COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 514

MARKET INDUCED WELFARE OPTIMA

Rolf R. Mantel

January 31, 1979

MARKET INDUCED WELFARE OPTIMA

by

Rolf R. Mantel

Universidad Católica Argentina
and
Instituto Torcuato Di Tella

July 1973

The present investigation has been carried out while the author was Visiting Professor at the Cowles Foundation for Research in Economics at Yale University. The finantial support of that institution and of the National Science Foundation of the United States of America, which allowed this research to be carried out, and of the Consejo Nacional de Investigaciones Científicas y Técnicas of the Argentine Republic during the preparation of the final version are gratefully acknowledged. I wish to thank my colleages from the Cowles Foundation, from the Universities of Minnesota, Northwestern, Washington and Tucumán, and from the Instituto Torcuato Di Tella for their patience in listening to the first versions, and for their criticism and suggestions which permitted a considerably improved final product.

MARKET INDUCED WELFARE OPTIMA

by Rolf Mantel

a. Introduction

We are observing the competive equilibrium solution of a multimarket economy; the invisible hand has done its job. For some such a solution is optimal. If so, what does the invisible hand maximize? A correct answer is, the negative of the distance to equilibrium. This is obvious but hardly helpful, since the equilibria have to be known to apply such a concept of optimality. A minimum requirement for a criterion for optimality or social welfare function is that it be a monotone increasing function of the utility levels attained by the individual consumers. This type of welfare function is associated with the names of Bergson and Samuelson; its monotonicity with that of Pareto. The problem becomes then one of determining an increasing function of the utilities which when maximized leads to the set of competitive equilibria of the standard Arrow-Debreu general equilibrium model. Such a function could then be analyzed so as to have an indication of the welfare judgments involved implicitly in accepting the existing wealth distribution (statu-quo). Perhaps it would also show a way to compute such equilibria, using nonlinear programming methods.

b. Description ot the Model and Tools

The model analyzed is the standard Arrow-Debreu general equilibrium model of a private ownership economy as expounded by Debreu (1959). Since we are looking for a function of the utility levels, the model will be examined in utility space. Thus for the present purpose, the actual consumption and production allocations are immaterial, as are prices of commodities and incomes of the agents. The following statement will be used in order to justify the analysis of only the simplest pure trade version of the model.

Proposition: The standard Arrow-Debreu model can be approximated, from the point of view of competitive equilibrium utility allocations, by the following pure trade economy. There are m traders (consumers), with initial endowments w^i in their consumption sets, which are just the non-negative orthant R^n_+ of commodity space, and with preferences represented by utility functions $u^i : R^n_+$ to R_+ which are continuous, concave, linear homogeneous and monotone.

This proposition, not being central to the ensuing arguments, will not be proved here. Intuitively, production can be assigned to households, and induced preferences defined on trades. Each trader's utility function is then "homogenized" by introducing a new commodity, his "ego" or "essence". After a convenient displacement, his consumption set can then be extended to R^n_+. The sense of the approximation is given by the following assumption.

Assumption on Preferences: They can be represented by concave, continuous, functions. Granted the other usual assumptions in general equilibrium theory this is not too strong, since such utility functions can approximate convex, continuous preferences to any degree, as has been shown in Mantel (1967), Kannai (1972), and Mas-Colell (1972).

Once the model has been reduced to the simple pure trade model described in the Proposition, define the utility possibility set U as follows.

$$U = \{ u \in R_+^m : u_i \leq u^i(x^i) \; ; \; X \geq 0 \; ; \; Xe = We \} \; ,$$

where the consumption allocation X is a matrix with n rows and m columns of non-negative elements. Its i-th column x^i represents the consumption bundle assigned to the i-th consumer. The m-vector e has all coordinates equal to unity. If a is an m-vector of positive welfare weights, maximizing $a \cdot u$ on U leads to a P to optimum. Under our assumptions, all Pareto optima can thus be sustained by some vector a of welfare weights. Since the competitive equilibrium is a Pareto optimum, it could be hoped that a way to determine such an equilibrium would be by selecting appropriate welfare weights and the subsequent maximization of the weighted average of individual utility levels. Negishi (1961) was the first to show that if these weights are adjusted taking into account the budget surpluses of the consumers at the optimum achieved, evaluating the optimum trades at the implicit prices associated with the resource constraints, the resulting process has a stationary point that corresponds to the welfare weights sought. Since the budget constraints are then

