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SOME RELATIONSHIPS BETWEEN DISCRETE AND CONTINUOUS MODELS OF AN URBAN ECONOMY

Richard P. McLean and Thomas J. Muench

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by

Richard P. McLean[1] and Thomas J. Muench[2]

In a classic article by Beckmann and Koopmans [10] (B-K), it is shown that in a class of location-assignment shipping models, there does not necessarily exist a set of decentralizing prices. This is due to the indivisibility or discreteness of agents, i.e., one and only one agent can be assigned to a location. In a more recent book by Mills [12], it is shown that in a simple urban setting with a continuum of agents, there are decentralizing prices. However, Mills allowed fractional allocations of agents.

In order to compare these results,it is necessary to set up a Mills type B-K problem. It can be easily shown that this discretized Mills problem has the B-K property of no decentralizing prices in Mills' "integrated" city case (where there are actual fractional allocations). Now we can think of a continuum city as a limit of a seuqence of cities with a finite number of locations and agents. Thus the above two results indicate a kind of discontinuity in the limit for location problems.

The purpose of this paper is to reconcile the B-K and Mills results (for the integrated city case) and thus eliminate this discontinuity.

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[1]Cowles Foundation for Research in Economics, Box 2125, Yale Station, New Haven, CT 06520.

[2]Department of Economics, SUNY at Stony Brook, Stony Brook, NY 11794.

In particular we show that, first, the Mills optimal allocation is the limit of a sequence of (approximately optimal) allocations in a sequence of B-K problems and, second, that the Mills decentralizing prices are approximately decentralizing for a sequence of B-K problems when a natural cutting plane is added to each. The dual variable associated with this extra constraint can be interpreted as a subsidy to commuting (of workers). This is not needed in the limit because fractional allocations mean (formally) that no commuting takes place. It does, however, appear in the limit in the relationships between the regular dual prices.

Thus Mills solution hides an important component of the decentralizing mechanism (in any finite realization of the Mills city) while the B-K model usually has no such decentralizing prices. Our reconciliation brings out the full implications of both the B-K and Mills results.

Although we have handled only a special case of the class of problems covered by B-K, we believe our paper indicates an extremely useful method of getting economic insight from B-K problems in general. (i) Look at sequences of B-K problems which have a "limit." (ii) Find natural cutting planes which approximately decentralize the allocation problem.

Though we will use the duality theory of linear programming, we will not emply any duality results for integer programming. For an interesting discussion of the economic aspects of duality in integer programming, see the paper by Gomory and Baumol [5]. Also, we will not discuss any specific solution procedures here. For a treatment of algorithms for the quadratic assignment problem, the interested reader is referred to Lawler [11], Hillier and Connors [9], and Graves and Whinston [6]. For a general bibliography on location problems, see [4].

We now proceed to discussion of the mathematical models of Mills and Koopmans and Beckmann.

The Koopmans-Beckmann Problem

The quadratic assignment problem may be described as follows. It is necessary to assign n facilities to n locations so as to minimize the sum of location costs and total interplant transportation costs. Define the cost coefficients and parameters as follows. Let c_{ij} be the cost of shipping one unit of a commodity from location i to location j. Let b_{st} be the known requirement of plant t for the output of plant s. Finally, let a_{ij} be the cost of locating plant i on site j. Define the location variable x_{ij} as

$$x_{ij} = \begin{cases} 1, \text{ if plant } i \text{ locates at site } j \\ 0, \text{ if plant } i \text{ does not locate at site } j \text{ .} \end{cases}$$

In addition, each site can accommodate only one plant and each plant must be allocated to a site. We then have the following quadratic assignment problem:

$$\text{minimize} \quad \sum_i \sum_j a_{ij} x_{ij} + \sum_i \sum_j \sum_s \sum_t c_{ij} b_{st} x_{si} x_{tj}$$

$$\text{subject to} \quad \sum_i x_{ij} = 1$$

$$\sum_j x_{ij} = 1$$

$$x_{ij} \in \{0,1\} \text{ .}$$

Koopmans and Beckmann reformulate the problem as an equivalent linear mixed integer program by letting $b_{st} x_{si} x_{tj} = y_{stij}$ and adding materials balance constraints at each location. The modified problem is presented as:

$$(P_0) \quad \begin{aligned} \text{minimize} \quad & \sum_i \sum_j a_{ij} x_{ij} + \sum_i \sum_j \sum_s \sum_t c_{ijst} y_{ijst} \\ \text{subject to} \quad & \sum_i x_{ij} = 1 \\ & \sum_j x_{ij} = 1 \\ & x_{si} b_{st} + \sum_{j \neq i} y_{stji} = x_{ti} b_{st} + \sum_{j \neq i} y_{stij} \\ & x_{ij} \in \{0,1\} , \quad y_{ijst} \geqq 0 . \end{aligned}$$

With the introduction of the binary location variables x_{ij} , we are confronted by the most intractable of non-convexities, that of indivisibility. Indeed, Koopmans and Beckmann provide a very simple example in which no decentralizing prices exist that will lead each firm to maximize profit at the location it chooses and leave no firm with an incentive to move to another location. In the jargon of linear programming, we cannot find values for the dual variables to problem (P_0) (considered as a linear programming problem) which, together with the optimal integer assignment, will satisfy the complementary slackness conditions. This is certainly not unusual for integer programming problems which are usually solved by complicated combinatorial procedures precisely because their non-convex feasible sets preclude the use of more standard mathematical programming techniques.

Our purpose in this paper is to investigate certain equilibrium problems of a location theoretic nature of which the Koopmans-Beckmann problem is an example. More recently, Mills discusses a model of location in the context of an urban economy and we shall briefly describe some of his results.

The Mills Urban Model

A central delivery point is specified and the surrounding land is available for the production of two commodities, an output good $(i = 1)$ which must be shipped to the central delivery point and a housing good $(i = 2)$. Define

$L_i(u)$ = land allocated to the production of good i at a distance u units from the central delivery point;

$X_i(u)$ = production of commodity i at a distance u units from the center.

A simple fixed coefficients technology is assumed with land as the only input to housing and production of the output good scaled so that each unit of output good requires one worker. If X_0 is the known exogenous requirement of the output good, then we may summarize these assumptions as follows:

$$X_1(u) = a_1 L_1(u) \ , \quad 0 \leqq u \leqq \bar{u}$$

$$X_2(u) = a_2 L_2(u) \ , \quad 0 \leqq u \leqq \bar{u}$$

$$L_1(u) + L_2(u) = 2\pi u \ , \quad 0 \leqq u \leqq \bar{u}$$

$$\int_0^{\bar{u}} X_1(u)du = \int_0^{\bar{u}} X_2(u)du = X_0 \ .$$

Here, $\bar{u}$ is the radius of the city and feasibility requires that $\bar{u} = (2(a_1 + a_2)X_0/a_1 a_2 \pi)^{1/2}$. Note that land and production are considered perfectly divisible so that X_1 , X_2 , L_1 , L_2 are density functions. It is desired that we allocate firms and workers so as to minimize the sum of commuting costs and shipping costs. An elementary argument indicates that no outward commuting will take place, i.e., the "number" of firms at a distance u units from the center is at least as great as the number of workers residing u units from the center, for all distances $0 \leq u \leq \bar{u}$. If t_1 is the cost of shipping a unit of output good one unit distance and t_2 is the cost of round trip of commuting, then the model is

$$\text{minimize} \quad t_1\int_0^{\bar{u}} uX_1(u)du + t_2\int_0^{\bar{u}}\int_0^{u}[X_1(u') - X_2(u')]du'du$$

$$\text{subject to} \quad X_1(u) = a_1L_1(u) \ , \quad 0 \leq u \leq \bar{u}$$

$$X_2(u) = a_2L_2(u) \ , \quad 0 \leq u \leq \bar{u}$$

$$L_1(u) + L_2(u) = 2\pi u \ , \quad 0 \leq u \leq \bar{u}$$

$$\int_0^{\bar{u}} X_1(u)du = \int_0^{\bar{u}} X_2(u)du = X_0$$

$$\int_0^{u} [X_1(u') - X_2(u')]du' \geq 0 \ , \quad 0 \leq u \leq \bar{u} \ .$$

If commuting costs are sufficiently high relative to shipping costs, the so called integrated solution obtains. Specifically, if $(a_1/[a_1 + a_2])t_1 \leq t_2$, then the optimal solution is

$$L_1(u) = \frac{2\pi a_2}{a_1 + a_2}u \ , \quad 0 \leq u \leq \bar{u}$$

$$L_2(u) = \frac{2\pi a_1}{a_1 + a_2}u \ , \quad 0 \leq u \leq \bar{u} \ .$$

Mills shows that the decentralizing prices that could be used to achieve this allocation in a competitive market are given by:

$$p_1^0 = t_1\bar{u} + W \qquad \text{(price of output good at delivery point)}$$

$$R(u) = \left(\frac{a_1 a_2}{a_1 + a_2}\right) t_1(\bar{u}-u) \qquad \text{(price of land)}$$

$$p_1(u) = W + t_1(\bar{u}-u) \qquad \text{(price of output good)}$$

$$p_2(u) = \left(\frac{a_1}{a_1 + a_2}\right) t_1(\bar{u}-u) \qquad \text{(price of housing)}$$

$$W(u) = W + \left(\frac{a_1}{a_1 + a_2}\right) t_1(\bar{u}-u) \qquad \text{(wage rate)}$$

