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A STRATEGIC MARKET GAME

WITH PRICE AND QUANTITY STRATEGIES

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by

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1. INTRODUCTION

The noncooperative game theoretic approach to exchange, in contrast with general equilibrium theory requires the complete specification of the market mechanism through which trade is conducted. Elsewhere several different market mechanisms have been investigated. In particular a "sell-all" model has been considered by Shubik [1] and Shapley and Shubik [2]; Dubey and Shubik [3] have investigated a "bid-offer" model of trade and Shubik has considered a simple market with price-quantity strategies [4] and it has been suggested that there are only a limited number of market mechanisms involving simultaneous bidding in terms of money, goods or prices [5]; and which have intrinsically symmetric roles for all individuals.

A model related to the one suggested here is that of Wilson [8], however he has a special nonsymmetric role for the auctioneer.

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## 2. THE DOUBLE AUCTION MARKET

A mechanism is described for a single market and then it is generalized for  $m$  markets. Suppose that  $n$  traders have endowments of two commodities, one of which serves as a means of payment and a numeraire. We fix the price of a unit of numeraire at 1.

All traders are required to move simultaneously without knowledge of each other's actions by bidding and offering in a market. The endowment of trader  $i$  is given by  $(a_1^i, a_2^i)$ ; a move by trader  $i$  (which is also his strategy) is described by four numbers  $(p^i, q^i; \tilde{p}^i, \tilde{q}^i)$  which are interpreted as follows:

$p^i$  = the maximum price (in terms of the numeraire) that  $i$  will pay to buy  $q^i$  or fewer units of the first good. We require that  $p^i q^i \leq a_2^i$ , i.e., he cannot bid more money than he has on hand. We assume  $q^i \geq 0$ .

$\tilde{p}^i$  = the minimum price that  $i$  will accept to sell  $\tilde{q}^i$  or fewer units of the first good. We require that  $0 \leq \tilde{q}^i \leq a_1^i$ , i.e., he cannot offer for sale that which he does not possess.

Although it may be unlikely that a trader will wish both to sell and buy the same item at the same time, there is no a priori reason to rule this behavior out, hence the strategies employed here permit an individual to be on both sides of the market if he so chooses.

Figures 1a, b, and c show possible configurations of supply and demand for the market in aggregate. The supply schedules are obtained by ranking the offer prices in ascending order and cumulating supply; similarly demand prices are ranked, but in descending order.

The market mechanism works as follows. The aggregate supply and demand schedules are calculated and the market price is fixed at the price

