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COMPARISON OF PUBLIC CHOICE SYSTEMS

A. Comparison of Voting Systems

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COMPARISON OF PUBLIC CHOICE SYSTEMS

The area of social decision-making is one in which the disciplines of game theory, economics, and political science all meet. One of the simplest decisions to be faced is the election of one candidate (or the choice of one alternative) from several. Various voting systems have been proposed for such elections. In order to compare these systems, the three papers collected here develop a measure of "effectiveness," which indicates the extent to which a voting system selects candidates representative of the voters' preferences.

The first paper, "Comparison of Voting Systems," defines effectiveness in terms of the expected social utility of the candidate elected from a "random" society. Working in the same terms, the second paper, "Multiply-weighted Voting Systems," develops an equivalence relation among voting systems under the assumption of optimal (equilibrium) voter behavior. The last of the three papers, "Reproducing Voting Systems," gives an alternative derivation of the effectiveness measure from the viewpoint of a single voter. It is seen that a voting system of high effectiveness in terms of social utility is of equal effectiveness in satisfying the desires of each individual voter. An appendix to this third paper presents a characterization of the uniform probability distribution which motivates the use of that distribution in the definition of a random society.

A number of specific issues are also dealt with. In the first paper, several recent elections are reviewed to illustrate drawbacks of the standard voting system (in which each voter casts a single vote, and the candidate receiving the greatest vote total is elected). It is shown that as the number of candidates becomes large, the standard system becomes totally ineffective. The Borda system (in which the voters are called upon to rank the candidates, and then votes are awarded in proportion to the ranks assigned) is seen to be the most effective of all voting systems in which the votes received by candidates depend solely on the voters' rankings. We introduce the approval voting system (in which each voter may cast one vote for every candidate of whom he "approves"), and show that for three-candidate elections it is slightly more effective than the Borda system (which is in turn substantially more effective than the standard system). A variety of practical considerations are presented, which argue in favor of the implementation of approval voting.

The second paper treats several families of voting systems related to the approval voting system, and introduces the important concept of a "reproducing" voting system. The effectiveness of particular three- and four-candidate systems is determined; the three-candidate system is the most effective we have discovered to date.

A general treatment of reproducing voting systems is presented in the third paper. A formula for the effectiveness of any reproducing system is derived, and is applied to voting systems which allow a voter to vote for a fixed number of candidates, and to the Borda system. It is found that, in contrast to the standard voting system, the Borda system is asymptotically (for elections involving many candidates) as effective as any system can be.

The papers presented here are the first in a series concerning the comparison of public choice systems. Subsequent work will center on three topics: the search for specific highly-effective voting systems, the study of the merits of multistage election processes, and consideration of elections meant to yield several winners or to lead to proportional representation (such as the apportionment of convention delegates among the candidates).

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COMPARISON OF VOTING SYSTEMS

by

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Abstract. There are various methods by which a group of voters may elect one candidate from several. Among these are the standard "one man, one vote" system and other weighted ranking systems, and the approval voting system (in which a voter may cast one vote for each of the candidates of which he "approves"). In order to compare these systems in terms of their tendency to elect a candidate representative of the voters' preferences, a measure of "effectiveness" is developed. As a result, the approval voting system is established as a desirable alternative to the standard system.

This paper is an extension and revision of an earlier paper with the same title.

COMPARISON OF VOTING SYSTEMS

Introduction. A democratic society frequently faces the task of choosing one from a group of several candidates. If the choice is between only two candidates, there is little argument against letting each member of the society vote for one of the two, and then choosing the candidate who receives the greater number of votes. However, the selection of a procedure for choosing among three or more candidates is open to controversy.

Among the many methods which have been proposed are the standard voting system (in which each voter may vote for a single candidate), the Borda voting system (in which each voter ranks the candidates, and then each candidate is awarded a number of votes proportional to his ranking), and the approval voting system (in which each voter may cast one vote for each of as many candidates as he wishes). These methods can be compared on various heuristic grounds or with respect to specific historical examples of elections, but such comparisons are inconclusive. An important criterion is that a voting system should be likely to lead to the election of a candidate who represents the desires of the voting public. In this paper, we develop a measure of the effectiveness of a voting system in meeting this goal. The various voting systems can then be subjected to direct, objective comparison in terms of this measure.

Central to the definition of effectiveness is the concept of a "random" society. In such a society, the utilities of the candidates to the voters are random variables. As the utilities vary independently over their range, every conceivable society (thought of as a collection of voter preferences) is generated. Hence, by computing expected social utility in the random society, an average of social utilities over all possible societies is obtained. The effectiveness of a voting system is defined as a comparison between the expected social utility of the socially-optimal candidate and the expected social utility of the candidate actually elected under the system. (An alternative derivation of this measure, which does not employ the notion of social utility, is also available.)

Applying this idea, we explicitly determine the effectiveness of the standard voting system for elections involving any number of candidates. It is shown that the Borda voting system is the most effective among all systems in a class which includes the standard system. Finally, for the case of three-candidate elections, it is demonstrated that the Borda system is substantially more effective than the standard system, and that the approval voting system has a slight edge on the Borda system. Because the approval voting system (which is, of course, equivalent to the standard system in the two-candidate case) can be used in conjunction with existing voting machines, we conclude that its implementation merits serious consideration.

Voting Systems. Consider an election in which a group of voters is to select one from a group of m candidates. A voting system is a collection of (one or several) sets of m weights. Each voter assigns the weights from one of the sets to the candidates. (The weight assigned to a particular candidate by a voter is the number of "votes" cast by that voter for the candidate.) The candidate who is assigned the greatest total weight (that is, who receives the greatest vote total) is the winner of the election.

The standard voting system, used in almost all one-winner elections in the United States, consists of the single set of weights $(1,0,\ldots,0)$. Under this system, each voter has one vote, which he may cast for the candidate of his choice. This system has several practical strengths, among them the facts that it is easily comprehended by the electorate, and that the vote is easily tallied by mechanical devices (voting machines). However, the standard system has historically led to a number of candidate-selections which were (apparently) not representative of the voters' preferences.

Two recent examples come from the State of New York. In 1970, the race for a seat in the U.S. Senate was contested by three major candidates. James Buckley was endorsed by the Conservative party, Charles Goodell by the Liberals and the Republicans, and Richard Ottinger by the Democrats. Public opinion polls indicated that politically liberal voters were in the majority in the state, but that their sympathies were divided between the latter two candidates. Politically conservative voters were solidly behind Buckley. The outcome of the election is indicated in Table 1. It appears unlikely that the public desire was well-satisfied by this outcome.

<u>Table 1</u>

The 1970 U.S. Senate contest in New York

Buckley	2,288,190	38.82%
Goodell	1,434,472	24.34%
Ottinger	2,171,232	36.84%

This example has a traditional interpretation. If it may be presumed that either of the losing candidates would have beaten Buckley in a two-way race, then one of them (the winner of a head-to-head contest) was preferred to each of his opponents by a majority of the voters. Hence, the actual outcome violated the well-known Condorcet principle, that a candidate who would win all two-way elections (if such a candidate exists) is most representative of the preferences of the voters. Although there are good objections to a general application of the Condorcet principle, its violation frequently indicates misrepresentative election results.

Another example arises from the 1971 mayoral contest in Ithaca, N.Y. The outcome is indicated in Table 2. Conley and McNeill were the Democratic and Republican candidates, respectively, Bangs was a Democrat running independently, and the other candidates were independent Republicans. In this case, it is not clear which candidate would have best represented the desires of the voters. However, it is apparent that the presence of several candidates kept the standard voting system from providing definitive information concerning voter preferences.

Table 2

The 1971 mayoral contest in Ithaca, N.Y.

Bangs	653	10.2%
Conley	1862	29.1%
Johns	1151	18.0%
McNeill	1853	28.9%
Rumph	886	13.8%

Both of these examples indicate a political application of the "divide-and-conquer" rule: the shortest route to victory may lie in encouraging your opposition to run several similar candidates against you. A strong argument against the standard system is that it leads to this conclusion.

In view of these examples, it is of interest to investigate alternative voting systems. One class of systems, which includes the standard system as a special case, results when a ranking of the candidates is elicited from each voter, and each candidate is then assigned a weight dependent only on his rank. Formally, a _weighted ranking voting system_ is any voting system consisting of a single set of weights $(w_1,\ldots,w_m)$. A special case which will be of interest later is the Borda voting system, which corresponds to the set of weights $(m, m-1, \ldots, 1)$.

Another kind of voting system consists of the $m-1$ sets of weights $(1,0,\ldots,0)$, $(1,1,0,\ldots,0),\ldots,(1,\ldots,1,0)$. This, which we have named the approval voting system, permits a voter to cast one vote for each of the candidates of which he "approves". It may be noted that the addition or subtraction of a fixed constant, from every weight in one of the sets of weights of a voting system, yields an equivalent voting system. Hence, in the three-candidate case, the approval voting system is equivalent to the system with weight sets $(1,0,0)$ and $(0,0,-1)$, under which each voter may choose to vote either "for" or "against" a single candidate. Recall the Buckley-Goodell-Ottinger example. It seems reasonable to assume that a number of supporters of the latter two candidates would have welcomed the opportunity to vote for both of these men (or, equivalently, to vote against Buckley). Therefore, the outcome of that contest would probably have been different had the approval voting system been in effect.

With such a variety of voting systems at hand, the question of comparisons naturally arises. A discussion of relevant social, economic, and political factors (such as ease of implementation) will be presented later. However, one of the most important areas of comparison must be that of "representativeness". That is, which systems are most likely to lead to the election of a candidate representative of the preferences of the voters? It is this question to which we address ourselves.

Random societies. Consider a random society of voters facing a choice among m candidates. Assume that, in this society, the utility of any given candidate to any given voter is a random variable, uniformly distributed on the unit interval, and is independent of all other utilities of candidates to voters. (In other words, for each voter, his m-dimensional vector of utilities of the candidates is randomly selected from the unit m-cube. When a voter is called upon to vote, he bases his action on the specific values assumed by his random utilities.) The social utility of a candidate is the sum of his utilities to the voters. It is our belief that an equally-weighted sum of commonly-scaled utilities yields a social utility function which represents well the egalitarian goals of a democratic society. It should be noted that this definition reflects the intensity of interest various voters have in the outcome of the election, to the extent that a voter to whom the candidates have approximately equal utility has little impact on the relative social utilities of the candidates. (For a further discussion of the definitions of a random society and of social utility, see the appendix. It should be noted that another approach to the comparison of voting systems, which yields the same results as those below but does not employ the concept of social utility, is presented in [13].)

If there are n voters in a random society, the social utility of any given candidate is the sum of n independent uniformly-distributed random variables. By the Central Limit Theorem, this sum will be asymptotically normal, with mean $\frac{n}{2}$ and variance $\frac{n}{12}$. In particular, consider the candidate of greatest social utility. His expected social utility is asymptotic to

$$\frac{n}{2} + \sqrt{\frac{n}{12}} \cdot \mathrm{Norm}_{max}(m) ,$$

where $\mathrm{Norm}_{max}(m)$ is the expected value of the maximum of m independent unit normal random variables (see Table 3). This provides an (asymptotic) upper bound on the expected social utility of the candidate elected under any particular voting system.

Table 3

The expected value of the maximum of m independent unit normal random variables.

m	$\mathrm{Norm}_{max}(m)$	m	$\mathrm{Norm}_{max}(m)$
1	0.0	7	1.35218
2	0.56419	8	1.42360
3	0.84628	9	1.48501
4	1.02938	10	1.53875
5	1.16296	20	1.86748
6	1.26721	30	2.04276

Two-candidate elections. Consider the case of a two-candidate election. If it is assumed that each voter acts so as to maximize the expected utility (to him) of the elected candidate, then all voting systems are essentially equivalent to the standard system (in the sense that all voting systems will yield the same result). The expected social utility of the elected candidate, in a random society of n voters, is

$$\sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} \left(\frac{2}{3} \max\,(k,n-k) + \frac{1}{3} \min\,(k,n-k) \right)$$

$$= \begin{cases} \dfrac{n}{2} + \dfrac{n}{3 \cdot 2^n} \dbinom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\[2ex] \dfrac{n}{2} + \dfrac{n}{3 \cdot 2^{n+1}} \dbinom{n}{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

$$\approx \frac{n}{2} + \sqrt{\frac{n}{18\pi}} \quad .$$

Since a randomly-selected candidate has expected social utility $\frac{n}{2}$, this result suggests defining the "effectiveness" of a two-candidate voting system as the ratio between the coefficients of $\sqrt{n}$ in this formula and in the formula for the expected social utility of the socially-optimal candidate. Then the effectiveness of every two-candidate voting system will be $\sqrt{\frac{2}{3}} = 0.8165$, since $\mathrm{Norm}_{max}(2) = 0.56419 = \frac{1}{\sqrt{\pi}}$. In other words, it is impossible to find a two-candidate voting system that falls much less than 20% short of perfection.

It is not surprising that no perfect voting system exists. The expected social utilities of the candidates are evaluated over a random society which has an infinitude of realizations. Consider one particular instance, a society in which a slight majority of the voters marginally prefer one candidate while the remaining voters strongly prefer the other. If each voter acts in his own self-interest, the result of the election (that is, the election of the first candidate) will not be socially optimal. Indeed, only by external measurements -- such as the use of a polygraph in the voting booth, or the weighting of votes according to the payment of a voluntary voting fee -- can strengths of preference be estimated in a two-candidate race. And the employment of such measurement methods is, of course, open to many severe objections.

Optimal voting strategies. When more than two candidates are under consideration, the determination of a winning candidate depends on the voting strategies employed by the various voters. For this reason, we turn our attention to the topic of voter behavior.

The behavior of a real-world voter will likely depend upon the information available to him. If he knows that there are only two candidates with any chance of being elected, he may ignore his preferences among all other candidates and vote in a manner designed to maximize the probability that the more-preferred of the two is elected. Generally, predicting behavior in the face of public opinion polls is quite difficult. For example, in the final weeks of the Buckley-Goodell-Ottinger senatorial contest, most polls presaged a Buckley victory. In spite of this, the supporters of Goodell and Ottinger failed to coordinate their actions in a manner sufficient to avert the predicted result.

We will assume that each voter in a random society, besides knowing his own preferences, knows only that the utilities of the candidates to the other voters are all independent and uniformly distributed (that is, he knows that the rest of his society is "random"), and that his fellow voters will base their behavior solely on their own expectations. Under these assumptions, consider the probability that a number v of votes cast by one voter for a particular candidate is critical (that is, whether or not the candidate receives these votes determines whether or not he wins the election). If the number of voters in the society is large, then this probability depends (approximately) linearly on v .

