

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 517

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

COOPERATIVE GAME SOLUTIONS:

AUSTRALIAN, INDIAN AND U.S. OPINIONS

by

Martin Shubik

February 28, 1979

COOPERATIVE GAME SOLUTIONS:
AUSTRALIAN, INDIAN AND U.S. OPINIONS*

by
Martin Shubik

ABSTRACT

As part of several lectures the audience in Australia, India and the U.S. were asked how the gain from an agreed upon cooperation should be divided among three individuals. The responses are considered in the context of various cooperative solutions which have been suggested in the theory of games. More questions are raised than answered.

1. INTRODUCTION

In 1973 at several Australian universities several different sets of individuals were asked to give their opinions on how a certain three-person nonconstant sum game should be played. The individuals were presented with a diagram as shown in Figure 1.

This figure shows various solutions to a three-person nonconstant sum game with the following characteristic function:

*This work relates to Department of the Navy Contract N00014-77-C-0518 for the Project Center for the Study of Competitive and Conflict Systems issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

The United States Government has at least a royalty-free, nonexclusive and irrevocable license throughout the world for Government purposes to publish, translate, reproduce, deliver, perform, dispose of, and to authorize others so to do, all or any portions of this work.

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = 1, \quad v(13) = 2, \quad v(23) = 3$$

$$v(123) = 4.$$

On the figure various suggested solutions have been noted. These solutions are discussed in Section 3.