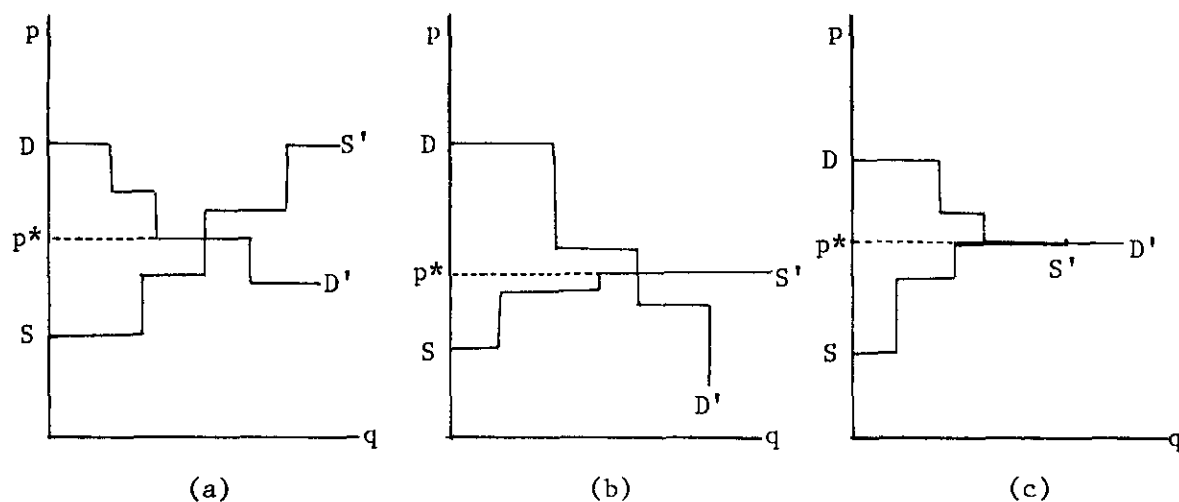


FIGURE 1

given by their intersection. All suppliers who require a higher price sell nothing; all buyers who require a lower price buy nothing. At the margin, as is shown in Figures 1a and c there may be some buyers willing to pay the market price of  $p^*$  for whom there is not enough supply. Similarly there may be sellers willing to supply more at  $p^*$  than the buyers wish to buy. This is shown in Figure 1b.

Figure 2 illustrates four further possibilities. In Figure 2a there will be no trade as there is no  $p^*$  for which both sides of the market will be active. Figure 2b illustrates an instance where there is an open range of prices at which effective demand will equal supply. Figure 2c shows an instance where all buyers and sellers have their demands and offers met.

When as in Figures 1a and 1c there is excess demand at  $p^*$  or, as in Figure 1b there is excess supply we adopt the convention that the marginal buyers or sellers are rationed in proportion to their demands or offers. In a market such as that shown in Figure 2a no trade will take place. In markets such as shown in 2b and 2d we assume that a price

