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Box 2125, Yale Station
New Haven, Connecticut 06520

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Robert James Weber

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Robert James Weber
Yale University
New Haven, Connecticut U.S.A.

The recent axiomatic study of probabilistic values of games has clarified the relationship between various valuation methods and the players' subjective perceptions of the coalition-formation process. This has important bearing upon the increasingly-common use of the Banzhaf value in measuring the apportionment of power among the players in voting games. The incompatibility of the players' hypothesized subjective beliefs (under the Banzhaf valuation scheme) leads to the strange phenomenon of "pitfall" points (points of value discontinuity) in weighted majority games with several major players and an ocean of minor players. Such results argue against the use of the Banzhaf value (or indeed, of any value other than the Shapley-Shubik index) in the measurement of power in weighted voting systems.

Introduction. Let N be a finite set of players. A simple game v is a (0-1)-function on the subsets (coalitions) of N, which satisfies $v(\emptyset) = 0$ and $v(S) \geq v(T)$ for all $S \supset T$. The collection of all simple games on N is denoted $SG(N)$. We often think of a simple game as representing the decision rule of a political body: coalitions S for which $v(S) = 1$ are said to be winning, and those for which $v(S) = 0$ are said to be losing.

The weighted voting game $[q: w_1,\ldots,w_n]$ is a simple game defined on the player set $N = \{1,\ldots,n\}$, in which a coalition S is winning if and only if $w(S) = \Sigma_{i \in S} w_i \geq q$. The quantity q is the quota of the game, and $w_1,\ldots,w_n$ are the players' voting weights.

Many organizations are formally administered as weighted voting games. (The stockholders of a corporation, for example, are traditionally accorded voting weights equal to the number of shares they own.) Other institutions use decision rules which, while not explicitly formulated as weighted voting games, are nevertheless equivalent to such games. (The United Nations Security Council passes measures with the approval of at least nine of its fifteen members; each of the five permanent members has the right to veto any measure. This situation is fully represented by a weighted voting game in which the quota is 39, the permanent members have voting weights of 7 each, and the remaining ten members have weights of 1 each.)

Legislative bodies in which the legislators represent districts of unequal populations are frequently organized as weighted voting games. How should the voting weights of the various legislators be determined? An obvious approach is to make the weights proportional to the district populations. However, this can lead to highly unsatisfactory results. For example, assume that four districts respectively contain 30, 30, 30, and 10 percent of the total population. An assignment of weights yielding a game such as [51: 30,30,30,10] will leave the residents of the fourth district without effective representation; their legislator will never be an essential member of a winning coalition. Again, assume that three districts respectively contain 45, 45, and 10 percent of the population. Although the third district is much smaller than the other two, the proportionally-weighted voting game [51: 45,45,10] is actually symmetric; all coalitions of two or more legislators are winning.

Situations analogous to those just given (although perhaps a bit more subtle) have arisen in a number of municipal legislative bodies in the United States (for example, see [5]). The courts have generally ruled that such situations violate the "equal representation" principles of the U.S. Constitution, and have required that voting weights be reallocated in order to provide a more equitable distribution of influence.

These same courts have shown a willingness to accept the idea of a measure of "power" of the players in a simple game. If the relative power of the legislators in a weighted-voting legislative body is roughly proportional to the population of the legislators' districts, then the situation is deemed satisfactory.

In order to facilitate comparisons among various measures of power, we shall present several properties which might be desired of such measures. Two particular measures, the Shapley-Shubik and Banzhaf power indices, have received much attention. We will find that both of these reside within a common axiomatic framework which provides a natural interpretation of them in terms of the players' subjective perceptions of the process of coalition formation.

Probabilistic values. As a fixed player $i \in N$ varies his attention over the games in $SG(N)$, he will perceive himself as having greater influence in some games than in others. A value for i on $SG(N)$ is a real-valued function $\mu_i : SG(N) \to R$ which indicates the subjective assessment, by player i, of his power in the various games.