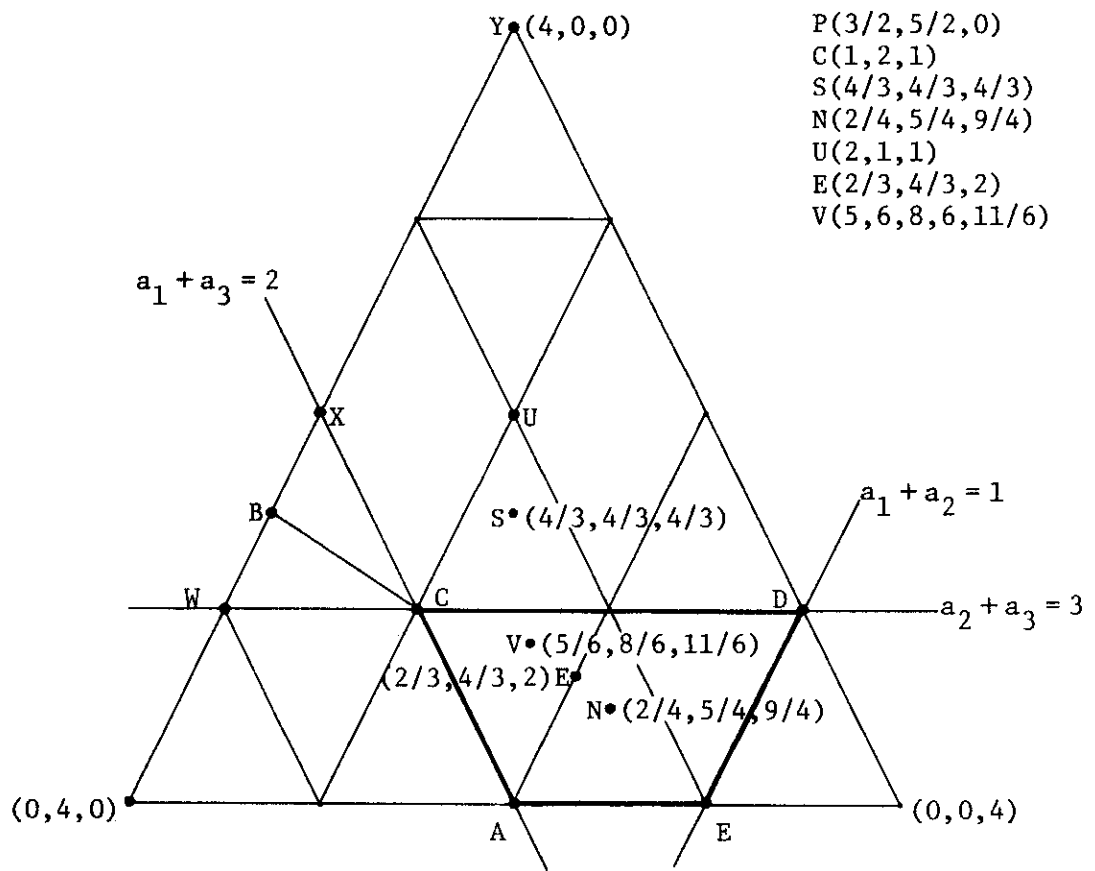


FIGURE 1

In 1977 the same game was run with a group of students in a game theory class at Yale. The results from these investigations were reported in a paper in 1978.* In December 1978 and January 1979 it was possible

*Shubik (1978).

to run one of the versions of this game again together with two other highly related games for a much larger sample size. The results of these new investigations are reported here and compared with the previous ones. The new games were run in India.

2. BRIEFING AND RESPONDANTS

In Australia the game was run six times, four times with an abstract scenario and twice with an explicitly economic, or productivity scenario. Three of the four Australian audiences for the abstract scenario were composed of economics graduate students and faculty. The fourth was mathematics graduate students and faculty. The other two were social scientists and economists.

The Americans were given the same briefing as the Australian economic briefing. They were undergraduate and graduate students in game theory with backgrounds in economics or other social sciences at Yale.

The Indians were all given a productivity briefing which differed from that described in Figure 1. No diagram was supplied nor were any outcomes suggested. Instead a relatively simple verbal briefing as is indicated in Appendix A was given to all.

The game was used seven times in India in conjunction with several other games which are noted later. The groups were as follows: 20 faculty and graduate students in economics at the University of Dehli; 49 third and second year students with some faculty included) in mathematics and economics, Saint Steven's College, Dehli; 52 first and third year students in economics at Presidency College, Calcutta; 11 graduate economists and members of the staff of the Indian Statistical Institute, Calcutta; 59 students and faculty of economics at the University of Madras; 41 students

in economics and faculty of the University of Madurai and 12 faculty members of the Indian Institute of Management at Bangalore.

In all instances in Australia, the U.S. and India the game served as an introduction to a lecture on game theory. The lecture began with the briefing and the playing of the game and the players were informed that the game would serve as a useful basis for understanding the lecture.

3. SOLUTIONS

The core of an n-person game is a peculiarly economic solution. It has the property that any imputation or division of all proceeds that is within the core satisfies:

Individual rationality--no individual obtains less
than he could get by himself

Group rationality--no group obtains less than
they could get by themselves

Societal rationality or
Pareto optimality--All together waste nothing

In Figure 1 the vertices represent the three points where one individual obtains all and the other two nothing. The side of the triangle opposite any vertex is where the individual favored at that vertex (individual 1 at vertex Y) obtains nothing. Thus along the base of the triangle individuals 2 and 3 vary their returns leaving nothing for individual 1.

The core is the area bounded by ACDE. No individual or pair of individuals can obtain more by independent action than they can obtain when offered any imputation in the core. An imputation $a = (a_1, a_2, a_3)$ in the core satisfies the inequalities:

$$a_1 + a_2 \geq 1, \quad a_1 \geq 0$$

$$a_1 + a_3 \geq 2, \quad a_2 \geq 0$$

$$a_2 + a_3 \geq 3, \quad a_3 \geq 0$$

$$\text{and } a_1 + a_2 + a_3 = 4.$$

All three person games with a core may be regarded as having arisen from an underlying economic structure involving trade and prices.* Work in economics has shown that all trading economies when portrayed as cooperative games give rise to games with a core and the trade called forth by a price system distributes wealth in a manner that the outcome lies within the core.**

Unfortunately without further economic information there is no unique way that we can pick out the full economic structure of the game.

In some instances the core may not exist, in other instances it is large. Shapley*** has suggested a single point solution to a cooperative game which always exists. This is the value. In essence it awards each individual the expected value of his marginal contributions to all coalitions on the "sociologically neutral" assumption that the entry of any individual into any coalition is perfectly random.