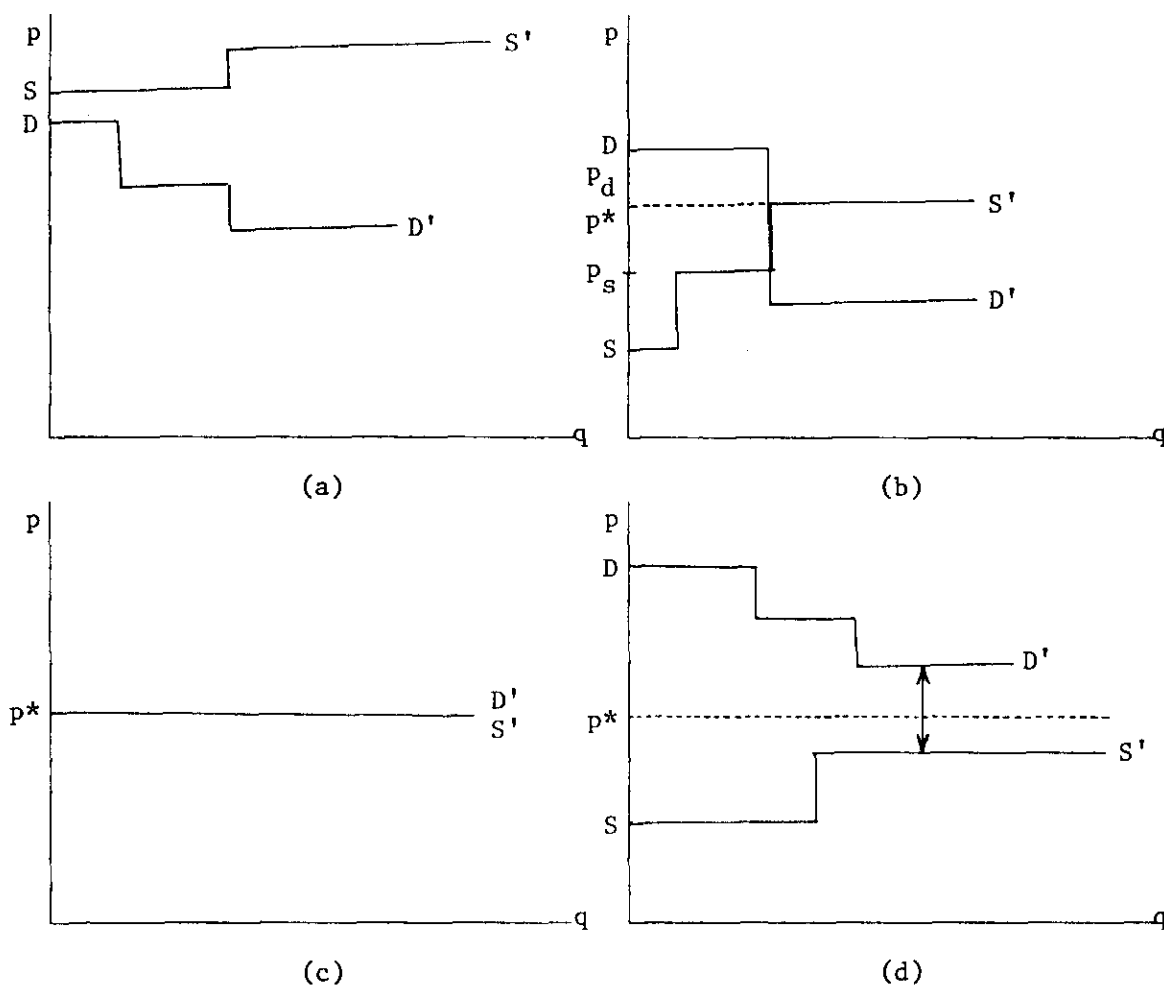


FIGURE 2

interior to the range  $p_s p_d$  is selected by some convention, for example  $p^*$  may be selected as the midpoint of the indeterminate range.

### 2.1. Variations

Before we set up and analyze the  $m$  commodity market structure several points are made concerning variants of this model.

In contrast with the mechanisms analyzed in the papers already noted [1, 2, 3] the payoff functions arising from the game are highly discontinuous as functions of the strategies. Intuitively this is the same distinction that can be made between the Bertrand-Edgeworth and the

Cournot models of duopolistic competition. Even here, the distinction in results obtained by Cournot and Bertrand are preserved. The Cournot noncooperative equilibria converge slowly towards the competitive equilibria (under the appropriate circumstances) but in general for a finite number of traders the noncooperative equilibria are not efficient [9]. In contrast, under the appropriate circumstances, as was shown by Bertrand full competition may begin with two competitors; i.e. there may be noncooperative equilibria which are also competitive equilibria when there are as few as two competitors on each side of the market.

In actual markets sometimes a trader may buy or sell shares or other items through several accounts, or under several names. This proxy account trading behavior is frequently associated with struggles for control, however even setting aside the corporate control features of stock trading there is no a priori reason to rule out multiple account trading. In a separate discussion [10] we have shown that the assumption that there is no gain to be had from multiple account trading is equivalent to showing that the game has strategies which can be aggregated, i.e. individuals are concerned only with their own moves and the sum of the moves of the others.

The model of trade above is only one from a fairly natural class of models which include the conventions of:

- (a) maximize trade;
- (b) maximize the take of a middleman;
- (c) maximize the surplus of one side.

In the first instance in a situation such as that shown in Figure

la the highest priced seller who would otherwise be extramarginal could be matched with the highest priced demander.

In the second instance we could introduce a middleman who buys from the sellers and sells to the buyers and keeps the "spread" for himself.

A third convention is to have the sellers sell up to the supply-demand intersection with the buyers all buying at the price they named; i.e. all the surplus goes to the sellers. For example, we may begin by matching the lowest priced seller with the highest priced buyer; if they do not quite match the residual goes to the second seller or buyer and so on.

These somewhat different models are examined elsewhere [11].

### 3. THE NONCOOPERATIVE EQUILIBRIA OF A PRICE-QUANTITY STRATEGY GAME

Let there be  $n$  individuals trading in  $m$  commodities using credit as a means of payment. Each trader  $i$  has an endowment of  $(a_1^i, a_2^i, \dots, a_m^i)$  where  $a_j^i \geq 0$  for all  $i = 1, \dots, n$ ;  $j = 1, \dots, m$ , and at least one  $a_j^i > 0$  for  $j = 1, \dots, m$ . Also assume that for each  $j$  there is some  $u^j$  which is strictly increasing in the  $j^{\text{th}}$  variable.