Indeed, let $f:[0,1]^m \to R$ be a density function, with $f(x_1,\ldots,x_m)$ proportional to the probability that each candidate c receives x_c percent of the maximum vote total attainable by a single candidate. If the percentage received by a particular candidate (say, candidate 1) is increased by ε , the probability that this changes the outcome of the election (by moving that candidate into the lead) is

$$\begin{aligned}
&\int_0^1\int_0^{t+\varepsilon}\cdots\int_0^{t+\varepsilon} f(t,x_2,\ldots,x_m)\; dx_2\ldots dx_m dt \\
&\qquad - \int_0^1\int_0^t\cdots\int_0^t f(t,x_2,\ldots,x_m)\; dx_2\ldots dx_m dt \\
&= \varepsilon(m-1)\int_0^1\int_0^t\cdots\int_0^t f(t,t,x_3,\ldots,x_m)\; dx_3\ldots dx_m dt + o(\varepsilon) \;.
\end{aligned}$$

(The symmetry of f in all m arguments has been used to simplify the last expression.) This final expression is essentially linear in ε .

Consider a voter to whom the utilities of the candidates are $u_1, \ldots, u_m$, and assume that he casts $w_1, \ldots, w_m$ votes for the candidates. If, as a result of this voter's actions, candidate c moves past candidate d and wins the election, then his actions yield him a utility gain of $u_c - u_d$. From the preceding paragraph, we know that the probability of this event is asymptotically (when n , the number of voters, is large) proportional to $\max(0,\ w_c - w_d)$. Therefore, the voter's subjective expected gain from voting is asymptotically proportional to

$$\sum_{c \neq d} (u_c - u_d)\max(0,\ w_c - w_d) = m[\sum_c w_c(u_c - \bar{u})] \ ,$$

where $\bar{u} = \frac{\Sigma u_c}{m}$. This motivates the following definition. Given a particular voting system, an <u>optimal voting strategy</u> for a voter with utilities $u_1, \ldots, u_m$ is an assignment of weights (votes) $w_1, \ldots, w_m$ to the candidates which maximizes $\Sigma w_c(u_c - \bar{u})$.

The election can be viewed as an n-person game. An initial chance move assigns utility m-vectors to the voters, then each voter casts votes for the candidates. In these terms, the unique symmetric equilibrium point of the game arises (asymptotically) when the voters follow their optimal voting strategies (as defined above). Furthermore, a voter's optimal voting strategy is asymptotically his best response to any symmetric (n-1)-tuple of opposing strategies. This indicates that real-world voters might arrive at their optimal strategies through adaptive behavior.

For weighted ranking voting systems, this definition yields the anticipated results: the optimal voting strategy is to assign the greatest weight to the most-preferred candidate, and generally to assign the weights in correspondence with the preference ranks of the candidates (for example, if

$u_1 \geq \dots \geq u_m$, assign the weights so that $w_1 \geq \dots \geq w_m$). For the approval voting system, the result is not quite so obvious: the optimal voting strategy is to vote for every candidate of utility greater than $\bar{u}$, the average utility over all candidates (that is, take $w_c = 1$ if $u_c - \bar{u} > 0$, and $w_c = 0$ otherwise). Throughout the remainder of this paper, it is assumed that voters always follow optimal voting strategies.

The effectiveness of a voting system. Recall the previous discussion of "effectiveness". In an n-voter random society, the expected social utility of the socially-optimal candidate in an m-candidate election exceeds $\frac{n}{2}$ (the expected social utility of a "typical" candidate) by an amount asymptotic to $\sqrt{\frac{n}{12}} \cdot \mathrm{Norm}_{max}(m)$. Now that optimal voting strategies have been specified, it is also possible to determine, for any given voting system, the expected social utility of the elected candidate. We define the effectiveness of a voting system for an m-candidate election as the asymptotic (for large societies) difference between the expected social utilities of the elected candidate and a typical candidate, normalized by the maximum possible difference. Formally, let $E_{maximal}(m,n)$ be the expected social utility of the best of m candidates in an n-voter random society, and for a given voting system let $E_{elected}(m,n)$ be the expected social utility of the candidate elected under this system when all of the voters follow optimal voting strategies. The effectiveness of this voting system for m-candidate elections is

$$\lim_{n \to \infty} \frac{E_{elected}(m,n) - \frac{n}{2}}{E_{maximal}(m,n) - \frac{n}{2}} .$$

This definition at last provides a method for comparing voting systems, in terms of the extent to which they lead to the election of a candidate representative of the preferences of the voters.

<u>The standard voting system</u>. Consider the standard voting system, employed in an m-candidate election. As indicated previously, the optimal strategy for a voter is to vote for the candidate of greatest utility to him. The expected utility of that candidate to the voter is $\frac{m}{m+1}$ (this is the expected value of the maximum of m independent uniform random variables). The expected utility of each other candidate to this same voter is $\frac{m}{2(m+1)}$. (This is the expected value of a uniformly-distributed random variable, given that it is not the greatest of m independent such variables.)

The voters can be partitioned into m classes, each according to the identity of his most-preferred candidate. Each voter in a random society is equally likely to fall into any of the m classes. Therefore, in an n-voter random society, the probability that the candidates respectively receive $k_1,\ldots,k_m$ votes is $\frac{1}{m^n}\binom{n}{k_1,\ldots,k_m}$. Given this distribution of votes, the expected social utility of the elected candidate (the candidate with the greatest number of votes) is $\frac{m}{m+1}\max(k_1,\ldots,k_m) + \frac{m}{2(m+1)}(n - \max(k_1,\ldots,k_m))$. Therefore, for the standard voting system,

$$
\begin{aligned}
E_{\text{elected}}(m,n) &= \sum_{k_1+\ldots+k_m=n} \frac{1}{m^n}\binom{n}{k_1,\ldots,k_m}\Big[\frac{m}{m+1}\max(k_1,\ldots,k_m) \\
&\qquad\qquad + \frac{m}{2(m+1)}(n-\max(k_1,\ldots,k_m))\Big] \\
&= \frac{m}{2(m+1)}\sum \frac{1}{m^n}\binom{n}{k_1,\ldots,k_m}[n+\max(k_1,\ldots,k_m)] \\
&= \frac{m}{2(m+1)}\Big[n + \sum \frac{1}{m^n}\binom{n}{k_1,\ldots,k_m}\max(k_1,\ldots,k_m)\Big] \\
&= \frac{n}{2} + \frac{m}{2(m+1)}\Big[\sum \frac{1}{m^n}\binom{n}{k_1,\ldots,k_m}\max(k_1,\ldots,k_m) - \frac{n}{m}\Big] \\
&= \frac{n}{2} + \frac{m}{2(m+1)}\Big[\mathrm{Mult}_{\max}(m,n) - \frac{n}{m}\Big],
\end{aligned}
$$

where $\mathrm{Mult}_{\max}(m,n)$ is the expected value of the maximum of m random variables with a joint multinomial distribution, with equal cell probabilities and summing to n .

$Mult_{max}(m)$ is asymptotic (for large n) to $\frac{n}{m} + \sqrt{\frac{n}{m}}\, Norm_{max}(m)$. This can be verified by approximating the multinomial distribution with an equicorrelated multivariate normal distribution, and then expressing the expected value of the maximum order statistic of that distribution in terms of the expected value of the maximum order statistic of an uncorrelated multivariate normal distribution (see [11] for details). Hence, for the standard voting system,

$$E_{elected}(m,n) \approx \frac{n}{2} + \frac{\sqrt{nm}}{2(m+1)}\, Norm_{max}(m) .$$

This immediately yields the result: <u>the effectiveness of the standard voting system for an m-candidate election is $\frac{\sqrt{3m}}{m+1}$</u> . Observe that in the two-candidate case, this result indicates an effectiveness of $\sqrt{\frac{2}{3}}$, as was demonstrated earlier. Also observe that the effectiveness of the standard voting system decreases as the number of candidates increases.

[As a matter of curiosity, one might ask how this result is affected if it is assumed that the utilities of candidates to voters in the random society are independent and identically distributed, but not necessarily uniformly distributed. Assume that the new common distribution has mean $\frac{1}{2}$ and standard deviation σ , and let $Dist_{max}(m)$ be the expected value of the maximum of m independent observations under this distribution. In this case, the effectiveness of the standard voting system for m-candidate elections is $\frac{\sqrt{m}}{(m-1)\sigma} \cdot (Dist_{max}(m) - \frac{1}{2})$.]

<u>The Borda voting system</u>. We now widen our attention to weighted ranking

voting systems in general, which of course include the standard voting system as a special case. A system which is the best among all voting systems of this type can be explicitly characterized. Recall that the Borda voting system for an m-candidate election consists of the single set of weights $(m, m-1, \ldots, 1)$. The Borda voting system has the greatest effectiveness of all weighted ranking voting systems. A verification of this statement follows.

For each voter in an n-voter society, there is an ordering of the candidates corresponding to their ranks on his utility scale. Enumerate the $m!$ possible orderings, and let $(k_1, \ldots, k_{m!})$ represent the profile of a society in which k_i voters rank the candidates according to ordering i . Let $v_q(c; k_1, \ldots, k_{m!})$ be the number of voters for whom candidate c stands in position q in their orderings. The expected social utility of candidate c , in a random society with profile $(k_1, \ldots, k_{m!})$, is

$$E(c; k_1, \ldots, k_{m!}) = \frac{m}{m+1} v_1(c; k_1, \ldots, k_{m!}) + \ldots + \frac{1}{m+1} v_m(c; k_1, \ldots, k_{m!}) .$$

Under the weighted ranking voting system with weights $(w_1, \ldots, w_m)$ satisfying $w_1 \geq \ldots \geq w_m$, candidate c will receive

$$V(c; k_1, \ldots, k_{m!}) = w_1 v_1(c; k_1, \ldots, k_{m!}) + \ldots + w_m v_m(c; k_1, \ldots, k_{m!})$$

votes. Let $\bar{c}(k_1, \ldots, k_{m!})$ be the candidate who receives the greatest number of votes. Then the expected social utility of the elected candidate in a random n-voter society is

$$E_{\text{elected}}(m,n) = \sum_{k_1+\ldots+k_{m!}=n} \frac{1}{(m!)^n} \binom{n}{k_1,\ldots,k_{m!}} \cdot E(\bar{c}(k_1,\ldots,k_{m!});k_1,\ldots,k_{m!}) \; .$$

Certainly, no weighted ranking voting system can yield a greater expected social utility than a system for which $E(c;k_1,\ldots,k_{m!})$ is maximized by $c = \bar{c}(k_1,\ldots,k_{m!})$ for every profile $(k_1,\ldots,k_{m!})$. This is precisely what happens under the Borda voting system, since in that case $V(c;k_1,\ldots,k_{m!}) = (m+1)E(c;k_1,\ldots,k_{m!})$ and therefore the candidate $\bar{c}$ who receives the greatest vote total in a society with a given profile simultaneously is the candidate of greatest expected social utility for a society with that profile. Hence, the Borda voting system is the most effective of all weighted ranking voting systems.

It can be shown [12] that the effectiveness of the Borda system for m-candidate elections is $\sqrt{\frac{m}{m+1}}$. Therefore, this system differs substantially from the standard system; it is nearly 100%-effective for many-candidate elections.

The approval voting system. The various weighted ranking voting systems depend only on ordinal voter preferences, in the sense that two voters with the same ordering of candidates on their utility scales will cast the same pattern of votes. On the other hand, the approval voting system brings a cardinal element into consideration, since the votes optimally cast by a voter indicate the locations of the candidates on his utility scale relative to the average utility of all candidates to him. One might hope that this introduction of a qualitatively different aspect into the choice among voting strategies would yield a system which compares favorably with the weighted ranking voting systems.

In order to compute the effectiveness of approval voting for m-candidate elections, several quantities must be determined. One is the probability p_k

that a random voter will cast k votes. (This is, of course, the probability that precisely k of m independent, uniformly distributed random variables are greater than the average of all m.) In a three-candidate election, $p_1 = p_2 = \frac{1}{2}$, because the middle-ranked candidate is equally likely to be of utility greater than or less than the average of the utilities of the other two. It is also necessary to determine the expected utilities a_k and d_k of approved and disapproved candidates to a voter, given that the voter casts k votes. When there are three candidates, $a_1 = \frac{3}{4}$, $d_1 = \frac{5}{16}$, and $a_2 = \frac{11}{16}$, $d_2 = \frac{1}{4}$. (For example, if it is given that three independent uniform random variables satisfy $u_1 \geq u_2 \geq u_3$ and $2u_2 \geq u_1 + u_2$, then their expectations are respectively $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{1}{4}$. Therefore, $a_2 = \frac{1}{2}(\frac{3}{4} + \frac{5}{8}) = \frac{11}{16}$ and $d_2 = \frac{1}{4}$.)

The voters can be partitioned into 2^m-2 classes, according to the set of candidates for whom they vote. With the preceding information at hand, the expected utility of the elected candidate in a random society of given voter profile can be determined. Then, by averaging expectations over all social profiles, the expected social utility of the elected candidate in a general random society can finally be determined. For the three-candidate case, the result of such computations is reported below.

Three-candidate elections. Consider an election involving three candidates. The effectiveness of the standard voting system for such an election is $\frac{\sqrt{3m}}{m+1} = 75\%$. From our earlier results, it is known that the Borda voting system will be more effective than this; the approval voting system may be better still. Exact computations have been carried out for a number of specific values of n. These are reported in Table 4, along with results for the standard voting system which are included for purposes of comparison. Lest it be thought that the differences between the entries in the three columns are insignificant, it should be noted that in a random society it is very likely that one candidate is so highly favored by the voters as to be chosen under almost any reasonable voting system. Therefore, a major component of the expected social utility of the elected candidate corresponds to noncontroversial situations. It is, however, the difference between systems in controversial situations which is of great interest, and the numerical differences in the table reflect precisely these situations.

Table 4

The expected social utility of the elected candidate, under three voting systems.

Number of voters	Standard voting system	Borda voting system	Approval voting system
2	1.2500	1.2917	1.2917
3	1.8333	1.8750	1.8646
4	2.3889	2.4236	2.4213
5	2.9167	2.9765	2.9726
10	5.5975	5.6706	5.6719
15	8.2245	8.3206	8.3245
20	10.8328	10.9472	10.9531
25	13.4328	13.5588	13.5662
30	16.0190	16.1597	16.1684

It seems reasonable to assume that for each of the three voting systems, the expected social utility of the elected candidate in an n-voter random society can be expanded in powers of $n^{-1/2}$; that is, generally, for an m-candidate election

$$E_{\text{elected}}(m,n) = \frac{n}{2} + An^{1/2} + B + Cn^{-1/2} + Dn^{-1} + \dots .$$

(The plausibility of this expansion follows from results on high-order terms in the Central Limit Theorem; see [5].) Under this assumption it is possible to estimate the value of A for each system, by extrapolation from the table. We obtain the following results. <u>For a three-candidate election, the Borda voting system is 86.6% effective, and the approval voting system is 87.5% effective</u>. These figures are significantly higher than the 75% effectiveness of the standard voting system.