satisfied, that point corresponds to a competitive equilibrium with equilibrium prices equal to the implicit prices that have been mentioned. Unfortunately that process is not stable in general (Mante, 1971). Neither do the values of the welfare weights at an equilibrium relate in any simple way to the initial data of the problem. Thus the linear welfare function is not what we are looking for, especially taking into account that in general it may lead to several maximizing utility allocations, not all of which correspond to equilibria, while other equilibria will not maximize such a simple function without a modification of the weights.

Note that the utility possibility set does not hold sufficient information to locate competitive equilibria. For the latter it is necessary to know the distribution of initial holdings of goods and services. One way of retaining more information is to use the utility transformation cone

$$T = \{ (u,v) \geq 0 : u_i \leq u^i (x^i) \; ; \; X \geq 0 \; ; \; Xe \leq Wv \}$$

which can be interpreted as a set of processes which transform people -- the numbers v_i of consumers of type i -- into utility levels. Of course in the end we are interested in the original economy, in which there is only one consumer of each type obtained by setting $v = e$. Note that T is defined in a way similar to U ; the latter now becomes the section of the cone T corresponding to one person of each type. This cone has been introduced by Mantel (1965), together with the offer set to be defined subsequently. It has been shown there that T has all the properties usually assigned to the technology of the von Neumann growth model.

Let the polar cone -- or dual cone -- of T be defined as follows.

$$T^{o} = \{ (a,b) \geq 0 : au - bv \leq 0 \quad \text{for all} \quad (u,v) \in T \} .$$

Then the offer set, in the space of utility levels, is defined by $T^{*}(e)$.

$$T^{*}(e) = \{ u : \text{there exists } (a,b) \in T^{o} \text{ such that } b_i = a_i u_i \text{ for all } i \}$$

The name assigned to this set comes from its boundary when there are two traders and two commodities, which consists then of utility allocations corresponding to pairs of consumption allocations on each consumer's offer curve corresponding to some price vector. The present use of the term "offer set" should be distinguished from the related definition used previously by the author (1975, remark 5).

As has been shown by Mantel (1965) the intersection of the offer set with the utility possibility set is the set of competitive utility allocations; these allocations were there designated as coordinate-wise value-preserving. The existence of competitive equilibrium was there demonstrated by showing that it is possible to determine in a finite number of steps a point in the utility possibility set which is at a preassigned distance from the offer set, and then letting this distance tend to zero. That proof was not constructive in the sense of converging to the intersection; but it did provide an approximate solution of the general equilibrium model in the sense of present day fixed point methods.

The utility transformation cone is closely related to other concepts in economic and game theory. If e^{i} represents the i^{th} unit vector in

R^m and if $e^S = \sum_{i \in S} e^i$, where S is a coalition (i.e., subset) of consumers M, we can consider the sections of the utility transformation cone T.

$$T(v) = \{ u : (u,v) \in T \} .$$

As noted before, $T(e^M)$ is the usual utility possibility set. The restriction of the correspondence $T(\cdot)$ to subsets of M leads to market games in characteristic function form, extensively analyzed in a sequence of articles by Billera and Bixby and one by Mas-Colell (1975). The restriction of the same correspondence to arguments which have non-negative integer coordinates not exceeding some integer k gives the market game associated with the k-fold replica of an economy, concept useful in the Scarf-Debreu limiting theorems. Finally, the restrictions to the convex hulls of coalitions give Aubin's fuzzy games.

