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AN EVEN MORE ELEMENTARY "CALCULUS" PROOF

OF THE BROUWER FIXED POINT THEOREM

by

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1. The Brouwer fixed point theorem is usually proved by means of either combinatorial arguments, homology theory, differential forms, or methods from differential topology, see [1], [4], [5], [8]. J. Milnor has suggested recently [6] an analytic proof. This proof, however, was somewhat involved and "strange." The proof given in [2], while analytic and entirely elementary, uses a seemingly artificial homotopy in order to get the desired contradiction via (n+1)-dimensional integration. We offer here a self-contained proof, inspired by the one in [2], of the "no differentiable retraction" theorem, a proof which employs only "engineering type" Advanced Calculus concepts. The motivation for the computation is clear.

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2. We prove in the present section the following:

"No Differentiable Retraction" Theorem. There exists no twice differentiable map f of the unit ball B in R^n into its boundary S, such that $f(x) = x$ for all $x \in S$.

Proof. Let f be such a retraction, $f(x) = (f_1(x), \ldots, f_n(x))$. Let $J(x)$ denote the Jacobian determinant of f at x. Expanding $J(x)$ by the first column, we get

$$(1) \qquad J(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial f_1}{\partial x_i} E_i(x)$$

where $E_i(x)$ is the determinant of the matrix obtained from the matrix

$$(2) \qquad M(x) = \begin{pmatrix} \frac{\partial f_2}{\partial x_1}, & \ldots, & \frac{\partial f_n}{\partial x_1} \\ \vdots & & \\ \frac{\partial f_2}{\partial x_n}, & \ldots, & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

by omitting the i-th row. Note that $J(x)$ vanishes identically on B, as the n scalar functions $f_1, \ldots, f_n$ satisfy the functional relation $\sum_{i=1}^{n} f_i^2(x) \equiv 1$. (Note that we use here only the easy part of the vanishing Jacobian theorem.) Integrating $J(x)$ over B, we find, using integration by parts and (1), that

$$0 = \int_B \cdots \int J(x)dx_1 \cdots dx_n = \int_S \cdots \int \sum_{i=1}^n (-1)^{i+1} f_1(x) E_i(x) dx_1 \cdots \tag{3}$$

$$\cdots dx_{i-1} dx_{i+1} \cdots dx_n + \int_B \cdots \int \sum_{i=1}^n (-1)^i f_1(x) \frac{\partial E_i}{\partial x_i} dx_1 \cdots dx_n \ .$$

According to a well-known theorem of Jacobi [3], [7] (used in the proof of the Brouwer fixed point theorem in [2]),

$$\sum_{i=1}^n (-1)^i \frac{\partial E_i}{\partial x_i}(x) \equiv 0 \ . \tag{4}$$

(If $n = 2$ then (4) reduces to the equality of the mixed derivatives.) To prove (4), let $c_{i,j}(x)$, $i \neq j$, denote the determinant of the matrix obtained from $M(x)$ by omitting the i-th row and replacing the row $\left(\frac{\partial f_2}{\partial x_j}, \ldots, \frac{\partial f_n}{\partial x_j}\right)$ by $\left(\frac{\partial^2 f_2}{\partial x_i \partial x_j}, \ldots, \frac{\partial^2 f_n}{\partial x_i \partial x_j}\right)$. Applying the rule for differentiating determinants we see that $\frac{\partial E_i}{\partial x_i} = \sum_{j \neq i} c_{i,j}$. The equality of the mixed derivatives $\frac{\partial^2 f_k}{\partial x_i \partial x_j} = \frac{\partial^2 f_k}{\partial x_j \partial x_i}$ implies that $c_{j,i} = (-1)^{j-i-1} c_{i,j}$ as the row of the second order derivatives get shifted $j-i-1$ rows when one passes from $c_{i,j}$ to $c_{j,i}$ if $i < j$, and $i-j-1$ rows otherwise Hence

$$\sum_{i=1}^n (-1)^i \frac{\partial E_i}{\partial x_i} = \sum_{i=1}^n (-1)^i [\sum_{j<i} c_{i,j} + \sum_{j>i} c_{i,j}]$$

$$= \sum_{j<i} (-1)^i c_{i,j} + \sum_{j>i} (-1)^i (-1)^{j-i-1} c_{j,i} = 0 \ .$$

Substituting (4) in (3), we find that a contradiction would follow once we prove that

(5) $$I = \int_S \cdots \int f_1(x) \sum_{i=1}^n (-1)^{i+1} E_i(x) dx_1 \ldots dx_{i-1} dx_{i+1} \cdots dx_n \neq 0 .$$