For any $i \in N$, consider the simple game on N in which the winning coalitions are precisely those containing i. Player i is a dictator in this game; it is difficult to imagine him in a more powerful position. On the other hand, since all games in $SG(N)$ are monotonic (if $S \supset T$ and T is winning, then S is also winning), player i's membership can never hurt a coalition. Therefore, his weakest position arises when he cannot contribute anything to any coalition. In this case, when $v(S \cup i) = v(S)$ for all $S \subset N|i$, we say that i is a zero-dummy of v. (For notational convenience, we will often omit the braces when indicating one-element sets.) Generally, if $v(S \cup i) = v(S) + v(i)$ for all $S \subset N|i$, then i is a strategic dummy in v; both dictators and zero-dummies are strategic dummies.

Combining these observations, we impose the following normalization requirement upon a value μ_i.

(P1) For any $v \in SG(N)$, $0 \le \mu_i(v) \le 1$. If i is a strategic dummy in v, then $\mu_i(v) = v(i)$.

A simple game on N is completely characterized by its collections of winning and losing coalitions. Hence, a particularly elementary measure of player i's power in a game would arise from simply tallying the winning and losing coalitions to which he belongs. This idea is embodied in the next requirement.

(P2) There are constants $\{a_T : T \subset N\}$, $\{b_T : T \subset N\}$ such that for every $v \in SG(N)$, $\mu_i(v) = \sum_{\substack{T \text{ wins} \\ \text{in } v}} a_T + \sum_{\substack{T \text{ loses} \\ \text{in } v}} b_T$.

There are a number of equivalent formulations of this requirement, some less transparent than others. For example, for any games u and w in $SG(N)$ define the games $u \vee w$ and $u \wedge w$ by $(u \vee w)(S) = \max(u(S), w(S))$ and $(u \wedge w)(S) = \min(u(S), w(S))$. These two new games are also in $SG(N)$. Recently, attention has been given to the requirement that a value satisfy $\mu_i(u) + \mu_i(w) = \mu_i(u \vee w) + \mu_i(u \wedge w)$ for all games u and w under consideration; this is a lattice-theoretic analogue of linearity. In essence, this says that the transfer of winning coalitions from one game to another does not affect the total value of the two games. (P2) is a direct consequence of this.

A probabilistic value is a value satisfying (P1) and (P2). These two properties together imply that probabilistic values have a very special form.

Theorem. A value μ_i for i on $SG(N)$ is a probabilistic value if and only if there is a collection $\{P_T : T \subset N|i\}$ of nonnegative constants satisfying $\Sigma P_T = 1$, such that for every $v \in SG(N)$,

$$\mu_i(v) = \sum_{T \subset N|i} P_T[v(T \cup i) - v(T)] .$$

Proof. It follows from (P2) that for any v,

$$\mu_i(v) = \sum_{v(T)=1} a_T + \sum_{v(T)=0} b_T = \sum_{T\subset N} (a_T-b_T)v(T) + \sum_{T\subset N} b_T .$$

For any $T\subset N$, let $\hat{v}_T \in SG(N)$ be the game in which the winning coalitions are precisely those S for which $S \supsetneq T$. If T is nonempty, let $v_T \in SG(N)$ have as winning coalitions those S for which $S\supset T$. The game $\hat{v}_N$ is identically zero. Player i is a zero-dummy in this game; it follows from (P1) that $\Sigma b_T=0$. For every $T\subset N$, define $c_T = a_T - b_T$. Then $\mu_i(v) = \Sigma c_T v(T)$.

