COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

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New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 539

MORSE PROGRAMS

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October 23, 1979

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Abstract

Spingarn and Rockafellar [13] showed in a program

$$(Q_v^u) \quad \text{minimize } \{f(x) - u^T x \ \text{ subject to } \ g(x) \le b + v\}$$

where $f : R^n \to R$, $g : R^n \to R^m$; $f \in C^2$, $g \in C^n$; $u \in R^n$, $v \in R^m$; $n \geq m$; that at any local minimum point x of (Q_v^u) the Jacobian matrix of g at x has full rank, strict complementary slackness holds and the second order sufficiency conditions hold, for almost every $\begin{pmatrix} u \\ v \end{pmatrix}$ in $R^n \times R^m$ (Lebesgue measure).

The purpose of this paper is to explicate the geometry underlying their work and to exploit this geometry in the generic analysis of constrained optimization problems. Namely we show that their work can be reduced to the study of minimizing a Morse function on a manifold with boundary.

We follow a classical tradition of first studying an equality constrained program and then reducing inequality constrained programs to a finite family of equality constrained programs, through the device of active (or binding) constraints.

*This research was supported in part by National Science Foundation Grants ENG-78-25182 and SOC 77-03277.

Our main concerns are the second order sufficiency conditions (Theorem A, Theorem I, Theorem J, Theorem N); sensitivity analysis (Theorem B, Theorem G, Theorem H); generic properties of smooth nonlinear programs (Theorem C, Theorem D, Theorem K, Theorem M, Theorem O, Theorem P, Theorem Q); global duality (Theorem E); local uniqueness (Theorem F); strict complementary slackness (Theorem L).

In the final section of this paper, we consider a smooth nonlinear program defined by a set of variable constraints (constraints which we are allowed to perturb) and fixed constraints (constraints which are not subject to perturbation).

The generic strong second order sufficiency conditions for this type of program are derived in Theorem R. The first result of this kind is due to Spingarn [11], [12].

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I. Basic Definitions and Notation

A property that holds except on a subset of R^n whose Lebesgue measure is zero is said to hold almost every $u \in R^n$. The complement of the above measure zero set of R^n is said to have full measure in R^n.

The Jacobian matrix and the Hessian matrix of f at x are denoted by $Df(x)$ and $D^2f(x)$ respectively.

Let $f : M \to R^m$ be a C^γ map from a k-dimensional C^γ manifold M with boundary ∂M in R^n. Let (ϕ, U) be a local parametrization of M at x such that $x = \phi(u)$, $u \in U \subseteq H^k = \{x \in R^k | x_k \geq 0\}$. The tangent space T_xM of M at x is defined to be the image of $D\phi(u) : R^k \to R^n$. A point $x \in M$ is a regular point of f if $D(f\phi)(u) : R^k \to R^m$ is surjective, if $D(f\phi)(u)$ is not surjective x is a critical point of f. A critical point x of $f : M \to R^1$ is nondegenerate if the $k \times k$ matrix $D^2(f\phi)(u)$ is nonsingular. It is easily shown that the above definitions do not depend on the choice of local parametrization. A point $y \in R^m$ is a regular value of f, denoted by $f \pitchfork y$, if every $x \in f^{-1}(y)$ is a regular point of f, otherwise y is a critical value of f. $f : M \to R^1$ is a Morse function if all critical points of f are nondegenerate.

Let $f : M \to N$ be a C^γ map, $A \subseteq N$ be a C^γ submanifold of N. f is transversal to A, denoted by $f \pitchfork A$, if for every $x \in f^{-1}(A)$, $\text{Image}Df(x) + T_{f(x)}A = T_{f(x)}N$ holds, where $Df(x) : T_xM \to T_{f(x)}N$ is the derivative of f. Two submanifolds A, B of M are transversal denoted by $A \pitchfork B$, if $i \pitchfork B$ where $i : A \to M$ is the inclusion map. f is an immersion if for every $x \in M$, $Df(x) : T_xM \to T_{f(x)}N$ is injective. f is a submersion if $Df(x)$ is surjective for every $x \in M$. f is proper if the preimage of every compact set in N is compact in M . An immersion that is injective and proper is called an embedding.

We refer the interested reader to Guillemin and Pollack [5] for an introduction to the concepts of differential topology that will be used in this paper. Those theorems of elementary differential topology which are used in the body of this paper are stated in the appendix. The proofs of those theorems can be found in Gillemin/Pollack [5] and Hirsh [7].

II. Equality Constraints

1. Properties of Morse Programs

We consider a program

$$(P) \quad \text{minimize } \{f(x) \quad \text{subject to} \quad g(x) = b\}$$

and a perturbation of (P)

$$(P_v^u) \quad \text{minimize } \{f(x) - u^T x \quad \text{subject to} \quad g(x) = b + v\}$$

where $f : R^n \to R$, $g : R^n \to R^m$; $f, g \in C^2$; $u \in R^n$, $v \in R^m$; $n \geq m$.

Definition. A program (P) is a Morse program if $g \pitchfork b$ and f is a Morse function on $g^{-1}(b)$.

Definition. A point $x \in g^{-1}(b)$ is a critical point of (P) if x is a critical point of f on $g^{-1}(b)$.

It is easily verified that nondegenerate critical points are isolated (cf. Guillemin/Pollack [5]). Hence each critical point of a Morse program (P) is isolated. By the Morse Lemma (Appendix (1)) the local behavior of a function at a nondegenerate critical point is completely determined, i.e., at any critical point of a Morse program (P) f is a local minimum, a local maximum, or a saddle point.

If $g \pitchfork b$ and $g \in C^\gamma$ then $g^{-1}(b)$ is (n-m)-dimensional C^γ submanifold of R^n (Appendix (5)).

A Morse program has three distinguishing properties;

(a) The second order sufficiency conditions hold at every critical point of a Morse program (P) (Theorem A).

(b) If x is a critical point of a Morse program (P), then the matrix $\begin{pmatrix} D^2f(x) + \sum_{i=1}^{m} \lambda_i D^2 g_i(x) & Dg(x)^T \\ Dg(x) & 0 \end{pmatrix}$ is nonsingular (Theorem B).

(c) Generically (P) can be considered a Morse program, namely (P^u_v) is a Morse program for almost every $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$ (Theorem C).

Suppose $g \pitchfork b$ and $g \in C^\gamma$ $(\gamma \geq 2)$. Then $M = g^{-1}(b)$ is (n-m)-dimensional C^γ submanifold of R^n and at each point $x \in M$ $Dg(x)$ has full rank, hence $R^n = \text{Ker } Dg(x) \oplus \text{Im } Dg(x)^T$ and $T_xM = \text{Ker } Dg(x)$ (because differentiating $g\phi \equiv b$ on U, where (ϕ, U) is a local parametrization of M at $x = \phi(u)$, we obtain $T_xM = \text{Im } D\phi(u) \subseteq \text{Ker } Dg(x)$. Comparing dimensions of both sides we have $T_xM = \text{Ker } Dg(x)$). A point $x \in M$ is a critical point of f on M iff $Df(x)^T \perp T_xM$, because $D(f\phi)(u)R^{n-m} = Df(x)D\phi(u)R^{n-m}$

$= Df(x)T_xM = \{0\}$ iff $Df(x)^T \perp T_xM$. Then $Df(x)^T \in \operatorname{Im} Dg(x)^T$.

Hence we have

Lemma 1.* If $g \pitchfork b$, then $x \in g^{-1}(b)$ has a Lagrange multiplier iff x is a critical point of f on $g^{-1}(b)$. Moreover the Lagrange multiplier is uniquely determined.

The next proposition gives a representation of the Hessian matrix of f on $g^{-1}(b)$, at $x \in g^{-1}(b)$, in terms of the second derivative of the Lagrangian at x .

Lemma 2.* Let $g \pitchfork b$, x be a critical point of $M = g^{-1}(b)$ with the associated Lagrange multiplier λ , $\mathcal{L}(x) = D^2f(x) + \sum_{i=1}^{m} \lambda_i D^2 g_i(x)$ and (ϕ,U) be a local parametrization of M at x such that $\phi(u) = x$ and $u \in U \subseteq R^{n-m}$. Then $D^2(f\phi)(u) = D\phi(u)^T\mathcal{L}(x)D\phi(u)$.

Proof.** By the chain rule we have

$$(1) \qquad D^2(f\phi)(u) = D\phi(u)^T D^2 f(x) D\phi(u) + \sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} D^2\phi_j(u) .$$

Differentiating $\sum_{i=1}^{m} \lambda_i(g_i\phi) \equiv \sum_{i=1}^{m} \lambda_i b_i$ on U , we have

$$(2) \qquad D\phi(u)^T\left(\sum_{i=1}^{m} \lambda_i D^2 g_i(x)\right)D\phi(u) + \sum_{j=1}^{n}\left(\sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j}\right) D^2\phi_j(u) = 0 .$$

Adding (2) to (1) and taking account $Df(x) + \sum \lambda_i Dg_i(x) = 0$. We obtain $D^2(f\phi)(u) = D\phi(u)^T\mathcal{L}(x)D\phi(u)$.

Q.E.D.

*These facts have been pointed out previously by Tanabe ([14] Proposition 1, [15] Lemma 5.4, respectively).