<u>The case for approval voting</u>. There are a number of factors which deserve consideration before a particular voting system is adopted for use. Certainly, the system should be one which is easily understood by the general public, and the choice among voting strategies should not be too complicated. In addition, the system should be perceived by the voters as being likely to yield a result representative of their desires. Both the Borda voting system and the approval voting system satisfy these criteria. They are simple to comprehend, and the decision of how to vote, given one's preferences, is not overly complex. The results of this paper make it seem likely that the voting public could be convinced that either system is more representative (more effective) than the standard voting system.

Most municipalities already have a substantial capital investment in voting machines. These machines are designed to tally the results of standard voting. For implementation of a new voting system to be economically feasible, it is necessary that the system be compatible with these machines, requiring at most minor modifications to them. The Borda voting system fails to satisfy this criterion. However, many voting machines can be adjusted to allow the casting of multiple votes, one for each of several candidates in a single race. This feature permits the use of the instruction, "vote for k

or less of the m candidates", in board or council elections in which several (usually k) "at large" seats are at stake. By setting these machines to correspond to the instruction, "vote for m or less of the m candidates", the results of approval voting can be tallied. Therefore, the approval voting system has the potential for low-cost implementation.

In our earlier discussion, the approval voting system for m-candidate elections was defined as a collection of $m-1$ sets of voting weights. The weight sets $(1,\ldots,1)$ and $(0,\ldots,0)$ were omitted from consideration because they lack theoretical interest: no optimal voting strategy need involve their use. However, an actual implementation of approval voting should allow the use of either set. A winning candidate often claims not only his victory, but also a "mandate" from the voters. By allowing a voter to indicate approval (or disapproval) of all of the candidates, we give him the option of strengthening (or weakening) that mandate, irrespective of the outcome of the election.

Consider an election involving two major-party candidates and a number of minor-party or one-issue candidates. Under the standard voting system, a voter may face a difficult decision. If he wishes to express his sympathy with one of the third-party causes, he must sacrifice the opportunity to vote for his more-preferred of the leading candidates. In contrast to this situation, both of his desires can be satisfied under the approval voting system. As a result, the winning candidate is provided with another indication of the nature of his mandate, since the election results display the relative support received by minor parties and by individual issues.

Another disadvantage of the standard voting system is particularly apparent in the case of primary elections. Consider the results of the 1976

Massachusetts Democratic presidential primary, as indicated in Table 5. The total vote was divided among so many candidates that no clear-cut winner emerged. (Interestingly, this worked to the advantage of Jimmy Carter, who found himself with little competition for the attention he won the week before in New Hampshire and the next week when he beat Wallace in Florida.) Under approval voting, the number of votes cast for a candidate, expressed as a percentage of the number of people voting, is a clear indication of the candidate's popularity.

Table 5

The 1976 Massachusetts Democratic presidential primary.

Bayh	34,963	5%
Carter	101,948	14%
Harris	55,701	8%
Jackson	164,393	23%
McCormick	25,772	4%
Shapp	21,693	3%
Shriver	53,252	7%
Udall	130,440	18%
Wallace	123,112	17%

This last example suggests two arenas for the testing of public reaction to the approval voting system. A number of states, such as New Hampshire and Pennsylvania, conduct two simultaneous presidential primaries: one to select convention delegates, and a "beauty contest" to indicate the relative popularity of the candidates. Adoption of the approval voting system for the latter type of contest would provide the public with contact with the system, without binding anyone by the results. The other arena is that of the public opinion polls. One of the expressed goals of the major polling organizations is to provide a public service by accurately reporting the sentiment of the populace. If they were to regularly report approval/disapproval ratings for each candidate during the primary season (rather than merely estimating the likely result of a primary election using the standard voting system), this public service might be more successfully performed.

A number of the practical advantages of the approval voting system have been discussed in the preceding paragraphs. Furthermore, a major purpose of this paper has been to precisely document the significantly greater effectiveness of this system than that of the system currently in use. In view of all of this, and particularly in view of the possibility of implementation at little expense, the case for the trial adoption of the approval voting system appears quite strong.

Related work. Our concern with the comparison of voting systems, and with approval voting in particular, dates back to the elections of 1970 and 1971 cited in the beginning of this paper. The method of comparison presented here was first investigated and privately discussed in 1973. Recently, interest in the approval voting system has arisen from several sources. Fishburn and Gehrlein [6] have investigated the probability that various voting systems (including the approval voting system) select the candidate satisfying the Condorcet principle, when such a candidate exists. Since they deal only with ordinal voter preferences, they are forced to treat voting strategies in an ad hoc manner. It would be interesting to duplicate their work in the context of random societies, in which optimal voting strategies are well-defined. Brams [2] and Brams and Fishburn [3] have investigated strategic properties of voting systems (again, in terms of ordinal preferences), and have obtained several results which confirm the desirability of the approval voting system. Kellett and Mott [8] have considered the potential use of approval voting in presidential primaries.

Merrill has independently taken a cardinal approach to voting systems [9], and has obtained results concerning strategic considerations which are similar to those derived here. Using these results he has examined cardinal preference data from the 1972 Democratic presidential primaries [10], and has contrasted the probable outcomes of those primaries under various voting systems.

Appendix: Social utility and random societies. The concept of social utility has a history of controversy. In essence, one wishes to take as given the utility functions of a number of voters (which are determined only up to arbitrary changes of origin and scale), and to construct from these a single utility function which represents certain ideals of social welfare. To do this, one must establish a method for comparing the relative intensities of preference of the voters. Many social theorists deny that such interpersonal comparisons can be made in practice.

As an article of faith, we believe that the concept of interpersonal comparisons is meaningful. However, we agree with the traditionalists that there may be no methods by which a scale for such comparisons can be constructed. (This situation is somewhat analogous to that in mathematics: within a given logical framework there may be theorems which are true, but for which no proofs exist.) Indeed, our inability to compare the preference intensities of voters is the source of many non-representative election outcomes.

For example, assume (as earlier in the paper) that a slight majority in an electorate favors one candidate, while the remaining voters favor a second. We believe that there are situations (such as when the latter group of voters is by far the more fervent) in which the second candidate is more representative of the society's desires. We do not assert that these situations can always be recognized (although, in practice, extreme cases are readily apparent).

In our work we make the egalitarian assumption that all voters have the same capacity for intensity of preference. That is, we assume that the utilities of a candidate to the voters are distributed on a common underlying scale. It is known (Harsanyi [7]) that under certain plausible assumptions any social utility function must be a weighted sum of individual utility functions; with the further assumption of a common scale, the weights are naturally taken to be equal. In consequence, we treat social utility as the unweighted sum of commonly-scaled individual utilities.

Again, we emphasize that assuming the existence of a common underlying utility scale does not entail assuming that such a scale can actually be constructed. Assume that the utilities of the candidates to the voters are independent, identically distributed random variables on the underlying scale. Let the true utilities of the candidates to a specified voter be $u_1 \leq \cdots \leq u_m$. Although these cannot be directly observed, the "normalized" utilities $0, (u_2 - u_1)/(u_m - u_1), \ldots, (u_{m-1} - u_1)/(u_m - u_1), 1$ can be measured by asking the voter to choose among various lotteries over the candidates. If these normalized utilities provide information concerning the origin and scale of the true utilities (that is, information concerning u_1 and u_m), then interpersonal comparison of utilities is feasible. What assumption concerning the distribution of the utilities must be made, in order that no interpersonal comparisons are possible? It is shown in [13] that u_1 and u_m will be independent of the normalized utilities only if the probability distribution is uniform. Hence, our use of the uniform distribution in the definition of a random society conforms with our desire to avoid assuming that interpersonal utility comparisons can actually be made.

Our interpretation of expected social utility in a random society is as the average of social utilities over societies of all possible preference profiles (and furthermore, over all conceivable combinations of individual preference intensities). Therefore, without ever being able to determine the relative social utilities of the candidates in any particular election, we can nevertheless distinguish voting systems which, on the average, lead to the election of candidates of high social utility.

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MULTIPLY-WEIGHTED VOTING SYSTEMS

by

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Abstract. The area of social decision-making is one in which the disciplines of game theory, political science, and economics all meet. One of the simplest decisions to be faced is the election of one candidate from several. Various voting systems have been proposed for such elections. In order to compare these systems in terms of their tendency to elect a candidate representative of the voters' preferences, a measure of "effectiveness" has been developed.

The study of the effectiveness of voting systems is continued in this paper. In particular, two families of voting systems are found to contain three- and four-candidate systems which are more effective than any previously discussed.

Introduction. One of the simplest choices which a democratic society may face is the election of one candidate (or the selection of one alternative) from several. Because the voters are likely to have differing preferences for the candidates, it is desirable to have available a procedure for making a choice which, in some sense, represents the overall preferences of the electorate.

When the choice is between two candidates, the generally-accepted procedure is to let each voter indicate his more-preferred candidate, and then to select the candidate preferred by the majority. However, it has long been recognized that, when an election involving three or more candidates is faced, the "standard" system of voting (in which each voter casts a single vote, and the candidate receiving the greatest vote total is elected) may fall far short of the democratic ideal (see, for example, the historical cases discussed in [3]).

There are several approaches to the construction of more effective methods for selecting from among a group of candidates. One is to hold the election in several stages, gaining information about the preferences of the voters from the outcome at each stage. For example, several candidates may be eliminated from consideration after an initial stage, with the ultimate winner selected in a subsequent run-off election. Another approach is to allow greater variety in the responses of the voters. It is this approach which will be taken in this paper. As examples, one could consider the Borda voting system (in which the voters are called upon to rank the candidates, and then votes are awarded in proportion to the ranks assigned), or the approval voting system (in which each voter may cast one vote for every candidate of which he "approves").

How should a choice among voting systems be made? Practical considerations, such as expense of implementation, sensitivity to strategic considerations, and the complexity of elicited responses, are important. But first and foremost, the system adopted must be effective in making a socially-representative choice among the candidates.

A measure of effectiveness was proposed and developed in [3]. It centers on the idea of a "random" society. In such a society, the utilities of the candidates to the voters are random variables. As these utilities vary independently over their range, every conceivable society (as determined by individual preferences) is generated. Therefore, expected social utility in the random society corresponds to an average of social utilities over all possible situations. The "effectiveness" of a particular voting system is defined as a comparison between the expected social utility of the candidate elected under the system, and the expected social utility of the socially-optimal candidate.

In other works ([3], [4]), we have shown that the effectiveness of the standard voting system decreases as the number of candidates under consideration increases. In contrast, the Borda system becomes progressively more effective. For three-candidate elections, the approval voting system is slightly superior to the Borda system, which in turn is substantially better than the standard system.

In this paper, several additional aspects of the comparison of voting systems are discussed. Two specific results are obtained. A three-candidate voting system is found which is more effective than any system previously discussed, and a four-candidate system more effective than the four-candidate Borda system is also discovered.

Definitions. We direct our study to one-stage elections, in which the voters cast varying numbers of votes for the candidates, with the candidate who receives the greatest vote total being declared the winner. Formally, an m-candidate voting system is a collection of (one or several) sets of m weights. Each voter assigns the weights from one of the sets to the candidates. (The weight assigned to a candidate by a voter corresponds to the number of "votes" cast by the voter for that candidate.) The candidate assigned the greatest total weight is the winner of the election.

The most familiar type of election arises from the standard voting system, with the single weight set $(1,0,\ldots,0)$. Under this system, each voter may cast one vote for the candidate of his choice. More generally, a weighted ranking voting system is any voting system consisting of a single set of weights $(w_1,\ldots,w_m)$. In essence, such a system elicits a preference ordering of the candidates from the voters by having them assign weights dependent solely on the candidates' ranks. The Borda voting system, with weight set $(m,m-1,\ldots,1)$, is an example of this type.

Much of our attention in this paper will be directed toward voting systems consisting of several weight sets. One such system, which has recently drawn substantial interest, is the approval voting system, with the $m-1$ weight sets $(1,0,\ldots,0)$, $(1,1,0,\ldots,0),\ldots,(1,\ldots,1,0)$. Under this system, a voter may cast one vote for every candidate of whom he "approves." In a sense which will be made precise later, the three-candidate approval voting system is essentially equivalent to the positive/negative voting system with weight sets $(1,0,0)$ and $(0,0,-1)$. This system permits a voter to vote either "for" or "against" a single candidate. Two classes of multiple-weight-set voting systems, which generalize this system, form major objects of study in this paper.

Voting systems may be compared in terms of their relative effectiveness in electing a candidate representative of the desires of the public. To this end, we define a random society as a group of voters for which the utility of any particular candidate to any particular voter is a random variable, uniformly distributed on the unit interval. If an n-voter society is choosing among m candidates, there are nm of these random variables; we take them to be mutually independent. Hence, each voter's m-vector of utilities is chosen at random from the unit m-cube.

In an n-voter random society, the social utility of a candidate is the sum of his utilities to the voters (and is therefore itself a random variable). By the Central Limit Theorem, the social utility of any given candidate is asymptotically (in n) normally distributed, with mean $\frac{n}{2}$ and variance $\frac{n}{12}$. The expected social utility of a candidate may be regarded as an average of social utilities over all possible societies. In particular, the expected social utility of the socially-best candidate in an m-candidate election is

$$E_{\text{maximal}}(m,n) \simeq \frac{n}{2} + \sqrt{\frac{n}{12}} \cdot \text{Norm}_{\max}(m) ,$$

where $\text{Norm}_{\max}(m)$ is the expected value of the maximum of m independent standard (mean 0 , variance 1) normal random variables.

We work in terms of social utility for the sake of notational convenience. However, it is possible to construct our model, and to study effectiveness as defined below, in terms of the utilities of the various outcomes to a single preselected voter. Due to the symmetry of our considerations, the choice of any other voter would lead to the same measure; hence, a voting system which is highly effective in terms of social utility is of equal subjective effectiveness to each individual voter. Furthermore, the use of the uniform probability distribution in the definition of a random society can be justified by the assumption that interpersonal utility comparisons cannot be made in practice. These issues are discussed in detail in [6].

