

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

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COWLES FOUNDATION DISCUSSION PAPER NO. 509

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John Sutton

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A model of a quasi-competitive industry is constructed, in which the firm's sales are described by a random variable whose expected rate of change depends on price. It is shown that a stationary (non-degenerate) market distribution of prices results, so that price differences persist over time.

Considerable attention has been devoted in recent years to the study of markets which are quasi-competitive in the sense that they retain the notion of a large number of firms selling a homogeneous product, but depart from perfect competition in relaxing the assumption that consumers are perfectly informed as to the prices of the various firms.

The initial surge of interest in this type of model was motivated by the need, first noted by Arrow (1956), to deal with the firm, even in a competitive environment, as a price setter, in order adequately to tackle the analysis of disequilibrium behavior. Thus early work in the field, beginning with Fisher ((1970), (1972), (1973)) focussed on the question of whether an initial market distribution of prices would, over time, converge to a unique equilibrium price.

More recent work has, however, developed the idea that market equilibrium might be characterized by a persistent distribution of prices. That this is more reasonable in the light of "the variety and volatility of prices (which is) the commonplace of our experience" was argued by Rothschild (1973). A further, and theoretically more compelling, reason for exploring this question, however, is provided by what is probably the most striking aspect of the literature on these markets: the fact that for a very wide range of apparently quite reasonable assumptions, the distribution of prices converges to the monopoly price (Diamond (1971), Hey (1974)). Indeed, where prices do converge, they converge to the competitive price only under very strong conditions: for example, where firms are artificially constrained to behave as if they were perfect competitors (Fisher, Rothschild, op.cit.). Thus it would seem that in order to tackle the question of whether, under conditions of imperfect price information,

any competitive features of the market may be preserved, we are compelled to examine market equilibria of this more general class.

Such price dispersion as is empirically observed in many markets undoubtedly owes its origin to a wide range of contributory factors. This suggests representing the firm as experiencing a succession of exogenous random shocks, as in Lucas and Rapping (1974). An alternative approach is to explore the possibility that firms set a range of suboptimal prices via their various estimates of actual demand conditions, as deduced by following an optimal estimation procedure (stopping rule), as explored by Rothschild (1974). More germane to our present concerns as to whether the range of actual prices, or their average, might be drawn by competitive pressures below the monopoly price, is the more recent work which begins from the notion that consumers differ in their costs of acquiring information, so that firms partition themselves permanently into subgroups patronized predominantly by different mixtures of consumer types; the "better informed consumers" being associated, as it were, with the "lower price firms." (Salop and Stiglitz (1978), Axell (1977).)

The present model adopts a rather different type of approach; we aim to model equilibrium in the quasi-competitive economy as an ongoing process, in which firms continually compete with each other to increase their respective sales to a number of identical customers.

1. BACKGROUND

The present approach to the problem rests on two fundamental notions. The first of these is that a consumer will not always select a supplier at random in making each purchase, but will show some tendency to return to his previous supplier. This is reasonable in a model with imperfect informa-

tion, insofar as the consumer, being unaware of rival prices, may be indifferent between his old supplier and a randomly chosen alternative, as will be the case in the present model. (Alternatively, such persistence might derive from the notion that the consumer remains with his old supplier until he hears of a lower price offer elsewhere.) Such persistence of patronage also derives indirect support from the evidence on the analogous problem of brand choice in the marketing literature (for example, Brown (1953).) Its role in the model is to make current sales depend not only on current prices, but also, indirectly, on past prices via their effect on current patronage. In the present context we will introduce this assumption in a very simple form analogous to that of the marketing models, in which a fixed proportion of the firms' consumers leave each period.

Our second basic notion concerns the responsiveness of consumers to price in their choice of a new supplier. We require that the rate of arrival of new (potential) customers is greater, according as the price (currently) set by the firm is less.

This may seem not only eminently reasonable, but rather innocuous. In fact, however, its implications are far reaching; in particular, it is not true in the usual sequential search models, and it is this which leads to the tendency for price dispersion in such models to collapse (Butters (1977a), (1977b)). The idea, in the sequential search model, is that consumers select each potential supplier at random, and decide on the basis of the price then observed whether to proceed further. If a range of prices $[P_{\min}, P_{\max}]$ persists, it follows that it is never optimal for a firm to charge P_{\min} , for no potential customer will refuse a price offer of $P_{\min} + \frac{1}{2}c$ where c is the (lowest) search cost; the potential gains certainly fall short of c .

One way out of this difficulty is to assume that some consumers have $c = 0$, which is equivalent to assuming that they are perfectly informed, or enjoy free information. Such models as have been constructed suggest that this is not enough--the distribution of search costs over consumers must exhibit a "heavy concentration," or an "atom" of probability, at $c = 0$ (Axell (1977), Salop and Stiglitz (1978)). This approach to the problem is then, arguably, an uncomfortably strong one--for in attempting to relax the notion of "perfect information" in competitive theory, we in fact retain it among a (substantial) subset of agents.

Our present "solution" is less drastic; we merely assume directly that the arrival rate is price dependent--this does involve the notion that some free channel of information exists, but we do not require the stronger property that some consumers enjoy unlimited access to such information.

Our motivation for this procedure is the observation that the standard search model actually understates the volume of information which is in practice available to the consumer, whether via "passing by stores," "reading advertisements," or "hearing of price offers" from fellow consumers.

We will introduce here only such a partial broadening of the available channels of information flow as is essential to our analysis. What we have in mind is the notion that, in casual encounters, some individuals,

knowing of a low price offer, will mention it to acquaintances whom they know are interested in making a purchase in the current period. Clearly, one does not tell one's acquaintances of all the prices one observes; there is a disutility to be set against the potential value of the information involved. Hence we may reasonably assume that news of lower prices, being more valuable, will be more widely communicated.

This leads directly to our second basic notion, that the rate of arrival of new customers is greater, the lower the price.

2. THE DYNAMICS OF CONSUMER FLOW

The treatment of the consumer choice problem in the present analysis is rather primitive. We merely extend the usual simplistic approach of the current literature to an environment in which the consumer makes multiple purchases.

We assume the good in question to be one which the consumer purchases at certain regular intervals ("consumption planning periods".) Within each of these intervals, he may or may not consume one unit of the good in question--some imperfect substitutes are available to him, and, on the basis of his income and (average) prices in other markets, he will choose not to purchase if the price exceeds some ceiling price P_0 , which we may identify with the "monopoly price."

The notion that the consumer derives utility from one unit of the commodity (per period), but not from additional units, is of course very restrictive, but is not unreasonable for a range of actual commodities. It is useful as an assumption in the present context only in that it allows us to focus attention on those aspects of the model which are novel.

That the consumer makes his purchases one unit at a time we might associate either with income constraints within each "consumption planning period," or with the notion that the good is perishable--we shall prefer simply to appeal directly to the latter notion here.

We now proceed to an informal discussion of the dynamics of consumer flow in the model, which provides a motivation for our formal assumption on the demand conditions faced by the firm in the next section. As we already noted above, we intend to introduce some persistence of patronage by simply assuming that some fixed fraction, say γ , of the firm's customers leave per unit time. We now consider the selection of an alternative supplier by these "floating" consumers.

It will be a property of our model that each firm's price fluctuates over time in a similar manner, and our consumers will be armed with imperfect information concerning the prices of particular firms. Thus they face a problem in general of choice between probability distributions which is

intrinsically rather complex. We now introduce an assumption which seems to play no important economic role in what follows, but which is useful in simplifying the analysis. This is the notion that the frequency of purchase of the commodity in question is low, so that the interval between successive purchases is long.

One--rather trivial--consequence of this is that we may then assume that the rate of departure of customers is price independent, for the customer will regard the price charged by his previous supplier as a poor guide to his current price; his previous supplier is then characterized by the same expected price for the current period as a randomly chosen firm (i.e. the consumer does not benefit from any serial correlation in the individual firm's price.)

A more important consequence of our "infrequent purchases" assumption is simply that it allows us to confine the consumer's choice to a choice between prices known with certainty, and a single price distribution characterizing "all other firms".

Our assumption on the dissemination of price information is that a certain fraction α of our "floating" consumers hear of the price currently offered by the firm; we take it that information is "sparse" in that we ignore the possibility of hearing of several offers.

Our consumer, thus informed, makes a choice between this offer of P or a random choice. (In terms of the standard sequential search model, we may characterize him as facing a choice, at his first "visit," between two types of firms: that firm whose price he knows, and all other firms, the latter having unknown prices described by identical probability distributions.)

We will assume our consumers to be risk neutral, so that a price is preferred to a random offer according as it lies below the average market price \bar{P} .

The value of news of a price offer may be measured by the expected saving $\bar{P} - P$; we specify the fraction of "floating" consumers who hear of the firm's price offer as $\alpha(\bar{P} - P)$, which is zero at $\bar{P} \geq P$ (news of zero value is not communicated) and we assume that returns to price reductions are diminishing, in the sense that $\alpha(\bar{P} - P)$ is concave.

Those consumers who do not hear of some specific supplier's price are now assumed to choose a new supplier at random, armed with a reservation price reflecting the market distribution of prices and the consumer's (constant) unit search cost. It will be an immediate conclusion of our analysis of the firm's optimal pricing policy that the maximum market price will coincide with this common reservation price P_x of our identical consumers.