**The idea for this proof was first given by Luenberger [9].

The $n \times n$ matrix $\mathcal{L}(x)$ gives a homomorphism on R^n, hence for $v \in T_x M \subseteq R^n$, $\mathcal{L}(x)v \in R^n$. In general $\mathcal{L}(x)v$ is not necessarily in $T_x M$. But if x is a nondegenerate critical point of f on M, then $\mathcal{L}(x)v \in T_x M$ and $\mathcal{L}(x)$ induces an isomorphism on $T_x M$, namely we have

<u>Lemma 3</u>. <u>Let</u> $g \pitchfork b$ <u>and</u> x <u>be a critical point of</u> f <u>on</u> $M = g^{-1}(b)$, <u>then</u> x <u>is nondegenerate iff</u> $\mathcal{L}(x)T_x M = T_x M$.

<u>Proof</u>. Let (ϕ, U) be a local parametrization of M at x such that $\phi(u) = x$ and $u \in U \subseteq R^{n-m}$. Note that $T_x M = \operatorname{Im} D\phi(u)$ and $R^n = \operatorname{Im} D\phi(u) \oplus \operatorname{Ker} D\phi(u)^T$.

Suppose x is nondegenerate, then $D^2(f\phi)(u) = D\phi(u)^T \mathcal{L}(x) D\phi(u)$ (Lemma 2) is nonsingular hence $\mathcal{L}(x)$ is 1-1 on $T_x M = \operatorname{Im} D\phi(u)$.

If $\mathcal{L}(x)T_x M \cap \operatorname{Ker} D\phi(u)^T \neq \{0\}$, then $\dim \operatorname{Im} D\phi(u)^T \mathcal{L}(x) D\phi(u) = \dim D\phi(u)^T \mathcal{L}(x) T_x M < n-m$ which contradicts the nonsingularity of $D\phi(u)^T \mathcal{L}(x) D\phi(u)$. Therefore $\mathcal{L}(x)T_x M \cap \operatorname{Ker} D\phi(u)^T = \{0\}$, hence $\mathcal{L}(x)T_x M \subseteq \operatorname{Im} D\phi(u) = T_x M$. Since $\mathcal{L}(x)$ is 1-1 on $T_x M$, this implies $\mathcal{L}(x)T_x M = T_x M$.

Conversely suppose $\mathcal{L}(x)T_x M = T_x M = \operatorname{Im} D\phi(u)$. Then $\operatorname{Ker} D\phi(u)^T \cap \mathcal{L}(x)T_x M = \{0\}$ hence $\operatorname{Ker} D\phi(u)^T \mathcal{L}(x) D\phi(u) = \{0\}$. Therefore $D\phi(u)^T \mathcal{L}(x) D\phi(u)$ is nonsingular and hence x is nondegenerate.

Q.E.D.

<u>Theorem A</u>. <u>Let</u> (P) <u>be a Morse program and</u> x <u>be a critical point of</u> (P). <u>Then we have</u>

(a) $Dg(x)$ <u>has full rank</u>

(b) <u>there exists a unique</u> $\lambda \in R^m$ <u>such that</u> $Df(x)^T + Dg(x)^T \lambda = 0$

(c) $\mathcal{L}(x) = D^2 f(x) + \sum_{i=1}^{m} \lambda_i D^2 g_i(x)$ <u>induces an isomorphism on</u> $T_x M$ <u>where</u> $M = g^{-1}(b)$.

(d) <u>on</u> T_xM , $\mathcal{L}(x)$ <u>is positive definite iff</u> x <u>is a local minimum</u>; <u>negative definite iff</u> x <u>is a local maximum</u>; <u>indefinite iff</u> x <u>is a saddle point</u>.

<u>Proof</u>. (a), (b), and (c) follow from $g \pitchfork b$, Lemma 1 and Lemma 3. (d) positive (negative) definite $\Longrightarrow$ local minimum (maximum) is obvious. If x is a local minimum (maximum), then $\mathcal{L}(x)$ is positive (negative) semidefinite. However by (c) $\mathcal{L}(x)$ is nonsingular on T_xM , hence $\mathcal{L}(x)$ must be positive (negative) definite on T_xM . The saddle point case is an immediate consequence of the preceeding argument.

Q.E.D.

Now let us vary $b \in R^m$ and consider a critical point x of (P) as a function of b , $x(b)$. A sufficient condition that $x(\cdot)$ is a C^1 function of b is the nonsingularity of the matrix

$$(3) \qquad \begin{pmatrix} \mathcal{L}(x) & Dg(x)^T \\ Dg(x) & 0 \end{pmatrix}$$

(this follows from the implicit function theorem).

Consider the function $F_b : R^n \times R^m \to R^n \times R^m$ where $F_b(x,\lambda) = \begin{pmatrix} Df(x)^T + Dg(x)^T\lambda \\ g(x) - b \end{pmatrix}$. Then the nonsingularity of (3) is equivalent to $F_b \pitchfork 0$ which is equivalent to f being a Morse function on $g^{-1}(b)$; namely we have

<u>Theorem B</u>. <u>If</u> $g \pitchfork b$, <u>then</u> $F_b \pitchfork 0$ <u>iff</u> f <u>is a Morse function on</u> $M = g^{-1}(b)$.

<u>Proof</u>. (If) Let $(x,\lambda) \in F_b^{-1}(0)$, then x is a nondegenerate critical point of f on M (Lemma 1) hence $\mathcal{L}(x)T_xM = T_xM$ (Lemma 3). Suppose $\begin{pmatrix} \mathcal{L}(x) & Dg(x)^T \\ Dg(x) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{L}(x)u + Dg(x)^T v \\ Dg(x)u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $u \in \text{Ker } Dg(x) = T_xM$ and $\mathcal{L}(x)u \in \mathcal{L}(x)T_xM = T_xM = \text{Ker } Dg(x)$. Since $\mathcal{L}(x)u = -Dg(x)^T v \in \text{Ker } Dg(x) \cap \text{Im } Dg(x)^T = \{0\}$, $\mathcal{L}(x)u = 0 = Dg(x)^T v$ hence $u = 0$ and $v = 0$ because $\mathcal{L}(x)$ is 1-1 on T_xM and $Dg(x)^T$ is 1-1 . This implies $\text{Ker } DF_b(x,\lambda) = \{0\}$, namely $DF_b(x,\lambda)$ is nonsingular therefore $F_b \pitchfork 0$.

(Only if) Let x be a critical point of f on M . Then there exists $\lambda \in R^m$ such that $F_b(x,\lambda) = 0$ by Lemma 1. Let (ϕ,U) be a local parametrization of M at x such that $\phi(u) = x$ and $u \in U \subseteq R^{n-m}$. Suppose $D\phi(u)^T\mathcal{L}(x)D\phi(u)v = 0$ for some $v \in R^{n-m}$. Let $w = D\phi(u)v$; then $D\phi(u)^T\mathcal{L}(x)w = 0$ hence $\mathcal{L}(x)w \in \text{Ker } D\phi(u)^T$ $= \text{Im } Dg(x)^T$ because $R^n = \text{Im } D\phi(u) \oplus \text{Ker } D\phi(u)^T = \text{Ker } Dg(x) \oplus \text{Im } Dg(x)^T$ and $\text{Im } D\phi(u) = T_xM = \text{Ker } Dg(x)$ imply $\text{Ker } D\phi(u)^T = \text{Im } Dg(x)^T$. Hence $\mathcal{L}(x)w = Dg(x)^T\mu$ for some $\mu \in R^m$. Then

$$\begin{pmatrix} \mathcal{L}(x) & Dg(x)^T \\ Dg(x) & 0 \end{pmatrix}\begin{pmatrix} w \\ -\mu \end{pmatrix} = \begin{pmatrix} \mathcal{L}(x)w - Dg(x)^T\mu \\ Dg(x)w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{since } w = D\phi(u)v \in T_xM$$

$= \text{Ker } Dg(x)$. Since $F_b \pitchfork 0$, $DF_b(x,\lambda)$ is nonsingular hence $w = 0$ and $v = 0$ because $D\phi(u)$ is 1-1 . Therefore $D\phi(u)^T\mathcal{L}(x)D\phi(u)$ is nonsingular and by Lemma 2 x is nondegenerate.

Q.E.D.

The third property of Morse programs is genericity. In general (P) is not necessarily a Morse program, but (P_v^u) is a Morse program for almost every $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$. Thus generically we can assume that (P) is a Morse program. Moreover, our perturbation enables us to guarantee at most one global solution, namely we have,

<u>Theorem C.</u> <u>If</u> $f \in C^2$ <u>and</u> $g \in C^{n-m+1}$, <u>then for almost every</u> $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$ (P_v^u) <u>is a Morse program having at most one global solution</u>.

<u>Proof</u>. By Sard's Theorem (Appendix (2)) if $g : R^n \to R^m$ is of class C^{n-m+1}, then $g \pitchfork b+v$ for almost every $v \in R^m$. For a C^2 manifold $X \subseteq R^n$ and a C^2 map $h : X \to R^1$, $h(x) - u^T x$ is a Morse function for almost every $u \in R^n$ (Appendix (6)). Therefore for $v \in R^m$ such that $g \pitchfork b+v$, $f(x) - u^T x$ is a Morse function on $g^{-1}(b+v)$ for almost every $u \in R^n$. By Araujo and Mas-Colell ([1], Theorem 1),* we have

> Fix any $v \in R^m$, then for almost every $u \in R^n$ (P_v^u) has at most one global solution.

Let $A = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| (P_v^u) \text{ is not a Morse program or } (P_v^u) \text{ has multiple global solutions} \right\}$. Let $v \in R^m$ be such that $g \pitchfork b+v$. Then $A_v = \{u | (P_v^u)$ is not a Morse program or (P_v^u) has multiple global solutions$\}$ is of measure zero in R^n. Since for almost every $v \in R^m$ we have $g \pitchfork b+v$, by Fubini's theorem (Appendix (3)) A has measure zero in $R^n \times R^m$.