The outcome of an election depends on the strategies employed by the voters. Assume that the utilities of the candidates to a specific voter are $u_1, \ldots, u_m$, and let $\bar{u} = \frac{\Sigma u_c}{m}$ be the average of these utilities. An <u>optimal voting strategy</u> for this voter is an assignment of weights $w_1, \ldots, w_m$ to the candidates which maximizes $\Sigma w_c(u_c - \bar{u})$. (This quantity is shown in [4] to be asymptotically (for large societies) proportional to the voter's expected gain from casting the votes $w_1, \ldots, w_m$ rather than abstaining. Hence, the optimal voting strategies correspond (asymptotically) to an equilibrium point of the "voting game.") For a weighted ranking voting system, this yields the anticipated result: an optimal voting strategy is to assign the weights to accord monotonically with the utilities (for example, if $u_1 \geq \ldots \geq u_m$, take $w_1 \geq \ldots \geq w_m$). For the approval voting system, the result is not quite so obvious. A candidate c is said to be of "greater-than-average" utility to a voter if $u_c > \bar{u}$. Under the approval voting system, it is optimal to vote for all candidates of greater-than-average utility (that is, take $w_c = 1$ if $u_c - \bar{u} > 0$, and $w_c = 0$ otherwise).

In the case of approval voting, there are several distinct optimal strategies if a voter's utilities are such that $u_c = \bar{u}$ for some candidate c . However, in a random society such a relationship holds with zero probability. We shall therefore, at no essential cost, frequently disregard the possible nonuniqueness of optimal strategies in subsequent discussions. Throughout the remainder of this paper, it is assumed that all voters follow optimal voting strategies.

We are now prepared to formally define the "effectiveness" of a voting system. Consider a particular m-candidate voting system. Let $E_{\text{elected}}(m,n)$ be the expected social utility of the candidate elected under this system in an n-voter random society (when all of the voters follow optimal strategies). This quantity is bounded above by $E_{\text{maximal}}(m,n)$ (which is, of course, independent of the voting system under consideration). The expected social utility of a randomly-selected candidate is $\frac{n}{2}$. The <u>effectiveness</u> of the voting system is the (asymptotic) difference between the expected social utility of the elected candidate and a "typical" candidate, normalized by the upper bound on this difference; that is,

$$\text{Effectiveness} = \lim_{n \to \infty} \frac{E_{\text{elected}}(m,n) - \frac{n}{2}}{E_{\text{maximal}}(m,n) - \frac{n}{2}} .$$

This measure provides a means for comparing voting systems, in terms of their tendencies to lead to the election of candidates representative of the voters' desires.

A more extensive discussion of all of these definitions appears in [5].

Essentially equivalent voting systems. Consider the two-candidate standard voting system, with the single weight set $(1,0)$. If this is compared with the voting system with weight set $(2,1)$, it is apparent that there is no essential difference between the two systems (in the sense that, if all of the voters behave optimally, the same candidate will win under either system). Similarly, the votes cast by the voters under the voting system with weight set $(2,0)$ differ from the votes cast under the standard system only by a scaling factor. And the voting system with the two weight sets $(1,0)$ and $(\frac{1}{2},0)$ can be expected to yield the same results as the standard system, since no optimally-behaving voter need ever use the second set of weights.

Generally, the outcome of an election under a given voting system will not be changed by adding a fixed constant to all of the weights in a particular weight set, or by multiplying all of the weights in all of the weight sets by a fixed positive scaling factor, or by introducing or eliminating weight sets which need never be used in optimal strategies. Therefore, we define two m-candidate voting systems to be essentially equivalent if they satisfy the following property: there is a $K > 0$ such that, for every m-tuple $(u_1,\ldots,u_m)$ of utilities, there is a constant d and there are optimal assignments of weights $(w_1,\ldots,w_m)$ and $(w'_1,\ldots,w'_m)$ to the candidates under the respective systems, for which $w_c = K(w'_c+d)$ for all c.

For example, for two-candidate elections all (nontrivial) voting systems are essentially equivalent to the standard system. The only strategic consideration is to select a weight set (if there is more than one) for which the difference between the two weights is maximal, and then assign the greater weight to the more-favored candidate.

Consider the three-candidate approval voting system, with the two weight sets $(1,0,0)$ and $(1,1,0)$. As was previously indicated, this is essentially equivalent to the positive/negative voting system with weight sets $(1,0,0)$ and $(0,0,-1)$. Consider the three-candidate voting system with the infinite family of weight sets $\{(1,a,0)\colon 0 \le a \le 1\}$. If the candidates have utilities $u_1 \ge u_2 \ge u_3$ to a voter, he will seek the value for a which maximizes $(u_1-\bar{u}) + a(u_2-\bar{u})$. This can be done while restricting a to the extreme values 0 and 1. Therefore, this system is also essentially equivalent to the approval system.

Along slightly different lines, it also follows that the m-candidate approval voting system, which has $m-1$ weight sets, is essentially equivalent to the voting system with the $m+1$ weight sets $(0,\ldots,0)$, $(1,0,\ldots,0),\ldots,$ $(1,\ldots,1,0)$, $(1,\ldots,1)$; these are the weight sets of the approval system in combination with two weight sets in which all of the weights are equal. Under this system, a voter has the additional options of expressing either "approval" or "disapproval" of all of the candidates. Although it is never optimal to avail oneself of these options, an implementation of the approval voting system would most likely take this latter form.

Using the idea of essential equivalence, we can define a canonical form for voting systems. A voting system is 0-normalized if the least weight in each weight set is 0 , and is (0,1)-normalized if, furthermore, the greatest of all weights (over all weight sets) is 1 . Finally, a voting system is minimal if, for every weight set of the system, there is at least one vector of utilities for which the weight set must be used in the corresponding optimal strategy. Every nontrivial voting system is essentially equivalent to a unique minimal, (0,1)-normalized voting system. Because of this, we will frequently refer to a single voting system, when the context makes it clear that we are actually referring to a family of essentially equivalent systems.

Basic results. The study of the effectiveness of various voting systems was begun in [3]. It was shown that the effectiveness of the m-candidate standard voting system is $\frac{\sqrt{3m}}{m+1}$. Since all two-candidate voting systems are essentially equivalent to the standard system, it follows that every two-candidate voting system has effectiveness $\sqrt{\frac{2}{3}} = 81.65\%$. In other words, no two-candidate system can be much less than 20% short of perfection.

In the same paper, it was shown that the Borda voting system is the most effective of all weighted ranking voting systems. Similar results, characterizing certain voting systems as the most effective within particular classes, will be presented in this paper. In [4], it is shown that the Borda system has effectiveness $\sqrt{\frac{m}{m+1}}$.

Observe the qualitative difference between the standard and Borda systems. The former is of decreasing effectiveness as the number of candidates increases, while the latter is nearly 100%-effective for many-candidate elections. Several natural questions suggest themselves. Are there any voting systems which are more effective than the Borda system for elections involving relatively few candidates? Are there any which approach 100%-effectiveness for many-candidate elections more rapidly than the Borda system? Can any such highly-effective systems be practically implemented?

An attack on these questions was initiated in [3], where the approval voting system was studied. It was shown that the three-candidate approval voting system is 87.5% effective, in contrast to the 86.6% effectiveness of the Borda system. In addition, the feasibility and desirability of implementing the approval voting system were discussed. In the next few sections, we shall make further progress by considering several generalizations of the three-candidate approval voting system. We will show that the three-candidate system with weight sets $(4,3,0)$ and $(4,1,0)$ is $\frac{\sqrt{13}}{4} = 90.14\%$ effective, and the four-candidate system with weight sets $(6,5,3,0)$ and $(6,3,1,0)$ is $\frac{\sqrt{21}}{5} = 91.65\%$ effective.

Average-dependent voting systems. The exercise of an optimal voting strategy consists of two phases. First, a weight set must be chosen from the several weight sets constituting the voting system, and then the weights from the selected set must be assigned to the candidates. However, the optimal assignment of weights is straightforward: they should be assigned to accord monotonically with the utilities of the candidates. Therefore, it is the choice among weight sets which is of primary strategic concern.

Consider the approval voting system. An optimal strategy for a voter under this system is to vote for all candidates of utility greater that $\bar{u}$, the average of all of the candidates' utilities to him. In other words, the choice among weight sets is based solely on the number of candidates of greater-than-average utility to him. We generally define a voting system to be average-dependent if the optimal choice among weight sets similarly depends only on the number of candidates of greater-than-average utility.

The average-dependent voting systems can be explicitly characterized. Every m-candidate average-dependent voting system is essentially equivalent to a system with $m-1$ weight sets $w^{(1)},\dots,w^{(m-1)}$. The weight sets are of the form

$$w^{(k)} = (a_1,\dots,a_k, b_{k+1},\dots,b_m)$$

for all $1 \leq k \leq m-1$, where the sequences $\{a_1,\dots,a_{m-1}\}$ and $\{b_2,\dots,b_m\}$ satisfy the inequalities $a_1 \geq \dots \geq a_{m-1} \geq b_m$, $a_1 \geq b_2 \geq \dots \geq b_m$, and $a_k \geq b_k$ for all $2 \leq k \leq m-1$. Observe that all weighted ranking voting systems, as well as the approval voting system, are average-dependent; accordingly, they are instances of this result.

The remainder of this section is devoted to a verification of the characterization. Consider any particular m-candidate average-dependent voting system. There are $m-1$ strategically distinct situations which a voter might face, depending on the number of candidates of greater-than-average utility to him. If there are k such candidates, let $w^{(k)} = (w_1^{(k)},\dots,w_m^{(k)})$ be the corresponding set of weights used in an optimal stategy. Clearly, no generality is lost by assuming that the voting system under consideration consists of precisely these $m-1$ weight sets.

We further assume the system under consideration to be 0-normalized, so that $w_m^{(k)} = 0$ for all k. (This corresponds to taking $b_m = 0$ in the characterization; no generality is lost in our arguments.) Given a vector of utilities $u = (u_1,\dots,u_m)$ with $u_1 \geq \dots \geq u_m$, and given a weight m-vector w, let $E(u;w) = \Sigma w_c(u_c-\bar{u})$. The optimal choice of a weight set, by a voter to whom the candidates have utilities $u_1,\ldots,u_m$, is that $w^{(k)}$ which maximizes this function. Consider a particular vector u for which $u_k = \bar{u}$. From the continuity of $E(u;w)$ in u, it follows that both $w^{(k-1)}$ and $w^{(k)}$ are optimally-chosen weight sets for a voter with utilities u; that is, $E(u;w^{(k-1)}) = E(u;w^{(k)})$.

For example, let $u = (1,\frac{1}{2},\ldots,\frac{1}{2},0)$. Then we must have $E(u;w^{(1)}) = \ldots = E(u;w^{(m-1)})$. Since $\bar{u} = \frac{1}{2}$, $E(u;w^{(k)}) = w_1^{(k)}(1 - \frac{1}{2})$. Therefore, all of the $w_1^{(k)}$ are equal. (This common value will be a_1 in the characterization.)

In order to complete the justification of the characterization, we will show that, for any $2 \leq \ell \leq m-1$,

$$w_\ell^{(1)} = \ldots = w_\ell^{(\ell-1)} \leq w_\ell^{(\ell)} = \ldots = w_\ell^{(m-1)} .$$

(The first of these common values will be b_ℓ; the second will be a_ℓ.) Assume for inductive purposes that $w_k^{(k)} = \ldots = w_k^{(m-1)}$ $(= a_k)$ has been established for all $1 \leq k \leq \ell$. Let $u = (1,\ldots,1,\frac{\ell+1}{\ell+2},\ldots,\frac{\ell+1}{\ell+2},0)$, where the first $\ell+1$ components of u are equal. Then $\bar{u} = \frac{\ell+1}{\ell+2}$. We must have $E(u;w^{(\ell+1)}) = \ldots = E(u;w^{(m-1)})$. But, applying the inductive hypothesis,

$$\begin{aligned} E(u;w^{(t)}) &= w_1^{(t)}(1-\bar{u}) + \ldots + w_\ell^{(t)}(1-\bar{u}) + w_{\ell+1}^{(t)}(1-\bar{u}) \\ &= a_1(1-\bar{u}) + \ldots + a_\ell(1-\bar{u}) + w_{\ell+1}^{(t)}(1-\bar{u}) , \end{aligned}$$

for all $\ell+1 \leq t \leq m-1$. Therefore, $w_{\ell+1}^{(\ell+1)} = \ldots = w_{\ell+1}^{(m-1)}$.

Similarly, assume that $w_k^{(1)} = \ldots = w_k^{(k-1)}$ $(= b_k)$ has been established for all $\ell \leq k \leq m$. Let $u = (1,\frac{1}{m-\ell+3},\ldots,\frac{1}{m-\ell+3},0,\ldots,0)$, where the last $m-\ell+2$ components of u are equal. Then $\bar{u} = \frac{1}{m-\ell+3}$, and we must have $E(u;w^{(1)}) = \ldots = E(u;w^{(\ell-2)})$. But

$$E(u;w^{(t)}) = w_1^{(t)}(1-\bar{u}) + w_{\ell-1}^{(t)}(0-\bar{u}) + w_\ell^{(t)}(0-\bar{u}) + \dots + w_m^{(t)}(0-\bar{u}) \ ,$$

$$= a_1(1-\bar{u}) + w_{\ell-1}^{(t)}(0-\bar{u}) + b_\ell(0-\bar{u}) + \dots + b_m(0-\bar{u}) \ ,$$

for all $1 \le t \le \ell-2$. Therefore, $w_{\ell-1}^{(1)} = \dots = w_{\ell-1}^{(\ell-2)}$.

Finally, for any $2 \le k \le m-1$, let $u = (u_1,\dots,u_m)$ satisfy $u_1 \ge \dots \ge u_{k-1} > \bar{u} > u_k \ge \dots \ge u_m$. Then $w^{(k-1)}$ will be an optimal set of weights, and consequently

$$E(u;w^{(k-1)}) - E(u;w^{(k)}) = (b_k-a_k)(u_k-\bar{u}) \ge 0 \ .$$

Therefore, $a_k \ge b_k$. This completes our verification.

Computing the effectiveness of an average-dependent voting system. Given a particular m-candidate average-dependent voting system, it is possible, at least in principle, to determine the effectiveness of the system. An efficient approach requires the establishment of some terminology.

Consider a voter in an n-voter random society. He can be classified according to the ordering π of the candidates on his utility scale and the number $1 \le k \le m-1$ of candidates of greater-than-average utility to him. Therefore, the society can be partitioned into $m! \cdot (m-1)$ classes. Let $n(\pi;k)$ represent the number of voters in a specific class. Then $\{n(\pi;k)\}$ represents a profile of the n-voter society. (For example, in the three-candidate case there are 12 classes, and $\{n(\pi;k)\} =$ $\{n(123;1),n(123;2),n(132;1),n(132;2),\dots\}$.)