A strategy  $s^i$  of player  $i$  is to announce four vectors in  $R_+^m$ :  $p^i$ ,  $q^i$ ,  $\tilde{p}^i$ ,  $\tilde{q}^i$ . How this leads to the disbursement of commodities has already been discussed. In addition we can compute the net credit of  $i$  as: money obtained from sales minus money spent on purchases. Suppose  $i$  ends up with the final bundle  $x^i \in R_+^m$ , and the credit  $\beta^i$ . Then his payoff  $\Pi^i(s^1, \dots, s^n)$  in the game is  $u^i(x^i) + \lambda^i \min[0, \beta^i]$ , where  $\lambda^i$  is a preassigned positive constant. This simply says that while having surplus fiat money is useless, being

in debt involves a "bankruptcy penalty" which has disutility for the trader (e.g. some of his commodities may be confiscated).

The strategy set  $S^i$  of trader  $i$  is thus a subset of  $R_+^m \times R_{++}^m \times R_+^m \times R_{++}^m$  (Here  $R_{++}^m$  is the interior of  $R_+^m$ , i.e., traders are not allowed to name zero prices.) Put  $S = S^1 \times \dots \times S^N$ . For any  $s = (s^1, \dots, s^N) \in S$ ,  $M \subset N$  and  $e = \{e^i : i \in M\} \in \prod_{i \in M} S^i$ , let  $(s|e)$  denote the element of  $S$  obtained from  $s$  by replacing  $s^i$  by  $e^i$  for each  $i \in M$ . Define  $s$  to be M-efficient if there does not exist any  $e \in \prod_{i \in M} S^i$  such that:

$$\Pi^i(s|e) \geq \Pi^i(s), \text{ all } i \in M;$$

$$\Pi^l(s|e) > \Pi^l(s), \text{ some } l \in M.$$

If  $s$  is  $\{i\}$ -efficient for each  $i \in N$ , we call it a non-cooperative equilibrium (N.E.); if it is  $N$ -efficient, we call it simply efficient; if it is  $M$ -efficient for all  $M \subset N$ , we call it a strong non-cooperative equilibrium.

This game has certain trivial N.E., e.g. those in which the traders announce that they will buy and sell nothing for some subset of the trading-posts. We focus our attention henceforth on non-trivial N.E. i.e. those in which trade actually occurs in each post.

Finally, recall that  $(p; x^1, \dots, x^N)$  is a competitive equilibrium (C.E.) of the market (where  $p$  and  $x^i$  are in  $R_+^m$ ) if

$$\sum_{i=1}^n x^i = \sum_{i=1}^n a^i \text{ and } x^i \text{ maximizes } u^i \text{ on } i\text{'s budget set}$$

$$B^i(p) = \{x \in R_+^m : p \cdot x \leq p \cdot a^i\} \text{ i.e. } x^i \in \tilde{B}^i(p) = \{x \in B^i(p) : u^i(x) = \max_{y \in B^i(p)} u^i(y)\}.$$

Also note that with each C.E. we can associate



shadow prices of income  $\lambda^1, \dots, \lambda^n$  where  $\lambda^i \geq 0$  is chosen to ensure that

$$\max_{y \in R_+^m} u^i(y) + \lambda^i(p \cdot a^i - p \cdot y)$$

has  $x^i$  as a solution.

We are ready to state our main result:

Theorem. Let  $(\hat{p}; \hat{x}^1, \dots, \hat{x}^n)$  be the prices and allocation produced at an N.E. Then the set of traders who are not optimal on their budget sets i.e.  $\{i \in N : \hat{x}^i \notin \hat{B}^i(p)\}$ , has at most  $2k$  members. Moreover, each C.E. has corresponding to it a strong N.E. which produces the same prices (up to scaling) and allocation.