If $(a_1/[a_1+a_2])t_1 \geqq t_2$, then a "segregated" solution results. This case will not be analyzed in detail in this paper, so we refer the interested reader to Mills [11], pp. 87-92 for a discussion of this case.

A Simple Linear City

Before proceeding directly to the discretized model, we will discuss a one-dimensional version of a Mills type Koopmans-Beckmann model which lends insight to the solution of the more complex model. Consider the following simple location problem in which n workers and n firms desire to arrange themselves on a one-dimensional plot of land with 2n possible locations so as to minimize combined shipping and commuting costs. We may visualize the land as a line segment divided into 2n cells with 0 designated as the city center (see Figure 1):

FIGURE 1

Let t_1 be the cost of shipping one unit of output good and t_2 be the round trip commuting cost. Each firm requires one unit of land and one worker for each unit of output good produced. Each unit of housing requires one unit of land and each house contains one worker. Also, the number of firms k units from the center must be at least as great as the number of houses at the same distance from the center. (Note that these are Mills conditions.) Define

x_i = number of workers residing at site i

y_i = number of firms at site i

so the model may be stated as:

$$\text{minimize} \quad t_1 \sum_{i=1}^{2n} i y_i + t_2 \sum_{k=1}^{2n} \sum_{i=1}^{k} (y_i - x_i)$$

$$\text{subject to} \quad x_i + y_i = 1$$

$$\sum_{i=1}^{2n} y_i = n$$

$$\sum_{i=1}^{k} (y_i - x_i) \geqq 0 \;, \quad k = 1,\ 2n-1$$

$$\sum_{i=1}^{2n} (y_i - x_i) = 0$$

$$x_i,\ y_i \in \{0,1\} \;.$$

We modify the problem by defining $z_k = \sum_{i=1}^{k} (y_i - x_i)$ as the excess of jobs over available workers. The variable z_k may be interpreted as the number of commuters who must pass from site k+1 to site k on their way to work. Note that we are not concerned with determining which commuters but only how many. Making this substitution we have:

$$\text{minimize} \quad t_1 \sum_{i=1}^{2n} i y_i + t_2 \sum_{k=1}^{2n} z_k$$

$$(P_1) \qquad \begin{aligned} \text{subject to} \quad & x_i + y_i = 1 && (R_i) \\ & \sum_{i=1}^{2n} y_i = n && (p_0) \\ & y_1 - x_1 = z_1 && (W_1) \\ & y_k - x_k + z_{k-1} = z_k \,, \quad k = 2, \ldots, 2n-1 && (W_k) \ (\alpha) \\ & y_{2n} - x_{2n} + z_{2n-1} = 0 && (W_{2n}) \\ & x_i,\ y_i \in \{0,1\} \,, \quad z_i \geqq 0 \end{aligned}$$

where R_i , p_0 , W_i represent the associated dual variables. If we replace the integer restrictions with simple non-negativity constraints we have a linear program whose associated dual is:

$$\text{maximize} \quad np_0 - \sum_{i=1}^{2n} R_i$$

$$(D_1) \qquad \begin{aligned} \text{subject to} \quad & p_0 - R_i - W_i \leqq it_1 && (y_i) \ (1) \\ & W_i - R_i \leqq 0 && (x_i) \ (2) \\ & W_i - W_{i+1} \leqq t_2 && (z_i) \ (3) \end{aligned}$$

where y_i , x_i , z_i are the associated dual variables. We interpret p_0 as the price of the output good at the city center, R_i as the price of land at site i and W_i as the wage rate at site i . The dual constraints (and associated complementary slackness conditions) have the following interpretations. Equation (1) states that potential profits must be non-positive and that in order for a firm to locate at site i , profit at site i must be zero (profit = $p_0 - R_i - W_i - it_1$). Equation (2) says simply that wages net of commuting costs are less than rent at site i and if someone resides at site i , then they are equal. Finally, equation (3) states that as one moves closer to the city center, the difference in wages cannot exceed the cost of commuting. If one or more workers actually commutes from site $k+1$ to site k , the difference in wages at the two sites must exactly offset the cost of commuting.

The Integrated Case

What additional restrictions, if any, must be put on (P_1) so that the optimal solution to the unrestricted linear program satisfies the binary integer constraints? First, let us try an analog of Mills' integrated solution, i.e., we want $x_i = 1$ for all even i , $y_i = 1$ for all odd i , and $z_k = 1$ if k odd, $z_k = 0$ if k even. In order for this solution to be supported by decentralizing prices (i.e., in order for the dual variables of (D_1) to have values that complement the proposed values of x , y , z), the following complementary slackness conditions must be satisfied:

$$p_0 \begin{Bmatrix} = \\ \leq \end{Bmatrix} R_i + W_i + it_1 \ , \quad \text{if} \quad i \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$$

$$W_i \begin{Bmatrix} \leqq \\ = \end{Bmatrix} R_i \ , \quad \text{if} \quad i \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}$$

$$W_i - W_{i+1} \begin{Bmatrix} = \\ \leqq \end{Bmatrix} , \quad \text{if} \quad i \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} .$$

We now propose the following solution to this system of linear inequalities:

$$R_i = \frac{(2n-i)t_1}{2} \ , \quad \forall i \tag{4}$$

$$W_i = \frac{(2n-i)t_1}{2} \ , \quad \forall i \tag{5}$$

$$P_0 = 2nt_1 \ . \tag{6}$$

This solution is valid if and only if $t_2 = t_1/2$. Now it is easy to verify that $z_i = 0$, $x_i = 1/2$, $y_i = 1/2$, $\forall i$, is the optimal solution to the unrestricted linear program and (4)-(6) constitute the optimal solution to the dual linear program if and only if $t_2 \geqq t_1/2$. (This is Mills integrated solution.) Furthermore, we can show that no integer solution exists for the unrestricted linear program if $t_2 > t_1/2$, and the general procedures which exist for solving such integer programming problems yield no insight to the economic problem at hand. Indivisibility has created a problem for the competitive market mechanism analogous to those resulting from externalities.

To attack the problem, we make a simple observation. In the model (P_1) , each facility and each worker must be located on different sites so that in any optimal assignment of firms and workers to sites, someone must commute. This means that the z_i cannot all be zero, something which is true in the optimal solution to the relaxed linear program. In the simple model (P_1), an alternating arrangement of firms and workers requires that $z_i = 0$ if i is even and $z_i = 1$ if i is odd, so that total distance travelled by commuters must be at least n. Let us add the constraint (β) $\sum_{k=1}^{2n-1} z_k = n$ (with dual variable s) to (P_1). The equations of the dual remain the same except for the last which becomes:

$$W_i - W_{i+1} + s \leqq t_2 . \qquad (7)$$

With this new constraint, equations (4), (5), (6) will solve the dual problem if $t_2 \geqq t_1/2$. In this case, $W_i - W_{i+1} = t_1/2$ and the final constraint states that

$$t_1/2 + s = t_2 , \quad \text{if } i \text{ is odd}$$

$$t_1/2 + s \leqq t_2 , \quad \text{if } i \text{ is even.}$$

Hence, $s = t_2 - t_1/2$ is to be interpreted as a subsidy to commuters equal to the difference between the cost of commuting from site $k+1$ to site k and the increment to wages between site $k+1$ and site k. The externality caused by lumpy land and people is attacked by means of a subsidy to commuting.