Thus, in fact, we shall see below that those consumers who select a new supplier at random will not in fact encounter a price in excess of their reservation price, so that they will in either case make a purchase from the first supplier visited.

This is merely a simple consequence of our assumption of identical consumers. It is perhaps worth stressing at this point that it does not imply that their merely latent willingness to search has no effect--in fact it has, as we will see below, a crucial effect on the equilibrium price distribution.

The net flow of consumers is described, then, as the sum of three contributions; a fixed fractional rate of departure of existing consumers, a price dependent rate of arrival of informed consumers, and a rate of

arrival of consumers who select a new supplier at random. We may model this last contribution by assuming that each of our N firms enjoys the same probability $1/N$ of being selected so that the rate of arrival of new customers from this source is described by a binomial distribution, or, approximately, in the limit, a normal distribution with the same mean and variance.

Our use of two deterministic elements leads to a considerable simplification in the formal analysis; a framework in which all three contributions are probabilistic can be written down, however, and approximate solutions found. It seems, though, that there is little new, of economic interest, to be gained from such a generalization.

In the same spirit, we introduce the notion that, of the (identical) patrons of any firm (whose common-planned frequency of purchase is one unit every m periods, say,) a fraction of exactly $1/m$ make a purchase in any period. Thus "sales" in any period is simply a multiple $1/m$ of current "patronage," so that we may proceed to formulate our model in terms of a "sales" variable x , labelling our above contributions as "a flow of customers equivalent to sales of x units per period."

For any distribution of market prices (concentrated on $[0, P_0]$) our assumption on the individual demand schedules implies that total per period sales over all firms are constant, irrespective of the actual market price distribution; we will denote this level of total sales as $N\bar{x}$, thus defining \bar{x} as the average level of sales over all firms.

The first contribution to consumer flow corresponds to a rate of departure of a fixed fraction γ of the firm's patrons per unit time.

If the firm's sales are currently x , the component of the rate of change of sales via this contribution equals $-\gamma x$; while the total (gross) rate of outflow of consumers from their current suppliers equals $-\gamma N\bar{x}$.

We now proceed to the reallocation of these "floating" consumers to alternative suppliers. A fraction $\alpha(\bar{P} - P)$ of them hear of, and accept, the offer of any firm whose price is P , implying a rate of arrival of "informed" consumers of $\gamma N\bar{x}\alpha(\bar{P} - P)$; representing the probability distribution function of the market distribution of prices as $Y(p)$, this accounts for a fraction of

$$\int_0^{\bar{P}} \alpha(\bar{P} - P) N dY(P) = A(\bar{P}, Y(P)), \text{ say,}$$

of our $\gamma N\bar{x}$ floating consumers. Hence a total of $\gamma N\bar{x}(1 - A(\bar{P}; Y(P)))$ choose one of our N firms at random; our third contribution to consumer flow is represented by a binomial distribution with mean $\gamma\bar{x}(1 - A)$ and variance $(1 - 1/N)\gamma\bar{x}(1 - A)$, which, for reasonably large N , is well approximated by a normal² with the same mean and variance; we will avoid some clumsiness of expression in the sequel by dropping the factor $(1 - 1/N) \approx 1$ in the variance.

It is worth remarking at this point that it is the stochastic nature of this rate of arrival of new customers allied to our assumed persistence of patronage which drives the system, leading to the possibility of persistent price dispersion below.

We may thus model the firm's sales, over time, as a stochastic process such that if sales take the value x at time t , then sales at time $(t + \Delta t)$ are given by a normal distribution with mean $x + \mu\Delta t$ and variance $\sigma^2\Delta t$. Here μ represents the expected rate of change of sales, and we may write

$$\mu = -\gamma x + \mu_1(P)$$

where

$$\mu_1 = \gamma \bar{x}(1 - A(\bar{P}, Y(P))) + \gamma N \bar{x} \alpha (\bar{P} - P)$$

and

$$\sigma^2 = \gamma \bar{x}(1 - A(\bar{P}, Y(P)))$$

Here μ_1 represents the sum of our two rates of arrival, one--price dependent--contribution corresponding to informed consumers, the other corresponding to the expected rate of arrival of random searchers; while σ^2 is the variance associated with this latter contribution.

This process, in the limit $\Delta t \rightarrow 0$, may be represented as a Wiener process with "drift" μ and "variance" σ^2 .

We now consider the behavior of our process at the boundaries corresponding to zero sales, and to full capacity, respectively.

We will assume, technically, that our process has reflecting barriers at 0 and M, where full capacity corresponds to a level of output of M units per unit time. Such barriers carry the interpretation that "the process starts again" when one of these values is reached. Economically, we are in effect assuming that the firm cannot attract a level of patronage in excess of that corresponding to full capacity operation; explicitly, if the number of arrivals is such that planned sales exceed M per unit time, then some of these potential customers find the firm unable to supply, and we are in effect assuming that such consumers immediately switch their patronage to an alternative supplier.

The reallocation of these unsatisfied consumers to alternative suppliers is represented by the addition of a further (deterministic)

flow, via a constant "correction term" η to $\mu_1(P)$ above. This correction term, which will be small except where $\bar{x} \sim M$ (a point to which we return below), is defined implicitly by the requirement that industry sales coincide with total demand $N\bar{x}$.

3. THE FIRM

We may now proceed armed with the preceding motivation to specify the cost and demand conditions faced by the individual firm. We will assume throughout that the firm has constant marginal costs MC up to full capacity operation, and so declining average costs $AC(x)$ which take the value $AC(M)$ at full capacity. Empirical support for this specification is reviewed, for example, by Johnston (1960). We will further assume (to ensure that our model is viable) that our monopoly price P_0 exceeds $AC(M)$.

Turning now to the demand side, we may sum up our discussion of the dynamics of consumer flow in two stages:

Assumption 1 (Demand conditions): For all prices $P \leq P_r$, the reservation price, the firm's sales, per unit time, are described by a Wiener process between reflecting barriers at 0 and M with drift

$$\mu(P, x) = -\gamma x + \mu_1(P) \quad (1)$$

and constant variance σ^2 ; for $P > P_r$ sales are zero. Explicitly

$$\begin{aligned} \mu_1 &= \gamma \bar{x} (1 - A(\bar{P}, Y(P))) + \gamma N \bar{x} \alpha (\bar{P} - P) + \eta \\ &= \mu_0(P, x; A, \bar{P}, P_r) + \eta \end{aligned} \quad (2)$$

$$\sigma^2 = \gamma \bar{x} (1 - A(\bar{P}, Y(P))) \quad (3)$$

where

$$A(\bar{P}; Y) = \int_0^{\bar{P}} \alpha(\bar{P} - P) NdY(P) \quad (4)$$

It remains to specify the properties of the function $\alpha(\bar{P} - P)$, as illustrated in Figure 1.

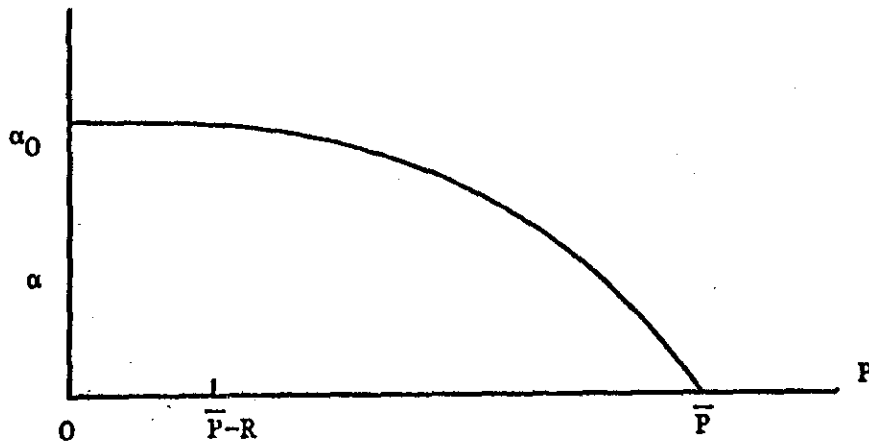


FIGURE 1. The properties of the function $\alpha(\bar{P} - P)$

We wish to assume that α is zero above \bar{P} , and that for $P \leq \bar{P}$ it is positive, and exhibits diminishing returns. Since, as is intuitively obvious, an optimal pricing policy will never involve reducing price below a point at which $\alpha(\bar{P} - P)$ becomes horizontal, we may conveniently exclude the appearance of negative prices in our analysis as follows: we note that, in seeking an equilibrium distribution of market prices, we shall be able, below, to confine ourselves to distributions characterized by average prices lying in the interval $MC - c \leq \bar{P} \leq P_0$ (here c is the unit search cost). Thus, introducing a "saturation level", $\bar{P} - P = R \leq MC - c$, such that further price reductions below this level are no more widely disseminated, it follows

that α is horizontal at all nonpositive P , for any \bar{P} in the appropriate range.

We may now state

Assumption 2 (Diminishing returns to price reductions): The function

$\alpha(\bar{P} - P)$ satisfies, for some $R \leq MC - c$,

$$\begin{aligned}
 \alpha &= 0, & \bar{P} - P &< 0 \\
 \alpha &= 0, \dot{\alpha} = \dot{\alpha}(0) < 0, & \bar{P} - P &= 0 \\
 \alpha &> 0, \dot{\alpha} < 0, \ddot{\alpha} < 0, & R > \bar{P} - P &> 0 \\
 \alpha &= \alpha_0, \dot{\alpha} = 0 & \bar{P} - P &\geq R
 \end{aligned} \tag{5}$$

The upper bound α_0 thus defined will be found useful below; we remark here that our assumption that information is sparse, i.e. no consumer hears of more than one firm's price, and our interpretation of α as the fraction of all consumers hearing of the price offered by any particular firm, imply that $N\alpha_0 \leq 1$.

