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PRICE-QUANTITY STRATEGIC MARKET GAMES

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# PRICE-QUANTITY STRATEGIC MARKET GAMES\*

by

Pradeep Dubey

## 1. Introduction

We consider a standard Walras exchange economy with a finite number of traders and commodities. This is recast as a game in strategic form in essentially two different ways. There is a trading-post for each commodity to which traders send contingent statements about how much they wish to buy and sell, and at what prices. In Models 1A and 1B, the trading point is determined by the intersection of the aggregate supply and demand curves. In models 2A and 2B, trade takes place so as to meet as many contingent statements as possible. Each buyer whose orders are filled pays the price he quoted, using a fiat money which can be borrowed costlessly and limitlessly. But after trade is over there is a settlement of accounts and a penalty is levied on those who are in debt in the form of a disutility.

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\*This paper would not have been written but for Martin Shubik. It was at his instigation--over a period of two years!--that a price-quantity model of a strategic market game [1, 2] was examined. All the models here are variants of it. I am also grateful to him for invaluable conversations in the writing of this paper.

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Our results are as follows. In 1A and 2A, the Noncooperative Equilibria (N.E.) of the market game coincide (in prices and allocations) with the competitive equilibria (C.E.) of the market. Furthermore, there is a subset of "tight" N.E. which also coincides with the C.E., and each N.E. in this subset is strong. In 1B and 2B, we find that the N.E., the strong N.E. and the C.E. all coincide.

The transition from 2A to 2B raises an interesting point. The game in 2A is not proof against the formation of proxies. For certain fixed choices of others' strategies, there may be motivation for a player to act as if he were more than one player. This leads us to vary the game just enough to make it "proxy-proof," to arrive at 2B. In our opinion the idea of "proxy-proofness" may be of independent interest beyond this model, for strategic market games in general. This will be explored in [4].

A strategic market game with analogous results (precisely: Nash Equilibria which yield efficient allocations) is presented in [7]. Unfortunately in that model the traders are not treated symmetrically, since one of them acts as an auctioneer and the others as bidders. Our mechanism, besides having symmetry, is much simpler and (we feel) corresponds more to familiar market processes. Also, the results are sharper in comparison, though mathematically trivial.

## 2. The Market $E$

$E$  consists of a set of traders  $N = \{1, \dots, n\}$ , where each  $i \in N$  is characterized by\* an initial endowment,  $a^i \in R_+^k$  and a utility function  $u^i : R_+^k \rightarrow R$ . We assume\*\* (i)  $u^i$  is continuous, concave and strictly increasing in at least one variable, (ii) for any commodity  $j$ , there exist at least two traders who are positively endowed with  $j$ , and at least two who "sufficiently desire"  $j$ , in the sense that  $u^{i-1}(c) \cap \{x \in R_+^k : x_j = 0\} = \emptyset$  for  $c \in R$ . For any  $p \in R_+^k$ , let

$$B^i(p) = \{x \in R_+^k : p \cdot x \leq p \cdot a^i\},$$

and

$$\hat{B}^i(p) = \{x \in B^i(p) : u^i(x) = \max_{y \in B^i(p)} u^i(y)\}.$$

A competitive equilibrium (C.E.) of the market  $E$  is a pair

$(p; x^1, \dots, x^n)$  of prices and an allocation such that each  $x^i$  is optimal in  $i$ 's budget set, i.e.,  $x^i \in \hat{B}^i(p)$  for  $i \in N$ , and

$$\sum_{i \in N} x^i = \sum_{i \in N} a^i.$$

Given a C.E.  $(p; x^1, \dots, x^n)$  we can associate with it shadow prices (of income)  $\lambda^1, \dots, \lambda^n$ . Here each  $\lambda^i$  is a nonnegative number such that  $x^i$  is a solution of

$$\text{Max}_{y \in R_+^k} u^i(y) + \lambda^i [p \cdot a^i - p \cdot y].$$

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\*  $R_+^k$  is the nonnegative orthant of Euclidean space of dimension  $k$ . For any  $x$  in  $R_+^k$ ,  $x_j$  is its  $j^{\text{th}}$  component.

\*\*The effect of dropping (ii) is discussed in Remark 2.

Observe that if  $(p; x^1, \dots, x^n)$  is a C.E. of  $E$  with shadow prices  $\lambda^1, \dots, \lambda^n$ , then so is  $(\frac{1}{t}p; x^1, \dots, x^n)$  with shadow prices  $t\lambda^1, \dots, t\lambda^n$ , for any  $t > 0$ . (We will identify C.E.'s which differ in this manner.) Also note that, by (i) and (ii), it follows that  $p > 0$ ,  $\lambda > 0$ , and there exist at least two traders  $i_1$  and  $i_2$  such that  $x_j^{i_1} > 0$ ,  $x_j^{i_2} > 0$ .