The Segregated Case

Before discussing the complex model, we shall give the results which correspond to Mills' segregated solution which holds when $t_2 \leqq t_1/2$, i.e., when shipping costs are sufficiently high relative to commuting costs. In this case, the solution to model (P_1) is

$$y_i = \begin{cases} 1, & 1 \leq i \leq n \\ 0, & n+1 \leq i \leq 2n \end{cases}$$

$$x_i = \begin{cases} 1, & n+1 \leq i \leq 2n \\ 0, & 0 \leq i \leq n \end{cases}$$

$$z_i = \begin{cases} 1, & i = 2n-1 \\ 2, & i = 2n-2 \\ \vdots & \\ n, & i = n \\ n-1, & i = n-1 \\ \vdots & \\ 1, & i = 1 \end{cases}$$

It is easy to verify that the following values for the dual variables, obtained by mimicking the Mills continuum solution, are the complements of the constraints given above:

$$p_0 = nt_1 + 2nt_2$$

$$R_i = \begin{cases} nt_2 + (t_1 - t_2)(n-i) , & i = 1, \ldots, n \\ t_2(2n-i) , & i = n+1, \ldots, 2n \end{cases}$$

$$W_i = t_2(2n-i) , \quad i = 1, \ldots, 2n .$$

By laboring through the complementary slackness conditions, it can also be shown that these conditions hold at the optimal solution if and only if $t_2 \leq t_1/2$, which is precisely Mills' condition for the segregated solution. Note that we do not need the extra condition $\sum_{k=1}^{2n-1} z_k = n$

because the firms will "bunch up" as close to the delivery point as possible, thus filling up the first n spaces and leaving no spaces for workers to occupy. Thus, the condition is ineffective and $s = 0$.

The City on a Plain

Formulation

We shall now present a more realistic discretization of Mills' model. Instead of density functions to describe the distribution of agents over a disk, consider the following grid of available sites of land (Figure 2):

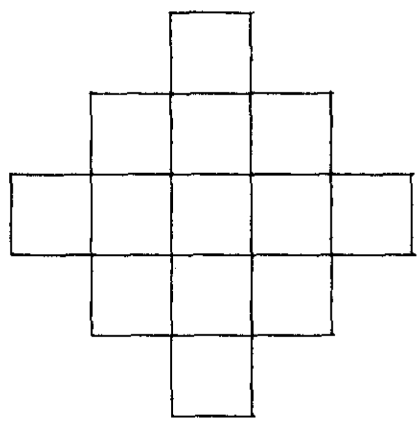

FIGURE 2

Each square (i.e., each site) is labelled by two coordinates. We use a metric which gives the distance between square (i,j) and square (s,t) as

$$d_{(st)}^{(ij)} = |i-s| + |j-t| \ .$$

All squares at a distance u units from the center form a ring, each of whose labels satisfies $|i| + |j| = u$. Let $I(u) = \{(i,j) \mid |i| + |j| = u\}$

be the set of indices of the squares on a ring u units from the center. Define the following variables:

y^1_{ij} = portion of site (i,j) devoted to production of the output good;

y^2_{ij} = portion of site (i,j) devoted to the housing good;

x_{ij} = amount of output good produced at (i,j) ;

r_{ij} = number of workers who live at (i,j) ;

$y_{(ij)(st)}$ = number of people who live at (i,j) but work at site (s,t) .

For simplicity, suppose that each house can accommodate no more than one worker and each factory can produce no more than one unit of output. The discretized model may be written as follows:

$$\text{minimize} \quad t_1 \sum_{(i,j)} d^{(ij)}_{(00)} x_{ij} + t_2 \sum_{(i,j)} \sum_{(s,t)} d^{(ij)}_{(st)} y_{(ij)(st)}$$

(P_2)

$$\text{subject to} \quad y^1_{ij} + y^2_{ij} = 1 \qquad R_{ij} \quad (8)$$

$$\sum_{(ij)} x_{ij} = x_0 \qquad p^1_0 \quad (9)$$

$$x_{ij} = y^1_{ij} \qquad p^1_{ij} \quad (10)$$

$$r_{ij} = y^2_{ij} \qquad p^2_{ij} \quad (11)$$

$$\sum_{(s,t)} y_{(st)(ij)} - \sum_{(s,t)} y_{(ij)(st)} = y^1_{ij} - r_{ij} \qquad W_{ij} \quad (12)$$

$$\sum_{(i,j)} \sum_{(s,t)} d^{(ij)}_{(st)} y_{(ij)(st)} \geqq m_0 \qquad S \quad (13)$$

$$y^1_{ij},\ y^2_{ij},\ x_{ij},\ r_{ij},\ y_{(ij)(st)} \in \{0,1\}$$

where

t_1 = cost per unit distance of shipping one unit of output good to the center;

t_2 = cost per unit distance of one round trip of commuting;

x_0 = exogenously determined requirement for the output good;

m_0 = minimum amount of commuting that <u>must</u> be undertaken;

n = the number of pairs of rings.

The dual variables are listed with their respective equations. Equation (8) says that the entirety of site (i,j) is used for housing and production. Equation (9) indicates that the total production over all sites must equal the exogenously determined demand for output. Equations (10) and (11) are "production functions" for output and workers. Specifically, (1) says that production at site (i,j) must equal productive capacity at that site and (11) says that the number of residents at (i,j) cannot exceed the capacity of the housing available at (i,j). Recall that, in the interest of simplicity, we have scaled the problem so that one unit of productive capacity results in one unit of output and each unit of housing can accommodate no more than one resident. Equation (12) is a generalization of equation (α) in the simple model of the previous section. It says that the supply of labor at site (i,j) must equal the demand for labor there. A remark is in order. In Mills' continuum model, no outward commuting is possible due to the constraint that says $\int_0^u [x_1(u^1) - x_2(u^1)]du^1] \geqq 0$. In the discrete analog, we do not prove this at the outset but rather leave that as a result of optimization. Finally, equation (13) is a cutting plane constraint, analogous to equation (β) in the simple model, which requires that the total distance commuted exceed some known minimum.

This constraint requires some justification. In the discrete problem, the fact that firms and workers must occupy different squares means that some commuting must take place. The minimum total distance that workers must cover in commuting to their jobs is denoted m_0 . The configuration of agents that results in $\sum_{(i,j)} \sum_{(s,t)} d^{(ij)}_{(st)} y_{(ij)(st)} = m_0$ may not be the optimal solution to P_2 , but this constraint does not exclude any feasible integer solutions for an integer program with only equations (8) to (12) as constraints. As in the simple model, the dual variable associated with this cutting plane is interpreted as a subsidy to commuting and in the limit results that will be described, it is this subsidy that accounts for the discrepancy in changes in wages between locations and travel costs between locations.

Problem (P_2) , without equation (13) is a Koopmans-Beckmann problem in a slightly modified form. If we let $x_{ij} = y^1_{ij}$ and $r_{ij} = y^2_{ij}$ then P_2 may be restated as:

$$\text{minimize} \quad t_1 \sum_{(i,j)} d^{(ij)}_{(00)} y^1_{ij} + t_2 \sum_{(i,j)} \sum_{(s,t)} d^{(ij)}_{(st)} y_{(ij)(st)}$$

$$\text{subject to} \quad y^1_{ij} + y^2_{ij} = 1$$

$$\sum_{(i,j)} y^1_{ij} = x_0$$

$$\sum_{(s,t)} y_{(st)(ij)} - \sum_{(s,t)} y_{(ij)(st)} = y^1_{ij} - y^2_{ij}$$

$$y^1_{ij},\ y^2_{ij},\ y_{(ij)(st)} \in \{0,1\} .$$

Note that by summing the last constraint over all (i,j) , we obtain

$\sum_{(i,j)} y^2_{(i,j)} = x_0$. Such a model corresponds to a quadratic assignment problem with two types of plants, "firms" and "workers." Each firm requires one unit of output of the workers and each worker requires zero units of output of each firm. We shall show that this Koopmans-Beckmann model can be approximately decentralized by Mills' optimal continuum prices.