The utility transformation cone can be defined for any game in characteristic function form if it is balanced, by applying the characterization theorems which imply that such games are generated by some economy; this permits extending economic concepts such as competitive equilibrium to games not originating in markets. Alternatively, if V is a characteristic function on coalitions to subsets of utility space, define T as the union os sums of the form $\sum_S d_S V(S)$ for non-negative weights d_S. The restriction of $T(\cdot)$ to the coalitions corresponds to the cover of the game, which coincides with the game if the latter is totally balanced. Here we use the strong concept of balanced games used by Billera and Bixby, as opposed to the concept of quasi-balanced games due to Scarf.

c. Special Cases for Which the Competitive Model Can Be Solved by Nonlinear Programming Methods

In the present section a classification of cases in which competitive solutions can be found by maximization will be given.

1. General not homothetic preferences.

   i. All consumers have the same preferences and endowments of commodities -- this is a special case of the next point --.

   ii. The initial endowments are distributed optimally according to the Pareto criterion. An obvious welfare function for this case is

$$b(u) = \min_i u_i \,/\, u^i\,(w^i)$$

2. Homothetic preferences.

   i. All consumers have the same preferences with possibly different endowments. The equilibrium prices are the marginal utilities of the aggregate endowment. Each individual's consumption bundle will be in proportion to this aggregate endowment, its level being consistent with the value of his initial endowment at those prices. Social welfare in this case can be taken to be the utility of aggregate consumption. In fact a more general from of this case is given by requiring that the utility transformation cone be the sum

$$T = \sum T^i$$

of the m intersections

$$T^i = T \cap \{ (u\ v) \in R^{2m} : u_k = 0 \quad \text{for} \quad k \neq i \}$$

representing the levels of resources needed for consumers of type i to attain given utility levels if there were no other consumption demand. Results are similar to those later described under item 2.iii.

ii. The relative income distribution is independent of market prices. This case has been analyzed by Eisenberg. Chipman, Sonnenschein and Shafer, Chipman and Moore have generalized the result. Gale studied the particular case in which the utility functions are linear. Under the present assumptions the aggregate economy behaves as if it were a single consumer. The welfare function maximized by the community is the average utility of the consumers, weighted by the relative income distribution. One has to take the geometric mean if utility functions are linear homogeneous; the arithmetic mean if they are logs of linear homogeneous functions. What makes this case especially attractive is the fact that the equilibrium solution can be found by maximizing this average using standard methods of nonlinear programming since the social welfare function is concave. Written out explicitly, if d_i represents the fraction of the aggregate endowment owned by the i-th consumer,

$$b(u) = \prod_i \left[u^i \right]^{d_i} .$$

iii. The market is perfectly balanced. This is the case when the utility transformation cone T is the sum

$$T = \sum_i T^i$$

of the m cones

$$T^i = T \cap \{ (u,v) \in R^{2m} : v_k = 0 \text{ for } k \neq i \}$$

Therefore, the utility levels that can be attained by the community are just sums of the levels that can be achieved separately by using only the endowment of each individual. It has been shown by the author (1978) that the offer set of perfectly balanced markets is convex, and that it is possible to construct a concave social welfare function whose maxima coincide with the set of competitive equilibria.

A special case of perfectly balanced markets has received some attention in the literature by Gale (1957, 1976), Eaves (1976), and Mantel (1976), under the name of linear exchange model. Linearity refers to the utility functions $u^i = c^i \cdot x^i$. As shown in the paper mentioned last, the unique competitive utility allocation can be obtained maximizing the concave, linear homogeneous, strictly monotone welfare function defined by

$$h(u) = \max_{p \in P} \min_{i,j} u_i p_j / [c^i_j (w^i \cdot p)]$$

on the utility possibility set U. The set P denotes the

$$\text{unit simplex } P = \{ p \in R^n_+ : e \cdot p = 1 \} .$$

The following sections will refer to the general case. It could be hoped a priori that the results might be similar to those listed in the present section. This could be inferred if one made a parallel between the development of the theory of the linear utility model and the general model.

The first equilibrium existence proof for the linear utility model has been provided by Gale (1957) using the fixed point theorem of Kakutami. Writing in 1975, Eaves (1976) provided an algorithm of the Lemke-Howson type which also led to an existence proof. Finally, the present author (1976) reduced the problem of existence of equilibria to a problem in non-linear programming applying the Kuhn-Tucken theory to the maximization of a convave welfare function subject to linear inequality constraints.