Let T be any nonempty coalition in $N|i$. Player i is a zero-dummy in v_T. Therefore, $\mu_i(v_T)=0=\Sigma_{T\subset S\subset N|i}(c_{S\cup i}+c_S)$. By induction, beginning with $T=N|i$ and proceeding to successively smaller coalitions, it follows that $c_{T\cup i}+c_T=0$. Define $P_T = c_{T\cup i} = -c_T$, and also define $P_\emptyset = c_i$. Then $\mu_i(v) = \Sigma_{T\subset N|i} P_T[v(T\cup i)-v(T)]$.

From (P1), it follows that $\Sigma P_T = \mu_i(v_i) = 1$. Furthermore, for any $T\subset N|i$, $P_T = \mu_i(\hat{v}_T) \geq 0$. This establishes the formula in the theorem. It is easily verified that any value defined in this manner indeed satisfies both (P1) and (P2). □

Our principle concern is with an interpretation suggested by the representation in the theorem. Player i can view the coefficients P_T as subjective probabilities. Coalitions form through a process of accretion. At some point, a coalition in $N|i$ will approach i and invite him to join; the probability that he is approached by T is P_T. The quantity $\mu_i(v)$ is then the probability that i is pivotal, converting the coalition he joins from losing to winning. Hence, the normalization and simplicity assumptions lead directly to a subjective model of coalition formation, in which a player's sole concern is with his marginal contribution to the coalition he joins. We shall hold this model in mind throughout the remainder of this paper.

Semivalues. When developing a measure to compare the influence of the various players in a game, it seems reasonable to adopt a symmetric point of view. In addition, it is desirable that the measurement method be applicable to all finite-player games (rather than merely to games on a fixed player set).

Let U be an infinite set, the universe of players. A simple game v on U is a monotonic (0,1)-function on the subsets of U which satisfies $v(\emptyset) = 0$. Any $N\subset U$ such that $v(S) = v(S\cap N)$ for all $S\subset U$ is a carrier of v. A finite simple game is a simple game with a finite carrier; the set of all such games is $SG(U)$. A value Ψ_i for $i\in U$ is a real-valued function on $SG(U)$. For any finite $N\subset U$, $SG(N)$ can be embedded in $SG(U)$ by treating the players in $U|N$ as zero-dummies. A value Ψ_i is said to satisfy (P1) and (P2) if its restriction to each $SG(N)$ satisfies these conditions. A *permutation* $\pi:U\to U$ is a one-to-one onto mapping. For any permutation π and game $v\in SG(U)$, define $\pi v \in SG(U)$ by $(\pi v)(S) = v(\pi S)$ for all $S\subset U$. A *semivalue* $\Psi = (\Psi_i)_{i\in U}$ is a collection of values satisfying (P1), (P2), and the following symmetry condition:

(P3) $\Psi_i(\pi v) = \Psi_{\pi i}(v)$ for all $i\in U$, all permutations π of U, and all games $v\in SG(U)$.

Let ξ be a probability distribution on [0,1]. The value Ψ^ξ on $SG(U)$ is defined for all $v\in SG(N)$ and $i\in N\subset U$ by

$$\Psi_i^\xi(v) = \sum_{S\subset N|i} P_s^n\,[v(S\cup i)-v(S)]\ ,$$

where $P_s^n = \int_0^1 t^s(1-t)^{n-s-1}d\xi(t)$. (Here, n and s generically denote the cardinalities of N and S.) Note that the definition of $\Psi_i^\xi(v)$ is independent of the selection of a carrier N of v. It is not difficult to verify that Ψ^ξ is a semivalue. The following result is derived in [3].

Theorem. Let Ψ be a semivalue on $SG(U)$. Then there is a probability distribution ξ on [0,1] such that $\Psi=\Psi^\xi$.

Adopting the interpretation of probabilistic values given in the previous section, we can view a semivalue in the following manner. Given $v\in SG(N)$, the players in $N|i$ are assigned random positions on [0,1] which are chosen independently and uniformly. The position of i is chosen according to the distribution ξ. Then $\Psi_i^\xi(v)$ is the expected marginal contribution of i (or equivalently, the probability that i is pivotal) when he joins the coalition of players whose positions precede his.