Q.E.D.

*Truman Bewley suggested the use of the Araujo/Mas-Colell theorem.

2. Properties of Proper Morse Programs

A map $g : R^n \to R^m$ is called proper if the preimage of every compact set in R^m is compact in R^n.

Definition (Brown, Heal, Westhoff [2]). A program (P) is called proper if g is proper.

It is easily shown that $g : R^n \to R^m$ is proper iff for any $\{x_k\} \subseteq R^n$ such that $\|x_k\| \to \infty$ in R^n, $\|g(x_k)\| \to \infty$ in R^m, where $\|\cdot\|$ is the Euclidean norm.

In this section we consider proper programs. A proper program has at least one solution since $g^{-1}(b)$ is compact. Hence by Theorem C we have

Theorem D. If (P) is proper, $g \in C^{n-m+1}$ then (P_v^u) is a proper Morse program having a unique global solution for almost every $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$.

Additional nice properties of a proper Morse program are global duality (Theorem E), local uniqueness of the critical points of (P) (Theorem F), locally Lipshitzness of the optimum value function (Theorem G) and differentiability of the function $w^u(v)$ (Theorem H), where

$$w^u(v) = \text{minimum } \{f(x) - u^T x \text{ subject to } g(x) = b + v\} .$$

Let us consider a program

$$(P_K) \quad \underset{x \in K}{\text{minimize}} \; \{f(x) \text{ subject to } g(x) = b\}$$

where f and g are of class C^2 and K is a compact set of R^n.

Let us define

$$(4) \qquad \phi_\sigma(\lambda) = \underset{x \in K}{\text{minimum}} \left\{ f(x) + \lambda^T (g(x) - b) + \frac{\sigma}{2} \| g(x) - b \|^2 \right\}$$

where $\lambda \in R^m$, $\sigma > 0$ and $\|v\|^2 = \sum_{i=1}^{m} v_i^2$ for $v \in R^m$.

Let $x_\sigma(\lambda)$ be a minimizer of $\phi_\sigma(\lambda)$. Since K is compact there exists such an $x_\sigma(\lambda) \in R^n$.

First of all let us study the properties of $\phi_\sigma(\lambda)$.

Lemma 4. (a) $\phi_\sigma(\cdot)$ is a concave function of λ for any $\sigma > 0$.

(b) supergradient of $\phi_\sigma(\lambda) = \partial\phi_\sigma(\lambda) \ni g(x_\sigma(\lambda)) - b$ for any $x_\sigma(\lambda) \in R^n$.

Proof. (a) is trivial and (b) is straightforward.

We will show $\phi_\sigma(\mu) \le \phi_\sigma(\lambda) + (\mu-\lambda)^T (g(x_\sigma(\lambda)) - b)$ for all $\mu \in R^m$.

By the definition of $x_\sigma(\cdot)$, we have

$$(5) \qquad \phi_\sigma(\mu) = f(x_\sigma(\mu)) + \mu^T (g(x_\sigma(\mu)) - b) + \frac{\sigma}{2} \| g(x_\sigma(\mu)) - b \|^2 ,$$

$$(6) \qquad \phi_\sigma(\lambda) = f(x_\sigma(\lambda)) + \lambda^T (g(x_\sigma(\lambda)) - b) + \frac{\sigma}{2} \| g(x_\sigma(\lambda)) - b \|^2 , \quad \text{and}$$

$$(7) \qquad \phi_\sigma(\mu) \le f(x_\sigma(\lambda)) + \mu^T (g(x_\sigma(\lambda)) - b) + \frac{\sigma}{2} \| g(x_\sigma(\lambda)) - b \|^2 .$$

Substituting (5) into (7) and rearranging (7) we obtain

$$(8) \qquad f(x_\sigma(\mu)) - f(x_\sigma(\lambda)) + \mu^T (g(x_\sigma(\mu)) - b) - \mu^T (g(x_\sigma(\lambda)) - b)$$
$$+ \frac{\sigma}{2} \| g(x_\sigma(\mu)) - b \|^2 - \frac{\sigma}{2} \| g(x_\sigma(\lambda)) - b \|^2 \le 0 .$$

By (5) and (6), (8) is equivalent to

$$\phi_\sigma(\mu) - \phi_\sigma(\lambda) - (\mu-\lambda)^T (g(x_\sigma(\lambda)) - b) \le 0 .$$

This completes the proof.

Q.E.D.

Theorem (Hestenes [6], Chapter 5, Theorem 4.4). *If* x^* *is a unique global solution of* (P_K) *satisfying the second order sufficiency conditions with an associated Lagrange multiplier* λ^* , *then there exist* $\sigma_0 > 0$ *and* $\tau > 0$ *such that for any* $x \in K$ *and* $\sigma \geq \sigma_0$, *we have*

$$(9) \qquad H_\sigma(x) \geq H_\sigma(x^*) + \tau\|x - x^*\|^2 = f(x^*) + \tau\|x - x^*\|^2$$

where

$$(10) \qquad H_\sigma(x) = f(x) + \lambda^{*T}(g(x) - b) + \frac{\sigma}{2}\|g(x) - b\|^2 .$$

Let (P) be a proper program. If we take $K = g^{-1}(b)$, then $(P_K) = (P)$. Moreover if (P) is a Morse program having a unique global solution x^* with the associated Lagrange multiplier λ^* (we can assume this generically as we have seen in Theorem D), then x^* satisfies the second order sufficiency conditions (Theorem A), hence we can apply the above theorem.

By (4), (9) and (10) we have for $\sigma \geq \sigma_0$, $x_\sigma(\lambda^*) = x^*$,

$$(11) \qquad \phi_\sigma(\lambda^*) = f(x^*)$$

and by Lemma 4(b)

$$(12) \qquad 0 = g(x_\sigma(\lambda^*)) - b \in \partial\phi_\sigma(\lambda^*) .$$

However (11), (12) and Lemma 4 imply

$$\phi_\sigma(\lambda^*) = \max_{\lambda \in R^m} \phi_\sigma(\lambda) = f(x^*) \quad \text{for} \quad \sigma \geq \sigma_0 ,$$

namely we obtain

Theorem E. *If* (P) *is a proper Morse program having a unique global solution* x^* *with the associated Lagrange multiplier* λ^* , *then there exists* $\sigma_0 > 0$ *such that for any* $\sigma \geq \sigma_0$ x^* *is a unique global solution of* $\phi_\sigma(\lambda^*)$ *and*

$$\phi_\sigma(\lambda^*) = \max_\lambda \phi_\sigma(\lambda) = \min\{f(x) \text{ subject to } g(x) = b\} = f(x^*) .$$

As we mentioned before, our assumption in Theorem E holds generically if $g \in C^{n-m+1}$ (Theorem D).

Let us consider the number of critical points of a proper Morse program (P) .

Lemma 5 (cf. Brown/Heal/Westhoff [2]). *If* g *is proper, then* $\{b \in R^m | g \pitchfork b,\ F_b \pitchfork 0\}$ *is open in* R^m .

Proof. First of all let us prove $\{b | g \pitchfork b,\ F_b \pitchfork 0\}$ is open in $\{b | g \pitchfork b\}$. Suppose it is not open at $b^0 \in \{b | g \pitchfork b,\ F_b \pitchfork 0\}$. Then there exist $b^n \in \{b | g \pitchfork b\}$, $(x^n, \lambda^n) \in F^{-1}_{b^n}(0)$ such that $b^n \to b^0$, $DF_{b^n}(x^n, \lambda^n)$ is singular. Let K be a compact neighborhood of b^0 , then for sufficiently large n , $x^n \in g^{-1}(K)$. Since g is proper $g^{-1}(K)$ is compact and there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $x^{n_j} \in g^{-1}(K)$, $x^{n_j} \to x^0$ for some $x^0 \in g^{-1}(K)$. Since $(x^{n_j}, b^{n_j}) \to (x^0, b^0)$, $g(x^{n_j}) = b^{n_j}$ we have $g(x^0) = b^0$. By $g \pitchfork b^n$, $\lambda^n = \lambda(x^n) = -(Dg(x^n)Dg(x^n)^T)^{-1}Dg(x^n)Df(x^n)^T$. Let $\lambda^0 = \lambda(x^0)$ then $(x^{n_j}, \lambda^{n_j}, b^{n_j}) \to (x^0, \lambda^0, b^0)$ and $0 = F_{b^{n_j}}(x^{n_j}, \lambda^{n_j}) \to F_{b^0}(x^0, \lambda^0) = 0$.

By $F_{b^0} \pitchfork 0$, $DF_{b^0}(x^0, b^0)$ is nonsingular. However we have also $DF_{b^{n_j}}(x^{n_j}, \lambda^{n_j}) \to DF_{b^0}(x^0, \lambda^0)$, hence $DF_{b^{n_j}}(x^{n_j}, \lambda^{n_j})$ is nonsingular for sufficiently large n_j , which contradicts our assumption. Therefore $\{b | g \pitchfork b, F_b \pitchfork 0\}$ is open in $\{b | g \pitchfork b\}$. By the similar argument we obtain $\{b | g \pitchfork b\}$ is open in R^m if g is proper. This completes the proof.

Q.E.D.

Let

$$P(b) \quad \text{minimize } \{f(x) \quad \text{subject to} \quad g(x) = b\}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $f \in C^2$, $g \in C^2$ and $n \geq m$. Then by Lemma 5 and Theorem B we obtain

Corollary 6. *If* g *is proper, then*

$$\{b \in R^m | P(b) \text{ is a proper Morse program}\}$$

is open in R^m .