For any given social profile, let $v_{q,\ell}(c;\{n(\pi;k)\})$ be the number of voters for whom candidate c lies in position q in their orderings and to whom precisely ℓ of the candidates have greater-then-average utility. The number of votes cast for candidate c (that is, the total weight assigned to him by the voters) is

$$V(c;\{n(\pi;k)\}) = \sum_{\ell=1}^{m-1} [\sum_{q\leq\ell} a_q v_{q,\ell}(c;\{n(\pi;k)\}) + \sum_{q>\ell} b_q v_{q,\ell}(c;\{n(\pi;k)\})] .$$

(In the three-candidate case, for example, $v_{1,2}(1;\{n(\pi;k)\}) = n(123;2) + n(132;2)$. Also, $V(1;\{n(\pi;k)\}) = a_1v_{1,1} + a_1v_{1,2} + b_2v_{2,1} + a_2v_{2,2} + b_3v_{3,1} + b_3v_{3,2}$, where all of the $v_{q,\ell}$ are evaluated at $(1;\{n(\pi;k)\})$.) Let $\overline{c}(\{n(\pi;k)\})$ be the candidate who maximizes V with respect to the given social profile. Then $\overline{c}$ is the candidate who will be elected by a society with this profile.

The expected utility $u_{q,\ell}$ of a particular candidate to a voter depends on the position q of the candidate in the voter's preference ordering, and on the number ℓ of candidates of greater-than-average utility to him. (In the three-candidate case, $u_{1,1} = u_{1,2} = \frac{3}{4}$, $u_{2,1} = \frac{3}{8}$, $u_{2,2} = \frac{5}{8}$, and $u_{3,1} = u_{3,2} = \frac{1}{4}$.) It follows that the expected social utility of candidate c , in a society with a given profile, is

$$U(c;\{n(\pi;k)\}) = \sum_{q,\ell} u_{q,\ell}v_{q,\ell}(c;\{n(\pi;k)\}) .$$

Let $p(\pi;k)$ be the probability that a randomly-chosen voter is a member of the indicated class. (In the three-candidate case, all twelve classes are equiprobable. Therefore, each $p(\pi;k) = \frac{1}{12}$.) The probability that the society has a particular profile is

$$P(\{n(\pi;k)\}) = [\prod_{(\pi;k)} p(\pi;k)^{n(\pi;k)}] \cdot \binom{n}{\{n(\pi;k)\}} ,$$

where the latter factor is a multinomial coefficient of the form $\binom{n}{n_1,\ldots,n_{m!\cdot(m-1)}}$. (In the three-candidate case, the first factor is simply $(\frac{1}{12})^n$.)

The expected social utility of the elected candidate, in an n-voter random society, can be expressed in terms of the quantities just defined. Summing over all social profiles,

$$E_{\text{elected}}(m,n) = \Sigma\ P(\{n(\pi;k)\}) \cdot U(\bar{c}(\{n(\pi;k)\});\{n(\pi;k)\}) \ .$$

In [3], estimates for the effectiveness of the three-candidate Borda and approval voting systems were obtained from expressions similar to these. The expected social utilities of the elected candidates were computed for relatively small values of n , and these results were then used in extrapolations.

The preceding formulation is valid for any m-candidate average-dependent voting system. For which such systems is $E_{\text{elected}}(m,n)$ relatively large? The only term in the expression for this quantity which varies with the choice of voting system is $\bar{c}(\{n(\pi;k)\})$. It follows that the best of all average-dependent voting systems would be one for which $U(c;\{n(\pi;k)\})$ always attains its maximum when $c = \bar{c}(\{n(\pi;k)\})$, if such a system exists.

Since $\bar{c}$ is defined as the candidate who maximizes V , it is certain that U will also be maximized by $\bar{c}$ if U is linearly related to V . In the three-candidate case, there is a voting system for which this is so. Simply take $a_1 = \frac{3}{4}$, $a_2 = \frac{5}{8}$, $b_2 = \frac{3}{8}$, and $b_3 = \frac{1}{4}$. For the corresponding average-dependent voting system with weight sets $(\frac{3}{4},\frac{5}{8},\frac{1}{4})$ and $(\frac{3}{4},\frac{3}{8},\frac{1}{4})$, the functions U and V are identical. Therefore, this system is the best of all three-candidate average-dependent voting systems. (An essentially equivalent system, which does not involve fractional weights, is the system with weight sets $(4,3,0)$ and $(4,1,0)$.) The approval voting system was the most effective three-candidate system discussed in [3]. Since the approval system is obviously average-dependent (after all, it was as a generalization of the approval system that average-dependent systems were originally defined), this new system is more effective than any three-candidate system previously studied.

It is natural to ask whether the best four-candidate average-dependent voting system can be similarly obtained. Unfortunately, our approach does not generalize. The values of the utilities $u_{q,\ell}$ are given in the table; they cannot be used as the weights of an average-dependent voting system. Therefore, in cases involving more than three candidates, another approach must be taken in order to determine the most effective average-dependent systems.

Table 1. The expected utility of a candidate to a voter in a four-candidate election, given the position q of the candidate in the voter's preference ordering and the number ℓ of candidates of greater-than-average utility to the voter.

		q = 1	2	3	4
ℓ	1	24/30	11/30	9/30	6/30
	2	24/30	19/30	11/30	6/30
	3	24/30	21/30	19/30	6/30

The effectiveness of the best three-candidate average-dependent voting system. Consider the voting system with weight sets $(\frac{3}{4},\frac{5}{8},\frac{1}{4})$ and $(\frac{3}{4},\frac{3}{8},\frac{1}{4})$.

For this system, $V(c;\{n(\pi;k)\}) = U(c;\{n(\pi;k)\})$ for all candidates and all social profiles. That is, the number of votes a candidate receives in a random society of given profile is equal to the expected social utility of that candidate.

In an n-voter random society, the twelve quantities $n(\pi;k)$ are random variables. They have a joint multinomial distribution, with equal cell probabilities. In other words, one method of generating a social profile for the random society is to assign each voter equiprobably to any of the twelve different classes. Therefore, $E_{\text{elected}}(3,n)$ is simply an expected value with respect to these jointly-distributed variables. Specifically,

$$E_{\text{elected}}(3,n) = E(\max_c V(c;\{n(\pi;k)\})) \ .$$

For notational convenience, let $Z_c = V(c;\{n(\pi;k)\})$, for $c = 1,2,3$. Then

$$\begin{aligned} Z_1 &= \tfrac{3}{4}\,(n(123;1) + n(123;2) + n(132;1) + n(132;2)) \\ &\quad + \tfrac{5}{8}\,(n(213;2) + n(312;2)) + \tfrac{3}{8}\,(n(213;1) + n(312;1)) \\ &\quad + \tfrac{1}{4}\,(n(231;1) + n(231;2) + n(321;1) + n(321;2)) \ . \end{aligned}$$

Z_2 and Z_3 can be similarly expressed as weighted sums of the random variables $n(\pi;k)$.

It is well-known that multinomially-distributed random variables have an asymptotic multivariate normal distribution. Although the twelve indicated variables have a degenerate joint distribution (their sum is constant), the three random variables Z_1 , Z_2 , and Z_3 can be written as independent linear combinations of some eleven of the variables (by expressing one of the $n(\pi;k)$ in terms of the others, and then substituting for it throughout).

These eleven will have a nondegenerate joint distribution which is asymptotically normal. It follows (see, for example, Corollary 2.1 on page 254 of [2]) that Z_1, Z_2, and Z_3 have a nondegenerate (asymptotic) multivariate normal distribution.

In order to determine the parameters of this final distribution, note first that for any π and k, $E(n(\pi;k)) = \frac{n}{12}$ and $E([n(\pi;k)]^2) = \frac{n(n-1)}{144} + \frac{n}{12}$. Furthermore, if $(\pi;k)$ and $(\pi';k')$ are distinct classes, $E(n(\pi;k) \cdot n(\pi';k')) = \frac{n(n-1)}{144}$. After some tedious term-counting, this leads to the following results: for all c, $E(Z_c) = \frac{n}{2}$, $E(Z_c^2) = \frac{n^2}{4} + \frac{3n}{64}$, and $Var(Z_c) = \frac{3n}{64}$. Also, if $c \neq c'$, $E(Z_c Z_{c'}) = \frac{n^2}{4} - \frac{n}{48}$, $Cov(Z_c, Z_{c'}) = -\frac{n}{48}$, and $Cor(Z_c, Z_{c'}) = -\frac{4}{9}$. These results specify the joint distribution possessed (asymptotically, with n) by Z_1, Z_2, and Z_3.

The expected value of the maximum of m jointly-normal equicorrelated random variables, with mean μ, standard deviation σ, and common correlation ρ, can be written as

$$\mu + \sigma(1-\rho)^{1/2} \cdot \mathrm{Norm}_{max}(m) ,$$

where, as before, $\mathrm{Norm}_{max}(m)$ is the expected value of the maximum of m independent standard normal random variables. (This result is developed in [1].) In our case, this implies that

$$E_{elected}(3,n) \simeq \frac{n}{2} + \sqrt{\frac{3n}{64}} \cdot \left(1+\frac{4}{9}\right)^{1/2} \cdot \mathrm{Norm}_{max}(3) = \frac{n}{2} + \frac{\sqrt{39n}}{24} \cdot \mathrm{Norm}_{max}(3) .$$

Also, as previously stated, it follows from the Central Limit Theorem that the expected social utility of the socially-best candidate is

$$E_{optimal}(3,n) \approx \frac{n}{2} + \frac{\sqrt{3n}}{6} \cdot \mathrm{Norm}_{max}(3) .$$

From the preceding two expressions we finally obtain the sought-after result. The effectiveness of the system under consideration (that is, the most effective three-candidate average-dependent voting system) is $\frac{\sqrt{13}}{4} = 90.14\%$. This new system therefore provides some increase in effectiveness (slightly more than 2.5%) over the three-candidate approval voting system, and is dramatically superior to the standard system. Unfortunately, broad implementation of this system is not practical, since it would require allowing twelve different types of voter responses. However, in limited situations, these results might justify adoption.

Midpoint-optimal voting systems. The family of average-dependent voting systems does not yield readily to analysis for elections involving more than three candidates. However, there is another naturally-defined family of systems which coincides with the average-dependent family in the three-candidate case. These "midpoint-optimal" voting systems form our next object of investigation.

Consider a voter facing a three-candidate election, to whom the candidates have utilities $u_1 \geq u_2 \geq u_3$. Under an average-dependent voting system, his optimal strategy (that is, his optimal choice of a weight set) will depend on whether u_2 is less or greater than $\bar{u} = \frac{u_1+u_2+u_3}{3}$. The relation between these quantities is the same as the relation between u_2 and $\frac{u_1+u_3}{2}$ (simply subtract $\frac{u_2}{3}$ from both, then multiply by $\frac{3}{2}$). Generalizing this, we define an m-candidate voting system to be midpoint-optimal if the optimal choice of a weight set, by a voter to whom the candidates have utilities $u_1 \geq \cdots \geq u_m$, depends solely on the relation between the average of the utilities of the m-2 centrally-ranked candidates, $\frac{u_2+\ldots+u_{m-1}}{m-2}$, and the average of the utilities of the most- and least-preferred candidates, $\frac{u_1+u_m}{2}$.

An explicit characterization of the midpoint-optimal voting systems can be given. Every m-candidate midpoint-optimal voting system is essentially equivalent to a voting system with two weight sets. These weight sets are of the form $w^{(1)} = (a_1,a_2,\ldots,a_{m-1},a_m)$ and $w^{(2)} = (a_1,a_2+c,\ldots,a_{m-1}+c,a_m)$, where the parameters satisfy the inequalities $a_1 \geq \cdots \geq a_m$ and $a_1 - a_2 \geq c \geq 0$. Observe that all weighted ranking voting systems are trivial instances of this result.

In order to verify this characterization, consider a particular m-candidate midpoint-optimal voting system. A voter to whom the candidates have utilities $u_1 \geq \cdots \geq u_m$ may face two strategically distinct situations, depending on whether $\bar{u}_{mid} = (u_2+\ldots+u_{m-1})/(m-2)$ is less than or greater than $\bar{u}_{ext} = (u_1+u_m)/2$. Let $w^{(1)}$ be the weight set selected by an optimal strategy in the former case (when $\bar{u}_{mid} < \bar{u}_{ext}$), and let $w^{(2)}$ be the weight set selected in the latter case. We lose no generality in assuming that the voting system under consideration consists of precisely these two weight sets.

Also without loss of generality, we assume the voting system under consideration to be 0-normalized, so that $w_m^{(1)} = w_m^{(2)} = 0$. (This corresponds to taking $a_m = 0$ in the characterization.) For any vector of utilities $u = (u_1,\ldots,u_m)$ and any weight m-vector w , let $E(u;w) = \Sigma\, w_c(u_c-\bar{u})$. If $u = (1,\frac{1}{2},\ldots,\frac{1}{2},0)$, then $\bar{u} = \bar{u}_{mid} = \bar{u}_{ext} = \frac{1}{2}$. Therefore, a voter to whom the candidates have these utilities will be indifferent in his choice of weight sets, and we must have $E(u;w^{(1)}) = E(u;w^{(2)})$. Equivalently, $w_1^{(1)}(1-\bar{u}) = w_1^{(2)}(1-\bar{u})$, so $w_1^{(1)} = w_1^{(2)}$. (This common value will be a_1 in the characterization.) If $m = 3$, this completes the verification.

Assume that $m \geq 4$. For any $1 \leq k \leq m-3$, and any small $\varepsilon > 0$, let

$u = (1, \frac{1}{2}+\varepsilon, \ldots, \frac{1}{2}+\varepsilon, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}-k\varepsilon, 0)$, where exactly k components of u are equal to $\frac{1}{2}+\varepsilon$. Then $\bar{u} = \bar{u}_{mid} = \bar{u}_{ext} = \frac{1}{2}$. It follows that $E(u;w^{(1)}) = E(u;w^{(2)})$; equivalently,

$$(w_2^{(2)}-w_2^{(1)})\varepsilon + \ldots + (w_{k+1}^{(2)}-w_{k+1}^{(1)})\varepsilon = (w_{m-1}^{(2)}-w_{m-1}^{(1)})(k\varepsilon) .$$

For $k = 1$, we obtain $w_2^{(2)}-w_2^{(1)} = w_{m-1}^{(2)}-w_{m-1}^{(1)}$. Continuing inductively (in k), we eventually obtain $w_k^{(2)}-w_k^{(1)} = w_{m-1}^{(2)}-w_{m-1}^{(1)}$ for every $2 \leq k \leq m-2$. This common value is the quantity c in the characterization.

A word is in order concerning the relation between $\bar{u}_{mid}$ and $\bar{u}_{ext}$. Consider m independent observations of a uniformly-distributed random variable (representing, of course, the utilities of m candidates to a random voter). Let u_{max} and u_{min} be the largest and smallest observations, and let the transformation $u \mapsto (u-u_{min})/(u_{max}-u_{min})$ be applied to the other $m-2$ observations. It is well-known that the $m-2$ transformed observations are independent and uniformly-distributed on the unit interval. As a particular application of this fact, the distribution of the average of the central $m-2$ observations with respect to $(u_{max}+u_{min})/2$ is linearly related to the distribution of the average of $m-2$ independent random variables, uniformly-distributed on the unit interval, with respect to $\frac{1}{2}$. Hence, concern about the relation between $\bar{u}_{mid}$ and $\bar{u}_{ext}$ arises naturally, rather than solely as an artifact. It is for this reason that we work in terms of the order relation between $\bar{u}_{mid}$ and $\bar{u}_{ext}$, rather than the equivalent relation between $\bar{u}$ and $\bar{u}_{ext}$.

