COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 476

GAME THEORY MODELS AND METHODS IN POLITICAL ECONOMY

Martin Shubik

GAME THEORY MODELS AND METHODS IN POLITICAL ECONOMY

GAME THEORY MODELS AND METHODS IN POLITICAL ECONOMY*+

by

Martin Shubik

SUMMARY

An overview of different models and solution concepts of game theory is given together with a sketch of the major areas of application to political economy, as well as with some indications of major open problems.

This piece is designed to complement the essay of Robert Aumann which is more directly concerned with the mathematical methods of game theory. Although many references are given the reader may find more detail and references in a series of Rand reports on Game Theory in Economics [1].

1. MODELLING METHODS

Perhaps the most important aspect of the theory of games as applied to political economy is that the methodology provided for constructing the mathematical models for the study of conflict and cooperation forces an explicitness found rarely even in the many mathematical investigations

---

*This work relates to Department of the Navy Contract N00014-77-C-0518 issued by the Office of Naval Research Under Contract Authority NR 277-239. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

To appear as a chapter in the Handbook of Mathematical Economics edited by Arrow and Intriligator.

+Much of this work is based upon portions of an unpublished manuscript by L. S. Shapley and M. Shubik.

of political economy. In particular the <u>extensive form</u> of a game calls for a complete process description.

A fully described game should be playable without too much difficulty by a group of students. If the game is well defined but difficult to play because of unreasonable demands on the time and data processing abilities of the individuals it well may be that it is not a good model of the economic process it purports to represent. Thus the discipline called for in building a well defined and playable game may enable the economist to formulate an operationally valuable critique of his initial model and to isolate factors which were not deemed to be important until the criterion of "playable" was applied.

There are three highly different representations of a game of strategy, the <u>extensive form</u>, the <u>strategic form</u> and the <u>cooperative form</u>. Each serves a different purpose, i.e. they are designed to answer different questions and hence utilize different descriptions of the phenomena being studied.

A game in extensive form can be used to specify the game in strategic form, and one in strategic form may be used to define the cooperative form, but the reverse does not hold true. There may be many different games in strategic form which give rise to the same cooperative form. It is best to think of the three forms as three independent formulations designed for different purposes, which when the occasion calls for it, can be related to each other.

Before we consider these forms we must note the assumptions made concerning preferences, utility and payoffs.

1.1. Preferences, Utility and Payoffs

In their book von Neumann and Morgenstern [2] presented axioms for the existence of a utility function defined up to a linear transformation, based upon considering gambles among a set of outcomes over which an individual has an ordering of preference. This utility measure was utilized by von Neumann and Morgenstern in their evaluation of the employment of mixed strategies in games in strategic form. Totally independent of this construction and its use was the assumption concerning the existence of some form of "U-money" or transferable utility which served to enable them to provide a particularly simple description of a game in cooperative form.

In many of the earlier critiques of the applicability of game theory to economics and other disciplines doubts were raised concerning the value of game theory to the behavioral sciences because of the two assumptions which were deemed to be highly unrealistic. As a greater understanding of the importance of decisionmaking under uncertainty and the strength of the various axiom systems [1] which lead to a utility measure has come about, so has the acceptance of the von Neumann-Morgenstern position.

Concerning cooperative games and transferable utility; as was noted by von Neumann and Morgenstern the assumption of transferable utility was a preliminary simplification made in order to start to open up analysis in what promised to be an extremely complex domain of mathematics. The conceptual framework of the various cooperative theories for the solution of games in no way depends critically upon this assumption. Although Shapley and Shubik [3] pointed out the possibility of studying solutions to cooperative games without sidepayments, it was not until the work of Aumann and Peleg that this became a practical possibility [4].

In general a game leads to some set of outcomes, and the individuals are assumed to have some sort of preferences with respect to those outcomes. The specialization of game theory in application to specific disciplines comes about to a great extent in making the appropriate assumptions which provide an appropriate structure to the set of outcomes. For example market games [5] reflect the special structure of a barter economy with individualistic preferences and simple games [6] have a natural interpretation in terms of voting.

1.2. The Extensive Form

The literature on oligopoly, auctions, bargaining and international trade especially whether verbal or mathematical is replete with partial or complete descriptions of process. Offers, counteroffers, threats, promises, demands, etc. are all critical features in the description of processes. Game theory provides a formal language for dealing with the description of the rules of the game which enables us to lay out the details of process with great precision. This is the language used to describe a game in extensive form.

The earliest descriptions of games in extensive form were given by von Neumann and Morgenstern [2], and then Kuhn [7]. Both deal with finite games, i.e. games in which the number of players, moves, and choices are all finite. Chess or Poker serve as examples. Many of the situations faced in economics or in politics are only crudely modelled as finite games. Usually they have continuous strategic possibilities, continuous time and the possibility of an indefinite continuation into the future.