The developments regarding the proof of existence of competitive equilibria in the general case from the point of view of welfare maximization start with Negishi's (1960) existence proof mentioned above applying Kakutami's fixed point theorem, deviating from the then fashionable and fruitful tradition of equating demand and supply. The present author (1965), using a Lemke-Howson type argument, followed essentially Negishi's formulation for another proof. Can it be inferred from here that the next step would again consist in reducing the problem to one of non-linear programming, where the Kuhn-Tucker theory can be applied? We will see that regretfully this is not the case.

d. Market-induced optima in utility space

It is now a simple matter to construct a monotone, linear homogeneous function of the utilities which when maximized leads to the set of competitive utility allocations. Define

$$b(u) = \sup \left[t : u \text{ is in } T^*(te) \right]$$

which under our assuptions is continuous on R^m_+, if we add one of the usual requirements for consumers to have positive incomes at quasi- or compensated equilibria. Explicitly, we assume Arrow and Hahn's resource relatedness assumption. In our terminology, this assumption is equivalent to the requirement that the utility possibility set can be expanded in the direction of the payoff tc some consumer in a coalition by proportionate increases in the endowments of consumers not in the coalition. Formally, if u is in $T(e)$, for any subset S of M there exists a non-negative v with $v_i = 1$ for i in S, and $\bar{u}$ in $T(v)$ such that $\bar{u} \geq u$ and $\bar{u}_S > u_S$. Where sets are used as indices, we mean the sum of the elements corresponding to the indices in the set, so that u_S stands for $\sum_{i \in S} u_i$. This assumption allows us to ignore the boundary of the welfare weight space, since competitive welfare weights will be strictly positive. By what has been said in section b., the intersection of $U = T(e)$ and $T^*(e)$ consists of the set of competitive utility allocations. Any such allocation u^* maximizes $b(u)$ subjet to u in $T(e)$, and $b(u^*) = 1$.

In the general case, even if $T^*(e)$, the offer set, is not convex,

the welfare function b will be well defined, continuous, homogeneous of degree 1, and strictly monotone on R^m_+. It will be maximized on $T(e)$, the utility possibility set, at any competitive utility allocation u^*, the value being $b(u^*) = 1$, whereas no other feasible utility allocation will achieve that value. Thus the set of maximizers will be the intersection of the utility possibility set with the offer set, that is to say, the set of competitive utility allocations. Since in general $T^*(e)$ is not convex, b need not be concave. In fact it cannot be concave if there exist several isolated competitive utility allocations.

Figure 1 shows how $T^*(e)$ is constructed in the general case. Point e is the origin for the Utility possibility set $U = T(e)$; the added dimension corresponds to changes in the relative numbers of v_1 and v_2. The cone T has been normalized so that the sum of the coordinates of v is constant to allow the representation. Corresponding to a vector of welfare weights a there is an efficient point u in U. The hyperplane H supports T at u, and is therefore a point in T^o. If $m = 2$, the intersection of H with the nonnegative orthant has four vertices. Consider those two labelled R and S. They define a "diagonal" -- really an m-dimensional linear subspace -- intersecting the flat defined by $v = e$ at the unique point $\bar{u}$ in $T^*(e)$. The nonnnegative orthant, displaced so that its origin coincides with $\bar{u}$, is a subset of $T^*(e)$. Since H need not be unique, the welfare weights a may generate several points such as $\bar{u}$.

In the two dimensional case it is quite obvious that a can be selected so that u and $\bar{u}$ coincide. For m larger than 2 fixed point

arguments are needed. Note that the "diagonal" through $\bar{u}$ does not meet the interior of T. This means that no point in the cone can dominate $\bar{u}$ in the sense that no point in T provides a higher u_i per unit of v_i for every i. No feasible transformation provides a higher expansion rate for all i.