Let $(\overline{P}_2)$ denote problem (P_2) without the integer restriction, i.e. $(\overline{P}_2)$ is a linear program. To obtain qualitative results, we must examine the dual to $(\overline{P}_2)$. This may be written:

$$\text{maximize} \quad x_0 p_0^1 + m_0 s - \sum_{(i,j)} R_{ij}$$

(D_2)

$$\begin{array}{lll}\text{subject to} & p^1_{ij} - R_{ij} - W_{ij} \leqq 0 & (y^1_{ij}) \quad (14) \\ & p^2_{ij} - R_{ij} \leqq 0 & (y^2_{ij}) \quad (15) \\ & W_{ij} - p^2_{ij} \leqq 0 & (r_{ij}) \quad (16) \\ & p^1_0 - p^1_{ij} \leqq t_1 d^{(ij)}_{(00)} & (x_{ij}) \quad (17) \\ & W_{st} - W_{ij} + s d^{(ij)}_{(st)} \leqq t_2 d^{(ij)}_{(st)} & (y_{(ij)(st)}) \quad (18) \\ & p^1_{ij},\ p^2_{ij},\ R_{ij},\ W_{ij},\ p^1_0 & \text{unrestricted in sign} \\ & s \geqq 0 & \end{array}$$

where y^1_{ij} , y^2_{ij} , r_{ij} , x_{ij} , $y_{(ij)(st)}$ are the natural primal variables associated with the appropriate dual constraints. The dual variables have the same interpretation as their counterparts in Mills' model except they hold by site rather than within rings. Equation (14) states that profit to producers of the output good must be non-positive at each site and zero if a plant is actually built. Equation (15) says that profit to producers of housing services must be non-positive and zero if housing services are provided there. Equation (16) means that housing prices must not be less than the wage rate at each site and they must be equal if someone lives at that site. From equation (17), we see that unless the price p^1_{ij} of output good at site (i,j) equals the "net" price $p^1_0 - t_1 d^{(ij)}_{(00)}$ of the good at the center, none of it will be shipped to the center. Finally, equation (18) states that in order for someone to live at (i,j) and work at (s,t), the difference in wages plus the subsidy to commuting must exactly equal the cost of commuting.

In this two dimensional framework, it is not immediately clear what kind of solution is the discrete analog of the Mills' continuous integrated solution. We could alternate firms and workers around each ring and have each worker commute to the firm on his right. Thus, no interring commuting would take place as in Mills' model. However, we do not make that assumption here. Instead, we consider alternate rings of firms and workers in which "most" workers commute one space inward on their jobs. The word "most" is important here because we cannot evenly distribute firms and workers in such a way that each worker need only commute one space to his job. We assume an even number of rings and we propose the following solution.

$$x_{ij} = y^1_{ij} = \begin{cases} 1 , & (i,j) \in I(u) \cup I^*(u+1) , \quad u \text{ odd} \\ 0 , & \text{otherwise} \end{cases} \tag{19}$$

$$r_{ij} = y^2_{ij} = \begin{cases} 1 , & (i,j) \in I(u) \setminus I^*(u) , \quad u \text{ even} \\ 0 , & \text{otherwise} \end{cases} \tag{20}$$

$$y_{(ij)(st)} = \begin{cases} 1\ , & (i,j) \in I(u)\backslash I^*(u)\ ,\quad u \text{ even and} \\ & (s,t) = (i,j+1) \text{ if } j < 0 \text{ or} \\ & (s,t) = (i,j-1) \text{ if } j > 0\ ; \\ 1\ , & (s,t) \in I^*(u)\ ,\quad u \text{ even and} \\ & (i,j) = (s-1,1) \text{ if } s > 0 \text{ and} \\ & (i,j) = (s+1,-1) \text{ if } s < 0 \\ 0\ , & \text{otherwise} \end{cases} \tag{21}$$

where $I^*(u) = \{(i,j) \mid |i| = u,\ j = 0\}$. Now consider the following proposal for a dual solution:

$$p_0^1 = 2nt_1 \tag{22}$$

$$W_{ij} = t_1/2(2n-u)\ ,\quad (i,j) \in I(u) \tag{23}$$

$$R_{ij} = t_1/2(2n-u)\ ,\quad (i,j) \in I(u) \tag{24}$$

$$p_{ij}^1 = t_1(2n-u)\ ,\quad (i,j) \in I(u) \tag{25}$$

$$p_{ij}^2 = t_1/2(2n-u)\ ,\quad (i,j) \in I(u) \tag{26}$$

$$S = t_2 - t_1/2\ . \tag{27}$$

The expressions are obtained by assuming that constraints (14)-(17) of the dual (D_1) hold with equality everywhere and solving the resulting system of simultaneous linear equations. This solution is not exact, however; not all dual constraints are satisfied. To see this, consider $W_{ij} - W_{st} = sd_{(st)}^{(ij)} - t_2 d_{(st)}^{(ij)}$. If (i,j) and (s,t) are in the same ring, then our dual solution implies that $s = t_2$. However, if (i,j) and (s,t) lie in adjacent rings, then $s = t_2 - t_1/2$. We show in the next section that it is an approximate solution, the advantages of which are two-fold. First, the resulting dual solution is an exact analog to the solution obtained by Mills for the continuous case. Second, the approximation is "good" in the sense that it works almost everywhere. Only at the corners (the elements of $I^*(u)$) do we encounter a difficulty in the sense that complementary slackness conditions are not satisfied.

One goal of this paper is to link the discrete location problem to the continuous. The approach we shall use is to study the limiting behavior of an urban economy as the number of rings increases but the size of the city remains constant. Note the contrast to limit theorems for the core and various treatments of Edgeworth's conjecture. In these problems, the number of agents grows large but each agent's consumption and demand sets are subsets of a commodity space of fixed dimension. In the problem at hand, the dimension of the problem changes at each stage. In addition, certain right hand side constants in the constraints of (P_2) are growing. Specifically, x_0 and m_0 are dependent on n. In order to obtain our results we require that $x_0 = 4n(2n+1)$ and $m_0 = 2n(2n+1)$. Clearly, any feasible integer solution must satisfy $\sum_{(i,j)} \sum_{(s,t)} d_{(st)}^{(ij)} y_{(ij)(st)} \geqq 2n(2n+1)$ where $2n(2n+1)$ is the number of workers in the economy. This is true since each worker must commute at least one space to work.

Approximate Equilibria for Large Cities

Before proceeding to the limit theorems we must discuss some preliminary ideas from mathematical programming. Let $S = \{x \in R_+^n | A_1 x = b_1, A_2 x \geqq b_2, x_i \in \{0,1\}\}$. Let $x_0 \in S$ and let $(\lambda_1, \lambda_2) \in R^{m_1} \times R_+^{m_2}$ be given. Note that $A_1x_0 - b_1 = 0$. Suppose that

$$c^Tx^* = \min_{x\varepsilon S} c^Tx \ .$$

Then the following sequence of inequalities must hold:

$$\begin{aligned}
c^Tx_0 \geqq c^Tx^* &\geqq c^Tx^* + \lambda_1^T(b_1 - A_1x^*) + \lambda_2^T(b_2 - A_2x^*) \\
&\geqq \min_{x\varepsilon S}[c^Tx + \lambda_1^T(b_1 - A_1x) + \lambda_2^T(b_2 - A_2x)] \\
&\geqq \min_{x_i\varepsilon\{0,1\}} [c^Tx + \lambda_1^T(b_1 - A_1x) + \lambda_2^T(b_2 - A_2x)] \\
&= \min_{x_i\varepsilon\{0,1\}} [(c - A_1^T\lambda_1 - A_2^T\lambda_2)^T x + y_1^Tb_1 + y_2^Tb_2 \\
&\qquad - (c - A_1^T\lambda_1 - A_2^T\lambda_2)^T x_0 + (c - A_1^T\lambda_1 - A_2^T\lambda_2)^T x_0] \\
&= \min_{\substack{x \\ x_i\varepsilon\{0,1\}}} [(c - A_1^T\lambda_1 - A_2^T\lambda_2)^T (x - x_0)] + c^Tx_0 + y_2^T(b_2 - A_2x_0) \\
&= c^Tx_0 + \Delta_1 + \Delta_2
\end{aligned}$$

where

$$\Delta_1 = \min_{\substack{x \\ x_1\varepsilon\{0,1\}}} (c - A_1^T\lambda_1 - A_2^T\lambda_2)^T (x - x_0)$$