Note that $dx_1 \ldots dx_{i-1} dx_{i+1} \cdots dx_n = x_i d\sigma$, where $d\sigma$ denotes the surface element ((n-1)-dimensional volume) on the unit sphere (x_i being equal to the projection of the unit normal of S on the i-th axis). Hence

(6) $$I = \int_S \cdots \int f_1(x) \sum_{i=1}^n (-1)^{i+1} x_i E_i(x) d\sigma .$$

In order to calculate I , observe that $f_i(x) \equiv x_i$ on S , $1 \leq i \leq n$. Hence grad f_i - grad x_i is perpendicular to S there. Thus there exist scalars λ_i (depending on x) such that grad $f_i(x)$ = grad $x_i + \lambda_i x$, and the matrix M can be written as

$$\begin{pmatrix} \lambda_2 x_1 & , \ldots , & \lambda_n x_1 \\ 1 + \lambda_2 x_2 & & \vdots \\ \vdots & & \vdots \\ \lambda_2 x_n & & 1 + \lambda_n x_n \end{pmatrix}$$

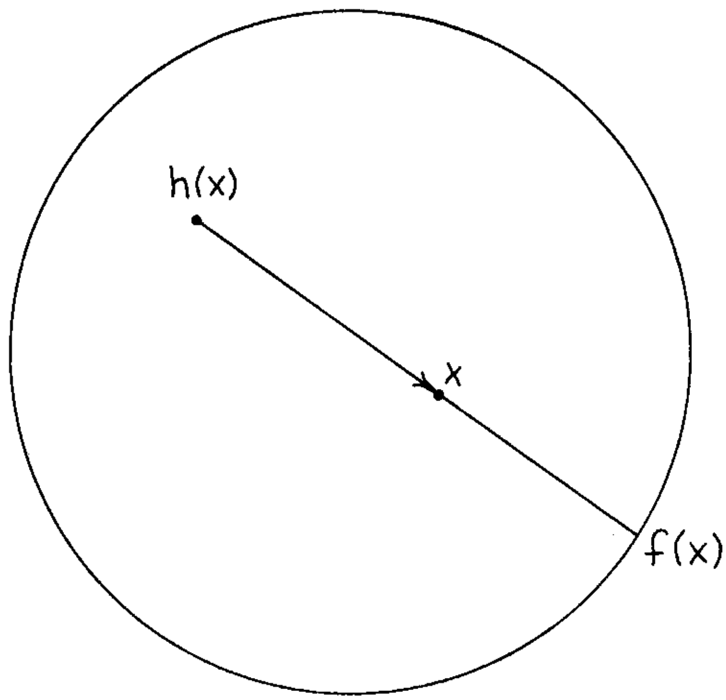
The sum $\sum_{i=1}^n (-1)^{i+1} x_i E_i(x)$ is equal to the determinant

$$\begin{vmatrix} x_1 & \lambda_2 x_1 & \cdots & \lambda_n x_1 \\ x_2 & 1+\lambda_2 x_2 & & \lambda_n x_2 \\ \vdots & & & \vdots \\ x_n & \lambda_2 x_n & & 1+\lambda_n x_n \end{vmatrix} = \begin{vmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 1 & & 0 \\ \vdots & & & \vdots \\ x_n & 0 & & 1 \end{vmatrix} = x_1 .$$

Moreover, $f_1(x) = x_1$ on S . Inserting these results in (6), we get the result $I = \int_S \cdots \int x_1^2 d\sigma > 0$, contradicting (3). (It is easy to compute that $I = \frac{1}{n} \int_S \ldots \int d\sigma = \mathrm{Vol}(B)$.)

Remark. The idea behind the preceding proof is, of course, that the signed measure of the image of B under a map f , such that f is a diffeomorphism on S , is equal to the measure of the figure bounded by $f(S)$-other regions cancelling each other out. It so happens that we can calculate that signed measure explicitly in the case relevant for us.

3. The Brouwer fixed point theorem follows from the no-differentiable retract theorem in a well-known way (see e.g. [2]). We sketch the argument for completeness. Suppose that $g : B \to B$ is a fixed point free continuous map. The compactness of B implies that $|g(x) - x| \geq \varepsilon > 0$ for $x \in B$. Let $h(x)$ be a C^2 function such that $|h(x) - g(x)| < \frac{\varepsilon}{2}$ on B (we can even let h be a polynomial). Then $h(x) \neq x$ for $x \in B$ and let $f(x)$ denote (for $x \in B$, $x \notin S$) the unique point on S such that $h(x)$, x and $f(x)$ lie on the same line and x is between $h(x)$ and $f(x)$, $f(x) = x$ for $x \in S$. Then $f(x)$ is a C^2 retraction, contradicting the theorem of Section 2.


h(x)
x
f(x)

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