### 3. The Market Games $\Gamma_A(E, \lambda)$ , $\Gamma_B(E, \lambda)$

A strategy\* by  $i \in N$  consists of  $\{\eta^i, p^i, q^i, \tilde{p}^i, \tilde{q}^i\}$  where:  
 $\eta^i \in R_+$ ,  $p^i \in R_{++}^k$ ,  $q^i \in R_+^k$ ,  $\tilde{p}^i \in R_{++}^k$ ,  $\tilde{q}^i \in R_+^k$ ,  $\tilde{q}_j^i \leq a_j^i$  (for  
 $j = 1, \dots, k$ ),  $\sum_{j=1}^k \tilde{p}_j^i \tilde{q}_j^i \leq \eta^i$ . We now interpret the symbols.  $\eta^i$  is  
the quantity of money borrowed by  $i$  (at zero rate of interest, from a  
bank in the background) in order to finance purchases.  $(p^i, q^i, \tilde{p}^i, \tilde{q}^i)$   
is shorthand for a statement made by  $i$ : "if the price of commodity  
 $j$  is  $p_j^i$  or less, then I am willing to buy up to  $q_j^i$ ; if its price  
is  $\tilde{p}_j^i$  or more, then I am willing to sell up to  $\tilde{q}_j^i$ ." (Note that we  
give  $i$  the strategic freedom to both buy and sell the same commodity.)

Denote the set of strategies of  $i$  by  $S^i$ . Put  $S = S^1 \times \dots \times S^n$ .  
We will define two distinct "outcome functions"  $g_A^i, g_B^i$  from  $S$  to  
 $R_+^k \times R$  for each  $i \in N$ . An "outcome"  $(x, \beta)$  will consist of the final  
bundle  $x \in R_+^k$  that accrues to  $i$  after trade, and of  $\beta \in R$  which  
represents his net credit.

Consider a choice of strategies  $(s^1, \dots, s^n) = s \in S$ . For each  
commodity  $j$ , this gives rise in an obvious way to supply and demand  
curves,  $S$  and  $D$ .  $S$  is obtained by ranking the selling prices  
 $\tilde{p}_j^1, \dots, \tilde{p}_j^n$  in ascending order and cumulating the supply quantities  
 $\tilde{q}_j^1, \dots, \tilde{q}_j^n$ ;  $D$  is obtained similarly with the buying prices ranked in  
descending order. The figures below essentially exhaust all possible  
configurations of the  $S$  and  $D$  curves.

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\*  $R_{++}^k$  is the relative interior of  $R_+^k$ .

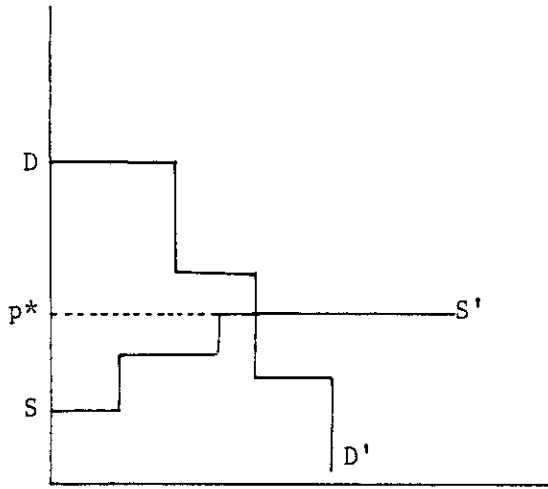


FIGURE (a)

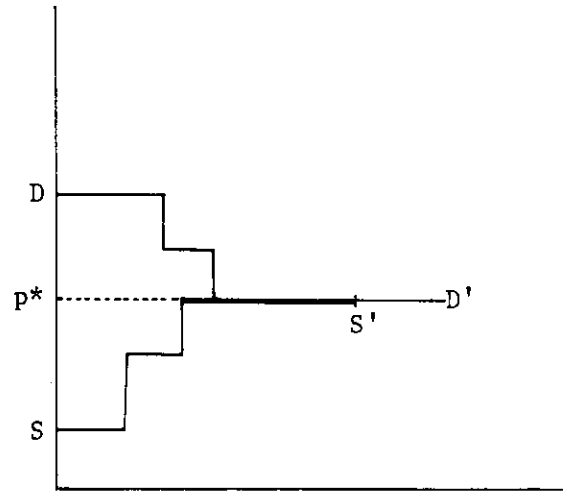


FIGURE (b)