and

$$\Delta_2 = y_2^T(b_2 - A_2x_0) \ .$$

The motivation for this analysis is as follows. We are confronted in our problem by a set of constraints of the form $A_1x = b_1$ (equations (8)-(12)) and $A_2x \geqq b_2$ (equation (13)). In the model presented here, $A_2x \geqq b_2$ consists only of a minimum travelling constraint. More generally, $A_2x \geqq b_2$ might include a system of cutting planes designed to close the duality gap that occurs in the presence of integer constraints. We have hypothesized a feasible solution x_0 and a set of dual variables (λ_1, λ_2) with certain desirable properties. Specifically (λ_1, λ_2) is a non-negative vector where λ_1 corresponds to the prices and λ_2 is the subsidy. However, $(x_0, \lambda_1, \lambda_2)$ does not constitute an optimal solution to the pair of dual linear programs $(\overline{P}_2, D_2)$, i.e., x_0 is not a solution to the location problem considered as a linear program. Unfortunately, x_0 is not the same as x^* and we may be unable to find x^* without recourse to certain integer programming techniques that do not have standard economic interpretations. Ideally, we would like to show that our problem falls in that class of linear programming models whose constraint sets have integer extreme points (e.g. transportation problems). Attempts in this direction have not been successful except in the one dimensional problem where the introduction of a cutting plane seems to make the problem work. In higher dimensions, however, this one extra constraint does not suffice due to certain irregularities (which we do not see how to remedy) in the structure of our urban economy. Hence a different approach is required and we shall employ a limiting argument. Consider a sequence of discrete cities of fixed area but increasingly many rings (hence increasingly many sites). As each plot of land becomes smaller, the discrete city will begin to resemble its perfectly divisible counterpart. Each city is characterized by $2n$ rings and ring u contains $4u$ plots of land for $u = 1, \ldots, 2n$. Such a city will be called a city of size n . The costs of traversing a unit distance will remain t_1 and t_2 . The metric, however, expressed as a function of the indice (i,j) becomes $d_{ij}^{st} = \frac{1}{2n}(|i-s| + |j-t|)$. In all sums over agents, the agents are weighted by $\pi \cdot [4n(2n+1)]^{-1} \equiv \pi \times$ (number of agents) .

We remark here that this way of connecting the continuum model with the discrete model is a standard approach. Another (equivalent) approach would be to allow the city to grow larger and keep the distance between rings constant. Any aggregate quantities obtained in the first (fixed area) approach would thus be average (per capita) values of the analogous quantities obtained in the second (fixed interring distance) approach. In particular, convergence in the former approach would be per capita convergence in the latter.

For each n , a vector λ_n of prices is proposed. Specifically, we propose the values of the dual variables given by equations (22) to (27) with u and $\bar{u}$ replaced by $u/2n$ and $\bar{u}/2n$ respectively. These particular values imply that the number of non-zero components in the vector $c - A_1^T\lambda_1 - A_2^T\lambda_2$ is small relative to the total number of components. In addition, we choose x_0 as given by equations (19) to (21) so that $A_2x_0 - b_2 = 0$. Thus, the complementary slackness conditions are satisfied at almost every location so that $(x_0, \lambda_1, \lambda_2)$ is "almost" optimal. It is necessary to formulate the notion of almost in the sense that it is used here.

The previous system of inequalities shows $c^Tx_0 \geqq c^Tx^* \geqq \min_{x \in S} F(x, \lambda_1, \lambda_2)$ where

$$F(x, \lambda_1, \lambda_2) = c^Tx + \lambda_1^T(b_1 - A_1x) + \lambda_2^T(b_2 - A_2x) .$$

Using $\min_{x \in S} F(x, \lambda_1, \lambda_2)$ as a lower bound is undesirable because it is no easier to compute than $c^T x^*$. However, $\min_{\substack{x \\ x_1 \in \{0,1\}}} F(x, \lambda_1, \lambda_2)$ is very easy to calculate and $\min_{\substack{x \\ x_1 \in \{0,1\}}} F(x, \lambda_1, \lambda_2)$ has been decomposed so that:

$$\min_{\substack{x \\ x_1 \in \{0,1\}}} F(x, \lambda_1, \lambda_2) = c^T x_0 + \Delta_1 + \Delta_2$$

where $\Delta_1 = \min_{\substack{x \\ x_1 \in \{0,1\}}} F(x, \lambda_1, \lambda_2) - F(x_0, \lambda_1, \lambda_2)$ and $\Delta_2 = \lambda_2^T (b_2 - A_2 x_0)$.

We would like to show that $|c^T x^* - c^T x_0|$ approaches zero as the number of sites becomes large. This is equivalent to showing that $\Delta_1 + \Delta_2$ approaches zero. Such an idea has been used to great advantage in general equilibrium theory where it is useful to examine the relationship between large, finite economies and an economy with a continuum of agents. These approaches generally entail defining a measure of non-competitiveness and then proceeding to show that this measure is bounded by a number that is independent of the number of agents in the economy. As the economy grows large, the average value of non-competitiveness goes to zero. In most limit type results, this measure is the number of agents for whom the continuum price system works poorly. We shall prove that the number of locations for which the continuum price system works poorly has measure zero in the limit economy. In addition, we show that the proposed allocation-price pair is (average) optimal in the limit. Both of these results depend on the idea that non-competitiveness is linear in n but the number of agents is quadratic in n . (The number of sites is $4n(2n+1)$ with

$2n(2n+1)$ each of firms and workers.) That is, for fixed n, we must prove that $\Delta_1 + \Delta_2$ is linear in n. For a treatment of pure exchange economies with indivisible commodities, see Dierker [3].

For notational convenience, let $x \in R_+^{16n^2+16n^4}$ and $(\lambda_1, \lambda_2) \in R_+^{(4n^2+1)+1}$ be defined as

$$x = ((y^1_{ij}), (y^2_{ij}), (x_{ij}), (r_{ij}), (y_{(ij)(st)})) ,$$

$$\lambda_1 = ((R_{ij}), (p^1_{ij}), (p^2_{ij}), (W_{ij}), (p^1_0)) ,$$

and

$$\lambda_2 = s .$$

<u>Lemma 1</u>: If $(x, \lambda_1, \lambda_2)$ is the proposed primal-dual solution described by equations (19) through (27), then $\Delta_1 + \Delta_2$ is a linear function of n.

<u>Proof</u>: Is is necessary to compute the difference $\Delta_1 = \min_{\substack{x \\ x_i \varepsilon \{0,1\}}} F(x, \lambda_1, \lambda_2)$ $- F(x_0, \lambda_1, \lambda_2)$. Now, $\Delta_1 = \min_{\substack{x \\ x_i \varepsilon \{0,1\}}} (c - A_1^T\lambda_1 - A_2^T\lambda_2)^T (x - x_0)$ and we have chosen (λ_1, λ_2) so that $c - A_1^T\lambda_1 - A_2^T\lambda_2 \geqq 0$. Hence, Δ_1 must be minimized by making $x_i - x_{0i}$ as small as possible whenever $(c - A_1^T\lambda_1 - A_2^T\lambda_2)_i > 0$ where $(c - A_1^T\lambda_1 - A_2^T\lambda_2)_i$ denotes the i^{th} component of $c - A_1^T\lambda_1 - A_2^T\lambda_2$. This can be accomplished by making $x_i = 0$ whenever $x_{0i} = 1$ and $x_i = 0$ whenever $x_{0i} = 0$. In the latter case $x_i - x_{0i} = 0$ so

$$\Delta_1 = - \sum_{\substack{i \\ x_{0i}=1 \\ c-A_1^T\lambda_1-A_2^T\lambda_2>0}} (c - A_1^T\lambda_1 - A_2^T\lambda_2) \ .$$

In the problem at hand, equations (14) to (17) are equalities so we need only be concerned with equation (18): $W_{ij} - W_{st} - sd_{(st)}^{(ij)} = -t_2 d_{(st)}^{(ij)}$. So the only component of $c - A_1^T\lambda_1 - A_2^T\lambda_2$ that could be positive is

$$\alpha_{(st)}^{(ij)} = t_2 d_{(st)}^{(ij)} + W_{ij} - W_{st} - sd_{(st)}^{(ij)} \ .$$