4. THE OPTIMAL PRICING POLICY

The expectation of the firm regarding the probability distribution of sales, after any time lapse τ , depends only on current sales, being independent of present time, i.e. the process defined above is Markovian. Hence, the optimal price depends only on x , and is time independent. We may therefore describe a price strategy as a function $P(x)$ defined on $0 \leq x \leq M$; our problem is to find the function which maximizes expected per period profits. We are thus assuming the firm to be a long-run profit maximizer, with zero discount rate.

It is convenient to work from this point forward in terms of the markup $p = P - MC$ over marginal cost, so that our optimal policy is expressed as $p(x)$. We may define the level of markup corresponding to \bar{P} and P_r as \bar{p} and p_r respectively, while $\alpha(\bar{P} - P)$ may equivalently be written as $\alpha(\bar{p} - p)$.

We now consider the probability distribution of sales associated with any price strategy. Let $y(x_0, x; t)$ be the conditional probability that sales lie in the interval $(x, x + \Delta x)$ at time t , given that $x = x_0$ at $t = 0$. Then $y(x_0, x; t)$ is described by the forward differential equation of the Wiener process

$$\frac{\sigma^2}{2} \frac{\delta^2 y}{\delta x^2} - \frac{\delta}{\delta x} \{\mu y\} = \frac{\delta y}{\delta t}$$

with boundary conditions for reflecting barriers at $x = 0, M$,

$$\frac{\sigma^2}{2} \frac{\delta y}{\delta x} - \mu y = 0; \quad x = 0, M$$

where the latter equation corresponds to the requirement that x takes values in the interval $0 \leq x \leq M$ only (Cox and Miller (1965), p. 223).

The process converges as $t \rightarrow \infty$ to a stationary distribution with density, up to a normalising constant,

$$\exp \frac{2}{\sigma^2} \int_0^x \mu(w) dw \quad (6)$$

so long as the integral of this expression over $[0, M]$ is bounded, as is certainly the case here, where $\sigma^2 > 0$ and μ is bounded above, by (1-5). (Mandl (1968), chapter IV). The expression (6) for the density $y(x)$ may be verified by setting the time derivative to zero in the forward differential equation. It indicates, for example, that for a "constant price" policy, where $\mu = -\gamma x + \mu_1(P)$, the sales distribution is a truncated normal.

It follows from the ergodic property of the stationary distribution (ex. Yaglom (1962)) that we may interpret $y(x)$ either as the cross-sectional distribution across firms at any point in time, or as the time average of any one firm over an arbitrarily long interval.

Differentiating this expression we obtain immediately the relationship for the density $y(x)$,

$$\dot{y}(x) = \frac{2}{\sigma^2} \mu(x) \cdot y(x)$$

or

$$\frac{\dot{y}}{y}(x) = \frac{2}{\sigma^2} \mu(x) \quad (7)$$

The normalizing constant is determined by the requirement that

$$\int_0^M y(x) dx = 1$$

In order to embody this property into our analysis below we introduce the distribution function

$$Y(x) = \int_0^x y(w) dw \quad (8)$$

The optimal pricing policy will now be described as the solution to a control problem; we may choose an appropriate control variable by reference to our equation for drift, in terms of p ,

$$\mu(p, x) = -\gamma x + \mu_1(p) \quad (2)'$$

where $\mu_1(p)$ is, by virtue of Assumption 2, positive, and exhibits diminishing returns up to \bar{p} , after which it is constant up to the "reservation price" p_r , and zero thereafter.

Our problem is to

$$\text{maximize}_{p(x)} \int_0^M p(x)y(x) dx$$

$$\text{subject to } p_r - p \geq 0$$

$$\dot{y} = \frac{2}{\sigma} \mu(p, x)y \quad (10)$$

$$\dot{Y} = y \quad (11)$$

end point condition

$$Y(0) = 0; \quad Y(M) = 1 \quad (12)$$

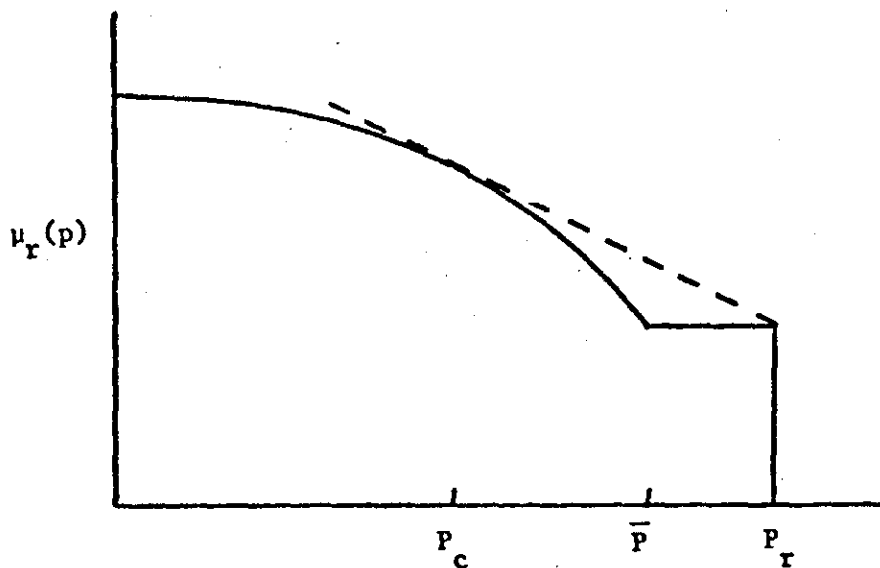


FIGURE 2. The function $\mu_1(p)$. The tangent at $p = p_c$ passing through p_r , $\mu_1(p_r)$ corresponds to the critical value of the costate variable λ , representing the shadow price of patronage, above which price reductions below p_r are profitable, as defined in the text.

A set of sufficient conditions for a maximum in the general control problem has been provided by Kamien and Schwartz (1971); in the present context they take the following form:

Define the Hamiltonian

$$H = p(x)y(x)x + \Lambda y + \lambda \frac{2}{\sigma} \mu(p(x), x)y(x) \quad (13)$$

Let $U(x, y^*(x), Y^*(x), \lambda(x), \Lambda(x))$ be the value of p which maximizes the Hamiltonian at x , given the (unknown) optimal values of the state variables y, Y and the associated Lagrange multipliers λ, Λ .

Then $p^*(x)$ is an optimal pricing policy if U as defined above is concave in the state variables y, Y and there exist functions $\lambda(x), \Lambda(x), w(x)$ such that $\lambda(x), \Lambda(x)$ are continuous and $w(x)$ is integrable and the following equations are satisfied:

$$p^*(x) = U(x, y^*(x), Y^*(x), \lambda(x), \Lambda(x)) \quad (14)$$

$$-\frac{\delta H}{\delta Y} = \dot{\Lambda} = 0 \quad (15)$$

$$-\frac{\delta H}{\delta y} = \dot{\lambda} = -px - \Lambda - \lambda \cdot \frac{2}{\sigma} \mu(p, x) \quad (16)$$

$$\frac{\delta H}{\delta p} = 0 = y(x)x + \lambda \cdot \frac{2}{\sigma} \frac{\delta \mu}{\delta p} y(x) + w(x) \quad (17)$$

$$w(x) \geq 0 \quad (18)$$

$$w(x)(p_T - p) = 0 \quad (19)$$

$$\lambda(x) \geq 0 \quad (20)$$

$$\lambda(0) = 0; \quad \lambda(M) = 0 \quad (21)$$

$$Y(0) = 0; \quad Y(1) = 1 \quad (22)$$

Here, equation (14) defines U , while (15)-(16), (20)-(21) are the usual differential equations for the costate variables λ and Λ , and

(17) is the condition for an internal maximum of the Hamiltonian over the control, modified by the inclusion of a shadow variable $w(x)$ which is positive only where $p = p_r$ by (18), (19).

The K-S conditions also require that the constraints on the control variable prescribe a feasible region which satisfies a constraint qualification; for the present case, where the feasible region is described by our single constraint $p_r - p \geq 0$ this is trivially satisfied.

The first of our differential equations for costate variables (15) implies that Λ is a constant; this is of course true in general for the costate variable (Lagrange multiplier) associated with an isoperimetric constraint (Bryson and Ho (1975), p. 90). In fact, the particular structure of the present problem allows for a simple interpretation of Λ as the negative of the maximand, expected profits per unit time, at the solution. This is established in Appendix 1.³

Turning now to the properties of the solution path to our set of equations (14)-(22), we first examine the requirement that p shall be that value which maximizes the Hamiltonian

$$H = pyx + \Lambda y + \lambda \cdot \frac{2}{\sigma} (-\gamma x + \mu_1(p))y \quad (13)'$$

given y , λ and Λ . From our assumed form of the drift function it follows immediately from inspection of (13)' that two cases arise depending on the values of λ and x .