Suppose that  $\{p^i, q^i, \tilde{p}^i, \tilde{q}^i\}_{i \in N}$  is an active N.E. of  $\Gamma(E, \lambda)$  which produces prices  $\hat{p}$  and the allocation  $\hat{x}^1, \dots, \hat{x}^n$ . First note that if  $\beta^1, \dots, \beta^n$  is the net credit of the traders, then

$\sum_{i=1}^n \beta^i = 0$ . If  $\beta^i > 0$  for some  $i$  then  $i$  could improve his payoff

by buying more of a commodity he desires without going bankrupt, a con-

tradiction. Hence  $\beta^i = 0$ , i.e.  $x^i \in B^i(p)$ , for each  $i$ . Call

a trader  $i$  "interior" at this N.E. if there is no commodity  $j$  at

which  $i$  is the only active marginal buyer or seller, i.e. at which:

$i$  actively buys (sells)  $j$ ,  $p_j^i = \hat{p}_j$ , and  $p_j^\ell > \hat{p}_j$  ( $p_j^\ell < \hat{p}_j$ ) for

all traders  $\ell$  who actively buy (sell)  $j$ . We will show that if  $i$

is interior then  $\hat{x}^i \in \hat{B}^i(p)$ . This is done by a contradiction. W.l.o.g.

let  $1$  be interior and suppose  $u^1(\hat{x}^1) < u^1(y)$  for some  $y \in \hat{B}^1(\hat{p})$ . Define

$$J = \{j : y_j - \hat{x}_j^1 > 0\}$$

$$J' = \{j : \hat{x}_j^1 - y_j < 0\}$$

$$T_j = \text{total active sale at trading-post } j$$

$$s_j = \text{sale of } l \text{ at trading-post } j$$

$$d_j = \text{purchase of } l \text{ at trading-post } j$$

$$d p_j = \max\{p_j^i : i \text{ actively buys } j\}$$

$$s p_j = \min\{\tilde{p}_j^i : i \text{ actively sells } j\}$$

Let  $0 < t < 1$  be chosen sufficiently small so as to ensure that:

$$T_j - d_j + t(y_j - \hat{x}_j^1) > 0 \text{ for } j \in J ;$$

$$T_j - s_j + t(\hat{x}_j^1 - y_j) > 0 \text{ for } j \in J' .$$

We now construct a strategy  $(*_p^1, *_q^1, \tilde{*_p}^1, \tilde{*_q}^1)$  for  $l$  as follows:\*

$$*_p^1 = \begin{cases} d p_j + \epsilon & \text{if } j \in J \\ p_j^1 & \text{otherwise} \end{cases}$$

$$\tilde{*_p}^1 = \begin{cases} s p_j - \epsilon & \text{if } j \in J' \\ \tilde{p}_j^1 & \text{otherwise} \end{cases}$$

$$*_q^1 = \begin{cases} q_j^1 + t(y_j - \hat{x}_j^1) & \text{for } j \in J \\ q_j^1 & \text{otherwise} \end{cases}$$

$$\tilde{*_q}^1 = \begin{cases} \tilde{q}_j^1 + t(\hat{x}_j^1 - y_j) & \text{for } j \in J' \\ \tilde{q}_j^1 & \text{otherwise.} \end{cases}$$

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\*For  $\epsilon$  small enough  $\tilde{*_p}^1 \in R_{++}^k$ , so the definition is viable.

Now suppose 1 deviates to the strategy  $(*_p^1, *_q^1, *_p^1, *_q^1)$  while others hold their strategies fixed. Then the commodity bundle obtained by 1 is  $\hat{x}^1 + t(y - \hat{x}^1) = z$ . At the same time the prices remain unaffected, hence his credit is  $\hat{p} \cdot z$ . But  $\hat{p} \cdot \hat{x}^1 = \hat{p} \cdot a^1$  by Fact 2, and  $\hat{p} \cdot y = \hat{p} \cdot a^1$  since  $y \in \hat{B}^1(\hat{p})$ . Hence  $\hat{p} \cdot z = \hat{p} \cdot a^1$ , i.e. the net credit of 1 remains 0 when he deviates. But  $u^1$  is concave, so  $u^1(z) > u^1(\hat{x}^1)$ . Consequently 1's payoff increases, a contradiction.

Clearly the maximum number of non-interior traders is obtained by having a distinct marginal buyer and seller at each of the  $k$  trading-posts. Thus the set  $\{i \in N : \hat{x}^i \notin \hat{B}^i(\hat{p})\}$  has at most  $2k$  members.