Now $s = t_2 - t_1/2$ so

$$\alpha_{(st)}^{(ij)} = W_{ij} - W_{st} + t_1/2 \ d_{(st)}^{(ij)}$$

$$= t_1/2\left(2n - d_{(00)}^{(ij)}\right) - t_1/2\left(2n - d_{(00)}^{(st)}\right) + t_1/2 \ d_{(st)}^{(ij)}$$

$$= t_1/2\left(d_{(00)}^{(st)} + d_{(st)}^{(ij)} - d_{(00)}^{(ij)}\right) \ .$$

We are interested in the case when $d_{(00)}^{(st)} + d_{(st)}^{(ij)} - d_{(00)}^{(ij)} > 0$ and $y_{(ij)(st)} = 1$. In our model, any interring commuting occurs between adjacent squares in an inward direction. Thus $d_{(00)}^{(st)} + d_{(st)}^{(ij)} - d_{(00)}^{(ij)} = 0$ and $y_{(ij)(st)} = 1$ in these cases. The remaining commuting occurs at the "corners" of the economy. Specifically, $y_{(ij)(st)} = 1$ for $(s,t) \in I^*(u)$, $(i,j) = (u-1,1)$, $(i,j) = (1-u,-1)$ and u even. In this case, $d_{(00)}^{(st)} + d_{(st)}^{(ij)} - d_{(00)}^{(ij)} = 2$ so that

$$\Delta_1 = - \sum_{\substack{u=1 \\ u \text{ even}}}^{2n} \sum_{\substack{(s,t)\in I^*(u) \\ (i,j)=(u-1,1) \\ (i,j)=(1-u,-1)}} (t_1/2)(2)$$

$$= - \sum_{\substack{u=1 \\ u \text{ even}}}^{2n} (t_1/2)(2)(2)$$

$$= -(2t_1)(n)$$

$$= -2nt_1 \ .$$

Now $A_2x_0 = 2n(2n+1) + 2n$, so $0 \geqq \lambda_2^T(b_2 - A_2x_0) = \Delta_2 \geqq -s2n$, so $|\Delta_2| \leqq 2ns$. Hence $|\Delta_1 + \Delta_2| \leqq 2nt_1 + 2ns$, a linear function of n .

Q.E.D.

<u>Theorem 1</u>: Let $(x_n, \lambda_{1n}, \lambda_{2n})$ be the proposed primal-dual solution for an economy of size n . Then $(x_n, \lambda_{1n}, \lambda_{2n})$ is average limit optimal i.e.

$$\frac{|c_n^T x_n^* - c_n^T x_n|}{\# E_n} \to 0 \quad \text{as} \quad n \to \infty$$

where x_n^* is optimal for the economy of size n .

<u>Proof</u>: The total number of agents in E_n is $\sum_{u=1}^{2n} 4u = 4n(2n+1) = \# E_n$.

Now $0 \geqq c_n^T x_n^* - c_n^T x_n \geqq \Delta_1 + \Delta_2$ so

$$0 \leqq \frac{|c_n^T x_n^* - c_n^T x_n|}{\# E_n} \leqq \frac{|\Delta_1 + \Delta_2|}{\# E_n} = \frac{2t_1 n + 2n(t_2 - t_1/2)}{4n(2n+1)}$$

and the result follows.

This result is not surprising since the number of agents for whom our proposed price system does not work is small relative to the total number of agents in the economy and this ratio goes to zero as the number of agents becomes large.

We present some convergence results. To relate the sequence of discrete problems to the continuous Mills' economy, it is necessary to describe the continuous representation of an economy of size n. Define $T = \{x \in R^2 \mid \|x\|_1 \leq 1\}$ where $\|-\|_1$ is the ℓ_1 norm on R^2. Next, define $T_{ij}(n)$ as follows:

$$T_{ij}(n) = \left\{x \in R^2 \,\middle|\, \frac{i}{2n} - \frac{1}{4n} \leq x_1 < \frac{i}{2n} + \frac{1}{4n}, \frac{j}{2n} - \frac{1}{4n} \leq x_2 \leq \frac{j}{2n} + \frac{1}{4n}\right\}$$

for i and j ranging over $-2n, \ldots, 2n$. Geometrically, $T_{ij}(n)$ is a small square centered at $(i/2n, j/2n)$ with area $1/4n^2$ whose top and right hand side are missing. Now define two functions from T to R as follows:

$$\Theta_n^1(x) = \sum_{(i,j)\in I_n} y_{ij}^1(n)\chi_{T_{ij}(n)}(x)$$

$$\Theta_n^2(x) = \sum_{(i,j)\in I_n} y_{ij}^2(n)\chi_{T_{ij}(n)}(x)$$

where $$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases},$$ the characteristic function of S.

The Θ's define step functions on T and we may define set functions ψ_n^1 and ψ_n^2 as follows: $\psi_n^i(S) = \int_S \Theta_n^i(x)d\mu$ where μ is Lebesgue measure on the measurable space $(T,B(T))$. Here $B(T)$ denotes the σ-algebra of Borel sets of T.

The set functions thus defined are measures (Halmos [7], pp. 97-98) that express the number of firms and homes in a Borel subset S of T. In Mills' continuum problem, the optimal number of firms, workers, etc., is expressed in terms of a density function. Specifically, the number of firms in a disk of radius r is $\int_0^r \pi u du = \pi r^2/2$ if the Euclidean metric is used and $a_1 = a_2 = 1$. More generally, we shall interpret the Mills' integrated continuum solution to be one in which the "number" of firms and the number of workers in any Borel subset S of T are equal. To be precise, the number of firms and workers in S is $\mu(S)/2$ where μ is Lebesgue measure. This means that throughout T, firms and workers are present in equal amounts and they are completely mixed. This corresponds directly to Mills' idea of an integrated solution. We would like to be able to show that our sequence of proposed solutions to the discrete problems converges (in an appropriate sense) to the optimal continuum solution. This requires the notion of weak convergence of measures. A sequence of measures $\{\nu_n\}$ on a metric space T is said to converge weakly to a measure ν on T if and only if $\int f d\nu_n \xrightarrow{n} \int f d\nu$ for every continuous, bounded function $f : T \to R$. Equivalently ν_n converges weakly to ν if and only if $\nu_n(S) \to \nu(S)$ for any subset S of T with $\nu(\partial S) = 0$, where ∂S denotes the boundary of S. (See Parthasarathy [13] or Billingsley [2].) We now state:

Theorem 2: ψ_n converges to $\mu/2$ weakly on T.

The proof of Theorem 2 has been reduced to a series of lemmas. First define $J_n(S)$ as the subset I_n such that $(i,j) \in J_n(S)$ if $T_{ij} \subsetneqq S$. Let $\sum_n(S) = \bigcup_{(i,j)\in J_n(S)} T_{ij}(n)$. We call $\sum_n(S)$ an inner approximation to S.

Lemma 2: Let S be open in T. Then for all points x in S $\exists\, n^* \ni \forall n > n^*$, $\exists\ (i,j) \in I_n \ni x \in \overline{T}_{ij}(n) \subseteqq S$.

Proof: Let $x = (a,b) \in S$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(a,b) = \{(x_1, x_2) \in T \mid \max\{|a - x_1|, |b - x_2|\} < \varepsilon\} \leqq S$ because S is open. That is, $B_\varepsilon(a,b)$ is an open square centered at (a,b) which is completely contained in S. Now $T \subseteqq \bigcup_{(i,j)\in I_n} T_{ij}(n)$, $\forall n$ so there exists a sequence $\{T_{i_n j_n}(n)\}_{n=1}^{\infty}$ such that $(a,b) \in T_{i_n, j_n}(n)$, $\forall n$. Hence, $\max\{|a - i_n/2n|, |b - j_n/2n|\} \leqq 1/4n$. Let $n^* > 1/2\varepsilon$. Choose $n > n^*$ and $(x,y) \in T_{i_n, j_n}(n)$. Then by the triangle inequality, it follows that

$$\begin{aligned}\max\{|a-x|,\ |b-x|\} &\leqq \max\left\{\left|a - \frac{i_n}{2n}\right|,\ \left|b - \frac{j_n}{2n}\right|\right\} + \max\left\{\left|\frac{i_n}{2n} - x\right|,\ \left|\frac{j_n}{2n} - y\right|\right\} \\ &\leqq \frac{1}{4n} + \frac{1}{4n} \\ &= \frac{1}{2n} \\ &\leqq \frac{1}{2n^*} \\ &< \varepsilon .\end{aligned}$$

Therefore, $(a,b) \in \overline{T}_{i_n, j_n}(n) \subseteqq B_\varepsilon(a,b) \subseteqq S$, $\forall n > n^*$.

Q.E.D.

Lemma 3: Let m be a positive integer. Then $\overline{\sum}_m \cap \overline{\sum}_{5m} = \overline{\sum}_m$ where $\overline{A}$ denotes the closure of A.