A simple duopolistic market is illustrated by a Kuhn game tree as a finite approximation.* Suppose that two firms must each select one among three levels of production simultaneously. Their levels of production determine the market outcome and the payoffs to both. Figure 1 presents an extensive form description of this game. The diagram shows

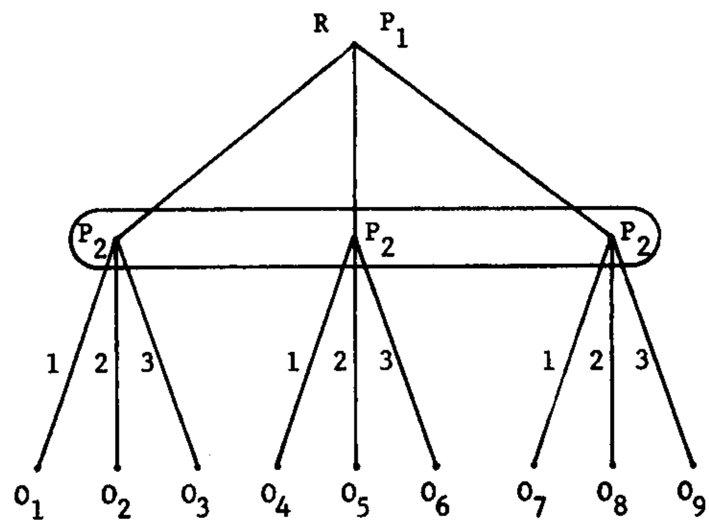

FIGURE 1

a <u>rooted tree</u> with the initial node marked by R. Each node represents a state in which the game might be found by an observer. Each node or vertex is labelled with a P_i or O_j indicating that it is either a decision point for a player or an outcome of the game. Any node labelled P_i is a decision point for player i. He must select one of the branches leading out of that node.

In the example above Player 1 has the first choice, he must select as his move one of the three branches leading out of the node marked P_1. After his move the play progresses to one of the three nodes labelled P_2. Player 2 makes his choice and the game will then reach one of the final

*Production is assumed to yield discrete rather than continuous levels of output.

nine nodes labelled O_j which are the outcomes. Any path from the initial node of the tree to a terminal node represents a possible play of the game.

In many situations we may require that players move simultaneously. In general our concern is not with the formality that they select their moves at the same moment, but that they select their moves without information of what the other is doing. It does not matter who goes first as long as the other is not informed. We can illustrate this lack of information on the game tree by encasing all of the nodes among which a player cannot distinguish in a closed contour which indicates that these choice points belong to the same information set (or more descriptively the same "lack-of-information set"). In Figure 1 the three nodes of Player 2 are encased in a single set which means that when he is called upon to choose he does not know what Player 1's move has been.

We may enlarge our description to take care of exogenous uncertainty by adding an extra player called nature, distinguished by the name P_0. Whenever this player is called upon to move it selects a branch with given probabilities. A simple example is given in Figure 2. The single player P_1 must choose after which Nature determines the outcome.

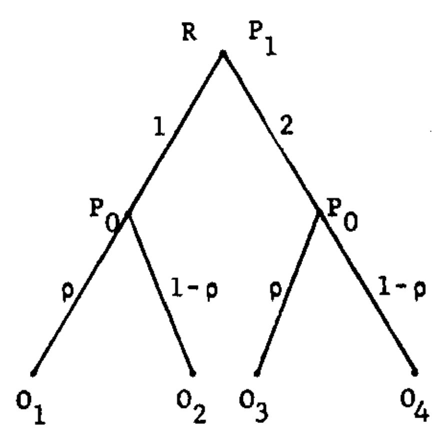

FIGURE 2

A game with one point information sets only is said to be a game with <u>perfect information</u>. Chess is such a game. At any point in the play all players know all the details of the path to that point. This is not true for Poker, or for sealed bid auctions.

A <u>strategy</u> of a player is a complete plan of action which specifies what he is to do under all contingencies. In terms of our description of a game tree we can describe it as follows:

> A strategy is a function that associates with each of a player's information sets one of the alternatives issuing from that set.

A brief contemplation of the size of the game tree for the game of chess and the size of the set of strategies for the players in chess will quickly convince anyone that except for games with few moves and choices the game tree is not in general of direct applied use. Furthermore the game theoretic definition of strategy is clearly different from the definition that would be used by a general or political strategist in the sense that they implicitly have in mind a considerable aggregation of detail, as well as delegation of decisionmaking to agents.

Even though it may not be possible to draw the game tree of a complex market process in detail, the formal method provided should serve to guide the modeller in making explicit the nature of his simplifications and abbreviations in his descriptions of process.

1.3. Games in Strategic Form

When a game is modelled in strategic form the details concerning moves and information are suppressed. Strategies are treated as primitive elements without any attempt being given to explain their genesis. The strategic form of a game which in extensive form is representable by a finite game tree is given by a set of n payoff matrices of dimension n . An example based on the game illustrated in Figure 1 will serve to illustrate the related form. In Figure 3 the numbers on the left of the

	1	2	3
1	0_1	0_2	0_3
2	0_4	0_5	0_6
3	0_7	0_8	0_9

FIGURE 3

matrix are the strategies of Player 1 (which because he has no information when he moves coincide with his moves). The numbers at the top of the matrix are the strategies for Player 2. The entries in the nine cells are the payoffs. We may consider that 0_j is a vector of n dimensions indicating the payoffs to each player, thus in Figure 3 $0_1 = (5,4)$ would be interpreted as a payoff of 5 is obtained by Player 1 and 4 by Player 2 if each uses his first strategy.