A semivalue Ψ^ξ gives each player in turn a distinguished treatment when his value is computed. A fully-symmetric treatment arises when ξ is the uniform distribution on [0,1]. This yields the *Shapley value* [8,9], with $P_s^n = s!(n-s-1)!/n!$. On the other hand, if ξ is the probability distribution concentrated at the point $\frac{1}{2}$ then the i-subjective viewpoint associated with Ψ_i^ξ is highly idiosyncratic; i considers himself likely to hold a central position among the players. This yields the *Banzhaf value* [1], with $P_s^n = 1/2^{n-1}$.

Consistency. The use of a semivalue Ψ to compare the relative influence of the players of a game is not affected by rescaling. Hence, one may work instead with the normalized value $\bar{\Psi}$, defined for all $v\in SG(U)$ by $\bar{\Psi}(v) = \Psi(v)/\Sigma\Psi_i(v)$. Historically, most applications of the Banzhaf value have employed this normalization.

A conceptual difficulty with this approach is that the sum in the normalizing factor combines the subjective probabilities of different individuals, a probabilistic analogue of adding apples and oranges.

<u>Theorem</u>. The Shapley value is the only semivalue Ψ for which $\Sigma\Psi_i(v) = 1$ for every nonzero $v \in SG(U)$.

This characterization of the Shapley value in the context of simple games first appeared in [2]. For an alternative derivation, let λ denote the Lebesque measure on [0,1] and let $\xi \neq \lambda$ be any other probability measure. Select $x \in [0,1]$ such that $(D\xi)(x) > 1$, or such that x is in the support of the component of ξ singular with respect to λ . Let v_k be a k-player game in which all coalitions of more than xk players win. Then for $x \neq 1$, sufficiently large values of k can be found so that $\Sigma\Psi_i^{\xi}(v_k) > 1$. (If $x=1$, the same result holds when v_k is a k-player unanimity game.)

The theorem provides a particular distinction to the Shapley value: it is the unique semivalue arising from consistent expectations. The effect of inconsistency on normalized values is discussed in the next section.

<u>Games with many minor players</u>. We consider weighted voting games consisting of a set $M = \{1,2,\dots,m\}$ of "major" players, and a large number k of "minor" players of total voting weight $\alpha > 0$. Let $v_k = [q:w_1,\dots,w_m,\frac{\alpha}{k},\dots,\frac{\alpha}{k}]$, and let $v_k' = (q:w_1,\dots,w_m,\frac{\alpha}{k},\dots,\frac{\alpha}{k})$; a coalition wins in the latter game if it has total weight strictly greater than q . The following results concern the semivalues (unnormalized and normalized) of the major players when k is large.

Let Ψ^t denote the semivalue associated with the probability distribution concentrated at t . Define $z_t = [q-t\alpha:w_1,\dots,w_m]$ and $z_t' = (q-t\alpha:w_1,\dots,w_m)$; if $q-t\alpha \leq 0$ then $z_t=0$, and if $q-t\alpha < 0$ then $z_t'=0$.

<u>Theorem</u>. For all $0<t<1$ and $i \in M$,

$$\lim \Psi_i^t(v_k) = \lim \Psi_i^t(v_k') = \tfrac{1}{2}\,\Psi_i^t(z_t) + \tfrac{1}{2}\,\Psi_i^t(z_t') \ .$$

<u>Theorem</u>. For all $0<t<1$ and $i \in M$,

$$\lim \bar{\Psi}_i^t(v_k) = \lim \bar{\Psi}_i^t(v_k') = \begin{cases} \bar{\Psi}_i^t(z_t) = \bar{\Psi}_i^t(z_t') & \text{if } t \notin P \\ 0 \ \text{ if } t \in P \ , \end{cases}$$

where $P = \{t : \text{for some } S \subset M \ , \ w(S)+t\alpha=q\}$.