Theorem F. *If* $P(b)$ *is a proper Morse program, then the number of critical points of* $P(b)$ *is finite and locally constant as a function of* b *on the open set* $\{b | g \pitchfork b, F_b \pitchfork 0\}$ *of* R^m .

Proof. Since $P(b)$ is a proper Morse program, $g^{-1}(b)$ is compact and each critical point of $P(b)$ is isolated. An isolated set in a compact set is finite, hence the number of critical points of $P(b)$ is finite. By the implicit function theorem, for each critical point x^i with the associated Lagrange multiplier λ^i $(i = 1, \ldots, k)$, there exist a neighborhood V^i of b and C^1 functions $x^i(\cdot)$, $\lambda^i(\cdot)$ defined on V^i such that $F_{b'}(x^i(b'), \lambda^i(b')) = 0$ for any $b' \in V^i \subseteq \{b | g \pitchfork b, F_b \pitchfork 0\}$. This implies $x^i(b')$ is a critical point of $P(b')$ for $b' \in V^i$. Taking a small neighborhood U of b such that $U \subseteq \bigcap_{i=1}^{k} V^i$, we have pairwise disjoint sets $x^1(U), \ldots, x^k(U)$. Therefore by the continuity of the function $F_{(\cdot)}(x(\cdot), \lambda(\cdot))$, the number of critical points of $P(b') = k$ for any $b' \in U$. This completes the proof.

Q.E.D.

Let $w(b)$ be the optimum value of $P(b)$, namely

$$w(b) = \text{minimum}\{f(x) \text{ subject to } g(x) = b\}. \tag{13}$$

Then by Theorem F, $w(b') = \min_{1 \leq i \leq k} f(x^i(b'))$ for all $b' \in U$.

<u>Definition</u>. A function $\phi : R^n \to R^m$ is a <u>locally Lipshitz</u> if for any $x^0 \in R^n$ there exist a neighborhood U of x^0 and a positive number K such that

$$\|\phi(x^0) - \phi(x)\| \leq K\|x^0 - x\|$$

for any $x \in U$.

It is easily verified that if $f^1, \ldots, f^k$ are of class C^1 then $g = \min\{f^1, \ldots, f^k\}$ is a locally Lipshitz. Since each $x^i(\cdot)$ is of class C^1, we obtain

<u>Theorem G</u>. <u>If</u> $P(b)$ <u>is a proper Morse program</u>, <u>then the optimum value function</u> $w(b)$ <u>defined by</u> (13) <u>is a locally Lipshitz function on the open set</u> $\{b \in R^m | g \pitchfork b,\ F_b \pitchfork 0\}$.

Let $x^u(v)$ be a unique global solution of (P^u_v) in Theorem D and $w^u(v)$ be the optimum value of (P^u_v) , namely

$$(14) \qquad w^u(v) = \text{minimum } \{f(x) - u^T x \ \text{ subject to } \ g(x) = b+v\} .$$

Then $w^u(v) = f(x^u(v))$. By Lemma 5 $W^u = \left\{v' \in R^m | g \pitchfork b+v',\ F_b \pitchfork \begin{pmatrix} u \\ v' \end{pmatrix}\right\}$ is open in R^m for a fixed $u \in R^n$, and by Theorem F for a sufficiently small neighborhood V^u ($\subseteq W^u$) of v , $x^u(v')$ is a unique global solution of $(P^u_{v'})$ for any $v' \in V^u$. Since $x^u(\cdot)$ is a C^1 function in the neighborhood of v , we obtain

<u>Theorem H</u>. <u>If</u> (P) <u>is a proper Morse program</u>, $g \in C^{n-m+1}$ <u>then the optimum value function</u> $w^u(v)$ <u>defined by</u> (14) <u>is a</u> C^1 <u>function in the neighborhood of</u> v <u>for almost every</u> $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$.

<u>Definition</u> (Brown/Heal/Westhoff [2]). A program (P) is called <u>regular</u> if $F \pitchfork 0$ where $F : R^n \times R^m \times R^m \to R^n \times R^m$ is defined by

$$F(x,\lambda,b) = F_b(x,\lambda) = \begin{pmatrix} Df(x)^T + Dg(x)^T\lambda \\ g(x) - b \end{pmatrix} .$$

Regularity is a generic property, namely if $f, g \in C^{m+2}$ (hence $F \in C^{m+1}$) then $F \pitchfork \begin{pmatrix} u \\ v \end{pmatrix}$ for almost every $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$ by Sard's theorem, hence (P^u_v) is a regular program for almost every $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$. Under the same assumptions $F \pitchfork 0$ implies $F_b \pitchfork 0$ for almost every $b \in R^m$ by virtue of parametric transversality theorem (Appendix (4)).

If $g \in C^{n-m+1}$ then $g \pitchfork b$ for almost every $b \pitchfork R^m$ by Sard's theorem, hence summarizing the above we obtain

<u>Proposition 7</u> (Brown/Heal/Westhoff [2]).

(a) <u>If</u> $f, g \in C^{m+2}$ <u>then</u> (P^u_v) <u>is a regular program for almost every</u> $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$.

(b) <u>If</u> (P) <u>is regular,</u> $f \in C^{m+2}$, $g \in C^{\max(m+2,n-m+1)}$ <u>then</u> $\{b | g \pitchfork b,\ F_b \pitchfork 0\}$ <u>has full measure in</u> R^m .

<u>Corollary 8</u> (cf. Brown/Heal/Westhoff [2]). <u>If</u> (P) <u>is a proper regular program,</u> $f \in C^{m+2}$, $g \in C^{\max(m+2,n-m+1)}$ <u>then</u> $\{b | g \pitchfork b,\ F_b \pitchfork 0\}$ <u>is open and has full measure in</u> R^m , <u>hence</u> $\{b \in R^m | P(b)$ <u>is a proper Morse program</u>$\}$ <u>is open and dense in</u> R^m .

III. <u>Inequality Constraints</u>

Let us consider a program

$$(Q) \quad \text{minimize } \{f(x) \quad \text{subject to} \quad g(x) \leq b\}$$

and a perturbation of (Q)

$$(Q^u_v) \quad \text{minimize } \{f(x) - u^T x \quad \text{subject to} \quad g(x) \leq b+v\}$$

where $f : R^n \to R$, $g : R^n \to R^m$; $f, g \in C^2$; $u \in R^n$, $v \in R^m$; $n \geq m$.

Let $I = \{1, 2, \ldots, m\}$, $J(x) = \{i \in I | g_i(x) = b_i\}$ for x such that $g(x) \leq b$. Let $g_J(x) = (g_j(x))_{j \in J}$, $b_J = (b_j)_{j \in J}$, $M_J = g_J^{-1}(b_J)$, $M_\phi = \{x | g(x) < b\}$, $J^C = I - J$ for every $J \subseteq I$.

1. Weak Morse Programs

We reduce the inequality case to the equality case by using the notion of active regular values.

Definition.* $b \in R^m$ is called an active regular value of (Q), denoted by $g \mathrel{\overline{\pitchfork}} b$, if $g_J \pitchfork b_J$ for every nonempty $J \subseteq I$.

Definition. A program (Q) is called a weak Morse program if $g \mathrel{\overline{\pitchfork}} b$ and f is a Morse function on $M_J = g_J^{-1}(b_J)$ for every $J \subseteq I$.

Definition. x is a critical point of (Q) if $g(x) \leq b$ and x is a critical point of f on $M_{J(x)}$.

However the notion of weak Morse programs is not completely satisfactory as we will see in the next theorem.

Theorem I. If (Q) is a weak Morse program and x is a critical point of (Q) with $J = J(x)$, then

(a) $Dg_J(x)$ has full rank

(b)**there exists a unique $\lambda \in R^m$ such that

$$Df(x)^T + Dg(x)^T\lambda = 0 \quad \text{and} \quad \lambda_{J^c} = 0 .$$

(c) $\mathcal{L}(x) = D^2f(x) + \sum_{i=1}^{m} \lambda_i D^2 g_i(x)$ induces an isomorphism on T_xM_J

(d) On T_xM_J, $\mathcal{L}(x)$ is positive definite if x is a local minimum; negative definite if x is a local maximum; indefinite iff x is a saddle point on M_J.

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*This notion is due to Smale [10].

** $\lambda \geq 0$ (or $\lambda \leq 0$) if x is a local minimum (or maximum) (see for example Luenberger [9]).

Proof. (a), (b), and (c) follow from $g \bar{\pitchfork} b$, Lemma 1 and Lemma 3. A local minimum (or maximum) point of (Q) is also a local minimum (or maximum) point on M_J, hence (d) follows from Theorem A(d).

Q.E.D.

2. Strong Morse Programs

In Theorem I(d), if we wish to have " $\mathcal{L}(x)$ is positive definite on $T_x M_J$ implies x is a local minimum," then we need the strict complementary slackness condition.

This leads us to the following definition,

Definition. A program (Q) is called a *strong Morse program* if

(a) $g \bar{\pitchfork} b$.

(b) f is a Morse function on M_J for every $J \subseteq I$.

(c) Every Lagrange multiplier associated with an active constraint is nonzero (namely, strict complementary slackness holds).

Strick complementary slackness guarantees that we need only consider the tangent space defined by the active constraints. The next theorem is, now, an immediate consequence of Theorem I and the definition of the strong Morse program.

Theorem J. If (Q) *is a strong Morse program and* x *is a critical point of* (Q) *with* $J = J(x)$, *then*

(a) $Dg_J(x)$ *has full rank*.