By the definition of H, corresponding to welfare weights a interpreted as "prices" on the "outputs" $\bar{u}$ one can associate "prices" with the "inputs" v, such that v_i carries the "price" $a_i\bar{u}_i$. In particular, at equilibrium, where $v = e$, this pricing relationship means that the value of output i, $a_i\bar{u}_i$, equals the value of the corresponding input, $(a_i\bar{u}_i) \times 1$, where the unity stands for the quantity of the input.

This property of the equilibrium utility allocation $\bar{u}$ has been designated as being coordinate-wise value-preserving in Mantel (1965).

e. Market-induced optima in commodity space

In order to understand better the meaning of the optimality concept involved, it is instructive to translate the concepts expressed in terms of utility allocations to their counterparts in commodity space.

It is well known that the counterpart of the utility possibility set is the familiar contract curve in the Edgeworth box diagram. As the usual definition of Pareto optimality goes, it is the set of allocations of trades $Y = X - W \geq -W$ such that Y satisfies two conditions.

i. Y is balanced, i.e., $Ye \leqq 0$.

ii. No balanced trade Z is Pareto superior to Y .

One says that the allocation Z_i is Pareto superior to the allocation Y if no trader i prefers his y^i to z^i, and some i prefers z^i to y^i .

In order to restrict the set of Pareto optima to the set of competitive equilibria, a larger comparison set is needed in ii. From the previous remark that competitive utility allocations have undominated "expansion rates" it is seen immediately that this means that per capita utilities cannot be increased for everybody by varying the number of consumers of each type. This can be rephrased as follows, to avoid the fiction of altering the number of consumers. Define the allocation Y to be a weakly balanced allocation of trades if it is individually feasible: $Y + W \geqq 0$, and if there exists a set of positive weights t such that $Yt \leqq 0$. That is to say that some other allocation which provides each trader with a trade $y^i t_i$ along the same ray as y^i is balanced. Define market-induced optimality by requiring that a feasible trade Y satisfies the following two conditions.

i. Y is balanced, so that $Ye \leqq 0$.

ii. No weakly balanced trade Z is Pareto superior to Y .

It can be shown that under the usual assumptions of the standard Arrow-Debreu model, the set of market-induced optima in fact coincides with the set of competitive allocations. This is a generalization of the previously stated results, which hold only for preferences with concave representation. Note that the set of weakly balanced trades is wider

than the alternative trades with which Schmeidler and Vind compare their trades to conclude that they are fair. These authors do not allow just any alternative weakly balanced trade, but only allocations which are integer multiples of the candidate fair allocation. This implies in particular that fair trades need not be optimal in the present sense, unless the candidate fair trade spans all weakly balanced trades.

From the definitions involved, it is easily seen that market-induced optima are Pareto optimal and fair.

It is interesting to analyze the meaning of condition ii. for a market-induced optimum. That is, we observe the set Q of feasible trades which are not Pareto dominated by weakly balanced trades. It can be shown that in the case in which the preferences have concave representations, the utilities associated with such trades are exactly the points in $T^*(e)$. Therefore we will call Q "offer set" in commodity space. Furthermore, the boundary of $T^*(e)$ consists of utility allocations in the offer set which do not dominate others in the same set; in commodity space this defines the set of trades which are not Pareto superior to weakly balanced trades, and do not dominate others with the same property. In the Edgeworth box diagram for the two trader, two commodity case, a point in this set is represented by two points, one for each trader, such that there is a line through the initial endowment point separating the corresponding two indifference curves. In other words, it consists of a pair of excess demands for some price vector, as could have been inferred from the fact that market-induced optima are competitive allocations: points on the boundary of the offer set are weakly balanced,

hence are competitive allocations, in an economy with a different origin for each trader, and therefore must be points on the offer curve for the competitive price system.

From these remarks we see that a maximum of the welfare function $b(u)$ defined above provides a utility allocation which is feasible -- is in $T(e)$ -- and in the offer set $T^*(e)$. The corresponding trade allocation is feasible -- in the Edgeworth box the points for the two traders coincide -- and a market induced optimum -- the two points are on the corresponding offer curves for some given price line through the initial endowment point --.