Next, let  $(\hat{p}, \hat{x}^1, \dots, \hat{x}^n)$  be a C.E. of  $E$  with shadow prices  $\mu = (\mu^1, \dots, \mu^n)$ . Pick  $\alpha > 0$  such that  $\alpha \mu^i < \lambda^i$  for each  $i \in N$ . Consider the  $n$ -tuple of strategies  $\{p^i, q^i, \tilde{p}^i, \tilde{q}^i\}_{i \in N}$  defined by:

$$p^i = \tilde{p}^i = \frac{1}{\alpha} \hat{p}^i$$

$$q_j^i = a_j^i$$

$$\tilde{q}_j^i = \hat{x}_j^i.$$

It is easy to check that these strategies constitute a N.E. of  $\Gamma(E, \lambda)$  and yield the prices  $\frac{1}{\alpha} \hat{p}$  and the allocation  $\hat{x}^1, \dots, \hat{x}^n$ .

It remains to show that this N.E. is strong. Suppose some condition  $T \subset N$  deviates to new strategies while all the players in  $N \setminus T$  hold theirs fixed. Then the members of  $T$  can effect two things: (a) trade among themselves, (b) buy from members of  $N \setminus T$  at prices  $p$  or more, or sell to them (as before) at prices  $p$ . Suppose  $T$  ends up with new trades  $\{t^i : i \in T\}$ . Here  $t^i = a^i - y^i$ , where  $y^i$  is the final

bundle of  $i \in T$  as a result of the deviation. We can decompose this trade into two parts: the trade  $\{\tilde{t}^i : i \in T\}$  which occurs among members of  $T$ , and the trade with members of  $N \setminus T$ . Suppose that the former results in the credit  $\{\gamma^i : i \in T\}$ . If  $\gamma^i = p \cdot \tilde{t}^i$  for each  $i \in T$ , then any trader in  $i \in T$  can do no better than procure the bundle  $\hat{x}^i$  with zero credit. (Recall that  $\hat{x}^i$  is optimal for  $i$  when he can buy and sell unrestrictedly at the prices  $p$ , with the rate of bankruptcy penalty equal to  $\mu^i$  or more, e.g.  $\lambda^i$ ). Thus  $T$  could not have improved, in this case. So suppose that it is not true that  $\gamma^i = p \cdot \tilde{t}^i$  for each  $i \in T$ . Then we claim that for at least one  $j \in T$ ,  $\gamma^j < p \cdot \tilde{t}^j$ . If not,  $0 = \sum_{i \in T} \gamma^i > p \cdot \sum_{i \in T} \tilde{t}^i = p \cdot 0 = 0$ , a contradiction. Consider the trader  $j$ . As a result of the deviation,  $j$  must be worse off than if he could buy and sell unrestrictedly at prices  $p$ , because his credit becomes less favorable. Thus  $T$  could not have improved in this other case either.

Q.E.D.

#### An Example of an N.E. That Is Not a C.E.

Consider two types of traders with two of each characterized by:

$$\text{Type 1: } u^1 = x_1^{2/3} y_1^{1/3} + \lambda_1 \min([6p_2 - p_1 x_1 - p_2 y_1], 0) \text{ with } (0,6)$$

and

$$\text{Type 2: } u^2 = x_2^{1/3} y_2^{2/3} + \lambda_2 \min([6p_1 - p_1 x_2 - p_2 y_2], 0) \text{ with } (6,0).$$

It is easy to see that the unique C.E. is given by  $p_1 = p_2 = 1$  (if we choose to normalize) and final holdings for Type 1 of  $(4,2)$  and  $(2,4)$ ; when  $\lambda_1 = \lambda_2 = (4)^{1/3}/3$ .

The following set of strategies also form an N.E.