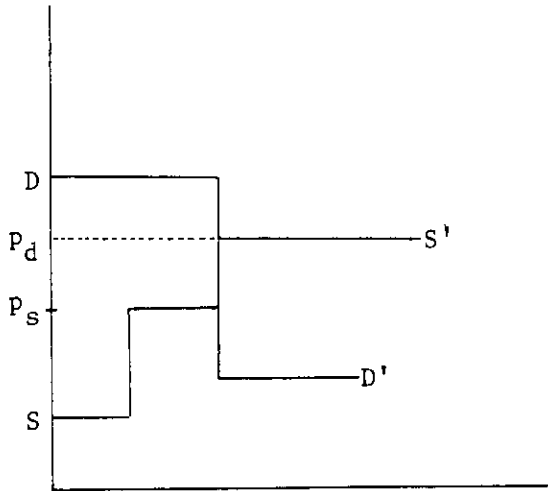


FIGURE (c)

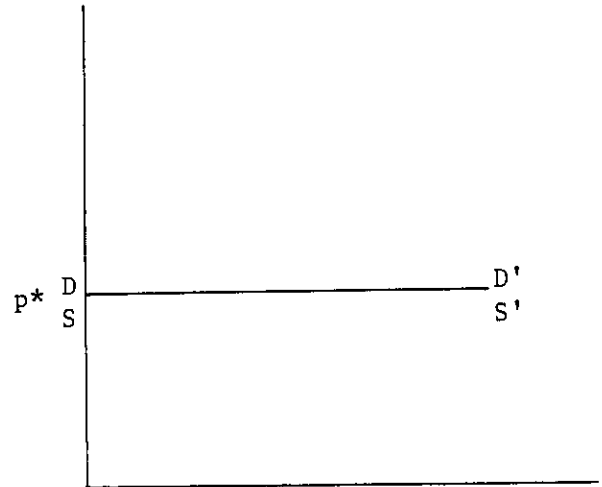


FIGURE (d)

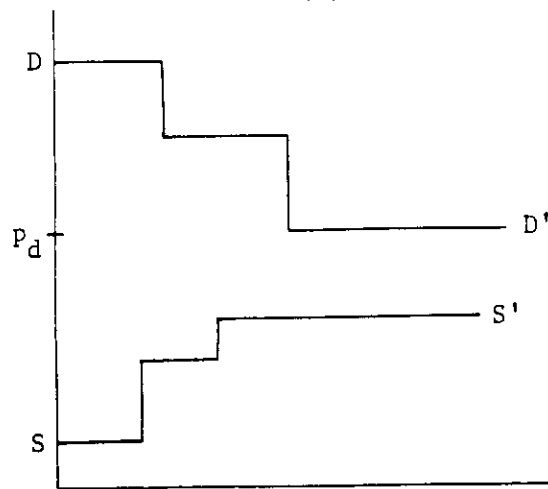


FIGURE (e)

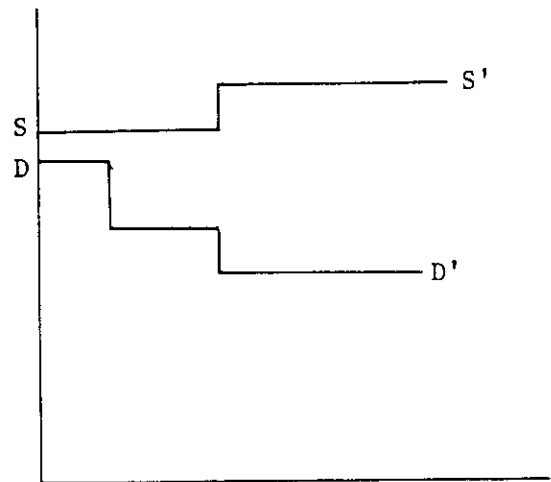


FIGURE (f)

The disbursement of commodities takes place as described below.

Case 1. The intersection of  $S$  and  $D$  determines a unique price  $p^*$  (Figures a, b, d). All sellers who quote a higher price sell nothing. All buyers who quote a lower price buy nothing. The rest buy and sell the quantities they quote. If there is excess demand at  $p^*$  (Figure b) or excess supply (Figure a) then the "marginal" buyers and sellers (i.e. those who quote the price  $p^*$ ) are rationed in proportion to their demands and supplies.