For values of λ/x sufficiently large the maximum is given by a point on the concave portion of the drift function such that

$$\frac{\delta H}{\delta p} = yx + \lambda \cdot \frac{2}{\sigma} \cdot \frac{\delta \mu}{\delta p} \cdot y = 0$$

or

$$\lambda = -\frac{\sigma^2}{2} \cdot x \cdot \frac{\delta p}{\delta \mu} \quad (23)$$

We define a critical value $(\lambda/x)_c$ such that for values $(\lambda/x) \geq (\lambda/x)_c$ a maximum is attained on the concave portion of the drift function; while for $(\lambda/x) < (\lambda/x)_c$ the maximum occurs at $p = p_r$. This critical value corresponds via (23) to the slope $\delta p/\delta \mu$ of the tangent passing through the point $p_r, \mu_1(p_r)$, as illustrated in Figure 2.

It follows from this that the value of the control variable p which maximizes the Hamiltonian depends only on λ, x , being independent of y ; and so the value of the Hamiltonian maximized over the control variable is linear, and so concave, in y . Moreover, the Hamiltonian is independent of Y , so that our problem satisfies the requirement of concavity of H in the state variables as demanded by the K-S conditions.

Turning to the two cases just identified, we note that our optimal pricing policy $p(x)$ consists of segments (i) along which price coincides with the reservation price, and where (using (17), (18), (19)) $(\lambda/x) \leq (\lambda/x)_c$, $w(x) > 0$, and (ii) along which price lies below the reservation price, and moreover, below average market price, where $(\lambda/x) > (\lambda/x)_c$ and $w(x) = 0$. The explicit form of the policy may be deduced from the solution paths of the differential equation

$$\dot{\lambda} = -px - \Lambda - \lambda \cdot \frac{2}{\sigma} \mu(p, x) \quad (16)'$$

$$\text{with } \lambda(0) = \lambda(M) = 0 \quad (21)'$$

as the (negative) parameter Λ varies.

The qualitative properties of the solution paths are deduced in Appendix 2; and formal proofs are supplied in Appendix 3. In view of the very lengthy nature of the mathematical development we merely state the form of the family of solution paths, parameterized by Λ , being the negative of expected profits per period.

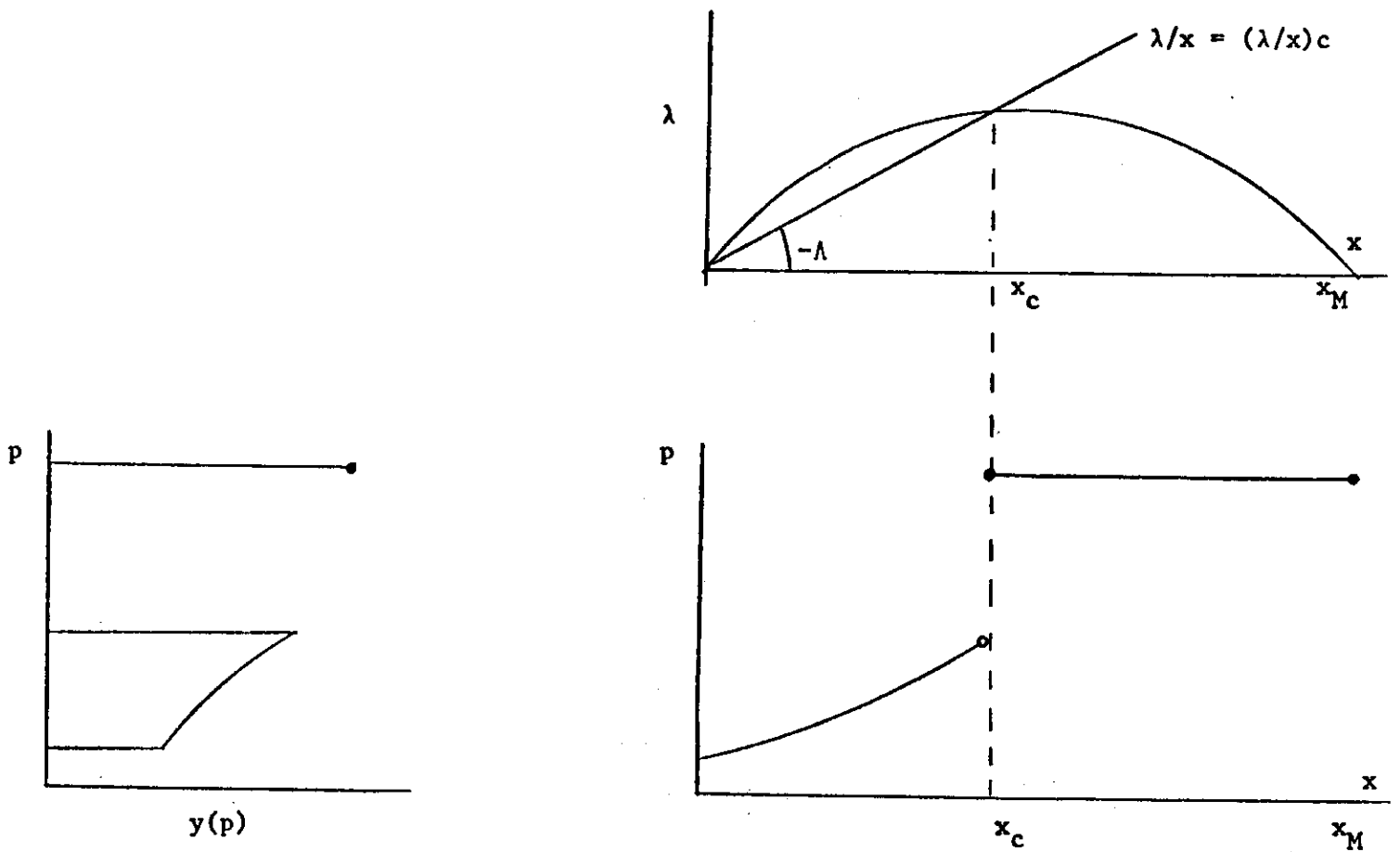


FIGURE 3. The optimal pricing policy $p(x)$, the shadow price of patronage $\lambda(x)$, and the probability density function of market prices $y(p)$.

From equation (16)', (21)', we have immediately for $x = 0$ that $\lambda(0) = 0$ and, $\dot{\lambda}(0) = -\Lambda$.

Hence, for values of Λ close to zero, the solution path $\lambda(x)$ lies below the critical ray $\lambda/x = (\lambda/x)_c$ in the region of the origin; it may further be shown to lie wholly below that ray. Thus the optimal pricing policy involves setting $p = p_r$, all x . The associated root x_M , such that $\lambda_c(x_M) = 0$ increases continuously as Λ declines from zero. The boundary condition $\lambda(M) = 0$ serves to identify uniquely the solution path corresponding to the given level of plant capacity.

For values of $-\Lambda$ lying above the critical value $(\lambda/x)_c$ however, the solution path $\lambda(x)$ lies, in the region of the origin, above the critical ray and the optimal pricing policy consists of two segments. Price increases strictly, and continuously, with current sales, up to some critical value x_c , though remaining below average market price; and thereafter jumps discontinuously to the reservation price p_r for all $x > x_c$. The associated root x_M increases continuously to $+\infty$ as Λ declines towards some limiting finite value corresponding to the expected profits of a firm with unlimited capacity facing the given demand conditions.

The increasing form of the pricing policy reflects the combination of two factors, both operating in the same direction. Firstly, price reductions are attractive insofar as they lead to increases in patronage, and so to increases in future, as well as present, sales. The value of such increases in patronage is greater, according as we are further below full capacity operation, and can meet such increases as result; the shadow price of patronage declines to zero at full capacity.⁴ Secondly, price reductions are less costly in terms of profits foregone according as sales are less; this latter factor implies the rather interesting property that

the (increasing) form of our optimal solution is preserved, even when the level of capacity becomes infinite.

The optimal pricing policy thus generates a distribution of market prices consisting of an atom at the reservation price P_r , and a continuous segment lying wholly below the average market price \bar{P} , and whose minimum may be shown to correspond to that price P_{\min} such that

$$\frac{2}{\sigma^2} \Lambda = \left. \frac{\delta p}{\delta \mu} \right|_{P = P_{\min}}, \quad \text{or} \quad \Lambda = \frac{1}{2} \left. \frac{\delta p}{\delta(\mu/\sigma^2)} \right|_{P = P_{\min}} \quad (24)$$

The interpretation of this expression is that price is reduced, as sales fall towards zero, to a minimum price which is lower according as $|\Lambda|$, expected profits per period, is greater (reflecting the greater returns from increasing the level of patronage), and according as the returns to price reductions, measured by $2(\mu/\sigma^2) = \dot{y}/y$, are more favourable.

To establish that, for any level of capacity, M , an associated optimal pricing policy of the form described does in fact exist, we invoke Lemma 1, which summarizes the properties of the solution paths, and whose proof is given in Appendix 3:

Lemma 1. The solution $\lambda(x; \Lambda)$ of equation (16)' has a root x_M , $\lambda(x_M) = 0$ which increases strictly and continuously from zero to $+\infty$ as the parameter Λ decreases from zero. Moreover, there exists a critical value of Λ , say Λ_c , such that for $0 > \Lambda \geq \Lambda_c$ the solution is of the form $p(x) = p_r$, all x ; while, for $\Lambda_c > \Lambda$ the solution consists of two segments such that $p(x)$ is strictly increasing up to some value of x , and thereafter jumps to p_r for all greater x .