(b)* *there exists a unique* $\lambda \in R^m$ *such that* $Df(x)^T + Dg(x)^T \lambda = 0$ *and* $\lambda_i \neq 0$ iff $i \in J$.

---

* $\lambda_J > 0$ if x is a local minimum, $\lambda_J < 0$ if x is a local maximum (see Luenberger [9]).

(c) $\mathcal{L}(x) = D^2 f(x) + \sum_{i=1}^{m} \lambda_i D^2 g_i(x)$ <u>induces an isomorphism on</u> $T_x M_J$.

(d) <u>On</u> $T_x M_J$, $\mathcal{L}(x)$ <u>is positive definite iff</u> x <u>is a local minimum</u>; <u>negative definite iff</u> x <u>is a local maximum</u>; <u>indefinite iff</u> x <u>is a saddle point on</u> M_J .

Let us show generic property of weak and strong Morse programs.

<u>Lemma 9</u>. <u>If</u> $g \in C^n$, <u>then</u> $g \pitchfork b$ <u>for almost every</u> $b \in R^m$.

<u>Proof</u> (Spingarn/Rockafellar [13], Theorem 1). Let $\phi \neq J \subseteq I$. By Sard's theorem, the set of critical values of g_J is of measure zero in $R^{|J|}$. So,

$$N(J) = \{b \in R^m | b_J \text{ is a critical value of } g_J\}$$

is of measure zero in R^m (a subset $A \subseteq R^m = R^{m-|J|} \times R^{|J|}$, whose $m-|J|$ "horizontal slices," each has measure zero considered as a subset of $R^{|J|}$, must itself have measure zero in R^m ; this follows from Fubini's theorem). Then $N = \bigcup_{\phi \neq J \subseteq I} N(J)$ is also measure zero.

Q.E.D.

For any fixed $b \in R^m$ such that $g \pitchfork b$ and for any $J \subseteq I$, $f(x) - u^T x$ is a Morse function on M_J for almost every $u \in R^n$ (Appendix (6)). Therefore, $f(x) - u^T x$ is a Morse function on each M_J for almost every $u \in R^n$.

By Fubini's theorem, we obtain

<u>Theorem K</u>. <u>If</u> $f \in C^2$ and $g \in C^n$, <u>then</u> (Q_v^u) <u>is a weak Morse program for almost every</u> $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$.

To show the generic property of strong Morse programs, we need some preliminary results.

Theorem L. Let $f : M \to R^1$ be a C^γ map of m-dimensional C^γ manifold $M \subseteq R^n$ with boundary ∂M. Then for almost every $u \in R^n$ $f(x) - u^T x$ has no critical point on ∂M.

Proof.* $x \in M$ is a critical point of $f(x) - u^T x$ iff $Df(x)^T - u \perp T_x M$ iff $u \in Df(x)^T + T_x M^\perp$ ($T_x M^\perp$ is the orthogonal complement of $T_x M$ in R^n). Let $E = \{(x,u) \in \partial M \times R^n | u \in Df(x)^T + T_x M^\perp\}$. Then E is (n-1)-dimensional $C^{\gamma-1}$ submanifold of $\partial M \times R^n$. Let us prove this fact. For any given $\bar{x} \in \partial M$ there exists an open set $\tilde{U}$ of R^n and a submersion $g : \tilde{U} \to R^{n-m}$ such that $U = M \cap \tilde{U} = g^{-1}(0)$ and $x \in \partial U = \partial M \cap \tilde{U}$ (Appendix (7)). Let $\Phi : \partial U \times R^{n-m} \to \partial M \times R^n$ be $\Phi(x,y) = (x,\ Df(x)^T + Dg(x)^T y)$, (ϕ, V) be a local parametrization of ∂M at $\bar{x}$ such that $\bar{x} = \phi(\bar{v})$, $\bar{v} \in V \subseteq R^{m-1}$, and $\Psi : V \times R^{n-m} \to \partial M \times R^n$ be $\Psi(v,y) = \Phi(\phi(v),\ y)$. Then $\Psi \in C^{\gamma-1}$ and for any $(v,y) \in V \times R^{n-m}$ we have

$$D\Psi(v,y) = \begin{pmatrix} D\phi(v) & 0 \\ * & Dg(\phi(v))^T \end{pmatrix}$$

Since $D\phi(v)$ and $Dg(\phi(v))^T$ is 1-1, $D\Psi(v,y)$ is 1-1 (i.e Ψ is an immersion) hence Φ is an immersion. Let $E(\partial U) = \{(x,u) \in \partial U \times R^n | u \in Df(x)^T + T_x M^\perp\} \subseteq \partial M \times R^n$, then $\Phi : \partial U \times R^{n-m} \to E(\partial U)$ is bijective and proper, hence Φ is an embedding of $\partial U \times R^{n-m}$ into $\partial M \times R^n$. Consequently $E(\partial U)$ is a $C^{\gamma-1}$ manifold (Appendix (8)) parametrized by Φ, with dimension

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*This line of argument was suggested by W. G. Dwyer.

$= \dim \partial U + n - m = m - 1 + n - m = n - 1$. Since every point of E has such a neighborhood, E is a $C^{\gamma-1}$ manifold. (Cf. Guillemin/Pollack [5], normal bundle on page 71.) Let $\pi : \partial M \times R^n \to R^n$ be a projection map. Then since E is (n-1)-dimensional, $\pi(E) \subseteq R^n$ has measure zero in R^n (Appendix (9)).

Since $\pi(E) = \{u \in R^n | u \in Df(x)^T + T_x M^{\perp}$ for some $x \in \partial M\}$, for almost every $u \in R^n$ (i.e., $u \notin \pi(E)$) every $x \in \partial M$ is not a critical point of $f(x) - u^T x$ on M .

Q.E.D.

Lemma 10. Let $g : R^n \to R^m$, $h : R^n \to R^1$, $b \in R^m$, $c \in R^1$, $X = \{x | h(x) \le c\}$. If $g \pitchfork b$, $h \pitchfork c$, $g|_{h^{-1}(c)} \pitchfork b$, then

(a) $M = g^{-1}(b) \cap X$ is (n-m)-dimensional manifold with boundary $\partial M = g^{-1}(b) \cap h^{-1}(c)$

(b) $T_x \partial M = T_x g^{-1}(b) \cap T_x h^{-1}(c) = \operatorname{Ker}\begin{pmatrix} Dg(x) \\ Dh(x) \end{pmatrix}$ for $x \in \partial M$

(c) $\begin{pmatrix} g \\ h \end{pmatrix} \pitchfork \begin{pmatrix} b \\ c \end{pmatrix}$ hence $T_x \partial M^{\perp} = \operatorname{Im}(Dg(x)^T, Dh(x)^T)$.

Proof. By $h \pitchfork c$, $X = \{x | h(x) \le c\}$ is n-dimensional manifold with boundary $\partial X = h^{-1}(c)$ (Appendix (10)). $g \pitchfork b$, $g|_{h^{-1}(c)} \pitchfork b$ imply $M = g^{-1}(b) \cap X$ is (n-m)-dimensional manifold with boundary $\partial M = g^{-1}(b) \cap h^{-1}(c)$ (Appendix (11)). Since $g|_{h^{-1}(c)} \pitchfork b$ iff $h^{-1}(c) \pitchfork g^{-1}(b)$ (Appendix (12)), $T_x \partial M = T_x g^{-1}(b) \cap T_x h^{-1}(c)$ $= \operatorname{Ker} Dg(x) \cap \operatorname{Ker} Dh(x)$ for $x \in \partial M$ (Appendix (13)). Since $\dim T_x \partial M = \operatorname{Ker}\begin{pmatrix} Dg(x) \\ Dh(x) \end{pmatrix} = n-m-1$, $\begin{pmatrix} Dg(x) \\ Dh(x) \end{pmatrix} : R^n \to R^m \times R^1$ is onto hence $\begin{pmatrix} g \\ h \end{pmatrix} \pitchfork \begin{pmatrix} b \\ c \end{pmatrix}$. $\partial M = \begin{pmatrix} g \\ h \end{pmatrix}^{-1} \begin{pmatrix} b \\ c \end{pmatrix}$, $\begin{pmatrix} g \\ h \end{pmatrix} \pitchfork \begin{pmatrix} b \\ c \end{pmatrix}$ imply $T_x \partial M^{\perp} = \operatorname{Im}(Dg(x)^T, Dh(x)^T)$.

Q.E.D.

Consider a program minimize $\{f(x)$ subject to $g(x) = b,\ h(x) \le c\}$ where $f, h : R^n \to R^1$, $g : R^n \to R^m$; $f, g, h \in C^1$.

<u>Proposition 11</u>. <u>Suppose</u> $g \pitchfork b$, $h \pitchfork c$, $g|_{h^{-1}(c)} \pitchfork b$, $M = g^{-1}(b) \cap h^{-1}(-\infty, c]$, <u>f on M has no critical point on</u> $\partial M = g^{-1}(b) \cap h^{-1}(c)$. If x^* <u>is a critical point of</u> f <u>on</u> ∂M, <u>then the Lagrange multiplier</u> μ <u>associated with the constraint</u> $h(x) = c$ <u>is nonzero</u>.