Trader 1 of Type 1  $p_1^1 = 10$  ,  $q_1^1 = 6$  ;  $\tilde{p}_2^1 = 1/2$  ,  $\tilde{q}_2^1 = 4$

Trader 2 of Type 1  $p_1^2 = 1$  ,  $q_1^2 = 6$  ;  $\tilde{p}_2^2 = 3/2$  ,  $\tilde{q}_2^2 = 4$

Trader 3 of Type 2  $\tilde{p}_1^3 = 1$  ,  $\tilde{q}_1^3 = 4$  ;  $p_2^3 = 3/2$  ,  $q_2^3 = 8/3$

Trader 4 of Type 2  $\tilde{p}_1^4 = 1$  ,  $\tilde{q}_1^4 = 4$  ;  $p_2^4 = 3/2$  ,  $q_2^4 = 8/3$  .

Figure 3 illustrates this unsatisfactory equilibrium where traders of type 1 are in excess demand in the first market (trader 1) and in excess supply in the second market (trader 1). If trader 1 would lower his bid price sufficiently in market 1 or raise his supply price sufficiently in market 2 the equilibrium would be destroyed.

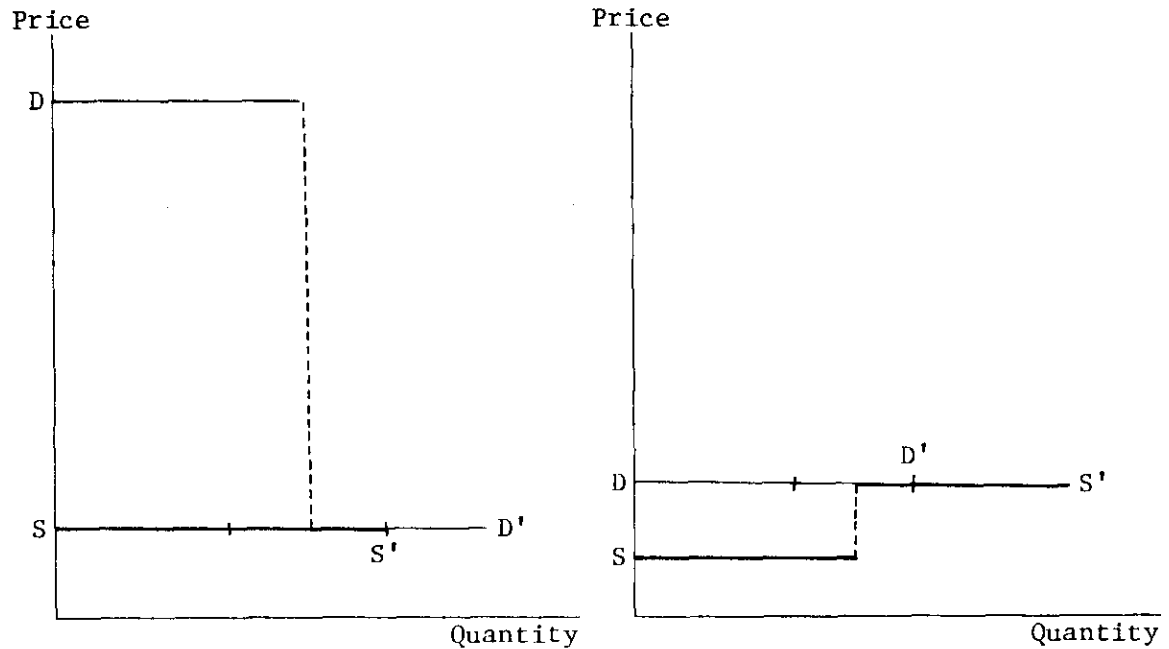


FIGURE 3

At equilibrium the payoffs to trader 1 (2) are given by:

$$(6)^{2/3}(2)^{1/3} = 4.16 \quad \text{for 1}$$

$$(2)^{2/3}(4-2/3)^{1/3} = 2.65 \quad \text{for 2}$$

At the C.E. the payoff to all are  $(4)^{2/3}(2)^{1/3} = 3.175$  .

Comment 1. In any economy where there are at least two traders of any type, any type symmetric N.E. will be a C.E.

Comment 2. In an economy with strictly concave utility functions, modelled as a cooperative market game the core will contain only type symmetric imputations, but the example of an N.E. shown above is not type symmetric hence is not in the (Edgeworth) core.

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