<u>The most effective four-candidate midpoint-optimal voting system</u>. The families of midpoint-optimal and average-dependent voting systems coincide for

three-candidate elections. Therefore, the most effective three-candidate midpoint-optimal system is the one with weight sets $(\frac{3}{4},\frac{5}{8},\frac{1}{4})$ and $(\frac{3}{4},\frac{3}{8},\frac{1}{4})$ (or, equivalently, $(4,3,0)$ and $(4,1,0)$), with effectiveness 90.14% . Consider an m-candidate midpoint-optimal voting system. Any voter to whom the candidates have utilities $u_1,\ldots,u_m$ can be classified according to the ordering of the candidates on his preference scale, and according to whether $\bar{u}_{mid}$ is less or greater than $\bar{u}_{ext}$. There are $2m!$ classes of voters, and a voter is equally likely to fall into any one of these classes.

We can follow an approach similar to that taken in our discussion of the average-dependent systems. Eventually, we will obtain a condition sufficient for a voting system to be the most effective of all m-candidate midpoint-optimal voting systems. This condition is that the number of votes received by a candidate from a random society of any given profile is equal to the expected social utility of the candidate to that society. Consider a random voter with utilities $u_1 \geq \cdots \geq u_m$. For the condition to be satisfied by the midpoint-optimal voting system with weight sets $w^{(1)}$ and $w^{(2)}$, we must have $w_k^{(1)} = E(u_k|\bar{u}_{mid} < \bar{u}_{ext})$ and $w_k^{(2)} = E(u_k|\bar{u}_{mid} > \bar{u}_{ext})$ for all k . This yields $w_1^{(1)} = w_1^{(2)} = \frac{m}{m+1}$, and $w_m^{(1)} = w_m^{(2)} = \frac{1}{m+1}$. For the system to actually be midpoint-optimal, we must of course have $w_k^{(1)}-w_k^{(2)}$ constant for all $2 \leq k \leq m-1$.

The sequence $\{w_k^{(1)}-w_{k+1}^{(1)}\}$ is strictly decreasing in k , and the sequence $\{w_k^{(2)}-w_{k+1}^{(2)}\}$ is strictly increasing. Assume that $m \geq 5$. Then $w_2^{(1)}-w_3^{(1)} > w_3^{(1)}-w_4^{(1)}$, and $w_2^{(2)}-w_3^{(2)} < w_3^{(2)}-w_4^{(2)}$. Subtracting the second inequality from the first, we obtain

$$(w_2^{(1)}-w_2^{(2)}) - (w_3^{(1)}-w_3^{(2)}) > (w_3^{(1)}-w_3^{(2)}) - (w_4^{(1)}-w_4^{(2)}) \ .$$

But, as just noted, if the system under consideration is midpoint-optimal, both sides of this inequality must be zero. This contradiction shows that our approach fails to isolate the most effective midpoint-optimal voting systems for elections involving more than four candidates.

However, in the four-candidate case we find success. We obtain weight vectors $w^{(1)} = (\frac{8}{10},\frac{5}{10},\frac{3}{10},\frac{2}{10})$ and $w^{(2)} = (\frac{8}{10},\frac{7}{10},\frac{5}{10},\frac{2}{10})$, which are indeed the weight sets of a midpoint-optimal voting system. Hence, these weight sets determine the most effective of all four-candidate midpoint-optimal voting systems. (An essentially equivalent voting system, without fractional weights, has weight sets $(6,3,1,0)$ and $(6,5,3,0)$.)

The effectiveness of the best four-candidate midpoint-optimal voting system. Consider the voting system with weight sets $(\frac{8}{10},\frac{5}{10},\frac{3}{10},\frac{2}{10})$ and $(\frac{8}{10},\frac{7}{10},\frac{5}{10},\frac{2}{10})$. For this system, the expected social utility of the candidate elected in an n-voter random society, $E_{\text{elected}}(4,n)$, is equal to the expected number of votes received by the winning candidate. If Z_c is the random variable representing the number of votes received by candidate c , then $E_{\text{elected}}(4,n) = E(\max_c Z_c)$.

For any fixed society, the voters can be partitioned into $2\cdot 4! = 48$ classes, according to their preference orderings and according to the relation between $\bar{u}_{\text{mid}}$ and $\bar{u}_{\text{ext}}$ for each. Let $n(\pi;1)$ and $n(\pi;2)$ be the number of voters with ordering π for whom $\bar{u}_{\text{mid}}$ is respectively less or greater than $\bar{u}_{\text{ext}}$. Then $\{n(\pi;k)\}$ represents a profile of the society. In an n-voter random society, the 48 quantities $n(\pi;k)$ are random variables with a joint multinomial distribution (summing to n , with equal cell probabilities). Each Z_c is a linear combination of these random variables, and consequently

(Z_1, Z_2, Z_3, Z_4) has, asymptotically in n, a multivariate normal distribution.

For any class $(\pi;k)$, $E(n(\pi;k)) = \frac{n}{48}$ and $E([n(\pi;k)]^2) = \frac{n(n-1)}{(48)^2} + \frac{n}{48}$. If $(\pi;k) \neq (\pi';k')$, then $E(n(\pi;k) \cdot n(\pi';k')) = \frac{n(n-1)}{(48)^2}$. Then, for any candidate c, it can be verified by extensive computation that $E(Z_c) = \frac{n}{2}$ and $E(Z_c^2) = \frac{n^2}{4} + \frac{11n}{200}$, so $\mathrm{Var}(Z_c) = \frac{11n}{200}$. Furthermore, if $c \neq c'$, $E(Z_c Z_{c'}) = \frac{n^2}{4} - \frac{3n}{200}$, so $\mathrm{Cov}(Z_c, Z_{c'}) = -\frac{3n}{200}$ and $\mathrm{Cor}(Z_c, Z_{c'}) = -\frac{3}{11}$.

Continuing along the lines of our treatment of the best three-candidate average-dependent (and midpoint-optimal) voting system, we find that the expected value of the maximum of the random variables Z_1, Z_2, Z_3, Z_4, which have a joint (asymptotic) normal distribution with common correlation $-\frac{3}{11}$, is

$$E_{\text{elected}}(4,n) \approx \frac{n}{2} + \sqrt{\frac{11n}{200}} \cdot \left(1+\frac{3}{11}\right)^{1/2} \cdot \mathrm{Norm}_{\max}(4) = \frac{n}{2} + \frac{\sqrt{7n}}{10} \cdot \mathrm{Norm}_{\max}(4) .$$

Combining this result with the asymptotic expression for $E_{\text{optimal}}(4,n)$, we obtain the desired result. <u>The effectiveness of the system under consideration (that is, the most effective four-candidate midpoint-optimal voting system) is</u> $\frac{\sqrt{21}}{5} = 91.65\%$. The effectiveness of the four-candidate Borda system, which is also midpoint-optimal, is $\sqrt{\frac{4}{5}} = 89.44\%$. Hence, this system provides a gain in effectiveness of more than 2% over the best system previously studied. (We do not currently know the effectiveness of the four-candidate approval voting system.)

<u>Summary</u>. In the preceding sections, we twice followed a paradigm for determining the effectiveness of certain voting systems. First, a family of voting systems is selected for study. The systems in the family are similar

in the sense that they induce the same partition of society into strategic classes. Next, for each voting system, the expected utility of a candidate to a typical voter in a particular class is compared with the number of votes which would be cast for the candidate by any voter in that class. If these two quantities are equal for each of the strategic classes, then the voting system under consideration is the best in its family; a voting system for which these equalities hold is said to be reproducing. For such a system, the expected social utility of the candidate elected by a random society is equal to the expected number of votes received by the winning candidate. But the number of votes received by any candidate depends linearly on the number of voters in the various classes. In consequence, straightforward computations permit the effectiveness of a reproducing voting system to be explicitly evaluated. A more extensive treatment of reproducing voting systems appears in [4].

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REPRODUCING VOTING SYSTEMS*

by

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Abstract. Voting systems can be compared in terms of their effectiveness in representing the preferences of the voters. Several generalizations of the standard voting system (in which each voter casts a single vote) are considered, as is the Borda system (in which the candidates receive votes proportional to the ranks assigned them by the voters). For elections involving three or more candidates, it is found that the Borda voting system is markedly more effective than the standard system. For many-candidate elections, the Borda system is essentially as effective as any voting system can be.

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REPRODUCING VOTING SYSTEMS

Introduction

Societies frequently face the need to choose among various alternative policies. The democratic approach to such a choice is to put the decision to a vote. But what should be the format of the voting? Should each voter be asked simply to indicate his most-preferred alternative, or should more detailed information be elicited? And how should this information be used in making the final choice?

An approach to these questions was first presented in [3]. An index of effectiveness of a voting scheme was defined, which measures the extent to which the scheme leads to selections which represent well the preferences of the voters. In that paper it was shown that the standard voting system, in which each voter indicates his most-preferred alternative and the plurality winner is selected, is only moderately effective when a choice among three alternatives is involved. Furthermore, the standard system becomes progressively less effective as the number of alternatives is increased.
In this paper we develop several general classes of voting systems, and show that the Borda system, which is defined below, is asymptotically as effective as any voting system can possibly be in choosing among a large number of alternatives.

Voting Systems

We shall investigate one-stage elections, in which the voters cast varying numbers of votes for the candidates, with the candidate who receives the greatest number of votes being declared the winner. Formally, an m-candidate *voting system* is a collection of (one or several) sets of m weights. Each voter independently selects one of the weight sets, and assigns the weights to the candidates. The candidate assigned the greatest total weight is the winner of the election.

The commonly-used *standard voting system* consists of the single weight set $(1, 0, \ldots, 0)$; each voter casts a vote for one candidate, and the plurality winner is elected. Generally, a *weighted ranking voting system* is any voting system consisting of a single weight set $(w_1, \ldots, w_m)$. For any $1 \leq k \leq m$, the *vote-for-k voting system* arises when $w_1 = \ldots = w_k = 1$, $w_{k+1} = \ldots = w_m = 0$; observe that the standard voting system is simply the case $k = 1$. The *Borda voting system*, proposed by Jean-Charles de Borda in 1781 [1], is the weighted ranking voting system with weight set $(m, m-1, \ldots, 1)$. Under this system each voter ranks the candidates, and they receive weights proportional to their rankings.

The *approval voting system*, which has attracted recent interest [3], consists of the $m-1$ weight sets $(1, 0, \ldots, 0)$, $(1, 1, 0, \ldots, 0)$, $\ldots$, $(1, \ldots, 1, 0)$. Under this system, a voter may cast one vote for every candidate of whom he "approves." We will not study this system directly, but will consider the *vote-for-or-against-k voting system* with weight sets $(w_1, \ldots, w_m)$ and $(w'_1, \ldots, w'_m)$, where $w_1 = \ldots = w_k = 1 = w'_1 = \ldots = w'_{m-k}$ and $w_{k+1} = \ldots = w_m = 0 = w'_{m-k+1} = \ldots = w'_m$. The three-candidate approval voting system is of this form, when $k = 1$. The vote-for-or-against-k system is essentially equivalent to the system with weight sets $(1, \ldots, 1, 0, \ldots, 0)$

and $(0, \ldots, 0, -1, \ldots, -1)$; both of the indicated sets have k nonzero components. (For a formal treatment of the equivalence of voting systems, see [4].)

The Effectiveness of a Voting System

In studying the relative merits of various m-candidate voting systems, we wish to take into account all possible combinations of individual preferences for the candidates. To this end, we define an n-voter *random society*. In such a society, the utilities of the candidates to any given voter are independent, identically uniformly-distributed random variables. The utilities to distinct voters are independent, and may come from distinct uniform distributions. (A discussion motivating this use of the uniform distribution in our work is presented in the appendix of this paper.) As the nm random utilities vary over their ranges, all conceivable preference profiles of the society are generated. When a voter is called upon to vote, he bases his action on the specific values assumed by his utilities, without knowledge of the values assumed by the utilities of the other voters.

Consider a particular voter to whom the candidates have utilities $u_1, \ldots, u_m$, and let $\bar{u} = \sum u_c/m$. Under any given voting system, an *optimal strategy* for this voter is to assign weights $w_1, \ldots, w_m$ to the candidates so as to maximize $\sum w_c(u_c - \bar{u})$. The election can be thought of as an n-person game, in which the payoff to any player is his utility for the elected candidate. It was shown in [3] that (asymptotically, for large values of n) the unique symmetric equilibrium point of this voting game arises when the players follow the optimal strategies defined above. (Indeed, as long as the other voters' strategies induce a probability distribution over vote totals which is symmetric with respect to the candidates, a voter's optimal strategy is, asymptotically, his best reply to their actions.)

Under the vote-for-k voting system, the optimal strategy of a voter is clear: vote for the k candidates of greatest utility. Under the Borda system, the optimal strategy for a voter is to assign the weights to accord monotonically with his utilities. The optimal strategy under the vote-for-or-against-k voting system is not quite so obvious. A voter should vote for his n-k most-favored candidates (rather than the k most-favored) if the utilities of the n-$2k$ centrally-ranked candidates average to at least $\bar{u}$. We shall return to these results later in this paper.

Assume that all n voters follow their optimal strategies. We wish to assess the effectiveness of a given voting system in yielding an outcome representative of the voters' preferences. Select a fixed voter k , and let $E^{(k)}_{elected}(m,n)$ be the expected utility of the elected candidate to this voter. (The expectation is taken over the nm random utilities which determine the preferences, and therefore the actions, of the voters.)

Define $E^{(k)}_{maximal}(m,n)$ to be the expected value of the maximum of m quantities, each the average of n independent random variables drawn from voter k's (uniform) utility distribution. This is an upper bound for $E^{(k)}_{elected}(m,n)$. In order to verify this, assume that all n voters' utilities are drawn from the same uniform distribution. (No generality is lost through this assumption, since the following argument applies in the general case after taking appropriate affine transformations.) Any particular realization of the random society corresponds to a collection of n utility m-vectors. Given this collection, no matter how these vectors are assigned to the voters, the use of a particular voting system will always lead to the election of the same candidate. Since voter k would be equally likely to be assigned any one of these vectors, he would most prefer a voting system which selects the candidate whose average utility, over the n vectors, is greatest. $E^{(k)}_{maximal}(m,n)$ is the expected utility of the elected candidate to this voter, under a hypothetical voting system which makes such a selection in every possible situation.