Both of these results follow in a straightforward manner from the central limit theorem, or from a few judicious applications of Stirling's formula. As k becomes large, each major player considers the total weight of the minor players in the coalition he will eventually join to be equally likely to be slightly more or slightly less than $t\alpha$; the first theorem is an immediate consequence of this. Fix a minor player j , and let L be the (binomially-distributed) number of other minor players in the coalition which j eventually joins. If $w(S)+t\alpha=q$, then $\Psi_j^t(v_k)\geq t^s(1-t)^{m-s}\Pr(L<tk\leq L+1)$; this probability is asymptotically proportional to $k^{-\frac{1}{2}}$. On the other hand, if $\epsilon = \min\{|t-t'|:t'\epsilon P\}>0$, then $\Psi_j^t(v_k) \leq \Pr(L<(t-\epsilon)k \text{ or } (t+\epsilon)k\leq L+1)$; this probability is asymptotically proportional to $k^{-\frac{1}{2}}r^k$, where $r = \exp(-\epsilon^2/[2t(1-t)])<1$. Consequently, in one case the sum of the values of the k minor players increases without bound, and in the other case the sum approaches zero. Hence, the normalized semivalues behave as indicated in the second theorem. (The cases $t=0$ and $t=1$ require separate treatment, because the distribution of L is degenerate. However, this degeneracy makes direct computation of the values Ψ^0 and Ψ^1 trivial.)

For any probability distribution ξ on [0,1], and any $v\epsilon SG(U)$ and $i\epsilon U$, $\Psi_i^\xi(v) = \int_0^1\Psi_i^t(v)d\xi(t)$. Since the integrand is nonnegative and bounded, the first limit theorem carries over to general distributions in an obvious manner.

The possible behavior of normalized values, as indicated in the second theorem, is best illustrated by an example. Let β denote the normalized Banzhaf value, and consider the games $v_k = [55: 40+\epsilon,30,20,\frac{10-\epsilon}{k},\ldots,\frac{10-\epsilon}{k}]$ for various values of ϵ . If $\epsilon=0$, $(\beta_1,\beta_2,\beta_3)(v_k)\to(0,0,0)$; that is, the minor players share essentially all the influence in the game. For any small $\epsilon>0$, $(\beta_1,\beta_2,\beta_3)(v_k)\to(\frac{3}{5},\frac{1}{5},\frac{1}{5})$. This discontinuity may seem unsurprising, since the voting weight of a major player has been increased at the expense of the minor players. But for any small $\epsilon<0$, we have $(\beta_1,\beta_2,\beta_3)(v_k)\to(\frac{1}{3},\frac{1}{3},\frac{1}{3})$; according to the normalized Banzhaf value, a sacrifice in voting weight can benefit the major players! (It can be shown that for a general distribution ξ and a game $v\epsilon SG(U)$, the occurrence of this type of "pitfall discontinuity" is related to the existence of a $t\epsilon P\cap(0,1)$ such that $d\xi/d\lambda$ either does not exist or is unbounded in every neighborhood of t .)

Of course, this strange behavior results from the inconsistency of the players' subjective expectations. If $\epsilon=0$, each minor player considers himself relatively more likely to be pivotal than any of his fellow minor players; for $\epsilon\neq0$, the opposite situation holds. And the normalized values of the major players depend critically upon this collective optimism or pessimism of the minor players.

References. Probabilistic values were first characterized in [11], and semivalues in [3]. Limiting results concerning weighted voting games are given for the Shapley value in [7], and for the Banzhaf value in [4]. These two papers allow for the unequal distribution of voting weights among the minor players in a game; the techniques used in these papers can be adapted to the problems treated here. Asymptotic properties of the relative values of the minor players are investigated in the paper by Neyman appearing elsewhere in this volume; see also [6]. A comparison of the Shapley and Banzhaf values, based on a probabilistic model different from that presented here, can be found in [10].

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