The value for this game is given by $V(5/6, 8/6, 11/6)$. The calculations for player 1 is indicated below.

*Shapley and Shubik (1976).

**Shubik (1959), Debreu and Scarf (1972).

***Shapley (1953).

All Orders		
$\overline{123}$	1 enters first and adds nothing	0
$\overline{132}$	1 enters first and adds nothing	0
$\overline{213}$	1 enters second and adds 2 - 0	1
$\overline{231}$	1 enters third and adds 4 - 3	1
$\overline{312}$	1 enters second and adds 2 - 0	2
$\overline{321}$	1 enters third and adds 4 - 3	1
		5

All orders have a probability of 1/6 of occurring, hence 1's expected return is 5/6.

A different one point solution has been suggested by Schmeidler.* We may imagine moving in all the "walls" of the core as shown in Figure 1 by rewarding all coalitions some bonus ϵ_1 . For example suppose $\epsilon_1 = 1/2$ then

$$a_1 + a_3 = 3\frac{1}{2}$$

$$a_1 = \frac{1}{2}$$

$$a_1 + a_3 = 2\frac{1}{2}$$

$$a_1 + a_2 = 1\frac{1}{2}.$$

The core of this game would be a line through the point N parallel to AE bounded by $(1/2, 1, 2)$ and $(1/2, 2, 1)$. We can repeat the rewarding of coalitions who form the walls of this new core with the proviso that we stop giving extra to coalitions if that would destroy the new

*Schmeidler (1969).

core completely. Thus for example

$$a_2 + a_3 = 3\frac{1}{2}$$

$$a_1 = \frac{1}{2}$$

tells us that $a_1 + a_2 + a_3 = 4$ hence any further subsidy to these is not possible. But we can continue subsidies to the coalitions (1,3) and (1,2). Let that be ϵ_2 . It is straightforward to check that the biggest we can let ϵ_2 become is $\epsilon_2 = 1/4$. For any ϵ_2 larger the core would disappear

$$a_1 + a_3 = 2\frac{3}{4}$$

$$a_1 + a_2 = 1\frac{3}{4}$$

but $a_1 = 1/2$ hence $(2/4, 5/4, 9/4)$ satisfies. This is the point N in Figure 1.

The point S $(4/3, 4/3, 4/3)$ is the point of pure symmetry, essentially ignoring questions of productivity.

Points in the triangle XWC could belong to the stable set solution suggested by von Neumann and Morgenstern.*

An important distinction between the groups in India and elsewhere is that the former were not given the geometrical diagram with cues as to possible outcomes.

Table 1 shows the responses to the game.

*von Neumann and Morgenstern (1944).

TABLE 1

	Australia Abstract Game						Econ Briefing Australia			U.S.		India								
	1	2	3	%	4	%	5	6	%	7	%	1	2	3	4	5	6	7		
S (4/3,4/3,4/3)	8	3	7	31.6	3	15	5	4	29			1		1	1	1°	3		2.8	7
E (2/3,4/3,2)	2	1	2	8.8	4	20	6	2	25.8	2	11.1	1	7		4	3	1	1	6.8	17
V (5/6,8/6,11/6)	1	2	1	7.0	2	10	2	3	16.1	4	22.2									
N (2/4,5/4,9/4)	2	3	1	10.5	3	15		1		6	33.3									
(1,1-1/2,1-1/2)	6		2	14.0						3	16.7	4	5	2	7	8	5		12.5	31
(1,4/3,5/3)		1	1		2	10	1							1	4				2.0	5
(0,2,2)	4			7.0											1 ⁺				1	1
(7/6,8/6,9/6)	1				1		1	1	6.5											
(1,1,2)		1	1				1					1	4	1	2	16	5	3	12.9	32
(0,3/2,3/2)			2																	
(1/2,3/2,2)													6		6	6	8		10.5	26
other points in core chosen	1	2	1	7.0	1		1						5	4	13	13	13	6	21.8	54
points not in core other than S	1														1	1	1		1.2	3
core as a whole										3	16.7	2								2
no reply or errors					4	20		3	9.7			11	22 ⁺	2 ⁺	17 ⁺	11	5	2	28.2	70
	26	13	18		20		17	14		18	126	20	49	11	56	59	41	12	248	