Unfortunately, there is no such voting system. For example, consider a two-candidate election. There is a realization of the random society corresponding to a collection of utility vectors, such that slightly more than half of the vectors show a slight preference for one candidate while the remaining vectors show a strong preference for the other. Voter k would most prefer a voting system which selects the minority choice in this case. However, no actual voting system can have this feature. Hence, the theoretical upper bound on $E^{(k)}_{elected}$ cannot be attained.

It should be noted that $E^{(k)}_{maximal}(m,n)$ is not only a bound for voter k's expectation from one-stage elections. If some other kind of choice system, such as a multistage election, is considered, the same bound will apply.

Let $E^{(k)}_{random}(m)$ be the expected utility to voter k of a randomly selected candidate. Clearly, we would hope that any reasonable voting system would out-perform random selection. We define the *effectiveness* of an m-candidate voting system to be:

$$\lim_{n\to\infty} \frac{E^{(k)}_{elected}(m,n) - E^{(k)}_{random}(m)}{E^{(k)}_{maximal}(m,n) - E^{(k)}_{random}(m)} ;$$

this is the relative advantage of the given voting system over random choice, normalized by an upper bound on that advantage.

It should be observed that this definition is independent of the specific uniform distribution from which the utilities of voter k are drawn, and hence is independent of k. Therefore, in the sequel we will assume, without loss of generality, that every random society contains a voter whose utilities are uniformly distributed on the unit interval. We will further assume that the various terms used to compute effectiveness are derived from the utilities of this *standard voter*. In this case, $E_{random}(m) = 1/2$. Furthermore, from the Central Limit Theorem we can conclude that

$$E_{maximal}(m,n) \approx \frac{1}{2} + \sqrt{\frac{1}{12n}} \cdot Norm_{max}(m) ,$$

where $Norm_{max}(m)$ is the expected value of the maximum of m independent normally-distributed random variables with mean 0 and variance 1.

The effectiveness of the m-candidate standard voting system was shown in [3] to be $\sqrt{3m}/(m+1)$. Since all two-candidate voting systems are essentially equivalent, this implies that no two-candidate voting system can be more than $\sqrt{2/3} = 81.65\%$ effective. For three-candidate elections, the standard voting system is only 75% effective; as m increases, the effectiveness of the standard system decreases to zero. By numerical methods, it was shown in [3] that for three-candidate elections the Borda system is 86.6% effective, and the approval system is 87.5% effective. We shall obtain these results as special cases of our work in this paper.

We have viewed effectiveness through the eyes of a single voter. A mathematically equivalent approach considers the utilities of the voters as being drawn from a common distribution, and defines the social utility of a candidate as the sum of his utilities to the n voters. $E_{elected}$, $E_{maximal}$, and E_{random} are expected social utilities, which are obtained from the expectations defined above after multiplication by n. Hence, the same measure of effectiveness is obtained through this alternative approach. This approach is developed and discussed in [3] and [4]; the individualistic approach presented here is an outgrowth of a conversation with David Schmeidler.

Reproducing Voting Systems

In a random society choosing among m candidates, each voter's utility m-vector is drawn uniformly from an m-dimensional hypercube. Consider a given voting system. It induces a partition of each voter's utility m-cube into a number of regions, each corresponding to those situations in which the voter casts a particular m-tuple of votes. These partitions are geometrically similar for all voters. Each region, or *strategic class*, has associated with it a probability, which is the likelihood that the voter's utility m-vector lies in this region; the probability is (due to the uniformity assumption) the relative volume of the region within the utility m-cube. Under the Borda voting system there are $m!$ equiprobable strategic classes, one for each ordering of the candidates. The vote-for-k system yields $\binom{m}{k}$ classes; the vote-for-or-against-k system, $2\binom{m}{k}$ classes; the approval voting system, $2^m - 2$ classes.

Let $\{1, 2, \ldots, T\}$ index the strategic classes. A voter is of *strategic type* t if his utility vector is in class t. Let $\{p_t\}$ be the probabilities associated with the strategic classes. Let $v_t(c)$ be the votes cast for candidate c by a voter of type t, and let $u_t(c)$ be the conditional utility of candidate c to the standard voter, given that he is of type t. For example, under the Borda system consider the strategic class composed of vectors $u = (u_1, \ldots, u_m)$ for which $u_1 \leq \ldots \leq u_m$. If this is class t, then $v_t(c) = c$ and $u_t(c) = c/(m+1)$. Under the standard voting system consider the class t of voters u for which $u_1 \geq u_2, \ldots, u_m$. In this case, $v_t(1) = 1$ and $u_t(1) = m/(m+1)$; for $c > 1$, $v_t(c) = 0$ and $u_t(c) = m/[2(m+1)]$.

Any particular collection of preferences for the n voters in a society will partition the voters among the strategic types. A *profile* $\{k_t\} = (k_1, \ldots, k_T)$ is an enumeration of the voters of each type; naturally, $\sum k_t = n$. In an n-voter random society, the T components of the profile are random variables, having a joint multinomial distribution with cell probabilities $\{p_t\}$. Let $v(c; \{k_t\}) = \sum k_t v_t(c)$ be the number of votes cast for candidate c by a society with profile $\{k_t\}$. The candidate $\overline{c}(\{k_t\})$ who maximizes $v(c; \{k_t\})$ is the candidate who will be elected by a society with this profile.

Given that a random society has profile $\{k_t\}$, the expected utility of candidate c to the standard voter is

$$\begin{aligned} u(c; \{k_t\}) &= \sum_t Pr(\text{standard voter is of type } t \,|\, \\ &\qquad\qquad \text{society has profile } \{k_t\}) \cdot u_t(c) \\ &= \sum \frac{k_t}{n} u_t(c) \; . \end{aligned}$$

Therefore, the expected utility of the elected candidate to the standard voter, when the voting system under consideration is used, is

$$\begin{aligned} E_{elected}(m,n) &= E_{\{k_t\}}(u(\overline{c}(\{k_t\}); \{k_t\})) \\ &= \sum_{\{k_t\}} \binom{n}{k_1, \ldots, k_T} p_1^{k_1} \cdots p_T^{k_T} u(\overline{c}(\{k_t\}); \{k_t\}) \; . \end{aligned}$$

A number of voting systems may induce the same strategic partition. For example, any weighted voting system which employs m distinct weights induces the same partition as the Borda system. Consider a family of voting systems, all of which lead to the same strategic classes. If we vary our consideration among the systems in this family, then only the function $\bar{c}$ will change in the expression for $E_{elected}(m,n)$. Therefore, no voting system in the family can be more effective than one for which $u(\bar{c}(\{k_t\}); \{k_t\}) = max_c\, u(c; \{k_t\})$ for every profile $\{k_t\}$.

Not every family of voting systems contains such a member. (The family of m-candidate average-dependent voting systems, for any $m \geq 4$, provides an example; see [4].) However, assume that there is an $a > 0$ and a collection of constants $\{b_t\}$, such that the voting system defined for all t and c by $v_t(c) = a \cdot u_t(c) + b_t$ is in the family; a voting system of this form is said to be *reproducing*. Under a reproducing voting system, the candidate c who maximizes $v(c; \{k_t\})$ for any given profile $\{k_t\}$ also maximizes $u(c; \{k_t\})$. Consequently, if a family of voting systems contains a reproducing member, that voting system must be the most effective in its family; for a reproducing voting system,

$$E_{elected}(m,n) = E_{\{k_t\}}(\max_c\, u(c; \{k_t\}))$$

$$= E_{\{k_t\}}\left(\max_c\, \sum \frac{k_t}{n} u_t(c)\right) .$$

In succeeding sections, we shall see that the vote-for-k, vote-for-or-against-k, and Borda voting systems are all reproducing, and we will determine their effectiveness. (The term "reproducing" suggests how a voting system can be tested for this property. One first determines the strategic partition induced by the system, and then computes the expectations $\{u_t(c)\}$. If an affine transformation of the expectations reproduces the original voting weights, then the system is indeed reproducing.)

The Effectiveness of a Reproducing Voting System

Consider a fixed reproducing voting system, which induces a strategic partition with parameters $\{u_t(c)\}$ and $\{p_t\}$. In order to determine the effectiveness of this voting system, it will suffice to obtain an asymptotic (in n) expression for $E_{elected}(m,n)$.

For each candidate c, let $Z_c = \frac{1}{n}\sum k_t u_t(c)$. It has been previously observed that the random variables $\{k_t\}$ have a joint multinomial distribution, summing to n with cell probabilities $\{p_t\}$. Hence, for any class t, $E(k_t) = np_t$ and $Var(k_t) = np_t(1-p_t)$. For $t \neq t'$, $Cov(k_t, k_{t'}) = -np_t p_{t'}$. It is well-known that jointly-multinomial random variables have an asymptotic joint normal distribution. Therefore, since each Z_c is a linear combination of asymptotically normal random variables, it is itself asymptotically normal.

For any candidate c,

$$\begin{aligned} E(Z_c) &= \frac{1}{n}\sum u_t(c)E(k_t) = \sum u_t(c)p_t \\ &= E(\text{utility of candidate } c \text{ to the standard voter}) \\ &= \frac{1}{2}\ . \end{aligned}$$

Also

$$\begin{aligned} Var(Z_c) &= \frac{1}{n^2}\Big[\sum_t (u_t(c))^2 Var(k_t) + \sum_{t \neq t'} u_t(c)u_{t'}(c)Cov(k_t,\ k_{t'})\Big] \\ &= \frac{1}{n}\sum_t (u_t(c))^2 p_t - \frac{1}{n}\sum_{t,t'} u_t(c)u_{t'}(c)p_t p_{t'}\ , \end{aligned}$$

and for $c \neq d$,

$$Cov(Z_c,\ Z_d) = \frac{1}{n^2}[\sum_t u_t(c)u_t(d)Var(k_t) + \sum_{t \neq t'} u_t(c)u_{t'}(d)Cov(k_t,\ k_{t'})]$$

$$= \frac{1}{n}\sum_t u_t(c)u_t(d)p_t - \frac{1}{n}\sum_{t,t'} u_t(c)u_{t'}(d)p_t p_{t'}\ .$$

The latter terms in the final expressions for both $Var(Z_c)$ and $Cov(Z_c,\ Z_d)$ can be factored:

$$\sum_{t,t'} u_t(c)u_{t'}(c)p_t p_{t'} = \sum_{t,t'} u_t(c)u_{t'}(d)p_t p_{t'}$$

$$= (\sum_t u_t(c)p_t)^2 = (\sum_t u_t(c)p_t)(\sum_t u_t(d)p_t) = \frac{1}{4}\ .$$

Furthermore, since the voting strategies available to the voters are symmetric with respect to the candidates, both $Var(Z_c)$ and $Cov(Z_c,\ Z_d)$ are independent of the choice of c and d .

Consider a family of m equicorrelated normally-distributed random variables, with common mean μ and variance σ^2 , and with correlation ρ . It was shown in [2] that the expected value of the maximum of the m variables is $\mu + \sigma(1-\rho)^{1/2}Norm_{max}(m)$. Therefore, using the preceding results we obtain the asymptotic expression

$$E_{elected}(m,n) = E_{\{k_t\}}(\max_c Z_c)$$

$$\approx \frac{1}{2} + \{Var(Z_c) - Cov(Z_c,\ Z_d)\}^{1/2}Norm_{max}(m)$$

$$= \frac{1}{2} + \{\frac{1}{n}\sum_t p_t u_t(c)[u_t(c) - u_t(d)]\}^{1/2}Norm_{max}(m)\ ,$$

where the last two expressions are both independent of the choice of $c \neq d$. Combining this with the previously-given expressions for $E_{random}(m)$ and $E_{maximal}(m,n)$, we can finally determine the effectiveness of the voting system.

Theorem 1. The effectiveness of a reproducing voting system, which induces a strategic partition with parameters $\{u_t(c)\}$ and $\{p_t\}$, is $\{12\sum_t p_t u_t(c)[u_t(c) - u_t(d)]\}^{1/2}$, where $c \neq d$ can be chosen arbitrarily.

The Vote-for-k Voting System

Using the preceeding theorem, it is relatively simple to evaluate the effectiveness of the m-candidate vote-for-k voting system. There are $T = \binom{m}{k}$ equiprobable strategic types under this voting system, each corresponding to a different cardinality-k collection S_t of candidates. Hence, each $p_t = 1/\binom{m}{k}$, and for the standard voter

$$u_t(c) = \begin{cases} \frac{1}{k}\left(\frac{m}{m+1} + \ldots + \frac{m-k+1}{m+1}\right) = \frac{2m-k+1}{2(m+1)} & c \in S_t \\ \frac{1}{m-k}\left(\frac{m-k}{m+1} + \ldots + \frac{1}{m+1}\right) = \frac{m-k+1}{2(m+1)} & c \notin S_t . \end{cases}$$

Consequently, if c and d are distinct candidates,

$$\sum_t p_t u_t(c)[u_t(c) - u_t(d)]$$

$$= \frac{1}{\binom{m}{k}}\left(\sum_{c,d\in S_t} 0 + \sum_{\substack{c\in S_t \\ d\notin S_t}} \frac{2m-k+1}{2(m+1)}\cdot\left(-\frac{m}{2(m+1)}\right) + \sum_{\substack{c\notin S_t \\ d\in S_t}} \frac{m-k+1}{2(m+1)}\cdot\left(\frac{m}{2(m+1)}\right) + \sum_{c,d\notin S_t} 0\right)$$

$$= \frac{1}{\binom{m}{k}}\left(\binom{m-2}{k-1}\cdot\frac{2m-k+1}{2(m+1)}\left(-\frac{m}{2(m+1)}\right) + \binom{m-2}{k-1}\frac{m-k+1}{2(m+1)}\cdot\left(\frac{m}{2(m+1)}\right)\right)$$

$$= \frac{mk(m-k)}{4(m-1)(m+1)^2} .$$

Therefore, the effectiveness of the m-candidate vote-for-k voting system is

$$\frac{1}{(m+1)}\left[\frac{3mk(m-k)}{m-1}\right]^{1/2} .$$

When $k = 1$, this reduces to the result already known for the standard voting system. The qualitative behavior of that system carries over for any fixed k : as m becomes large, the effectiveness of the vote-for-k system approaches zero.

It is interesting to observe that the vote-for-k and vote-for-$(n-k)$ voting systems are equally effective. The latter system is equivalent to the "vote-against-k" voting system. Hence, voting systems based on positive or on negative options appear equally desirable to the voters. (Of course, in *specific* elections the contrasted voting systems may lead to different results.)