<u>Proof</u>. By Lemma 10, M is $(n-m)$-dimensional manifold with boundary $\partial M = g^{-1}(b) \cap h^{-1}(c)$, $T_x M^{\perp} = \operatorname{Im} Dg(x)^T$ (because $h^{-1}(-\infty, c]$ is n-dimensional manifold, $T_x M = T_x g^{-1}(b) = \operatorname{Ker} Dg(x)$) and $T_x \partial M^{\perp} = \operatorname{Im}(Dg(x)^T, Dh(x)^T)$ for $x \in \partial M$.

x^* is a critical point of f on ∂M but not on M implies that $Df(x^*)^T \in T_{x^*}\partial M^{\perp}$ and $Df(x^*)^T \notin T_{x^*}M^{\perp}$, hence there exists a unique $\begin{pmatrix}\lambda \\ \mu\end{pmatrix} \in R^m \times R^1$ such that $Df(x^*)^T + Dg(x^*)^T\lambda + Dh(x^*)^T\mu = 0$, and $\mu \ne 0$ since $Df(x^*)^T \notin \operatorname{Im} Dg(x^*)^T = T_{x^*}M^{\perp}$. Q.E.D.

<u>Remark</u>. Proposition 11 provides a geometrical interpretation of strict complementary slackness, namely $\mu \ne 0$ iff x^* is a critical point of f on ∂M but not on M. The genericity of strict complementary slackness follows the genericity of Theorem L.

Let $B = \{b \in R^m | g_J \pitchfork b_J$ for every $J \subseteq I$, $g_J|_{g_i^{-1}(b_i)} \pitchfork b_J$ for every $J \subsetneq I$ and $i \in J^C\}$.

If $g \in C^n$ then B has full measure in R^m by Sard's theorem and Fubini's theorem (cf. Lemma 9).

Let $\overline{b} \in B$ be fixed and let

$$M(J,i) = \{x \mid g_J(x) = \bar{b}_J,\ g_i(x) \leq \bar{b}_i\} \quad \text{for} \quad J \subsetneq I \quad \text{and} \quad i \in J^C ,$$

$$M(I) = \{x \mid g(x) = \bar{b}\} = g^{-1}(\bar{b}) ,$$

$$M = \{x \mid g(x) < \bar{b}\} .$$

Then $\bar{b} \in B$ implies that $M(J,i)$ is $(n-|J|)$-dimensional C^n manifold with boundary $\partial M(J,i) = \{x \mid g_J(x) = \bar{b}_J,\ g_i(x) = \bar{b}_i\}$ (Lemma 10), $M(I)$ is $(n-m)$-dimensional C^n manifold and M is n-dimensional C^n manifold if these sets are not empty. By Theorem L, we obtain that for almost every $u \in R^n$, $f_{|M(J,i)}(x) - u^T x$ has no critical point on $\partial M(J,i)$ and $f_{|\partial M(J,i)}(x) - U^T x$ is a Morse function for every (J,i), $f_{|M(I)}(x) - u^T x$ and $f_{|M}(x) - u^T x$ are Morse functions (Appendix (6)).

Let $A_{\bar{b}}$ be a set of such u's in R^n.

Then a program

$$(\bar{Q}) \quad \text{minimize } \{f(x) - u^T x \quad \text{subject to} \quad g(x) \leq \bar{b}\}$$

where $\bar{b} \in B$, $u \in A_{\bar{b}}$; is a strong Morse program. It suffices to show $(\bar{Q})$ satisfies strict complementary slackness.

Let x be a critical point of $(\bar{Q})$ with $J = J(x) \neq \phi$, namely x is a critical point of $f - u^T x$ on $M_J = g_J^{-1}(b_J)$. Since $M_J = \partial M(J - \{j\}, j)$, x is not a critical point of $f - u^T x$ on $M(J - \{j\}, j)$ for any $j \in J$. Therefore by Proposition 11 we obtain that there exists a unique $\lambda_J \in R^{|J|}$ such that $Df(x)^T - u + Dg_J(x)^T \lambda_J = 0$ and $\lambda_j \neq 0$ for any $j \in J$. This completes the proof.

By Araujo/Mas-Colell ([1], Theorem 1), we have also

Fix any $v \in R^m$, then for almost every $u \in R^n$ (Q^u_v) has at most one global solution.

Therefore using Fubini's theorem we obtain

Theorem M (cf. Spingarn/Rockafellar [13], Corollary). *If $f \in C^2$, $g \in C^n$ then for almost every $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$, (Q^u_v) is a strong Morse program having at most one global solution.*

IV. Equality and Inequality Constraints

Let us consider a program

$$(R) \quad \text{minimize } \{f(x) - u^T x \quad \text{subject to} \quad g(x) \le b,\ h(x) = c\}$$

and a perturbation of (R)

$$(R^u_{v,w}) \quad \text{minimize } \{f(x) - u^T x \quad \text{subject to} \quad g(x) \le b+v,\ h(x) = c+w\}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $h : R^n \to R^p$; $f, g, h \in C^2$; $u \in R^n$, $v \in R^m$, $w \in R^p$; $n \ge p$.

Definition. $\begin{pmatrix} b \\ c \end{pmatrix} \in R^m \times R^p$ is called an *active regular value* of (R), denoted by $\begin{pmatrix} g \\ h \end{pmatrix} \overline{\pitchfork} \begin{pmatrix} b \\ c \end{pmatrix}$, if $\begin{pmatrix} g_J \\ h \end{pmatrix} \pitchfork \begin{pmatrix} b_J \\ c \end{pmatrix}$ for every $J \subseteq I = \{1, 2, \ldots, m\}$.

Definition. x is a *critical point of* (R) if $g(x) \le b$, $h(x) = c$ and x is a critical point of f on $M_J = \begin{pmatrix} g_J \\ h \end{pmatrix}^{-1} \begin{pmatrix} b_J \\ c \end{pmatrix}$ where $J = J(x)$.

Definition. A program (R) is a *strong Morse program* if

(a) $\begin{pmatrix} g \\ h \end{pmatrix} \overline{\pitchfork} \begin{pmatrix} b \\ c \end{pmatrix}$

(b) f is a Morse function on M_J for every $J \subseteq I$

(c) Every Lagrange multiplier associated with an active constraint is nonzero.

We have the following results which are easily verified.

Theorem N. *If* (R) *is a strong Morse program and* x *is a critical point of* (R) *with* $J = J(x)$, *then*

(a) $(|J| + p) \times n$ *matrix* $\begin{pmatrix} Dg_J(x) \\ Dh(x) \end{pmatrix}$ *has full rank*

(b)* *there exists a unique* $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in R^m \times R^p$ *such that* $Df(x)^T + Dg(x)^T\lambda + Dh(x)^T\mu = 0$, $\lambda_i \neq 0$ *iff* $i \in J$, $\mu_j \neq 0$ *for all* $j = 1, \ldots, p$

(c) $\mathcal{L}(x) = D^2f(x) + \sum_{i=1}^{m} \lambda_i D^2 g_i(x) + \sum_{j=1}^{p} \mu_j D^2 h_j(x)$ *induces an isomorphism on* $T_x M_J$

(d) *On* $T_x M_J$, $\mathcal{L}(x)$ *is positive definite iff* x *is a local minimum; negative definite iff* x *is a local maximum; indefinite iff* x *is a saddle point on* M_J .

Theorem O. *If* $f \in C^2$, $g, h \in C^n$ *then for almost every* $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in R^n \times R^m \times R^p$, $(R^u_{v,w})$ *is a strong Morse program having at most one global solution.*

Remark 1. To show all $\mu_j \neq 0$, we consider the following manifolds

$$N(J,K_j) = \{x | g_J(x) = b_J \ , \ h_{K_j}(x) = c_{K_j} \ , \ h_j(x) \leq c_j\}$$

for all J and $K_j = \{1, 2, \ldots, p\} - \{j\}$. Use the same argument given in the proof of Theorem M.

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* $\lambda_J > 0$ $(\lambda_J < 0)$ if x is a local minimum (maximum) (see Luenberger [9]).

Remark 2. Theorems A and C can be extended in the sense that each $\lambda_i \neq 0$, since each constraint is active (set $I = \phi$ in Theorems N and O).

V. Equality Constraints and One Regular Inequality Constraint

Let us consider a program

$$(\overline{P}) \quad \text{minimize } \{f(x) \text{ subject to } g(x) = b, \ \|x\|^2 \leq c\}$$

and a perturbation of $(\overline{P})$

$$(\overline{P}^u_v) \quad \text{minimize } \{f(x) - u^T x \text{ subject to } g(x) = b+v, \ \|x\|^2 \leq c\}$$

where $f : R^n \to R$, $g : R^n \to R^m$; $f, g \in C^2$; $u \in R^n$, $v \in R^m$; $n \geq m+1$. Let $h(x) = \|x\|^2$, $D = \{x \mid \|x\|^2 \leq c\}$, $S = \{x \mid \|x\|^2 = c\}$, then $h \pitchfork c$ for any $c \in R^1$ such that $c > 0$, i.e. h is a regular constraint. If $g \pitchfork b$, $g|_S \pitchfork b$ then $M = g^{-1}(b) \cap D$ is $(n-m)$-dimensional C^2 manifold with boundary $\partial M = g^{-1}(b) \cap S$ and $\begin{pmatrix} g \\ h \end{pmatrix} \pitchfork \begin{pmatrix} b \\ c \end{pmatrix}$ by Lemma 10.

Hence by Theorem O we have

Corollary 12. *If* $g \in C^n$, *then* $(\overline{P}^u_v)$ *is a strong Morse program for almost every* $\begin{pmatrix} u \\ v \end{pmatrix} \in R^n \times R^m$.

Let us consider the family of programs on $D = \{x \mid \|x\|^2 \leq c\}$

$$\underset{x \in D}{\text{minimize}} \ \{f(x) \text{ subject to } g(x) = b\}$$

for some $f : R^n \to R$, $g : R^n \to R^m$, $f \in C^2$, $g \in C^n$.