TABLE 1 (continued)

*The briefing of one of the first set of games in Australia contained a "should" and "would" version; i.e. the respondents were asked how they believed that the proceeds should be divided, then they were asked how they thought that it would be played without an outside arbitrator. This instance has been discussed by Shubik (1974) elsewhere. Two respondents confused the "should" and "would" scenarios and gave a two person coalition outcome for the game.

†The "no reply or errors" includes several cases when the respondents gave ordinal answers which though formally correct could not be easily coded. (For example $a_1 < a_2 < a_3$.)

°One respondent added "if I were only taking equity into account then instead of choosing what I have I would choose (4/3, 4/3, 4/3).

+One respondent noted that "I assume '1' is the lowest form--one need not give an inducement. Obviously there are ethical connotations of starving--we can give him a subsistence wage to live on. Inducements are most essential for the working of the system."

The groups in India were large, the lecturerroom culture is different, thus there was somewhat less control in eliciting responses from some of the audience.

There appear to be several items of interest which can be elicited from the data in Table 1. They are shown in Tables 2 and 3.

TABLE 2

	Aust 1	Aust 2	U.S.	India
(4/3,4/3,4/3)	28.8	29.0	--	3.9
other not in core	1.4	--	--	1.6
in core	69.8	71	100	94.5

Table 2 shows the percentage of respondents who selected (4/3,4/3,4/3) another point not in the core, or a point in the core. It should be noted that the points not in the core, other than (4/3,4/3,4/3) were all close to its boundary. The Australian are divided into the abstract and economic briefing.

TABLE 3

	Aust 1	Aust 2	U.S.	India
Value (5/6,8/6,11/6)	8.2	16.1	22.2	--
Nucleolus (2/4,5/4,9/4)	12.3	3.2	33.3	--
(2/3,4/3,2)	12.3	25.8	11.1	9.6
(1,3/2,3/2)	11.0	--	16.7	17.4
(1,1,2)	2.7	3.2	--	18.0
(1/2,3/2,2)	--	--	--	14.6

The samples from India were large, yet there is no indication of the value or nucleolus being selected. After the lectures there were indications that some of the audience was intrigued with the value as a normative criterion but without the prompting offered by Figure 1 to the Australian and U.S. respondents the value and nucleolus were not suggested independently.

The only overlap of significance appears to be $(2/3, 4/3, 2)$ and apart from the observation that the ratio 1:2:3 is the same as the ratio of the payoffs to the coalitions, no explanation or justification is offered here.

4. GAMES WITH AND WITHOUT CORES

The game used in all of the instances above has a large core. The data gathered appear to offer some evidence for the proposition that there is a high probability that a point in the core will be selected. A natural question to ask is how robust is the selection of a point in the core if we change its size? In order to obtain some insight into this question the last three groups in India were asked to make their decision on how to divide the proceeds from three different three person games. They are compared below.

The first is the game we have already described and investigated. The second is a game which has a one point core which gives nothing to the first two players. And the third has no core whatsoever.

Game 1

$$\begin{aligned} v(\bar{1}) &= v(\bar{2}) = v(\bar{3}) = 0 ; \\ v(\bar{12}) &= 1 , \quad v(\bar{13}) = 2 , \quad v(\bar{23}) = 3 ; \\ v(\bar{123}) &= 4 \end{aligned}$$

Game 2

$$\begin{aligned} v(\bar{1}) &= v(\bar{2}) = v(\bar{3}) = 0 ; \\ v(12) &= 0 , \quad v(13) = 4 , \quad v(23) = 4 ; \\ v(123) &= 4 \end{aligned}$$

Game 3

$$\begin{aligned} v(\bar{1}) &= v(\bar{2}) = v(\bar{3}) = 0 ; \\ v(12) &= 2\frac{1}{2} , \quad v(13) = 3 , \quad v(23) = 3\frac{1}{2} ; \\ v(123) &= 4 \end{aligned}$$