Case 2. The intersection of  $S$  and  $D$  determines an interval  $[p_d, p_s]$  of prices (Figure c). All sellers who quote a price higher than  $p_s$  sell nothing; all buyers who quote a price lower than  $p_d$  buy nothing. The rest buy and sell the quantities they quote.

Case 3.  $D$  lies above  $S$  (Figure e). Then move  $S$  up until it intersects  $D$ , at the price level  $p_d$ , and apply Case 1.

Case 4.  $S$  lies above  $D$  (Figure f). No trade takes place, i.e., buyers and sellers get back their quantities.

Case 5. One of the curves  $S$  or  $D$  is missing. No trade takes place.

We have yet to specify the rules of payment. Let us consider two conventions.

Convention IA. Buyers buy at the prices they quote. The highest buyer buys from the lowest seller. If he needs to buy more then he is serviced by the second-lowest seller (if there are several such they are rationed in proportion to their supplies); if he needs to buy less then the lowest



seller sells to the second-highest buyer (again rationing on both sides whenever necessary), etc.

Convention IB. The buyers pay as in IA, but the sellers sell at the prices they quote.

The net credit of the traders, after repaying  $\eta^i$  to the bank, is: money obtained from sales--money used for purchases. Thus we have defined the mappings  $g_A^i : S \rightarrow R_+^k \times R$ ,  $g_B^i : S \rightarrow R_+^k \times R$  for  $i \in N$ , where  $g_A^i$  ( $g_B^i$ ) computes the net credit in accordance with convention IA (IB).

To define the games  $\Gamma_A^1(E, \lambda)$  and  $\Gamma_B^1(E, \lambda)$  we need to specify the payoff functions from  $R_+^k \times R$  to  $R$  for each  $i \in N$ . Take any  $\lambda = (\lambda^1, \dots, \lambda^n)$ , with each  $\lambda^i > 0$ . Now let  $P_\lambda^i : R_+^k \times R \rightarrow R$  be given by

$$P_\lambda^i(x, \beta) = u^i(x) + \lambda^i \min[0, \beta]$$

for  $x \in R_+^k$ ,  $\beta \in R$ .

In the game  $\Gamma_A^1(E, \lambda)$  the payoff function of player  $i$  is  $\lambda^i \Pi_A^i : S \rightarrow R$  where  $\lambda^i \Pi_A^i(s) = P_\lambda^i(g_A^i(s))$ .  $\Gamma_B^1(E, \lambda)$  is defined similarly using  $g_B^i$ .

Observe that the term  $\lambda^i \min[0, \beta]$  simply says that if a trader ends up with a positive amount of money if has no utility to him; however if he is bankrupt, i.e.  $\beta < 0$ , then there is a penalty levied on him in the form of a disutility. The penalty function need not have the special linear form above; indeed, any harsher penalty will do. See Remark 1.

4. Noncooperative Equilibria of  $\Gamma_A^1(E, \lambda)$  and  $\Gamma_B^1(E, \lambda)$

For any  $s = \{s^i : i \in N\} \in S$ ,  $M \subset N$ , and  $e = \{e^i : i \in M\} \in \prod_{i \in M} S^i$ , let  $(s|e)$  denote the element of  $S$  obtained from  $s$  by replacing  $s^i$  by  $e^i$  for each  $i \in M$ . Define  $s$  to be M-efficient in  $\Gamma_A^1(E, \lambda)$  if there does not exist any  $e \in \prod_{i \in M} S^i$  such that

$$(*) \quad \begin{cases} \lambda \Pi_A^i(s|e) \geq \lambda \Pi_A^i(s), & \text{all } i \in M; \\ \lambda \Pi_A^\ell(s|e) > \lambda \Pi_A^\ell(s), & \text{some } \ell \in M. \end{cases}$$

If  $s$  is  $\{i\}$ -efficient for each  $i \in N$ , we call it a noncooperative equilibrium (N.E.); if it is  $N$ -efficient, we call it simply efficient; and if it is  $M$ -efficient for all  $M \subset N$ , we call it a strong noncooperative equilibrium.

$\Gamma_A^1(E, \lambda)$  has certain trivial N.E.'s, e.g.,  $q^i = \tilde{q}^i = 0$  for each  $i \in N$ , and  $n^i$ ,  $p^i$ ,  $\tilde{p}^i$  are arbitrary. Other trivial N.E.'s can also be constructed. We will focus our attention\* on those N.E.'s at which there is full-blown competition. To this end, call buyers and sellers who actually trade at any trading-post active at that post. Now define an active N.E. to be one in which, at each of the  $k$  trading-posts, there exist at least two active buyers and two active sellers. Finally an N.E. will be called tight if, at each trading post, all active traders quote the same price.