The boundary condition $\lambda(M) = 0$ is thus satisfied for exactly one number of this family; and this fixes both the pricing policy, and the associated value $|A|$ representing the expected per period profits associated with that policy. Moreover, as expected, the greater the level of capacity, the greater the level of expected profits. Less obviously, low values of capacity and so expected profits, are associated with a policy of no price reductions, $p = p_r$ all x . The intuitive interpretation of this is simply that, for a given response function, the attractiveness of price cuts is measured by the additional profits which higher sales imply; the lack of capacity to achieve such sales implies a reduced willingness to reduce prices so as to increase patronage. It is, however, the obverse of this view that is our prime focus of interest: for a given level of capacity, as we will see below, a greater responsiveness of consumers to price reductions enhances the attractiveness of price cuts.

The above Lemma immediately implies, then:

Theorem 1. For any given demand conditions satisfying Assumptions 1, 2, and where $P_r \geq MC$, there exists a unique optimal pricing policy $p(x)$ on $[0, M]$, where $p = P - MC \geq 0$, which maximises the profit functional $\int_0^M p(x)y(x)xdx$.

The optimal pricing policy $p(x)$ whose existence is thus established depends only on the parameters P_r , \bar{P} , A and η (which determine, via Assumptions 1, 2, the demand conditions faced by the firm. In particular, the function $\alpha(\bar{P} - P)$ being given, the dependence of the optimal pricing policy on the shape of the market distribution of prices is captured completely via the effect of the latter on the four parameters \bar{P} , P_r , A and η . Here A represents the fraction of all "floating" consumers who hear of a price $P \leq \bar{P}$ offered by some firm, i.e. the fraction of such

consumers making an informed choice, as opposed to a random choice, while the parameter η represents the "correction factor" associated with the requirement that total sales equal $N\bar{x}$ in each period. We now formally define the value of η generated by any market parameters (\bar{P}, P_r, A, η) , with associated values $\mu(p, x)$, σ^2 and optimal pricing policy $P^*(x)$, as that value which equates the expectation of the sales distribution, with probability density function, from (6) above,

$$C \exp \frac{2}{\sigma^2} \int_0^x (\mu_0(P^*(W), W; A, \bar{P}, P_r) + \eta) dW$$

(where μ_0 is defined by equation (2) above) with the given level of average sales, \bar{x} .

By invoking the bounds to μ, σ^2 implied by (1-3) we may, for any $\bar{x} < M$, deduce bounds to η such that $\underline{\eta} \leq \eta \leq \bar{\eta}$; explicit expressions for $\underline{\eta}, \bar{\eta}$ are given in Appendix 3.

Finally, regarding the family of functions $P(x)$ as a subset of the space of all (Lesbeque) integrable functions on $[0, M]$ equipped with the integral norm, we can further establish (Appendix 4):

Lemma 2. As the given market parameters \bar{P}, A and η , which (together with P_r) determine the demand conditions faced by the firm vary, the optimal pricing policy $P(x)$, and the associated values

$$P = \int_0^M P(x)y(x)dx \quad \text{and} \quad A = \int_0^P \alpha(\bar{P} - P)y(P)dP \quad \text{and} \quad \eta,$$

generated by that policy vary in a continuous manner with these market parameters.

5. INDUSTRY EQUILIBRIUM

Having established the nature of the optimal pricing policy of the individual firm, for given demand conditions, which reflect inter alia the actual distribution of market prices which prevails, we now turn to the question of whether, for some such market distribution of prices, the implied optimal pricing policy for the individual firm is such as to generate that distribution of market prices.

We have in mind here a Nash equilibrium: given that all other firms follow this pricing policy, each individual firm finds that the resulting market distribution of prices implies that its (unique) optimal strategy is to adopt that same pricing policy.

One point worth noting is that we ignore collusive behaviour, which is reasonable on the basis of our "large numbers" assumption. A second is that all firms adopt the same (uniquely defined) policy, though their actual prices will differ, reflecting fluctuations in patronage, at any point in time.

The results summarized in Lemma 2 establish a mapping from the space of market price distributions to the space of pricing policies. The choice of any given pricing policy by individual firms, in turn, generates a certain market distribution of prices. We aim to establish the existence of a fixed point in the mapping of the market price distribution into itself.

The fact that the firms pricing policy depends on the market distribution of prices only via the parameters \bar{P} , P_r , A and η , means that a considerable simplification of the existence proof is made possible by working directly in terms of these parameters.

We may further confine our search for an equilibrium by noting the form of the firm's optimal pricing policy developed above. This ensures that any equilibrium price distribution must take either the form of a degenerate distribution, or else a bimodal distribution with support $\{[0, \bar{P}], P_{\max}\}$. Given the unit search cost, c , the consumer computes the reservation price in the usual manner, viz P_r is the price such that, on obtaining an offer of P_r , the consumer is indifferent between accepting the offer and searching further, his expected gains from one further search

$$\int_0^{P_r} (P_r - P) dY(P)$$

coinciding with c . Now the price distribution generated by firms' optimal pricing policies has its maximum at P_r , so that a necessary condition for a market price distribution to be an equilibrium is that its maximum P_{\max} coincide with P_r . This in fact immediately allows a simple representation of P_r , viz

$$P_r = \min(\bar{P} + c, P_0) \quad (25)$$

To see this, consider a market distribution of prices of the form just described, for which $P_{\max} = \bar{P} + c \leq P_0$. Then, any offer above $\bar{P} + c$ will be rejected, since the expected gains from one further search, following a price offer of P , equal $P - \bar{P}$, which exceeds c if and only if the price P in question exceeds $\bar{P} + c$. Thus any price exceeding $\min(\bar{P} + c, P_0)$ will be rejected.

This representation of the reservation price allows us to characterize market equilibrium by the three parameters (\bar{P}, A, η) where P_r is given by (25).

We now proceed to define a (vector valued) mapping ϕ' of (\bar{P}, A, η) from the compact convex set

$$S = [0, P_0] \times [0, N\alpha_0] \times [\eta, \bar{\eta}]$$

in R^3 into itself.

We define the mapping in two stages, dealing first with the subset of the domain $[MC - c, P_0] \times [0, N\alpha_0] \times [\eta, \bar{\eta}]$. Here, we have from our above characterization of the reservation price that $P_r \geq MC$ or $p_r \geq 0$. Over this range, the optimal pricing policy of the firm is defined in the manner discussed above; we define $\phi(\bar{P}, A, \eta)$ as the corresponding triplet of values generated by the firm's optimal pricing policy for given market parameters \bar{P}, A, η . The continuity of the mapping is guaranteed by Lemma 2. That ϕ is into is assured since we have from the definition of A in (5) above that $0 \leq A \leq N\alpha_0$, for any price distribution while, for any optimal pricing policy $P(x)$ and any \bar{P} in the range $[MC - c, P_0]$ we have $0 \leq P(x) \leq P_0$ (as ensured by Assumption 2) so that the associated average price $\int_0^M P(x)y(x)dx$ is certainly in $[0, P_0]$.

(It is worth noting that it is not the case that $P(x) \geq MC - c$, necessarily; for, in principle, we might have an optimal pricing policy involving negative markups at low values of x , such that the average price across firms $\int P(x)y(x)dx$ lies below $(MC - c)$, though the average across sales $[\int P(x)y(x)xdx / \int y(x)xdx]$ does not, so that expected profits are positive. It is this rather pathological possibility that necessitates our extension of the domain of \bar{P} to $[0, P_0]$).

We now note that as $\bar{P} \rightarrow (MC - c)^+$, so that $p_r \rightarrow 0^+$, expected profits per period, being bounded above by $p_r M$, approach zero. It then follows from our characterization (24) of the minimum price that for P_r

sufficiently close to MC , the optimal pricing policy is of the form $P(x) = P_r$, all x , so that the average price generated by the firm's optimal pricing policies coincides with P_r .

We now extend the domain of the mapping ϕ to the set S as follows

$$\phi'(\bar{P}, A, \eta) = \begin{cases} \phi(\bar{P}, A, \eta) & , \quad \bar{P} \geq MC - c \\ \phi((MC - c), A, \eta) & , \quad \bar{P} \leq MC - c \end{cases}$$

Noting that this definition ensures the continuity of ϕ' at $\bar{P} = MC - c$, by virtue of our above remark, we have immediately that ϕ' is continuous over the entire domain S , and that it maps S into itself.

It thus follows immediately from Brouwer's theorem that the mapping possesses a fixed point, so that an equilibrium is guaranteed.

We may further note that this fixed point necessarily lies in the subset $P_0 \geq \bar{P} > (MC - c)$; for we have that if $\bar{P} \leq (MC - c)$ then ϕ' maps \bar{P} into MC , so that (\bar{P}, \cdot, \cdot) is not a fixed point of ϕ' .

We now turn to the question of whether such an equilibrium involves persistent price dispersion, or whether it is characterized by a single price. We first note an immediate result: since the maximum market price coincides with the reservation price, it follows that, if the market distribution of prices is degenerate, then it is degenerate at the monopoly price. For, suppose the market distribution is degenerate at $\bar{P} < P_0$; then $P_r = \min(\bar{P} + c, P_0) > \bar{P}$ so that the maximum market price exceeds \bar{P} , thus implying a contradiction.