The second characteristic function portrays a game in which player 3 is critical. Both 1 and 2 are equally productive in combination with 3. This game has a single point core yielding $(0,0,4)$ which satisfies:

$$\begin{aligned} a_1 &\geq 0 , \quad a_2 \geq 0 , \quad a_3 \geq 0 \\ a_1 + a_2 &\geq 0 , \quad a_1 + a_3 \geq 4 , \quad a_2 + a_3 \geq 4 \\ \text{and } a_1 + a_2 + a_3 &= 4 . \end{aligned}$$

The last game has no core whatsoever as can be seen from the following considerations:

$$a_1 \geq 0, \quad a_2 \geq 0, \quad a_3 \geq 0$$

$$a_1 + a_2 \geq 2\frac{1}{2}$$

$$a_2 + a_3 \geq 3\frac{1}{2}$$

$$a_1 + a_3 \geq 3$$

$$2(a_1 + a_2 + a_3) \geq 9$$

but

$$a_1 + a_2 + a_3 = 4$$

The three games are distinguished by a "fat" core; a single point core; and no core. All have single point values and each has a single point nucleolus. Table 4 shows all of these solutions.

TABLE 4

	<u>Core</u>	<u>Value</u>	<u>Nucleolus</u>
Game 1	"fat core"	(5/6, 8/6, 11/6)	(1/2, 3/2, 2)
Game 2	(0, 0, 4)	(4/6, 4/6, 16/6)	(0, 0, 4)
Game 3	no core	(13/12, 16/12, 19/12)	(5/6, 8/6, 11/6)

The conjecture made prior to running Game 2 was that less than 10% of the respondents would select the core even though it is a single point. The reasoning was that it is grossly **discriminatory** to the first two players. After all they are worth something to the production process. It is hard to believe that this point would not influence most respondents.

Tables 5 and 6 show results from Madras, Madurai and Bangalore.

TABLE 5

	5	India 6	7	%
(4/3,4/3,4/3)	--	--	--	
Core (0,0,4)	23	5	2	26.8
Value (2/3,2/3,8/3)				
(1,1,2)	8	2	4	12.5
(1/2,1/2,3)	5		1	5.4
(4/5,4/5,11/5)			1	.9
$0 < \epsilon < 2$ ($\epsilon, \epsilon, 4-\epsilon$)			1	.9
(0,2,2)	3	1		3.6
(1,3/2,3/2)	4			3.6
Others	2	3		4.5
No reply	14	30	3	42.0

Three points of interest can be noted. The percentage of nonrespondants in the audience rose in the three games. Limiting our statistics to the three institutions where the audience responded to all three games the percentages of nonrespondants changed from 16.1% to 42% to 82.1%. Hence for the latter (and to some extent more complex) games the respondents had been somewhat self selective.

Of the active replies, as can be seen from Table 6, 46.2% were at (0,0,4) , or in the core. 22 out of 65 or 33.8% were at an imputation which treats 1 and 2 symmetrically. The average for the 21 responses which were completely specified (i.e. leaving out ($\epsilon, \epsilon, 4-\epsilon$)) was (.85,.85,2.3) this compares with the value at (.66,.66,2.74) .

TABLE 6

	India 1-4 G1	India 5-7 G1	India 5-7 G2
percentage in core	96.2%	94.7%	46.2%

The fall off from the core is striking, but not as large as I had expected.

4.1. A Game without a Core

Only 20 out of 112 individuals responded to the third game. The data are displayed in Table 7. It is of importance to note that in a small

TABLE 7

	5	6	7	%
(4/3,4/3,4/3)	2			1.8
Nucleolus (5/6,8/6,11/6)			1	.9
Value (13/12,16/12,19/12)				
(1/2,3/2,2)	5			4.5
(1/2,1,2-1/2)		2		1.8
(1,3/2,3/2)	3	1		3.6
Other			6	
No Reply	48	38	3	82.1

seminar consisting of faculty at the Indian Institute of Management the size of respondents essentially did not shrink. The first game was answered by 10 out of 12; the second and third by 9 out of 12. This appears to be due to both the smallness in size and the sophistication of the audience.