N.E.'s, active N.E.'s, strong N.E.'s, tight N.E.'s of the games  $\Gamma_B^1(E, \lambda)$ ,  $\Gamma_A^2(E, \lambda)$ ,  $\Gamma_B^2(E, \lambda)$  --see below--are defined in exactly the same manner. Each of these games is afflicted with trivial N.E.'s. We

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\*Non-trivial, non-active N.E.'s are discussed in Section 6.

avoid them, and from now on, \* will mean an active N.E. by an N.E. To begin with, let us note some obvious facts.

Fact 1A. At any N.E. of  $\Gamma_A^1(E, \lambda)$  all active buyers (in any trading-post) quote the same price.

To see this, let  $p_j^*$  denote the maximum selling price for  $j$  quoted by the active sellers. Then each active buyer must be quoting  $p_j^*$  at an N.E., for if he quoted more he could quote  $p_j^*$ , buy the same amount of  $j$ , and use the credit thus saved to buy more of a commodity he likes (possibly  $j$ ). This would clearly improve his payoff, a contradiction. Similarly, we can show:

Fact 1B. Any N.E. of  $\Gamma_B^1(E, \lambda)$  is tight.

Note that in the light of Facts 1A and 1B we can talk of the prices produced at an N.E. of  $\Gamma_A^1(E, \lambda)$  or  $\Gamma_B^1(E, \lambda)$ , i.e. those quoted by the active buyers.

Fact 2. Suppose  $\beta^1, \dots, \beta^n$  is the credit of the traders at a N.E. of  $\Gamma_A^1(E, \lambda)$  or  $\Gamma_B^1(E, \lambda)$ . Then  $\beta^i = 0$  for each  $i \in N$ .

First observe that  $\sum_{i \in N} \beta^i = 0$  in either case. (For  $\Gamma_B^1(E, \lambda)$  this follows from the rules of payment; for  $\Gamma_A^1(E, \lambda)$  this follows from Fact 1B.) Hence it suffices to show that  $\beta^i \leq 0$  for each  $i \in N$ . Suppose  $\beta^\ell > 0$  for some  $\ell$ . Then trader  $\ell$  could buy some more of a commodity he likes (by demanding more of it) without going bankrupt. This would improve his payoff, a contradiction.

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\*Unless specified otherwise, i.e. in Section 6, and Remark 2.

Theorem 1A. Consider any  $E$  and any  $\lambda > 0$ . Then (a) the N.E. of  $\Gamma_A^1(E, \lambda)$  coincide with the C.E. of  $E$ ; (b) the tight N.E. of  $\Gamma_A^1(E, \lambda)$  coincide with the C.E. of  $E$ ; (c) every tight N.E. of  $\Gamma_A^1(E, \lambda)$  is also strong.

Proof. Let  $(\hat{p}, \hat{x}^1, \dots, \hat{x}^n)$  be a C.E. of  $E$  with shadow prices  $\mu = (\mu^1, \dots, \mu^n)$ . Pick  $\alpha > 0$  such that  $\alpha\mu^i < \lambda^i$  for each  $i \in N$ . Consider the n-tuple of strategies  $\{\eta^i, p^i, q^i, \tilde{p}^i, \tilde{q}^i\}_{i \in N}$  defined by:

$$p^i = \tilde{p}^i = \frac{1}{\alpha} \hat{p}$$

$$q_j^i = a_j^i$$

$$\tilde{q}_j^i = \hat{x}_j^i$$

$$\eta^i = \sum_{j=1}^k p_j^i a_j^i.$$

It is easy to check that these strategies constitute a tight N.E. of  $\Gamma_A(E, \lambda)$  and yield the prices  $\frac{1}{\alpha} \hat{p}$  and the allocation  $\hat{x}^1, \dots, \hat{x}^n$ .

Next suppose  $\{u^i, p^i, q^i, \tilde{p}^i, \tilde{q}^i\}_{i \in N}$  is an N.E. of  $\Gamma_A(E, \lambda)$  which produces prices  $\hat{p}$  and the allocation  $\hat{x}^1, \dots, \hat{x}^n$ . We have to show that  $\hat{x}^i \in \tilde{B}^i(\hat{p})$  for each  $i \in N$ . Suppose not. W.l.o.g. let  $\hat{x}^1 \notin \tilde{B}^1(\hat{p})$  i.e.  $u^1(\hat{x}^1) < u^1(y)$  for  $y \in \tilde{B}^1(\hat{p})$ . (Note that, by Fact 2,  $\hat{x}^1 \in B^1(\hat{p})$ .) Put  $J = \{j : y_j - \hat{x}_j^1 > 0\}$ ,  $J' = \{j : \hat{x}_j^1 - y_j > 0\}$ . Denote the total active sale (equivalently, purchase) at trading-post  $j$  by  $T_j$ , and the sale and purchase of  $l$  by  $s_j$  and  $d_j$ . (Note that we permit  $s_j$  and  $d_j$  to be zero i.e. trader 1 need not be active at any trading-post.) Since the N.E. is active, we have  $T_j > s_j$  and  $T_j > d_j$ . Let  $0 < t < 1$  be chosen sufficiently small so as to ensure that:

$$T_j - d_j + t(y_j - \hat{x}_j^1) > 0 \quad \text{for } j \in J ;$$

$$T_j - s_j + t(\hat{x}_j^1 - y_j) > 0 \quad \text{for } j \in J' .$$

Also let  $p_j^*$  be the minimum price quoted by the active sellers at the  $j^{\text{th}}$  trading-post.

We now construct a strategy  $(*_n^1, *_p^1, *_q^1, \tilde{*_p}^1, \tilde{*_q}^1)$  for 1 as follows:\*

$$*_p_j^1 = \begin{cases} \hat{p}_j + \varepsilon & \text{if } j \in J \\ \hat{p}_j & \text{otherwise} \end{cases} ,$$

$$\tilde{*_p}_j^1 = \begin{cases} p_j^* - \varepsilon & \text{if } j \in J' \\ \tilde{*_p}_j^1 & \text{otherwise} \end{cases}$$

$$*_q_j^1 = \begin{cases} q_j^1 + t(y_j - \hat{x}_j^1) & \text{for } j \in J \\ q_j^1 & \text{otherwise} \end{cases}$$

$$\tilde{*_q}_j^1 = \begin{cases} \tilde{*_q}_j^1 + t(\hat{x}_j^1 - y_j) & \text{for } j \in J' \\ \tilde{*_q}_j^1 & \text{otherwise} \end{cases}$$

$$*_n^1 = n^1 .$$

If 1 deviates to this strategy (while others hold theirs fixed) then his final bundle is  $\hat{x}^1 + t(y - \hat{x}^1) = z$ . Clearly, by the concavity of  $u^1$ , we have  $u^1(z) > u^1(\hat{x}^1)$ . Therefore the increase in 1's payoff is at least  $u^1(z) - u^1(\hat{x}^1) - \lambda^1 [\varepsilon \sum_{j=1}^k T_j]$ . For small enough  $\varepsilon$  this is positive, which contradicts that the original strategies constituted a N.E.

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\*For  $\varepsilon$  small enough,  $\tilde{*_p}^1 \in R_{++}^k$  so the definition is viable.

Finally, we have to show that every tight N.E. is also a strong N.E. So let  $\{\eta^i, p^i, q^i, \tilde{p}^i, \tilde{q}^i\}_{i \in N}$  be a tight N.E. which produces the prices  $p$  and the allocation  $x^1, \dots, x^n$ . As shown above  $\{p; x^1, \dots, x^n\}$  is a C.E. of  $E$ . Clearly, if  $\{\mu^1, \dots, \mu^n\}$  are the shadow prices at this C.E. then  $\mu^i \leq \lambda^i$  for each  $i \in N$ .

Now suppose some coalition  $T \subset N$  deviates to new strategies while all the players in  $N \setminus T$  hold theirs fixed. A moment's reflection reveals that by deviating members of  $T$  can effect two things: (a) trade among themselves, (b) buy from members of  $N \setminus T$  at prices  $p$  or more, or sell to them (as before) at prices  $p$ . Suppose  $T$  ends up with new trades  $\{t^i : i \in T\}$ . Here  $t^i = a^i - y^i$ , where  $y^i$  is the final bundle of  $i \in T$  as a result of the deviation. We can decompose this trade into two parts: the trade  $\{\tilde{t}^i : i \in T\}$  which occurs among members of  $T$ , and the trade with members of  $N \setminus T$ . Suppose that the former results in the credit  $\{\tilde{\beta}^i : i \in T\}$ . If  $\tilde{\beta}^i = p \cdot \tilde{t}^i$  for each  $i \in T$ , then any trader in  $i \in T$  can do no better than procure the bundle  $x^i$  with zero credit. (Recall that  $x^i$  is optimal for  $i$  when he can buy and sell unrestrictedly at the prices  $p$ , with the rate of bankruptcy penalty equal to  $\mu^i$  or more, e.g.  $\lambda^i$ .) Thus  $T$  could not have improved, in the sense of (\*), in this case. So suppose that it is not true that  $\tilde{\beta}^i = p \cdot \tilde{t}^i$  for each  $i \in T$ . Then we claim that for at least one  $j \in T$ ,  $\tilde{\beta}^j < p \cdot \tilde{t}^j$ . If not,  $0 = \sum_{i \in T} \tilde{\beta}^i > p \cdot \sum_{i \in T} \tilde{t}^i = p \cdot 0 = 0$ , a contradiction. Consider the trader  $j$ . As a result of the deviation,  $j$  must be worse off than if he could buy and sell unrestrictedly at prices  $p$ , because his credit becomes less favorable. Thus  $T$  could not have improved in this other case either.