For any value of m , the most effective vote-for-k voting systems occur when m is as near to $m/2$ as possible. The effectiveness of these "vote-for-half" systems is

$$\frac{\sqrt{3}}{2}\cdot\frac{m}{m+1}\cdot\left(\frac{m}{m-1}\right)^{1/2} \quad \text{if } m \text{ is even, and}$$

$$\frac{\sqrt{3}}{2}\cdot\left(\frac{m}{m+1}\right)^{1/2} \quad \text{if } m \text{ is odd.}$$

In either case, as m becomes large the vote-for-half voting systems approach $\sqrt{3}/2 = 86.6\%$ effectiveness. Hence, we see for the first time that there are voting systems which are dramatically more effective than the standard system for many-candidate elections.

The Vote-for-or-against-k Voting System

Consider an m-candidate election in which this system is employed, where $k < m/2$. Let S be any cardinality-k collection of candidates. There are two strategic types associated with S , corresponding to the strategies "vote for S " and "vote against S "; a voter is equally likely to be of either type. Let T_1 index the types of the first form, and T_2 the types of the second form. Then there are a total of $T = 2\binom{m}{k}$ strategic types, so each $p_t = \frac{1}{2\binom{m}{k}}$. The function $u_t(c)$ is tabulated below.

	$t \in T_1$: vote for S	$t \in T_2$: vote against S
m odd	$c \in S : a_1 = \frac{2m - \frac{3}{2}(k-1)}{2(m+1)}$	$c \notin S : a_2 = \frac{5m^2+2m+1-6k^2-2k}{8(m+1)(m-k)}$
	$c \notin S : b_1 = \frac{3m^2+6m-1+6k^2-6k-8mk}{8(m+1)(m-k)}$	$c \in S : b_2 = \frac{2+\frac{3}{2}(k-1)}{8(m+1)}$
m even	$c \in S : a_1 = \frac{2m - \frac{3}{2}(k-1)}{2(m+1)}$	$c \notin S : a_2 = \frac{5m^2+2m-6k^2-2k}{8(m+1)(m-k)}$
	$c \notin S : b_1 = \frac{3m^2+6m+6k^2-6k-8mk}{8(m+1)(m-k)}$	$c \in S : b_2 = \frac{2+\frac{3}{2}(k-1)}{2(m+1)}$

This table can be verified through geometric considerations. As an example, we derive the entries for m odd, $t \in T_1$. Consider the subset H of the unit m-cube consisting of vectors u for which $u_1 \geq \cdots \geq u_m$ and $(u_{k+1} + \ldots + u_{m-k})/(m-2k) \leq \bar{u}$. This region is a simplex, with extreme points $(0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $\ldots$, $(1, \ldots, 1, 0, 0, \ldots, 0)$, $(1, \ldots, 1, \frac{1}{2}, 0, \ldots, 0)$, $(1, \ldots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, 0)$, $\ldots$, $(1, \frac{1}{2}, \ldots, \frac{1}{2}, 0)$, $(1, \ldots, 1)$; each vector has one fewer zero components than its predecessor. Therefore, the centroid of H is $h = \frac{1}{(m+1)}\left(m,\ m-\frac{3}{2},\ \ldots,\ m-\frac{3}{2}\left(\frac{m-1}{2}\right),\ 1+\frac{1}{2}\left(\frac{m-3}{2}\right),\ \ldots,\ \frac{3}{2},\ 1\right)$.

The strategic class corresponding to $t \in T_1$, $S_t = \{1, 2, \ldots, k\}$ is the union of the $k!(m-k)!$ simplices obtained by independently permuting the first k, and the last $(m-k)$, components of the vectors in H. Hence $u_t(c)$ is the average of the first k components of h if $c \in S_t$, and is the average of the last $m-k$ components if $c \notin S_t$.

For any fixed m, let $w = a_1 - b_1 = a_2 - b_2$. (It is the fact that these differences are equal which ensures that the vote-for-or-against-k system is reproducing.) Then for any $c \neq d$,

$$\sum p_t u_t(c)[u_t(c) - u_t(d)] = \frac{1}{2\binom{m}{k}}\Big[\sum_{\substack{t\in T_1\\ c\in S_t\\ d\notin S_t}} a_1 w + \sum_{\substack{t\in T_1\\ c\notin S_t\\ d\in S_t}} b_1(-w) + \sum_{\substack{t\in T_1\\ c\notin S_t\\ d\in S_t}} a_2 w + \sum_{\substack{t\in T_1\\ c\in S_t\\ d\notin S_t}} b_2(-w)\Big]$$

$$= \frac{1}{2\binom{m}{k}}\left[\binom{m-2}{k-1}\cdot 2w^2\right]$$

$$= \frac{k(m-k)}{m(m-1)}w^2 .$$

Therefore, the effectiveness of the m-candidate vote-for-or-against-k voting system is

$$w\left[\frac{12k(m-k)}{m(m-1)}\right]^{1/2} = \begin{cases} \dfrac{5m^2+1-6mk}{4(m+1)}\left[\dfrac{3k}{m(m-1)(m-k)}\right]^{1/2} & \text{if } m \text{ is odd,}\\[2ex] \dfrac{5m^2-6mk}{4(m+1)}\left[\dfrac{3k}{m(m-1)(m-k)}\right]^{1/2} & \text{if } m \text{ is even.}\end{cases}$$

When m is even and $k = m/2$, the vote-for-or-against-k system coincides with the vote-for-k system. Our arguments can be modified to show that the formula just derived also holds for this case. Alternatively it can be checked that in this case the formula coincides with the formula for the effectiveness of the vote-for-half system.

For $k < m/2$, it is easily verified that the vote-for-or-against-k voting system is strictly more effective than the vote-for-k system. In particular, for three-candidate elections we see that the standard voting system (75% effective) is appreciably less effective than the approval voting system (87.5%).

What k yields the most effective vote-for-or-against-k voting system when m is large? Set $k = \alpha m$ and let $m \to \infty$. From either formula, odd or even, we obtain a limit of $\left\{\frac{5-6\alpha}{4}\right\}\left\{\frac{3\alpha}{1-\alpha}\right\}^{1/2}$. This is maximized when $\alpha = (9-\sqrt{21})/12 = .368$. Hence, the asymptotically-best of the vote-for-or-against-k systems arises when the voters are allowed to act for, or against, slightly more than a third of the candidates. The limiting effectiveness of such systems is $(42\sqrt{21}-138)^{1/2}/8 = 92.25\%$.

The Borda Voting System

The m-candidate Borda system differs from those previously discussed, in that a voter is called upon to differentiate among all m candidates (rather than to simply partition them into groups). Clearly, there are $m!$ strategic types under this system, each associated with a particular ordering of the candidates (say, in order of increasing utility). Hence each $p_t = 1/m!$ and, if candidate c has position $t(c)$ in ordering t, $u_t(c) = t(c)/(m+1)$.

For any $c \neq d$,

$$\sum p_t u_t(c)[u_t(c) - u_t(d)] = \frac{1}{m!}\cdot\frac{1}{(m+1)^2} \sum((t(c))^2 - t(c)t(d))$$

$$= \frac{1}{m!(m+1)^2}[(m-1)! \sum_{k=1}^{m} k^2 - (m-2)! \sum_{\substack{1\leq k,\ell\leq m \\ k\neq \ell}} k\ell]$$

$$= \frac{1}{m(m+1)^2}\cdot\frac{m(m+1)(2m+1)}{6} - \frac{1}{m(m-1)(m+1)^2}\left[\left(\frac{m(m+1)}{2}\right)^2 - \frac{m(m+1)(2m+1)}{6}\right]$$

$$= \frac{m}{12(m+1)} .$$

Therefore, the effectiveness of the m-candidate Borda voting system is

$$\left[\frac{m}{m+1}\right]^{1/2} .$$

It is important to observe that, as m becomes large, this formula approaches one. This permits two conclusions. For elections involving many candidates, the Borda system is essentially as good as any other voting system. Moreover, the Borda system is asymptotically as effective as any other type of social choice system could be, even if the voters' utilities could be directly observed and a candidate chosen accordingly.

Summary

We have presented a method for measuring the effectiveness of a voting system in representing the preferences of the voters. The measure is based on the performance of the system over all possible profiles of voter preferences, and can be interpreted either in terms of the expectations of individual voters, or in terms of a social utility function.

Using this measure, the effectiveness of the standard, vote-for-k, vote-for-or-against-k, and Borda voting systems was determined. The standard system is relatively ineffective for elections involving three or more candidates, and becomes totally ineffective when the number of candidates becomes large. In contrast, the Borda system is asymptotically (for elections involving many candidates) as effective as any system can be.

Two related questions are suggested by these results. What voting systems are highly effective for elections involving a small number of candidates? Are there other systems which asymptotically approach 100% effectiveness more quickly than the Borda system? Both of these questions will be addressed in subsequent papers.

The Effectiveness of Several Voting Systems

Number of Candidates	*Standard*	*Vote-for-half*	*Best Vote-for-or-against-k*	*Borda*
2	81.65%	81.65%	81.65%	81.65%
3	75.00	75.00	87.50	86.60
4	69.28	80.00	80.83	89.44
5	64.55	79.06	86.96	91.29
6	60.61	81.32	86.25	92.58
⋮				
10	49.79	82.99	88.09	95.35
⋮				
$m \to \infty$	0.00	86.60	92.25	100.00

APPENDIX

Individual Utility Functions, and a Characterization of the Uniform Probability Distribution

We imagine the utilities of the candidates, to the voters in a random society, as arising in the following manner. There is a (large) set of potential candidates (or, equivalently, of issues). Each voter has a von Neumann-Morgenstern utility function over this set of candidates. Hence, the random selection of a candidate generates a probability distribution over each player's utility space. A number of candidates are selected independently, to compete in a given election. Consequently, the utilities of the candidates to a particular voter are independent, identically-distributed random variables.

Most traditional theories of social choice assume that an individual's actual utilities cannot be measured. However, by observing his choices among various gambles or lotteries, we can determine the class of affinely-equivalent utility functions which contains his particular function. For example, assume that the utilities of m candidates to a voter are $u_1 \leq u_2 \leq \dots \leq u_m$. Then, although the specific vector $(u_1, u_2, \dots, u_{m-1}, u_m)$ cannot be directly observed, the normalized representation $(0, (u_2 - u_1)/(u_m - u_1), \dots, (u_{m-1} - u_1)/(u_m - u_1), 1)$ *is* experimentally obtainable.

Assume that a voter's probability distribution over candidate utilities is known. If m candidates, with utilities $u_1 \leq \dots \leq u_m$ to this voter, stand for election, the quantities $(u_2 - u_1)/(u_m - u_1), \dots, (u_{m-1} - u_1)/(u_m - u_1)$ may be directly measured. It is conceivable that these quantities, in turn, yield information concerning the original utilities. (For example, if the voter's probability distribution over utilities is concentrated at three points, the observation of a normalized utility strictly between 0 and 1 permits the identification of all of the actual utilities.) Generally, the normalized utilities will convey information about the location and scale of the actual utilities whenever the normalized utilities depend on u_1 and $u_m - u_1$ (or equivalently, on u_1 and u_m). When this is the case, we are led away from the realm of standard work on social choice, and into a much more complicated world involving such considerations as intertemporal or interpersonal comparison of preference intensities. Therefore, it is of interest to ask for conditions under which no location or scale information can be obtained from the normalized utilities.

Theorem 2. Let n independent observations be drawn from the real line according to the atomless probability distribution F , and order the observations so that $x_1 \leq \ldots \leq x_n$. The random variables x_1 and x_n are independent of the statistics $(x_2 - x_1)/(x_n - x_1), \ldots, (x_{n-1} - x_1)/(x_n - x_1)$ if and only if F is a uniform distribution.

Proof. Let $t = (x_2 - x_1)/(x_n - x_1)$, and assume that t is independent of x_1 and x_n . The support of F must be connected; otherwise the conditional distribution of t would have a connected support for some values of (x_1, x_n) , but not for other values.

Assume that $(x_1, x_n) = (A,B)$. The conditional distribution of t is then

$$H(t) = 1 - \left[1 - \frac{F(tB + (1-t)A) - F(A)}{F(B) - F(A)}\right]^{n-2} .$$

This must (by assumption) be independent of A and B . Hence

$$\frac{F(tB+(1-t)A) - F(A)}{F(B) - F(A)} = K(t) ,$$

and so

$$F(tB+(1-t)A) = (1 - K(t))F(A) + K(t)F(B) .$$

Let $A' = A + t\delta$ and $B' = B - (1-t)\delta$, where $0 \leq \delta < B-A$. Then $tB' + (1-t)A' = tB + (1-t)A$, and therefore

$$(1 - K(t))F(A+t\delta) + K(t)F(B-(1-t)\delta) = (1 - K(t))F(A) + K(t)F(B) ;$$

equivalently, for $0 \leq t < 1$,

$$F(A+t\delta) - F(A) = \frac{K(t)}{1 - K(t)}[F(B) - F(B-(1-t)\delta)] .$$

The right-hand side of the preceeding equation is independent of A ; perforce, the left-hand side is also independent of A . Therefore, by appropriately varying A and B we can define a function

$$g(p) = F(A+p) - F(A) ,$$

for all non-negative p which are no greater than the length of the support of F. Since

$$[F((A-p)+p) - F(A-p)] + [F(A+q) - F(A)] = F((A-p)+p+q) - F(A-p) ,$$

it follows that

$$g(p) + g(q) = g(p+q) ,$$

throughout the domain of g . F , and therefore g , are strictly increasing. The only additive, increasing functions are linear; hence there is a $C > 0$ such that $g(p) = Cp$. Consequently $F(x) = Cx + D$ throughout the support of F , and F is uniform, as claimed.

In order to verify the converse in the theorem, it suffices to note that, for any uniform distribution F, the n-2 random variables $t_2 = (x_2 - x_1)/(x_n - x_1)$, ..., $t_{n-1} = (x_{n-1} - x_1)/(x_n - x_1)$ have a joint distribution which is uniform over the region $0 \leq t_2 \leq \cdots \leq t_{n-1} \leq 1$. Clearly, this joint distribution is independent of x_1 and x_n. $\square$

The non-atomicity of F in the preceeding theorem corresponds to indifference between candidates being a probability zero event. Assume, to the contrary, that F has at least one atom. If F is concentrated at only one or two points (that is, if the individual under discussion sees the world in at most two colors), then the earlier discussion has little relevance; all observed preferences will be known to be of equal intensity. However, assume that there are at least three points in the support of F. Then the conditional distribution of $t = (x_2 - x_1)/(x_n - x_1)$, given x_1 and x_n, will depend upon the distribution of the atoms of F on the interval between x_1 and x_n, and this will depend on the specific values of x_1 and x_n.

In consequence of these arguments, the reason for the use of the uniform distribution in the formulation of a random society is clear. The use of any other distribution would imply that origin and scale information could be obtained from an individual's normalized utilities. And the availability of such information would facilitate the making of intensity-of-preference comparisons (from one election to the next, or even *between* individuals) which we naturally wish to avoid in social choice procedures.

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