We now proceed to develop conditions which exclude this possibility. Assuming that the unique market price is P_0 , we examine whether the associated optimal pricing policy for the firm involves setting a price of

P_0 for all x , as is required for consistency. We appeal to the condition that the optimal pricing policy involves price reductions according as

$$-\Lambda > (\lambda/x)_c$$

Appealing to our interpretation of Λ as the negative of expected profits per period, we have here that $\Lambda = -p_0 \bar{x}$. Using equation (23), our critical value $(\lambda/x)_c$ here takes the value $-(\sigma^2/2) \cdot (\delta p / \delta \mu) \Big|_{p = \bar{p}}$, where, by using the fact that $Y(p)$ here takes the form of an atom of probability at p , we have from (4) that $A = N\dot{\alpha}(0) = 0$, and so from (3) that $\sigma^2 = \gamma \bar{x}$, while differentiation of (2) yields

$$\frac{\delta \mu_1}{\delta p} = \gamma N \bar{x} \dot{\alpha}(\bar{p} - p) = \gamma N \bar{x} \dot{\alpha}(0)$$

Combining these results we have as our condition for a nondegenerate solution that

$$p_0 \bar{x} = -\Lambda > -\frac{\sigma^2}{2} \frac{\delta p}{\delta \mu} \Big|_{p = \bar{p}} = -\frac{1}{2N\dot{\alpha}(0)}$$

$$\text{or } N \bar{x} p_0 |\dot{\alpha}(0)| > \frac{1}{2}$$

We first note that $\dot{\alpha}(0)$ represents the rate of change of the fraction of consumers who hear of the firm's price per unit (absolute) reduction in price; thus $p\dot{\alpha}(0)$ simply represents the equivalent rate of response to fractional reductions in the markup p .

If this equation is satisfied, then price dispersion will persist, for the nature of our underlying Wiener process implies that the p.d.f. of sales is strictly positive at all x , $0 \leq x \leq M$ except in the extreme case where $\mu/\sigma^2 \rightarrow \infty$ implying that all firms operate always at full capacity; we will return to this point below. Otherwise, our above condition is sufficient to ensure price dispersion. There are two economically

interesting interpretations of the condition, as follows; for any level of industry demand $0 < \bar{x} < M$,

- (i) So long as the rate of response of sales to price reductions is sufficiently strong, price dispersion will persist, i.e. $\dot{a}(0)$ is large enough.

A much more appealing condition, however, is

- (ii) Given any nonzero rate of response $p_r |\dot{a}(0)|$, then, so long as the number of firms in the industry is sufficiently large, our condition is satisfied.

A simple intuitive interpretation of (ii) can be provided: the total volume of sales $N\bar{x}$ determines the "catchment area" of potential sales for the firm, in setting its pricing policy. Thus the smaller the firm in relation to the industry, the greater the fractional increase in its sales implied by a price cut—for it draws customers from a larger number of similar rivals.⁵

Thus we see, interestingly, that price dispersion is guaranteed precisely in the "atomistic" limit which we usually build into our definition of a competitive industry. In the present instance, we have merely preferred to proceed more slowly, in two steps—first using only the associated rationalization of the "no collusion" assumption via our Nash equilibrium definition, and only now strengthening our "large numbers" assumption further.

It is an important feature of equilibrium in the model that, so long as our above conditions on the availability of information are satisfied, the maximum market price lies below the monopoly price, and is "usually" independent of it. This is a feature of some importance, for it is the kind of result we expect in a competitive model. (It also means that we can now relax our notion that all consumers have the same ceiling

price; so long as N is large enough, our results are independent of this assumption.)

Finally, we remark on the role of excess capacity in the model. So long as the monopoly price P_0 exceeds $AC(M)$, then, if we allow free entry in the long run, excess capacity will tend to appear, and to persist. For, if the number of firms was so low as to eliminate excess capacity, so that $\bar{x} = M$, then expected profits per period would exceed total costs, so that (supernormal) profits are earned, and further entry induced. This entry of additional firms, though it (at least eventually) reduces expected (supernormal) profits to zero, does so only at the expense of creating excess capacity. Such excess capacity may be interpreted as a welfare loss associated with the lack of perfect information per se.

6. SUMMARY AND CONCLUSIONS

Equilibrium is characterized in the present model by persistent stochastic fluctuations at the level of the individual firm, while total industry demand remains constant. The driving force which keeps the system from collapsing to a degenerate equilibrium is endogenous: consumers, enjoying only partial price information, display, on the one hand, some responsiveness to price reductions, thus making such reductions worthwhile, but on the other hand some consumers, being currently uninformed of a more favourable offer, choose a new supplier at random. The firm, at each point in time, sets a price which, given its current level of patronage, is optimal; but the response to the price it chooses, in terms of the change in patronage which follows, is stochastic. Each of our firms, following the same optimal pricing policy, thus generates over time a different price history.

Equilibrium is characterized by the stationary distribution of the underlying stochastic process. This characterizes not only the cross sectional average over all firms at a point in time, but also the distribution generated by the price history of any individual firm over an arbitrarily long period.

This approach thus generates a "solution" to the problem of describing a "price dispersion equilibrium", the motivation for which we noted earlier, which is interesting in certain respects. Our consumers are identical, differing at any point in time only in their information sets; they do not differ, in particular, in their costs of acquiring information. This is closely associated with another property in which the present model differs from some recent work, which we noted above: firms do not separate permanently into "high price" and "low price" firms, but display, over time, identical characteristics.

The motivation for adopting the particular approach to the problem developed in the present paper lies in the observation that macroeconomic "equilibrium" is in practice characterized by a comparatively steady aggregate of the widely varying experiences of individual agents; and that an adequate description of certain macroeconomic phenomena may depend crucially on an explicit recognition of this fact (for example, Tobin (1972)). The divergence of experience across different firms has its origin in a wide variety of factors; what we have been concerned with here is the development of a coherent framework in which the minimal amount of dispersion associated only with the random choice of supplier by imperfectly informed consumers is endogenous to the model. The macroeconomic consequences of this feature of equilibrium will be the subject of future work.

APPENDIX I³THE INTERPRETATION OF THE LAGRANGE MULTIPLIER λ

Abstract: It is shown that the values of the Lagrange multiplier associated with the optimal pricing policy coincides with the negative of the maximand, π , being expected profits per period. This is established by noting that

the maximand $\int_0^M p(x)y(x)dx$, in the more general class of problems obtained by replacing our isoperimetric constraint (12) by $Y(M) = \int_0^M y(x)dx = k$, is linear in k ; and combining this with the standard interpretation of the Lagrange multiplier.

APPENDIX II

QUALITATIVE PROPERTIES OF THE OPTIMAL PRICING POLICY

We present here a qualitative analysis of the form of the solution paths to our basic differential equation,

$$\dot{\lambda} = -px - \Lambda - \lambda \cdot \frac{2}{\sigma^2} \mu(p, x) \quad (16)'$$

with $\lambda(0) = \lambda(M) = 0 \quad (21)'$

deferring all formal proofs to Appendices III and IV.

We noted in the main text that two cases arise as follows. Defining our critical value $(\lambda/x)_c$ we have, either,

$$(i) \quad 0 \leq \lambda/x \leq (\lambda/x)_c, \quad w(x) > 0, \quad \text{and} \quad p = p_r$$

or $(ii) \quad \lambda/x > (\lambda/x)_c, \quad w(x) = 0, \quad \text{and} \quad p < p_r. \quad (2.1)$

Again, as noted above, we have at $x = 0$ that $\lambda = 0$ and $\dot{\lambda}(0) = -\Lambda$, so that if

$$0 \geq \Lambda \geq -(\lambda/x)_c$$

our solution path (at least in the region of the origin) is of type (i), and our equation reduces to the special form

$$\dot{\lambda} = -p_r x - \Lambda - \lambda \cdot \frac{2}{\sigma^2} (\mu_r - \gamma x) \quad (2.2)$$

where $\mu_r = \mu_1(p_r)$.

The form of the solution path may here be deduced readily from the tangent diagram (Figure 2.1); the loci $\dot{\lambda} = 0$ take the form

$$\lambda = \frac{p_r x - |\Lambda|}{\frac{2}{\sigma} \gamma x - \frac{2}{\sigma} \mu_r}$$

being a right angled hyperbola. The solution path increases up to the point of intersection with this locus, and subsequently declines to cut the horizontal axis. The proofs of this are straightforward but tedious, and are given in Appendix III. We furthermore establish there that the root x_M , $\lambda(x_M) = 0$ increases strictly and continuously from the origin as Λ declines from zero.