Q.E.D.

Theorem 1B. Consider any  $E$  and any  $\lambda > 0$ . Then the N.E. of  $\Gamma_B^1(E, \lambda)$  are strong and coincide with the C.E. of  $E$ .

Proof. Recall that by Fact 1B every N.E. of  $\Gamma_B^1(E, \lambda)$  is tight. We now go through all the steps of the previous proof, replacing

$$" 0 = \sum_{i \in T} \hat{\beta}^i > p \cdot \sum_{i \in T} \hat{t}^i = p \cdot 0 = 0 " \text{ by } " 0 \geq \sum_{i \in T} \hat{\beta}^i > p \cdot \sum_{i \in T} \hat{t}^i = p \cdot 0 = 0 ."$$

Q.E.D.

5. The Market Games  $\Gamma_A^2(E, \lambda)$  and  $\Gamma_B^2(E, \lambda)$

First we define  $\Gamma_A^2(E, \lambda)$ . The strategy sets are exactly the same as before. However in the disbursement of commodities the aim is now to meet as many contingent statements as possible. This is not true of the previous mechanism. For instance if the supply and demand curves are identical [Figure (g)], then it is clearly possible for all buyers and sellers

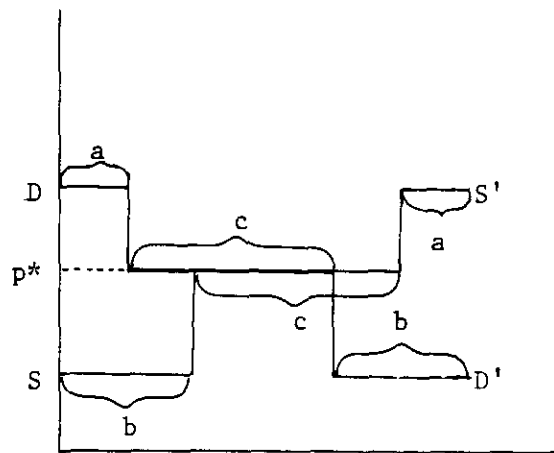


FIGURE (g)

to be fully active, but the "intersection method" does not permit this. To allow for the maximum compatible trade, first rank the buyers and sellers (as before) using the prices quoted. We will describe a finite sequence of imaginary trades which will terminate at a final actual trade. Start with the lowest buyer. If there are no sellers who quote an equal or lower price, then go to the next buyer. If there are such sellers, then start filling his demand with the lowest seller being services first, and rationing in case of ties. In general suppose the first  $j$  buyers have been scanned. Then start filling the  $(j+1)^{st}$  demand by the "surplus" supply below its price level, i.e., the supply which has not already been used to fill the first  $j$  demands. If the  $(j+1)^{st}$  demand can be met by



this surplus, then go to the  $j+2^{\text{nd}}$  demand. If not, then transfer to it the supply sent to the first  $j$  demands, again starting with the lowest supply, and rationing in case of ties. When the highest demand is scanned the actual trade gets defined. Clearly this maximizes the total trade under the constraint that no demand (supply) is serviced unless all higher-priced (lower-priced) demands (supplies) are serviced. To complete the definition of the game  $\Gamma_A^2(E, \lambda)$  we must specify the credit. Let us use convention 1A, i.e. that buyers pay the price they quote. As in the case 1A, we can show that all active buyers at any N.E. of  $\Gamma_A^2(E, \lambda)$  quote the same price, and we have (by essentially the same proof):

Theorem 2A. Consider any  $E$  and any  $\lambda > 0$ . Then (a) the N.E. of  $\Gamma_A^2(E, \lambda)$  coincide with the C.E. of  $E$ ; (b) the tight N.E. of  $\Gamma_A^2(E, \lambda)$  coincide with the C.E. of  $E$ ; (c) every tight N.E. of  $\Gamma_A^2(E, \lambda)$  is also strong.