A most interesting consequence of these investigations is that these considerations carry over to higher dimensional models. In particular, the offer set Q in commodity space provides an interesting generalization of the offer curves. The usual drawback of this tool is that it is not in general possible to determine competitive equilibria by their intersection in higher dimensions. This problem is resolved by the offer set by considering not each individual in isolation but by defining it as a set of allocations, one for each trader.

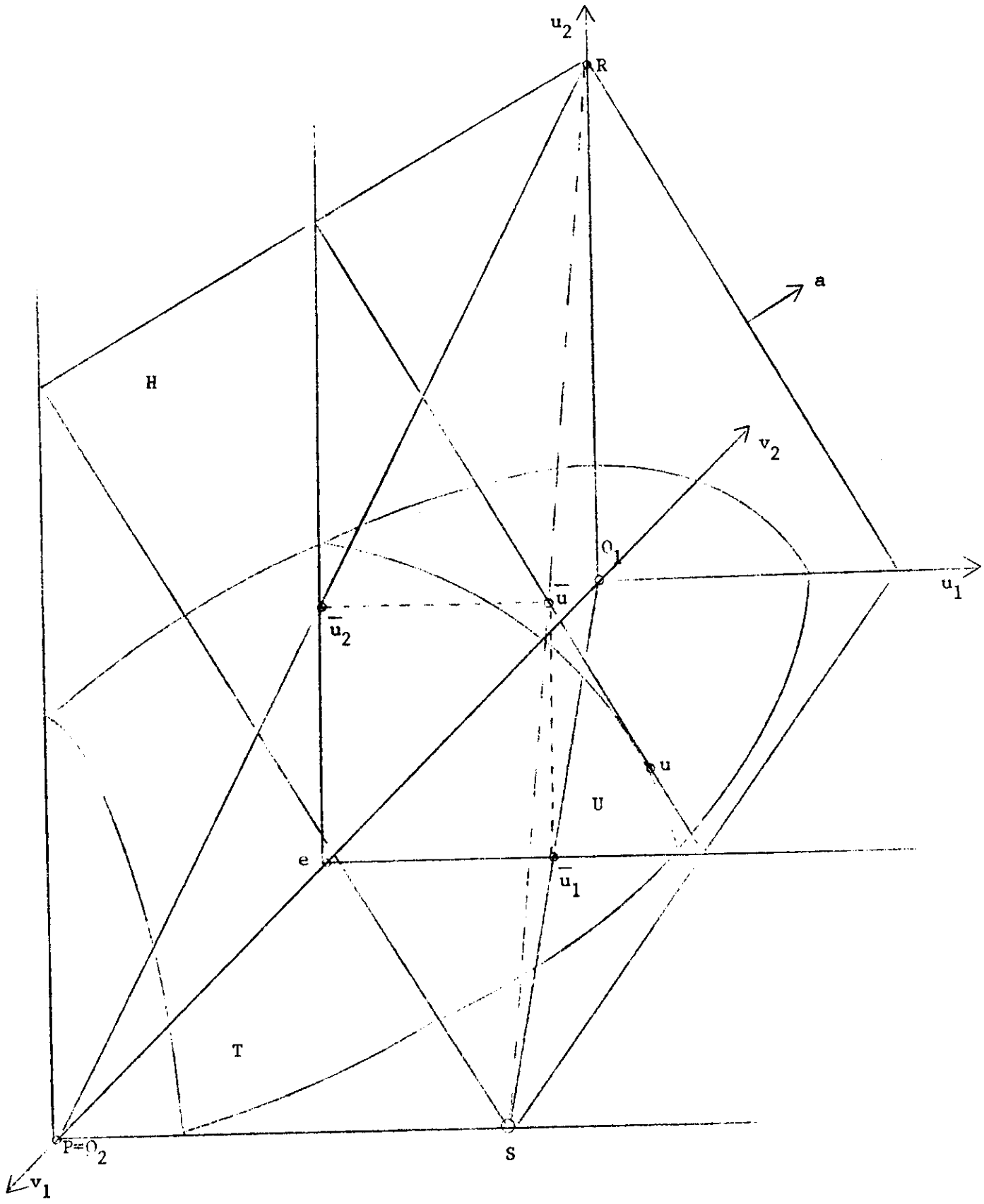

FIGURE 1

REFERENCES

Arrow, K. and F. Hahn (1971), General competitive analysis. San Francisco: Holden-Day.

