilon \overline{P}} \left(\frac{p\underline{\beta}}{\beta_0} \right) \Longrightarrow W^{\nu} > \frac{p\tilde{c} - 1}{\pi^{max}}$$

or

$$\text{(4)} \qquad \max_{\beta \varepsilon \overline{P}} \left(\frac{p\underline{\beta}}{\beta_0} \right) \geqq \frac{p\tilde{c} - 1}{\pi^{max}} .$$

Since $\overline{P}$ is a cone, we may normalize the left hand side of (4) by taking that action $\beta \varepsilon \overline{P}$ for which $\beta_0 = 1$ (that is, the maximal ratio $p\underline{\beta}/\beta_0$ is surely achieved at such an action, among others). But then (4) reads:

$$\text{(5)} \qquad \max_{\substack{\beta \varepsilon \overline{P} \\ \beta_0 = 1}} p\underline{\beta} \geqq \frac{p\tilde{c} - 1}{\pi^{max}} .$$

Now

$$\text{(6)} \qquad \Gamma \equiv \{\beta \varepsilon \overline{P} | \beta_0 = 1\} \supseteq \Theta^* .$$

In general Θ^* is a proper subset of Γ since the elements of Γ need not produce non-negative net output vectors, while the elements of Θ

do. From (6) we have:

$$\max_{\beta \varepsilon \Gamma} p\underline{\beta} \geqq \max_{\beta \varepsilon \Theta^*} p\underline{\beta} \ . \tag{7}$$

We proceed to show:

$$\max_{\beta \varepsilon \Theta^*} p\underline{\beta} \geqq \frac{p\tilde{c} - 1}{\pi^{max}} \ , \tag{8}$$

which, by (7) and (5), will prove the theorem. Notice $\pi^{max} p\underline{\beta} \equiv p\hat{\beta} - 1$ for any $\beta \varepsilon \overline{P}$, since both sides of this identity express the profits achieved at action $\beta \varepsilon \overline{P}$. Hence (8) is equivalent to:

$$\max_{\beta \varepsilon \Theta^*}(p\hat{\beta} - 1) \geqq (p\tilde{c} - 1) \ . \tag{9}$$

But (9) is true, since $\tilde{c} \varepsilon \Theta$.

Q.E.D.

It is worthwhile to alert the reader to the criticalness of taking the "right" definition of labor value, as the key juncture in the proof may have slipped by him/her. If we had adopted the other definition of labor value, then the choice of $\tilde{c}$ would have been from a larger set then Θ, namely from:

$$\hat{\Theta} = \{c \ \varepsilon \ R^n_+ | (\exists \alpha \ \varepsilon \ P)(\hat{\alpha} = c \text{ and } \alpha_0 = 1)\} \ .$$

(Compare this to Θ, where α ranges only over $\overline{P}$.) Then the bundle, which we might call $\hat{\tilde{c}}$, maximizing price per unit labor value would be:

$$\hat{\tilde{c}} \ \varepsilon \ \hat{\Theta} \text{ such that } p\hat{\tilde{c}} \geqq pc \quad \forall c \ \varepsilon \ \hat{\Theta} \ .$$

Then, when we arrive at inequality (9) in the proof, we could not assert that $\hat{\tilde{c}} \in \Theta$, and (9) would not necessarily hold. (I am confident, in fact, a counterexample can be constructed to (9) under the $\hat{\Theta}$ specification.) In words, the situation would be this. Consider the profits which can be gotten by operating activities requiring one unit of direct labor. (Profits are $p\hat{\beta} - 1$.) Consider the maximal profits from among those processes with highest profit rate on non-labor capital (i.e., π^{max} processes). Are those profits necessarily greater than profits from operating the process which maximize price per unit labor value? In general, the answer is no. The process which produces $\hat{\tilde{c}}$ as net output may entail a lower profit rate on capital but higher profits if the capital outlay for operating that process is sufficiently large.

Technical Note: The reader might ask whether the CECP is true in a model where wages are paid in advance, not at the end of the period, as in this model. If wages are paid in advance then the appropriate profit rate would be defined on capital inputs plus labor inputs, as follows:

$$\pi(p,\beta) = \frac{p\hat{\beta} - \beta_0}{p\underline{\beta} + \beta_0} .$$

I have worked through the model with the alternate specification of when wage payments take place, and where the profit rate is defined on wage plus non-wage capital, and the result is the same. The truth of the CECP depends on adopting the correct generalization of labor value when the technology is more general than Leontief.

A6. Conclusion

For a discussion of significance of this result, the reader is referred to the concluding section of the paper.

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