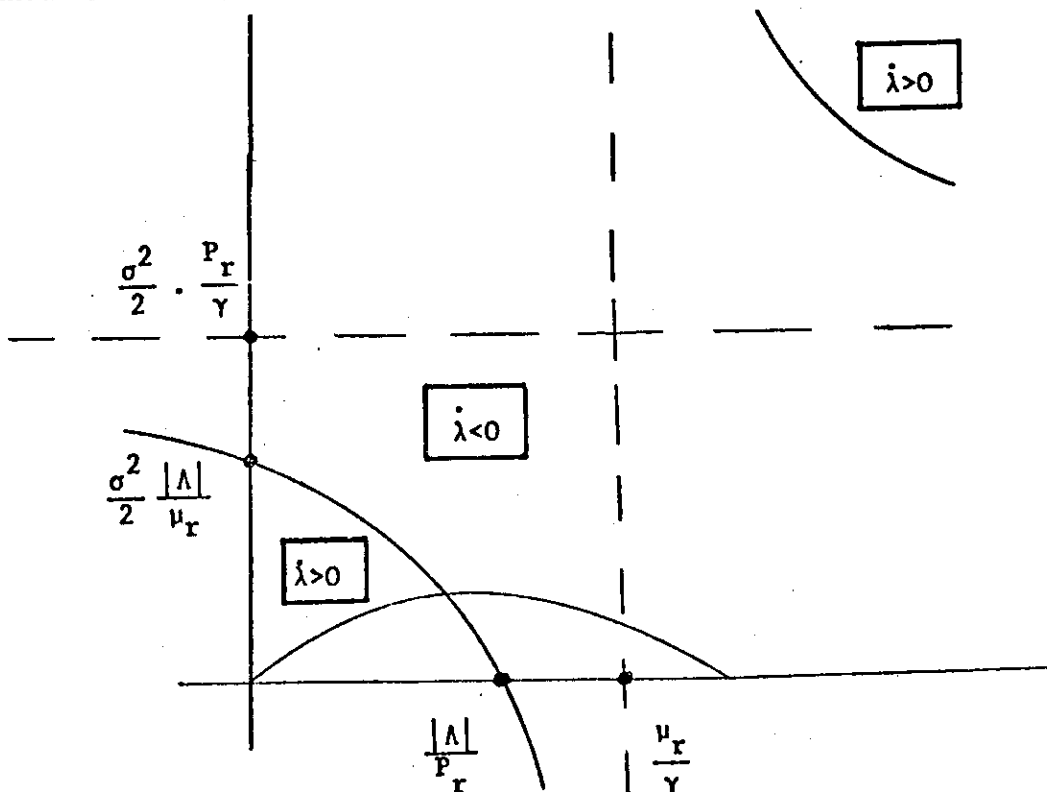


FIGURE 2.1. The solution path to the first form of the basic differential equation. (The case of $|\Lambda| < (\mu_r p_r / \gamma)$. For $|\Lambda| > (\mu_r p_r / \gamma)$ the branches appear in the second and fourth quadrant.)

We now turn to the alternative case, where

$$-(\lambda/x)_c > \Lambda$$

so that our solution path in the region of the origin is of type (ii); thus price lies below the reservation price when sales are sufficiently depressed.

Here $(\lambda/x) > (\lambda/x)_c$ and $w(x) = 0$, so that from (23) we have

$$\lambda = -\frac{\sigma^2}{2} x \frac{\partial p}{\partial \mu} = -\frac{\sigma^2}{2} x \frac{\partial p}{\partial \mu_1}.$$

Where the latter equality follows from the form (2) of our drift function.

It is convenient to introduce as an ancillary variable the value of $\partial p / \partial \mu_1$, defined over the relevant portion of the drift function, which we label ϕ . We then obtain on substituting into our differential equation,

$$\frac{d}{dx} \left[-\frac{\sigma^2}{2} x \phi \right] = -p(\phi) x - \Lambda - \left[-\frac{\sigma^2}{2} x \phi \right] \cdot \frac{2}{\sigma^2} \mu(x)$$

Carrying out the differentiation on the left hand side and simplifying, we obtain

$$\frac{d\phi}{dx} = \frac{\frac{2}{\sigma^2} \Lambda - \phi}{x} + \frac{2}{\sigma^2} (p - \mu \phi)$$

or $\dot{\phi} = f(\phi, x; \lambda) + g(\phi, x)$, say. (2.3)

The form of the first term on the right hand side implies that, for any solution nonsingular at the origin

$$\phi = \frac{2}{\sigma^2} \Lambda$$

We first investigate the derivative at the origin. Taking limits on both sides of the equation,

$$\lim_{x \rightarrow 0} \dot{\phi} = \lim_{x \rightarrow 0} \frac{\frac{2}{\sigma^2} \Lambda - \phi}{x} + \lim_{x \rightarrow 0} \frac{2}{\sigma} (p - \mu \phi) .$$

Applying l'Hospital's rule to the first term on the right hand side, we have

$$\dot{\phi}(0) = -\dot{\phi}(0) + \frac{2}{\sigma^2} (p - \mu \phi) \Big|_{x=0}, \quad \phi = (2/\sigma^2) \cdot \Lambda$$

or

$$\dot{\phi}(0) = \frac{1}{2} g(\phi, x) \Big|_{x=0}, \quad \phi = (2/\sigma^2) \cdot \Lambda .$$

We now formally define our nonsingular solution $\phi(x)$ on $[0, \infty)$ as:

$$\phi(x) = \frac{2}{\sigma^2} \Lambda ; \quad \dot{\phi}(x) = \frac{1}{2} g \left(\frac{2}{\sigma^2} \Lambda, 0 \right), \quad x = 0$$

$$\dot{\phi}(x) = \frac{\frac{2}{\sigma^2} \Lambda - \phi}{x} + g(\phi, x), \quad x > 0 . \quad (2.4)$$

The solution thus defined is differentiable, and hence continuous, at the origin.

The properties of the solution paths to (2.4) now follow from the properties of $g(\phi, x)$, which may be deduced from the assumed form of our drift function.

A useful graphical representation of $g(\phi, x)$ is illustrated in Figure 2.2, as the length of the horizontal segment at height γx between the vertical axis and the point of intersection with a tangent to the drift

function at ϕ , from which we may immediately see the following properties, whose proof is trivial:

- (i) (dependence on x): At $x = 0$, $g(\phi, 0)$ is positive from and decreases as ϕ increases towards ϕ_c .
As x increases $g(\phi, x)$ falls, the rate of decline being less as ϕ increases towards ϕ_c .

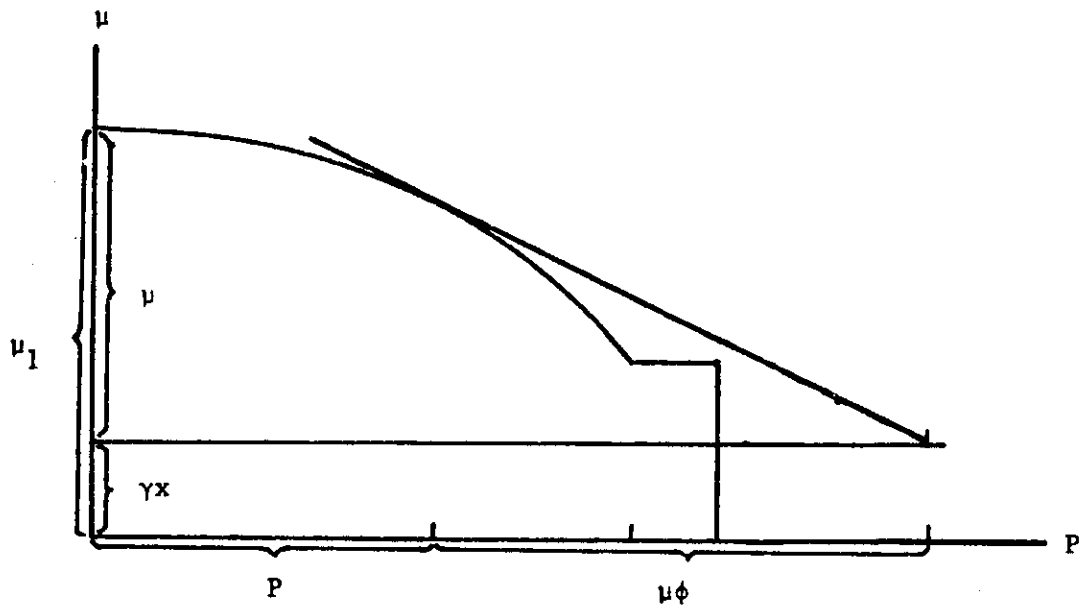


FIGURE 2.2. $(p - \mu\phi)$, where $\mu = \mu_1 - \gamma x$, $\phi = \partial p / \partial \mu_1$, as a function of ϕ , for $p \leq p_c$.

- (ii) (dependence on ϕ): At $x = 0$, $g(\phi, x)$ decreases with increasing ϕ , as noted already. This is true also for small x ; for sufficiently large x however, $g(\phi, x)$ increases with increasing ϕ . (This may be interpreted graphically in Figure 2.2 by considering a horizontal above the point of tangency at ϕ .)

From these properties of $g(\phi, x)$ the qualitative properties of our solution paths may be deduced using a tangent diagram, showing the sign of $\dot{\phi}$ as a function of ϕ and x , for any fixed value of Λ , over the range $\phi_c \geq \phi \geq (2/\sigma^2) \cdot \Lambda$; $x \geq 0$.

For any value of ϕ , we may investigate the value of the derivative as a function of x by reference to Figure 2.3 which indicates the graphs of the two terms $f(\phi, x; \Lambda)$ and $g(\phi, x)$ of equation (2.3).

Since $\phi > (2/\sigma^2) \cdot \Lambda$, $f(\phi, x; \Lambda)$ is negative and increasing in x , and $f(\phi, x; \Lambda) \rightarrow -\infty$ as $x \rightarrow 0+$ (rectangular hyperbola).

From our discussion of the properties of $(p - \mu\phi)$, we have that $g(\phi, x)$ is initially positive. For $\phi = 0$, $g(\phi, x)$ is constant. Here one, and only one root exists (Figure 2.3a). For $\phi < 0$, there are at most two roots x' and x'' (Figure 2.3b), the lesser of which we may identify with our unique root at $\phi = 0$ by noting that $\dot{\phi}$ is negative to the left of x' .