<u>Definition</u>. Let $C^2(D, R^k)$ be the set of all C^2 functions from D to R^k for some $k \geq 1$. C^2 <u>norm topology</u> $\|\cdot\|_2$ on $C^2(D, R^k)$ is defined by $\|\phi\|_2 = \max_{x \in D} \{\|\phi(x)\|, \|D\phi(x)\|, \|D^2\phi(x)\|\}$ for $\phi \in C^2(D, R^k)$ where $\|\cdot\|$ is the Euclidean norm (all $n \times m$ matrices are considered to be in $R^{n \times m}$).

<u>Lemma 13</u>. <u>If</u> $g|_D \pitchfork b$, $g|_S \pitchfork b$, $\|g^n - g\|_2 \to 0$ <u>then</u> $g^n|_D \pitchfork b$, $g^n|_S \pitchfork b$ <u>for sufficiently large</u> n .

<u>Proof</u>. It suffices to show that if $g|_D \pitchfork b$, $\|g^n - g\|_2 \to 0$ then $g^n|_D \pitchfork b$ for sufficiently large n . Suppose it is not true. Then there exists $x^n \in g^{n^{-1}}(b) \cap D$ such that $Dg^n(x^n)$ is not of full rank for sufficiently large n .

Since D is compact, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $x^{n_j} \to x^0$ for some $x^0 \in D$. However $\|g^n - g\|_2 \to 0$ implies $g^{n_j}(x^{n_j}) \to g(x^0)$, $Dg^{n_j}(x^{n_j}) \to Dg(x^0)$. Then $g(x^0) = b$, $Dg(x^0)$ is of full rank hence $Dg^{n_j}(x^{n_j})$ is of full rank for sufficiently large n_j which contradicts the assumption.

Q.E.D.

Under the same assumptions of Lemma 13, for sufficiently large ℓ let us define $M = g^{-1}(b) \cap D$, $M^\ell = g^{\ell^{-1}}(b) \cap D$. Then M and M^ℓ are $(n-m)$-dimensional manifolds with boundaries $\partial M = g^{-1}(b) \cap S$, $\partial M^\ell = g^{\ell^{-1}}(b) \cap S$ (Lemma 10). Then we have

<u>Lemma 14</u>. <u>If</u> f <u>is a Morse function on</u> M <u>and</u> $\|f^n - f\|_2 \to 0$, <u>then</u> f^n <u>is a Morse function on</u> M^n <u>for sufficiently large</u> n .

Proof. Let $F^n_b(x,\lambda) = \begin{pmatrix} Df^n(x)^T + Dg^n(x)^T\lambda \\ g^n(x) - b \end{pmatrix}$. By Theorem B (i.e. $F_b \pitchfork 0$ iff f is a Morse function on $g^{-1}(b)$, if $g \pitchfork b$), $F_b \pitchfork 0$. By Lemma 13 and by the same argument, we have $F^n_b \pitchfork 0$ for sufficiently large n, if $\|f^n - f\|_2 \to 0$ and $\|g^n - g\|_2 \to 0$. Hence by Theorem B, f^n is a Morse function on M^n for sufficiently large n.

Q.E.D.

Lemma 15. Under the same assumptions of Lemma 14, if f on M has no critical point on ∂M then f^n on M^n has no critical point on ∂M^n for sufficiently large n.

Proof. Suppose it is not true, then there exists $x^n \in \partial M^n$ such that x^n is a critical point of f^n on M^n for sufficiently large n. Then there exists a unique $\lambda^n \in R^m$ such that

$$Df^n(x^n)^T + Dg(x^n)^T\lambda^n = 0 .$$

Since D is compact, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $x^{n_j} \to x^0$ for some $x^0 \in D$. Since $\|f^n - f\|_2 \to 0$, $\|g^n - g\|_2 \to 0$ and $\|x^n\| = c$, we have $\|x^0\| = c$, $g(x^0) = b$ and $Df(x^0)^T + Dg(x^0)^T\lambda^0 = 0$ where $\lambda^0 = \lambda(x^0)$ (see Lemma 5). This shows $x^0 \in \partial M$ is a critical point of f on M which contradicts our assumption.

Q.E.D.

Combining Proposition 11, Corollary 12, Lemmas 13, 14, 15 we obtain

Theorem P.* In the C^2 topology, strong Morse programs are open and dense in the family of programs

$$\text{minimize } \{f(x) \text{ subject to } g(x) = b, \ \|x\|^2 \leq c\}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $f \in C^2$, $g \in C^n$.

Remark. When we consider equality and inequality constraints programs of type (R) in IV, we have the following result.

Theorem Q. The family of weak Morse programs is open and dense in the space of programs if the space of programs is given the Whitney C^2 topology.

Proof. The proof follows from the argument in Theorem P and the characterization of the Whitney C^2 topology found on page 43 of Golubitsky and Guillemin [4].

Q.E.D.

VI. Fixed and Variable Constraints

Consider a program

$$\text{(S)} \quad \underset{\substack{G(x)\leq 0 \\ H(x)=0}}{\text{minimize}} \{f(x) \text{ subject to } g(x) \leq b, \ h(x) = c\}$$

and a perturbation of (S)

*A special case of this theorem, Theorem 1 in [3], was first shown by Chichilnsky and Kalman. It follows from Theorem P that every critical point of a strong Morse program is strongly stable in the sense of Kojima [8].

$$(S^u_{v,w}) \quad \underset{\substack{G(x)\leq 0 \\ H(x)=0}}{\text{minimize}}\{f(x) - u^T x \ \text{ subject to } \ g(x) \leq b+v,\ h(x) = c+w\}$$

where $f : R^n \to R^1$, $g : R^n \to R^m$, $h : R^n \to R^p$, $G : R^n \to R^r$, $H : R^n \to R^s$; $f, g, h, G, H \in C^2$; $u \in R^n$, $v \in R^m$, $w \in R^p$.

We impose the following conditions (a)-(d) on G and H :

(a) $\begin{pmatrix} G_\alpha \\ H \end{pmatrix} \pitchfork \begin{pmatrix} 0_\alpha \\ 0 \end{pmatrix}$ for every $\alpha \subseteq \{1, \ldots, r\}$.

(b) $G_k \pitchfork 0$, $H_\ell \pitchfork 0$ for all k and ℓ .

(c) $\left.\begin{pmatrix} G_\alpha \\ H \end{pmatrix}\right|_{G_k^{-1}(0)} \pitchfork \begin{pmatrix} 0_\alpha \\ 0 \end{pmatrix}$ for every α and $k \notin \alpha$.

(d) $\left.\begin{pmatrix} G_\alpha \\ H_{\beta_\ell} \end{pmatrix}\right|_{H_\ell^{-1}(0)} \pitchfork \begin{pmatrix} 0_\alpha \\ 0_{\beta_\ell} \end{pmatrix}$ for every α and $\beta_\ell = \{1, \ldots, s\} - \{\ell\}$.

For example, if $G_k(x) = -x_k$ for $k = 1, \ldots, n$, then obviously G satisfies (a)-(d) and (S) becomes

$$\underset{x\geq 0}{\text{minimize}}\{f(x) \ \text{ subject to } \ g(x) \leq b,\ h(x) = c\} .$$

If $G, H \in C^n$ then generically G and H satisfy (a)-(d) (cf. Lemma 9). Let $N_\alpha = \{x | G_\alpha(x) = 0_\alpha, H(x) = 0\}$ for $\alpha \subseteq \{1, \ldots, r\}$, then N_α is $(n-|\alpha|-s)$-dimensional manifold in R^n . Let us consider all $\begin{pmatrix} b \\ c \end{pmatrix} \in R^m \times R^p$ that satisfy the following conditions (c1)-(c6)

(c1) $\begin{pmatrix} g_J \\ h \end{pmatrix} \pitchfork \begin{pmatrix} b_J \\ c \end{pmatrix}$, $\begin{pmatrix} g_J \\ h_{K_j} \end{pmatrix} \pitchfork \begin{pmatrix} b_J \\ c_{K_j} \end{pmatrix}$ for all $J \subseteq \{1, \ldots, m\}$

and $K_j = \{1, \ldots, p\} - \{j\}$.

(c2) $g_i \pitchfork b_i$, $h_j \pitchfork c_j$ for all i and j .

(c3) $\begin{pmatrix} g_J \\ h \end{pmatrix}\Big|_{N_\alpha} \pitchfork \begin{pmatrix} b_J \\ c \end{pmatrix}$, $\begin{pmatrix} g_J \\ h_{K_j} \end{pmatrix}\Big|_{N_\alpha} \pitchfork \begin{pmatrix} b_J \\ c_{K_j} \end{pmatrix}$ for all J , K_j and α .

(c4) $g_i\big|_{N_\alpha} \pitchfork b_i$, $h_j\big|_{N_\alpha} \pitchfork c_j$ for all i , j , and α .

(c5) $\begin{pmatrix} g_J \\ h \end{pmatrix}\Big|_{N_\alpha \cap g_i^{-1}(b_i)} \pitchfork \begin{pmatrix} b_J \\ c \end{pmatrix}$ for all J , $i \notin J$, and α .

(c6) $\begin{pmatrix} g_J \\ h_{K_j} \end{pmatrix}\Big|_{N_\alpha \cap h_j^{-1}(c_j)} \pitchfork \begin{pmatrix} b_J \\ c_{K_j} \end{pmatrix}$ for all J , K_j , and α .