Aubin, J. (1972), Equilibrium of a convex cooperative game. Wisconsin, Madison, Mathematics Center Report 1279.

Baudier, E. (1973), Competitive equilibrium in a game, Econometrica, 41, 1049-68.

Billera, L.J. (1974), On games without sidepayments arising from a general class of markets, Journal of Mathematical Economics, 1,129-39.

-------, and R.E. Bixby (1973), A characterization of polyhedral market game, International Journal of Game Theory, 2, 253-61.

Chipman, J.S. (1974), Homothetic preferences and aggregation, Journal of Economic Theory, 8, 26-38.

-------, and B.J. Moore (1975), paper presented to the Third World Congress of the Econometric Society, Toronto, Canada.

Debreu, G. (1959), Theory of value. New York: Wiley.

------, and H.S. Scarf (1963), A limit theorem on the core of an economy, International Economic Review, 4, 235-46.

Eaves, B.C. (1976), A finite algorithm for the linear exchange model, Journal of Mathematical Economics, 3, 197-203.

Eisenberg, E. (1961), Duality in Homogeneous programming, Proceedings of the American Mathematical Society, 12, 783-7.

Gale, D. (1957), Price equilibrium for linear models of exchange, Rand report P-1156.

----, (1960), The theory of linear economic models. New York: McGraw-Hill.

Kannai, Y. (1974), Approximation of convex preferences, Journal of Mathematical Economics, 1, 101-6.

Lemke, C.E. and J.T. Howson (1964), Equilibrium points of bimatrix games, SIAM Journal, 12, 412-23.

Mantel, R. (1965), Equilibrio de una economía competitiva, una prueba de su existencia. Mimeo, Instituto Di Tella.

Mantel, R. (1966), Toward a constructive proof of the existence of equilibrium in a competitive economy. Ph.D. dissertation, Yale University; Yale Economic Essays, 8 (1968), 155-196.

------, (1967), On the representation of preferences by concave utility functions. Mimeo, Instituto Di Tella.

------, (1971), The welfare adjustment process: Its stability properties, International Economic Review, 12, 415-430.

------, (1975), General Equilibrium and optimal taxes, Journal of Mathematical Economics, 2, 187-200.

------, (1976), The linear exchange model and induced welfare optima, Cowles Foundation Discussion Paper N° 428.

———, (1978), Perfectly balanced markets. Forthcoming.

Gale, D. (1976), The linear exchange model, Journal of Mathematical Economics, 3, 205-209.

Mas-Colell, A. (1974), Continuous and Smooth consumers: Approximation theorems, Journal of Economic Theory, 8, 305-336.

----------, (1975), A further result on the representation of games by markets, Journal of Economic Theory, 10, 117-122.

Negishi, T. (1960), Welfare economics and the existence of an equilibrium for a competitive economy, Metroeconomica, 12, 92-97.

Rader, T. (1963), Edgeworth exchange and general equilibrium, Ph.D. Dissertation, Yale University.

Samuelson, P.A. (1947), Foundations of Economic Analysis, Cambridge: Harvard University Press.

Scarf, H.E. (1967), The core of an n-person game, Econometrica, 35, 50-69.

———, (1973), The computation of economic equilibria. With the collaboration of T. Hansen. New Haven: Yale University Press.

Schmeidler, D. and K. Vind (1972), Fair net trades, Econometrica, 40, 637-642.

Shafer, W. and H. Sonnenschein (1975), Some theorems on the existence of competitive equilibrium, Journal of Economic Theory, 11, 83-93.

Shapley, L.S. and M. Shubik (1969), On market games, Journal of Economic Theory, 1, 9-25.

Wilson, R. (1967), Budgetary surplus adjustment, International Economic Review, 8, 103-108.

------, (1978), The bilinear complementarity problem and competitive equilibria of piecewise linear economic models, Econometrica, 46, 87-104.