Proof: Note that if $(i,j) \in I_m$, then $(5i, 5j) \in I_{5m}$ and these coordinates determine the same point in T (that is, $(i/2m, j/2m) = (5i/10m, 5j/10m)$). Let $T_{i_m j_m}(m)$ be an arbitrary element of $\sum_m$ and let $Q_m = \{(i,j) \in I_{5m} | \max\{|5i_m - i|, |5j_m - j|\} \leqq 2\}$. We shall prove the lemma by demonstrating that

$$\bigcup_{(i,j)\in Q_m} \overline{T}_{ij}(5m) = \overline{T}_{i_m j_m}(m) .$$

Suppose $(x,y) \in \bigcup_{(i,j)\in Q_m} \overline{T}_{ij}(5m)$. Then $(x,y) \in \overline{T}_{i^*j^*}(5m)$ for some $(i^*,j^*) \in Q_m$. Applying the triangle inequality we have

$$\begin{aligned} \max\left\{\left|x - \frac{i_m}{2m}\right|, \left|y - \frac{y_m}{2m}\right|\right\} &\leqq \max\left\{\left|x - \frac{i^*}{10m}\right|, \left|y - \frac{j^*}{10m}\right|\right\} \\ &\quad + \max\left\{\left|\frac{i^*}{10m} - \frac{5i_m}{10m}\right|, \left|\frac{j^*}{10m} - \frac{5j}{10}\right|\right\} \\ &\leqq \frac{1}{20m} + \frac{2}{10m} \\ &= \frac{1}{4m} . \end{aligned}$$

So $\qquad (x,y) \in \overline{T}_{i_m j_m}(m)$.

Now suppose $(x,y) \notin \bigcup_{(i,j)\in Q_m} \overline{T}_{ij}(5m)$ and recall that $|a| - |b| \leqq |a-b|$.

If $(i,j) \in I_{5m}$, then

$$\max\left\{\left|\frac{5i_m}{10m}-\frac{i}{10m}\right|,\ \left|\frac{5j_m}{10m}-\frac{j}{10m}\right|\right\} - \max\left\{\left|\frac{i}{10m}-x\right|,\ \left|\frac{j}{10m}-y\right|\right\}$$

$$\leqq \max\left\{\left|\frac{5i_m}{10m}-x\right|,\ \left|\frac{5j_m}{10m}-y\right|\right\} .$$

If $(x,y) \notin \bigcup_{(i,j)\varepsilon Q_m} \overline{T}_{ij}(5m)$, then $(x,y)\ \varepsilon\ \overline{T}_{\hat{i}\hat{j}}(5m)$ means that $\max\{|5i_m - \hat{i}|,\ |5j_m - \hat{j}|\} \geqq 3$ so $\max\{|5i_m/10m - \hat{i}/10m|,\ |5j_m/10m - \hat{j}/10m|\} \geqq 3/10m$. Also $-\max\{|\hat{i}/10m - x|,\ |\hat{j}/10m - y|\} \geqq -1/10m$. Combining these results we have $3/10m - 1/20m = 1/4m \leqq \max\{|5i_m/10m - x|,\ |5j_m/10m - y|\}$. Now suppose $1/4m = \max\{|5i_m/10m - x|,\ |5j_m/10m - y|\}$. Without loss of generality, let $1/4m = \max\{|5i_m/10 - x|,\ |5j_m/10m - y|\} = 5i_m/10m$. If $x > 5i_m/10m$, then $x - 5i_m/10m = 1/4m$. This can be rewritten as $x - [(5i_m + 2)/10m] = 1/20m$. If $x < 5i_m/10m$, $5i_m/10m - x = 1/4m$. This may be rewritten as $[(5i_m - 2)/10m] - x = 1/20m$. Now $|5j_m/10m - y| \leqq 1/4m$ implies $[(5j_m - 2)/10m] - 1/20m \leqq y \leqq [(5j_m + 2)/10m] + 1/20m$ so $x > 5i_m/10m$ means that (x,y) belongs to the boundary of some $T_{ij}(5m)$ with $\max\{|5i_m - i|,\ |5j_m - j|\} = 2$, thus contradicting the hypothesis that $(x,y) \notin \bigcup_{(i,j)\varepsilon Q_m} \overline{T}_{ij}(5m)$. Hence, $1/4m < \max\{|5i_m/10m - x|,\ |5j_m/10m - y|\}$ i.e., $(x,y) \notin \overline{T}_{i_m,j_m}(m)$. Now $\bigcup_{\substack{m \\ T_{i_m j_m}(m)\varepsilon\sum_m}} \left[\bigcup_{(i,j)\varepsilon Q_m} \overline{T}_{ij}(5m)\right] = \overline{\sum}_m \subseteqq S$

so $\bigcup_{\substack{m \\ T_{i_m j_m}\varepsilon\sum_m}} \bigcup_{(i,j)\varepsilon Q_m} \overline{T}_{ij}(5m) \subseteqq \overline{\sum}_{5m}$, so $\overline{\sum}_{5m} \cap \overline{\sum}_m = \overline{\sum}_m$.

Q.E.D.

<u>Lemma 4</u>: Let $S \subseteq T$ be open. Then $\forall \varepsilon > 0 \ \exists m$ and $\sum_m \ni \mu(S \setminus \overline{\sum_m}) < \varepsilon$, where μ denotes Lebesgue measure.

<u>Proof</u>: By Lemma 2, there exists k, an integer such that $\overline{\sum_k} \neq \emptyset$ and $\overline{\sum_k} \subseteq S$. By Lemma 3, there exists a sequence $\{\sum_m\}_{m=1}^{\infty}$ of inner approximations such that $\overline{\sum_m} \subseteq \overline{\sum_{m+1}} \subseteq S$. (For example take $\overline{\sum_m} \equiv \overline{\sum}_{(5^{m-1})\times(k)}$.) Clearly, $\bigcup_{m=1}^{\infty} \overline{\sum_m} \subseteq S$. According to Lemma 2, $x \in S \Rightarrow \exists m^* \ni \forall m > m^*$, $x \in \overline{\sum_m}$. Thus $S \subseteq \bigcup_{m=1}^{\infty} \overline{\sum_m}$. So $S = \bigcup_{m=1}^{\infty} \overline{\sum_m}$. The conclusion of the lemma follows from Halmos, Theorem D, p. 38.

The following result is the main lemma.

<u>Lemma 5</u>: Let m_0 be a positive integer and let $\mathcal{A} \subseteq \{T_{ij}(m_0) \mid (i,j) \in I_{m_0}\}$. Then $\forall \varepsilon > 0$, $\exists \hat{n} \ni \forall n > \hat{n}$, $|\psi_n(\mathcal{A}) - \frac{1}{2}\mu(\mathcal{A})| < \varepsilon$.

<u>Proof</u>: Let $\sum_n$ be an inner approximation to $\mathcal{A}$ and consider <u>one</u> of the elements of $\mathcal{A}$, say A. Now fix i at, say i_0 and define $K_A(i_0) = \{(i,j) \in I_n \mid i = i_0$ and $T_{ij}(n) \subseteq A\}$ i.e. $K_A(i_0)$ is the set of indices associated with a column of small squares of the inner approximation which are completely contained in a (generally) larger square A in $\mathcal{A}$.