An immediate property which follows from the form of $f(x; \phi, \Lambda)$ is that x'' increases as Λ declines (i.e. $|\Lambda|$ increases); this will be found useful later.

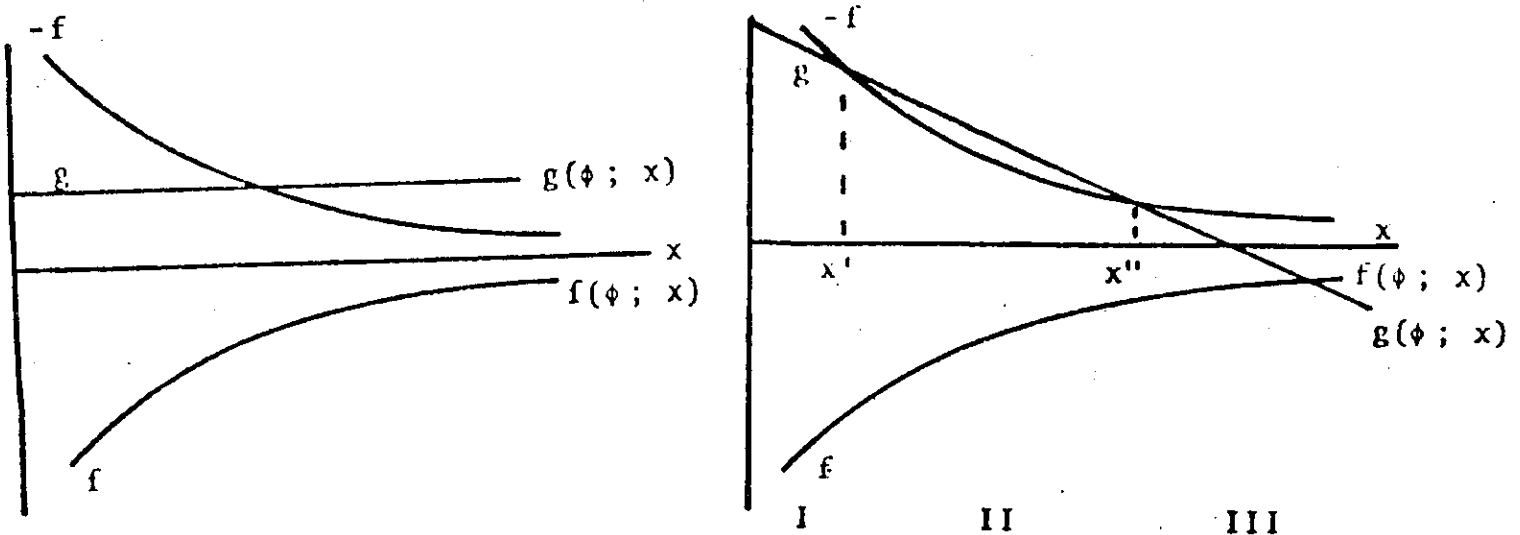


Figure 2.3. The roots of the differential equation $\dot{\phi}(x) = f(x, \phi; \Lambda) + g(x, \phi)$.

Our tangent diagram thus consists of three regions, as indicated in Figure 2.4. Noting that our nonsingular solution has $\dot{\phi} > 0$ at $x = 0$, it follows that there are two possibilities for the form of $\phi(x)$; either

(a) $\phi(x)$ increases monotonically with x ;

or (b) $\phi(x)$ increases initially with x , but intersects the boundary of regions II and III at some point, and thereafter decreases.

Thus, either $\phi(x)$ increases monotonically to $\phi = \phi_c$, or else there is no root $\phi = \phi_c$.

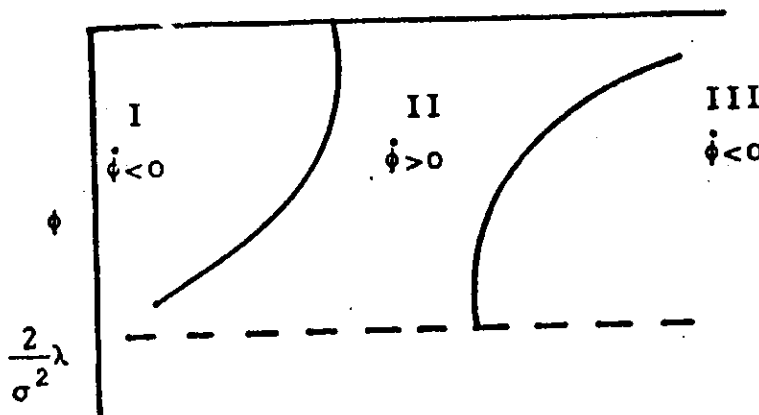


FIGURE 2.4. Tangent diagram for the differential equation $\phi(x) = f(x, \phi; \Lambda) + g(x, \phi)$.

(A further property illustrated in Figure 2.4 is the following: in a neighborhood of the point $(x = 0, \phi = (2/\sigma^2) \cdot \Lambda)$, $f \rightarrow \pm\infty$ as $x \rightarrow 0+$ according as $\phi \lesseqgtr (2/\sigma^2) \cdot \Lambda$. Hence as $x \rightarrow 0+$, the boundary of regions I and II approaches $(2/\sigma^2) \cdot \Lambda$.)

We demonstrate in Appendix III that for $\Lambda = -(\lambda/x)_c = \Lambda_c$, say, we have the trivial solution with root $x_c = 0$; and as Λ declines from Λ_c , the root x_c exists, and increases continuously with $|\Lambda|$, i.e., the solution is of type (a).

Combining this with our analysis of the first special form of the equation, we may form the composite solution path which characterizes the case $\Lambda < -(\lambda/x)_c$ and which is illustrated in Figure 2.5. The root x_c defines the initial conditions

$$\lambda(x_c) = -\frac{\sigma^2}{2} x_c \phi_c$$

for equation (2.2) over the interval $x \geq x_c$.

As Λ declines towards some critical value, $x_M \rightarrow +\infty$, corresponding to the level of per period profits associated with a firm having infinite capacity; and the solution path for λ below this value is of type (b); i.e., no root x_c exists.

These results, and those concerning the first form of the equation summarized above, combine to yield Lemma 1 of the text.

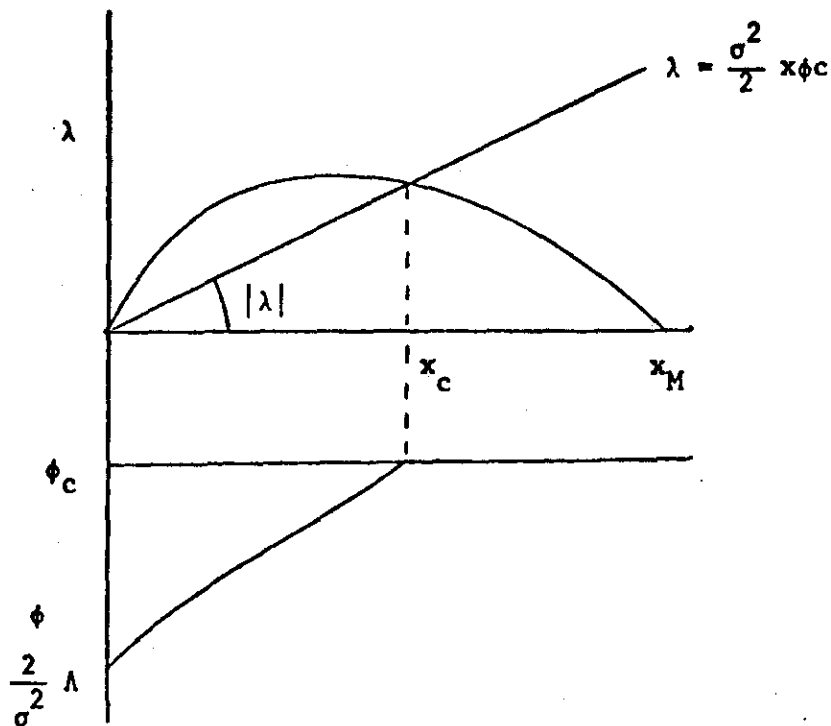


FIGURE 2.5. The nature of the solution path for the price reductions case.

NOTES

- 1 Thanks are due to Franklin Fisher, Katsuhito Iwai and John Hey, for a number of useful suggestions on the present model. I am also indebted to Koichi Hamada, Michael Jones-Lee, Christopher Dougherty, Christopher Pissarides and Anthony Sampson, and to an anonymous referee, for their many valuable comments on an earlier draft. The paper is based on work carried out during a visit to the Cowles Foundation which was supported financially by the SSRC.
- 2 Though it is worth noting that our motivating "story" somewhat understates the loss rate of consumers implied by the lower "tail" of the normal, an effect subsumed in our "correction term" below.
- 3 Appendices I, III and IV will be made available by the author on request.
- 4 An interesting analogy is provided by the analysis of optimal pricing where some persistence of patronage is present in the model of Phelps and Winter (1970), who consider a monopolist with a finite time horizon in a deterministic setting.
- 5 That the entire industry should constitute each firm's "catchment area" may seem a strong assumption; all that is required, however, is that the firm's catchment area becomes arbitrarily large in the limit $N \rightarrow \infty$, as for instance is the case where the catchment area corresponds to any fixed fraction of the total market.

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