Their answers are presented in detail in Table 8 below.

TABLE 8

Nucleolus	(.83, 1.33, 1.83)	1	
	(1 , 1.25, 1.75)	3	
	(1.2 , 1.3 , 1.5)	1	Average (1.05, 1.29, 1.66)
	(1.17, 1.35, 1.48)	1	
	(1 , 1.33, 1.66)	1	
	(1.1 , 1.3 , 1.60)	1	
	(1.17, 1.21, 1.61)	1	
Value	(1.08, 1.25, 1.58)	0	

Although the average is "not far" from the value, the sample size is too small to attach much significance to this.

5. CONCLUSIONS

There is a surprising difference in the selection of the equal split (4/3,4/3,4/3) outcome between the Australians on the one hand and the U.S. and Indian respondents on the other hand. The Australians were significantly more biased towards the equal split solution than the others.

Beyond the equal split outcome the core was an extremely good predictor of the responses to Game 1. However although when cues were supplied (Australian and U.S. sample) there was considerable selection of the value and the nucleolus; when the cues were not supplied there was no selection of either the value or nucleolus.

In the second game clearly the core at (0,0,4) flagrantly violates

equity. Furthermore if we considered players 1 and 2 as a syndicate or union the resultant "two"-person game becomes symmetric:

$$v(\overline{12}) = 0 , \quad v(\overline{3}) = 0$$

$$v(\overline{12},3) = 4$$

The core for this game stretches from $(0,4)$ to $(4,0)$. The results from India show clearly that the core selection drops off heavily; but that the symmetry between players 1 and 2 is preserved by many respondents.

The third game raises more questions than have been adequately answered. The specific game here was chosen to have no core and no symmetry. An innate ordering of productivity was selected and this was reflected in that 20 out of the 20 answers obtained at least weakly had the order. A computation of the mean and variance in a large sample size of responses to a three person nonsymmetric game without a core might well be suggestive to those who construct solution concepts.

REFERENCES

- Debreu, G. and H. Scarf. "The Limit Core of an Economy," in C. B. McGuire and R. Radner (eds.), Decision and Organization. Amsterdam: North Holland, 1972.
- Schmeidler, D. "The Nucleolus of a Characteristic Function Game," SIAM J. of Applied Math, 17 (1969), 1163-1170.
- Shapley, L. S. "A Value for n-Person Games," in H. Kuhn and A. W. Tucker (eds.), Contributions to the Theory of Games II, 1953, 307-317.
- _____ and M. Shubik. "Competitive Outcomes in the Cores of Market Games," International Journal of Game Theory, 4 (1976), 229-237.
- Shubik, M. "Edgeworth Market Games," in A. W. Tucker and R. D. Luce (eds.), Contributions to the Theory of Games IV, 1959, 267-278.
- _____. Games for Society, Business and War. Amsterdam: Elsevier, 1975.
- _____. "Opinions on How One Should Play a Three Person Nonconstant Sum Game," Games and Simulation (October 1978), 302-308
- von Neumann, J. and O. Morgenstern. The Theory of Games and Economic Behavior. Princeton: Princeton University Press, 1944.

APPENDIX A

A PROBLEM IN COOPERATION

Three individuals have the opportunity to work together on a task.
Call them 1, 2, and 3.

If they proceed independently each earns	0
If 1 and 2 work together they can earn	1
If 1 and 3 work together they can earn	2
If 2 and 3 work together they can earn	3
If 1, 2, and 3 all work together they can earn	4

Suppose that they decide that they should all work together to earn 4 but they come to you for advice as to how they should divide the 4 units between them. You are the judge.

Your task is to write down three numbers all greater than or equal to zero such that $\alpha_1 + \alpha_2 + \alpha_3 = 4$

$\alpha_1 =$

$\alpha_2 =$

$\alpha_3 =$

and to write down your explanation of why you chose these numbers. Write below and on the other side of the page if you need more space.