If i_0 is odd and (i_0, k) and $(i_0, k+1) \in K_A(i_0)$, then $y^1_{i_0,k} + y^1_{i_0,k+1} = 1$ where y^1_{ij} corresponds to the proposed solution for an economy of size n given by equation (19). (Note that we have suppressed the dependence of y^1_{ij} on n.) Hence $(y^1_{i_0,k} - 1/2) + (y^1_{i_0,k+1} - 1/2) = 0$. Thus if $\#K_A(i_0)$ is even, then $\left|\sum_{(i,j)\in K_A(i_0)} (y^1_{i,j} - 1/2)\right| = 0$ and $\#K_A(i_0)$ odd implies

$$\Big|\sum_{(i,j)\varepsilon K_A(i_0)} (y^1_{ij} - 1/2)\Big| = 1/2 \; .$$

Now suppose i_0 is even. In this case, we encounter the possibility that $y^1_{i_0,k} + y^1_{i_0,k+1} = 2$ (e.g. if $k = 0$) due to the "extra" firms that are located in the even numbered rings. If i_0 is even, then

$$\Big|\sum_{(i,j)\varepsilon K_A(i_0)} (y^1_{i,j} - 1/2) \leqq \Big|\sum_{\substack{(i,j)\varepsilon K_A(i_0)\\ j>0}} (y^1_{i,j} - 1/2)\Big|$$

$$+ \Big|\sum_{\substack{(i,j)\varepsilon K_A(i_0)\\ j<0}} (y^1_{i,j} - 1/2)\Big| + |y^1_{i_0,0} - 1/2| \; .$$

It is easy to verify that the right hand side of this inequality cannot exceed 3/2. We observe that each column of squares in A have no more than $4m_0 + 1$ elements of A. This enables us to put a bound on certain vertical sums in A. Specifically, i_0 even implies

$$\Big|\sum_{\substack{(i,j)\varepsilon J_n(A)\\ i=i_0}} (y^1_{ij} - \tfrac{1}{2})\Big| = \Big|\sum_{A\varepsilon A}\ \sum_{(i,j)\varepsilon K_A(i_0)} (y^1_{i,j} - \tfrac{1}{2})\Big| \leqq \tfrac{3}{2}(4m_0 + 1)$$

and, similarly, i_0 odd implies that

$$\Big|\sum_{\substack{(i,j)\varepsilon J_n(A)\\ i=i_0}} (y^1_{i,j} - 1/2)\Big| = \Big|\sum_{A\varepsilon A}\ \sum_{(i,j)\varepsilon K_A(i_0)} (y^1_{i,j} - 1/2)\Big| \leqq \tfrac{1}{2}(4m_0 + 1) \; .$$

We now combine these results. Choose $\hat{n}$ so that $(4m_0+1)(4\hat{n}+1)/2\hat{n}^2 < \varepsilon/2$ and $\mu(A\setminus\sum_n) < \varepsilon$. (The latter is possible by Lemma 4.) Let $n > \hat{n}$. Then we have the following sequence of inequalities in which the dependence of T_{ij} on n suppressed:

$$
\begin{aligned}
|\psi_n(A) - \tfrac{1}{2}\mu(A)| &= \Big| \sum_{(i,j)\in I_n} y^1_{i,j}\mu(T_{ij}\cap A) - \tfrac{1}{2}\mu(A)\Big| \\
&= \Big| \sum_{(i,j)\in I_n} y^1_{i,j}\mu(T_{ij}\cap A) - \tfrac{1}{2}\mu\Big(A\cap \bigcup_{(i,j)\in I_n} T_{ij}\Big)\Big| \\
&= \Big| \sum_{(i,j)\in I_n} (y^1_{i,j} - 1/2)\mu(T_{ij}\cap A)\Big| \\
&= \Big| \sum_{(i,j)\in J_n(A)} (y^1_{i,j} - 1/2)\mu(T_{ij}\cap A) \\
&\quad + \Big| \sum_{(i,j)\in I_n\setminus J_n(A)} (y^1_{i,j} - 1/2)\mu(T_{ij}\cap A)\Big| \\
&\leq \frac{1}{4n^2}\Big| \sum_{(i,j)\in J_n(A)} (y^1_{i,j} - 1/2)\Big| + \frac{1}{2}\Big| \sum_{(i,j)\in I_n\setminus J_n(A)} \mu(T_{ij}\cap A)\Big| \\
&\leq \frac{1}{4n^2}\Big| \sum_{\substack{i \\ i \text{ odd}}} \sum_{(i,j)\in J_n(A)} (y^1_{i,j} - 1/2)\Big| \\
&\quad + \frac{1}{4n^2}\Big| \sum_{\substack{i \\ i \text{ even} \\ (i,j)\in J_n(A)}} \sum_j (y^1_{i,j} - 1/2)\Big| + \tfrac{1}{2}\mu(A\setminus \textstyle\sum_n) \\
&\leq \frac{1}{4n^2}\Big[\sum_{\substack{i \\ i \text{ odd}}} \tfrac{1}{2}(4m_0+1)\Big] + \frac{1}{4n^2}\Big[\sum_{\substack{i \\ i \text{ even}}} \tfrac{3}{2}(4m_0+1)\Big] + \frac{\varepsilon}{2} \\
&\leq \left(\frac{1}{4n^2}\right)\left(\frac{4m_0+1}{2}\right)(4n+1) + \frac{1}{4n^2}\left(\frac{3}{2}\right)(4m_0+1)(4n+1) + \frac{\varepsilon}{2} \\
&= \frac{2(4m_0+1)(4n+1)}{4n^2} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon .
\end{aligned}
$$

Q.E.D.

We are at last ready to prove:

<u>Theorem</u> : ψ_n converges weakly to $\mu/2$ on T .

<u>Proof</u>: Let $S \varepsilon \mathcal{B}(T)$ and $\mu(\partial S) = 0$. If $\mathring{S} = \emptyset$, then $S \subseteqq \partial S$ so $\psi_n(S) = 0$, $\forall n$ and $\frac{1}{2}\mu(S) = 0$ so the result follows. If $\mathring{S} \neq \emptyset$, then $S = \mathring{S} \cup (S\backslash\mathring{S})$ so $\mu(S) = \mu(\mathring{S})$ because $S\backslash\mathring{S} \subseteqq \partial S$. Now choose $m \ni \mu(\mathring{S}\backslash\sum_m) < \varepsilon/3$. Also choose $n^* \ni \forall n > n^*$, $|\psi_n(\sum_m) - \frac{1}{2}\mu(\sum_m)| < \varepsilon/2$ as in Lemma 5. Then we have

$$
\begin{aligned}
|\psi_n(S) - \tfrac{1}{2}\mu(S)| &= |\psi_n(\mathring{S}) - \tfrac{1}{2}\mu(\mathring{S})| \\
&\leqq |\psi_n(\mathring{S}) - \psi_n(\textstyle\sum_m)| + |\psi_n(\textstyle\sum_m) - \tfrac{1}{2}\mu(\textstyle\sum_m)| \\
&\quad + |\tfrac{1}{2}\mu(\textstyle\sum_m) - \tfrac{1}{2}\mu(\mathring{S})| \\
&= \psi_n(\mathring{S}\backslash\textstyle\sum_m) + |\psi_n(\textstyle\sum_m) - \tfrac{1}{2}\mu(\textstyle\sum_m)| + \tfrac{1}{2}\mu(S\backslash\textstyle\sum_m) \\
&< \mu(\mathring{S}\backslash\textstyle\sum_m) + \tfrac{\varepsilon}{2} + \tfrac{1}{2}\mu(\mathring{S}\backslash\textstyle\sum_m) \\
&< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{2} + \tfrac{1}{2}\,\tfrac{\varepsilon}{3} \\
&< \varepsilon .
\end{aligned}
$$

Thus $\forall n > n^*$, $|\psi_n(S) - \frac{1}{2}\mu(S)| < \varepsilon$.

Q.E.D.

We have proved the theorem for the case of ψ_n^1 but a completely analogous result holds for ψ_n^2 with the obvious modifications. Suppose we define a continuous representation of the dual prices on T . Using the wage rate as an example, let

$$W_n(x) = \frac{t_1}{2}\left[1 - \frac{d^{(ij)}_{(00)}}{2n}\right] , \quad (i,j) \in \bigcup_{(i,j)\in I(u)} T_{ij}(n) .$$

Let $W(x) = (t_1/2)(1 - \|x\|_1)$ be the Mills' continuum price. We can easily show

Theorem: W_n converges to W uniformly on T .

Proof:
$$\sup_{x\in T}|W_n(x) - W(x)| = \max_{1\leqq u\leqq 2n} \sup_{x\in \bigcup_{(i,j)\in I(u)} T_{ij}(u)} |W_n(x) - W(x)|$$

$$= \max_{1\leqq u\leqq 2n} \frac{t_1}{4n}$$

$$= \frac{t_1}{4n}$$

and the uniform convergence of $\{W_n\}$ obtains.

Q.E.D.

For the sake of completeness, we shall briefly discuss the segregated solution for the discretized Mills' model. In this case, the optimal configuration will consist of three parts. Specifically, the "boundary" ring will separate a disk and an annulus. The disk will be filled with firms, and the annulus will be filled with workers. The separating ring will contain firms and workers. The analog of Mills' continuum optimal solution for the segregated case can be shown to decentralize the configuration just described everywhere except in the "boundary" ring. However, the number of sites in the boundary ring is linear in n so the set of sites at which the continuum prices do not work has measure zero in the limit economy.

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