If $g, h \in C^n$ then using Sard's theorem and Fubini's theorem the set of all $\begin{pmatrix} b \\ c \end{pmatrix}$ satisfying (c1)-(c6) has full measure in $R^m \times R^p$ (cf. Lemma 9). Let $M_J = \{x | g_J(x) = b_J,\ h(x) = c\}$ and $\begin{pmatrix} b \\ c \end{pmatrix}$ satisfying (c1)-(c6). Then for almost every $u \in R^n$, $f(x) - u^T x$ is a Morse function on $M_J \cap N_\alpha$ for all α , J and $f(x) - u^T x$ on $M_J \cap N_\alpha \cap g_i^{-1}(-\infty, b_i]$ (or $g_J^{-1}(b_J) \cap h_{K_j}^{-1}(c_{K_j}) \cap N_\alpha \cap h_j^{-1}(-\infty, c_j]$) has no critical point on $M_J \cap N_\alpha \cap g_i^{-1}(b_i)$ (or $g_J^{-1}(b_J) \cap h_{K_j}^{-1}(c_{K_j}) \cap N_\alpha \cap h_j^{-1}(c_j) = M_J \cap N_\alpha$) for all a , J , $i \notin J$ and K_j . (Appendix (6), Theorem L.)

Now let us fix $u \in R^n$ and $\begin{pmatrix} b \\ c \end{pmatrix}$ satisfying the above conditions. Let $x \in R^n$ be a feasible point of (S) and a critical point of $f(x) - u^T x$ on $M_J \cap N_\alpha$ where $J = \{i | g_i(x) = b_i\}$ and $\alpha = \{k | G_k(x) = 0\}$. By (c3) $\begin{pmatrix} g_J \\ h \end{pmatrix}\Big|_{N_\alpha} \pitchfork \begin{pmatrix} b_J \\ c \end{pmatrix}$ hence $N_\alpha \pitchfork M_J$ (Appendix (12)) and since

$x \in M_J \cap N_\alpha \neq \phi$, we have $|J| + p + |\alpha| + s \leq n$ and
$T_x(M_J \cap N_\alpha) = T_xM_J \cap T_xN_\alpha = \operatorname{Ker} Dg_J(x) \cap \operatorname{Ker} Dh(x) \cap \operatorname{Ker} DG_\alpha(x) \cap \operatorname{Ker} DH(x)$ (Appendix (13)). Hence by Lemma 1, there exist unique $\lambda \in R^m$, $\mu \in R^p$, $\xi \in R^r$, $\eta \in R^s$ such that

$$Df(x)^T - u + Dg(x)^T\lambda + Dh(x)^T\mu = -(DG(x)^T\xi + DH(x)^T\eta) \in T_xN_\alpha^\perp ,$$

$$\lambda_{J^c} = 0 , \quad \xi_{\alpha^c} = 0 .$$

Moreover using (c5), (c6), and the same argument in the proof of Proposition 11, we have

$$\lambda_i \neq 0 \text{ iff } i \in J \text{ and } \mu_j \neq 0 \text{ for all } j = 1, \ldots, p .$$

Hence we obtain

Theorem R. *Let* $g, h \in C^n$. *Then for almost every* $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in R^n \times R^m \times R^p$, $(S^u_{v,w})$ *has the following properties.*

If x is a feasible point of $(S^u_{v,w})$ *and a critical point of* $f(x) - u^Tx$ *on* $M'_J \cap N_\alpha$ *where* $J = \{i | g_i(x) = b_i + v_i\}$, $\alpha = \{k | G_k(x) = 0\}$, $M'_J = g_J^{-1}(b+v) \cap h^{-1}(c+w)$, $N_\alpha = G_\alpha^{-1}(0_\alpha) \cap H^{-1}(0)$, *then*

(a) $(Dg_J(x)^T, Dh(x)^T, DG_\alpha(x)^T, DH(x)^T)$ *has full rank.*

(b) *there exist unique* $\lambda \in R^m$, $\mu \in R^p$, $\xi \in R^r$, $\eta \in R^s$ *such that* $Df(x)^T - u + Dg(x)^T\lambda + Dh(x)^T\mu = -(DG(x)^T\xi + DH(x)^T\eta) \in T_xN_\alpha^\perp$; $\lambda_i \neq 0$ *iff* $i \in J$; $\mu_j \neq 0$ *for all* $j = 1, \ldots, p$; $\xi_k = 0$ *for* $k \notin \alpha$.

(c) $\mathcal{L}(x) = D^2 f(x) + \sum_i \lambda_i D^2 g_i(x) + \sum_j \mu_j D^2 h_j(x) + \sum_k \xi_k D^2 G_k(x) + \sum_\ell \eta_\ell D^2 H_\ell(x)$ <u>induces an isomorphism on</u> $T_x(M'_J \cap N_\alpha)$.

(d) <u>on</u> $T_x(M'_J \cap N_\alpha)$, $\mathcal{L}(x)$ <u>is positive definite if x is a local minimum; negative definite if x is a local maximum, indefinite iff x is a saddle point on</u> $M'_J \cap N_\alpha$.

<u>Remark</u>

Theorem R gives a geometrical interpretation of Spingarn ([12], Theorem 3.9, Theorem 4.7), where we restrict ourselves to manifolds with "corners."

In fact, if the fixed constraint set C can be reduced to considering at most a countable number of programs that are locally the same as (S) (these are special cases of cyrtohedra; see Spingarn [11]; [12], Lemma 3.7), then our analysis can be applied to nonlinear programs

$$(\bar{S}) \quad \underset{x \in C}{\text{minimize}}\{f(x) \quad \text{subject to} \quad g(x) \le b,\ h(x) = c\}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $h : R^n \to R^p$; $f, g, h \in C^2$; $n \geq p$.

<u>ACKNOWLEDGMENT</u>

The author is indebted to Professor D. J. Brown for several helpful conversations concerning the ideas in this paper.

VII. Appendix (Guillemin/Pollack [5], Hirsh [7])

(1) Morse lemma

Let $p \in M$ be a nondegenerate critical point of $f : M \to R^1$. Then there is a local coordinate system $(x_1, \ldots, x_m)$ in a neighborhood U of p such that

$$f = f(p) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_m^2$$

for some $0 \leq \lambda \leq m$.

(2) Sard's theorem (with boundary)

Let $f : X \to Y$ be a C^γ map of a C^γ manifold X with boundary ∂X into a boundaryless C^γ manifold Y. Then for almost every $y \in Y$ is a regular value of both $f : X \to Y$ and $f|_{\partial X} : \partial X \to Y$ if $\gamma > \max(0, \dim X - \dim Y)$.

(3) Fubini's theorem

Let $A \subseteq R^n \times R^m$ be a measurable set such that for almost every $v \in R^m$, $A_v = \{u \in R^n | \begin{pmatrix} u \\ v \end{pmatrix} \in A\}$ has measure zero in R^n. Then A has measure zero in $R^n \times R^m$.

(4) Parametric transversality theorem

Let $F : X \times V \to Y$ be a C^γ map of C^γ manifolds and A be any C^γ submanifold of Y. If $F \pitchfork A$ and $\gamma > \max(0, \dim X - \dim Y)$ then $F_v \pitchfork A$ for almost every $v \in V$ where $F_v(x) = F(x,v)$ for $x \in X$.

(5) Let $f : X \to Y$ be a C^γ map such that $f \pitchfork Z$ for a C^γ submanifold Z of Y, then $f^{-1}(Z)$ is a C^γ submanifold of X and $\dim f^{-1}(Z) = \dim X - \dim Y + \dim Z$. As a special case if $f \pitchfork y$ for

some $y \in Y$, then $f^{-1}(y)$ is a C^γ submanifold of X and $\dim f^{-1}(y) = \dim X - \dim Y$.

(6) Let $f : X \to R$ be a C^2 map of a C^2 manifold X in R^n. Then for almost every $u \in R^n$ the function $f(x) - u^T x$ is a Morse function on X.

(7) Let $X \subseteq R^n$ be m-dimensional manifold with boundary ∂X. Then for each point $x \in \partial X$, there exists an open set $\tilde{U}$ of R^n and a submersion $g : \tilde{U} \to R^{n-m}$ such that $U = X \cap \tilde{U} = g^{-1}(0)$ and $x \in \partial U = \partial X \cap \tilde{U}$.

(8) An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.

(9) Let X, Y be manifolds with $\dim X < \dim Y$. If $f : X \to Y$ is a C^1 map then $f(X)$ has measure zero in Y.

(10) Let $f : X \to R^1$ be a C^γ map such that $f \pitchfork c$ for some $c \in R^1$. Then $\{x | f(x) \leq c\}$ is a C^γ submanifold of X with boundary $f^{-1}(c)$.

(11) Let $f : X \to Y$ be a C^γ map of a C^γ manifold X with boundary ∂X onto a boundaryless C^γ manifold Y. If $f \pitchfork Z$, $f|_{\partial X} \pitchfork Z$ for a boundaryless submanifold Z of Y, then $f^{-1}(Z)$ is a C^γ submanifold of X with boundary $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$ and $\dim f^{-1}(Z) = \dim X - \dim Y + \dim Z$.

(12) Let $f : X \to Y$, $g : Y \to Z$ be C^γ maps and W be a C^γ submanifold of Z such that $g \pitchfork W$. Then $f \pitchfork g^{-1}(W)$ iff $gf \pitchfork W$.

(13) Let X, Z be submanifolds of Y such that $X \pitchfork Z$. Then $X \cap Z$ is again a submanifold of Y, $\dim(X \cap Z) = \dim X + \dim Z - \dim Y$ and $T_x(X \cap Z) = T_xX \cap T_xZ$ for any $x \in X \cap Z$.

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