To eliminate the non-tight N.E. and get the analogue of Theorem 1B, we could as before adopt the expedient of servicing the sellers at the prices they quote. However this has the undesirable feature (shared by the game  $\Gamma_B^1(E, \lambda)$ ) of leaving surplus fiat money in nonequilibrium positions of the game (taken in by imaginary brokers?). We will make a quite different alteration. Allow each trader to act as any (finite) number of traders as he wishes. Of course this is subject to constraints on his strategies. For instance if trader 1 announces that he will buy up to  $q_j$  at price  $p_j$  or less, and up to  $\hat{q}_j$  at  $\hat{p}_j$  or less, then we must require that  $q_j p_j + \hat{q}_j \hat{p}_j \leq \eta^1$ ; if he announces that he will sell (as three traders)  $q_j$ ,  $\hat{q}_j$ ,  $\tilde{q}_j$  then this is subject to  $q_j + \hat{q}_j + \tilde{q}_j \leq a_j^1$ . The rules of trade and payment are as before, and

the outcome to a player is now the sum of the outcomes of all his proxies (both for the final commodity bundle, and the credit). This defines the game  $\Gamma_B^2(E, \lambda)$ . It is straightforward as before to obtain:

Theorem 2B. Consider any  $E$  and any  $\lambda > 0$ . Then the N.E. of  $\Gamma_B^2(E, \lambda)$  are strong and coincide with the C.E. of  $E$ .

### 6. Non-active, Non-trivial Noncooperative Equilibria

The games we have described have other N.E. which need to be examined. Call an N.E. non-trivial if there is at least one active trader at each trading-post. The N.E. of interest that we have omitted are the non-active, non-trivial N.E. A bound on their departure from the C.E. of  $E$  is given by:

Theorem 3. Let  $(p; x^1, \dots, x^n)$  be the prices and allocation at a non-active, non-trivial N.E. of  $\Gamma_A^1(E, \lambda)$  or  $\Gamma_B^1(E, \lambda)$  or  $\Gamma_A^2(E, \lambda)$  or  $\Gamma_B^2(E, \lambda)$ . Then  $|\{i \in N : x^i \notin \tilde{B}^i(p)\}| \leq 2k$ .

Proof. Call a trader "interior" at the N.E. if he is not the sole active buyer or seller at any of the  $k$  trading-posts. By exactly the same argument as is used in the proof of Theorem 1A, we can show that if  $i$  is interior then  $x^i \in \tilde{B}^i(p)$ . Hence a trader can be non-interior only by being the sole seller or the sole buyer in some post. The number of such traders is maximized by having a distinct one in each of the  $k$  posts.

Q.E.D.

This shows that the non-active, non-trivial N.E., while they need not coincide with the C.E. (indeed need not even be efficient), are still close to the C.E. in large economies: the fraction of "non-optimal" traders is bounded above by  $2k/n$ , which goes to zero with  $n$ .

7. Remarks

- (1) Suppose the bankruptcy penalty is replaced by any other which satisfies (where  $x \in \mathbb{R}_+^k$ , and  $\hat{\lambda}$  is any positive number):

$$\Pi^i(x, \beta) = u^i(x) \quad \text{for } \beta \geq 0 ;$$

$$u^i(x) + \hat{\lambda}\beta < \Pi^i(x, \beta) < u^i(x) + \hat{\lambda}(\beta) \quad \text{for } \beta < 0 .$$

Then clearly all the results would continue to hold. (In fact we need the first inequality only for sufficiently small  $\beta$ .)

- (2) We introduced condition (ii) on  $E$  since it enabled us to deduce the existence of active N.E. What if we drop it, replacing it only by " $\sum_{i \in N} a^i > 0$ "? Here again, for each C.E. (in any of the market games) we can construct an N.E. (as in the proof of Theorem 1A) which coincides with it, and is also strong. Moreover, as is obvious, Theorem 3 can be restated.
- (3) The version of 1A in which all buyers pay the "intersection price"  $p^*$  has been examined in [2]. There the N.E. do not coincide with the C.E. but contain them as a strict subset. (They do however shrink to the C.E. as the player-set approaches a non-atomic continuum.)
- (4) In several other models ([5], [6]; see also [4]) of strategic market games, the C.E. and the N.E. are disjoint. Indeed the N.E. are generically inefficient [3] (though again they converge to the C.E., under appropriate conditions, as the player-set approaches a non-atomic continuum.) A critical difference between these models and the ones presented here (or in [1], [2]) is that here the payoff

functions are highly discontinuous in the strategies. Mathematically speaking, it is precisely this discontinuity which makes it not impossible for the N.E. to be efficient and strong.

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