Suppose that the information set for Player 2 in Figure 1 were replaced by two information sets indicating that if Player 1 chooses 1 Player 2 is informed, but otherwise he does not know whether Player 1 chooses 2 or 3 . The strategic form associated with this game is a matrix of size 3x9. The moves and outcomes are all the same as before, but the strategies for Player 2 now depend upon his extra knowledge. In particular he has 9 strategies which can be described as follows:

If Player 1 selects 1 then Player 2 selects i ,
if Player 1 selects 2 or 3 then Player 2 selects j .

Any $i = 1, 2, 3$ and $j = 1, 2, 3$ can be selected to give a strategy for Player 2 in this game.

Much of the experimental work on games has been devoted to experimentation with 2x2 matrix games. In particular many experiments have been run with the "Prisoners' Dilemma Game" [8], [9] which is a 2x2 matrix game where the payoffs are as indicated below in Figure 4.

	1	2
1	b_1, b_2	d_1, a_2
2	a_1, d_2	c_1, c_2

FIGURE 4

$$a_i > b_i > c_i > d_i \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad a_i + d_i < 2b_i \text{ .}$$

Rapoport and Guyer [10] have calculated that confining themselves to a strict orderings and eliminating symmetries there are 78 strategically different ordinal representations of a 2x2 matrix game. All of these games have been used for experimental purposes [9].

Simple 2x2 or 3x3 matrix games have been used considerably for expository and exploratory purposes as is evinced by the work of Luce and Raiffa [11] and Schelling [12].

Most duopoly models or other economic models tend to use continuous strategy sets, where in the simplest instances strategies and moves coincide. For example a Cournot [13] duopoly model calls for each player to select simultaneously a level of production, where the level of production may be any number within a range. Thus if Player 1 selects x where $0 \leq x \leq A$ and Player 2 selects y where $0 \leq y \leq B$ then the payoffs to 1 and 2 are given by two functions $f_1(x,y)$ and $f_2(x,y)$.

The use of continuous strategy sets, especially in economics comes about because frequently there is a natural structure present that is not present in most games in general. Chess for example cannot be modelled with continuous moves, but a wheat market can. Furthermore in many instances in economics there are natural ways to aggregate moves. Thus in a wheat market individual i may offer q_i units as his move but the outcome to him may depend only upon his offer and the total volume of wheat, or $q = \sum_{j=1}^{n} q_j$. In many games the addition of moves has no operational meaning.

1.4. Games in Cooperative or Coalitional Form

The stress in the presentation of a game in strategic form is upon the power of individuals in the sense of what they can obtain as a function of their strategies and the strategies of others. No particular attention is paid to explicit patterns of cooperation.

When we wish to study cartel formation, international trade or bargaining, or other group or sociological phenomena, the focus of attention may be upon the possible gains from coalition formation without paying particular attention to information conditions, the details of why or how various strategic options are available and the details and costs of coalition formation (provided they are deemed to be sufficiently low). Our attention may be focussed on the critical questions of how much groups have to gain from cooperation. This attention leads to formulating or presenting the game in cooperative or coalitional form.

As a simple illustration we may use the game given in strategic form in Figure 4. Two forms of this game in coalitional form are presented, the first makes use of an assumption of transferable utility; while the second does not use this assumption.

Let $v(s)$ stand for the amount that a coalition s of players can obtain together if they play as one. We denote the <u>characteristic function</u> by the letter v. It is a function from the subsets of players onto the real numbers. For an n person game there are $2^n - 1$ coalitions that are nonempty.

The notation $v(\overline{ij})$ is used to denote a specific coalition consisting of players i and j. The characteristic function for the Prisoners' Dilemma game shown in Figure 4 is as follows:

$$v(\theta) = 0 \qquad \text{where } \theta \text{ is the set of no players.}$$

$$v(\overline{1}) = c_1$$

$$v(\overline{2}) = c_2$$

$$v(\overline{1,2}) = b_1 + b_2 \ .$$

The characteristic function may be regarded as a "presolution" to a game inasmuch as the act of calculating it provides considerable insight into the structure of the game. In this example the values have been calculated by asking what is the most that any coalition can achieve by itself on the assumption that the remaining players will try to minimize its payoff. The best that Player 1 or Player 2 can do alone is to employ his second strategy (see Figure 4) and obtain c_1 or c_2. Together the players can obtain $b_1 + b_2$. In this instance it is fairly easy to see that it is reasonable to evaluate $v(\overline{1})$ at c_1 because Player 2 while minimizing the score of Player 1 is simultaneously optimizing his own score. This is not generally true as is shown in the game in Figure 5:

	1	2
1	5, 5	0, -100
2	10, 5	-1, -1000

FIGURE 5

Here the characteristic function is given by:

$$v(\theta) = 0$$
$$v(\overline{1}) = 0$$
$$v(\overline{2}) = 5$$
$$v(\overline{12}) = 16\ .$$

But it seems somewhat odd that this appears to portray Player 2 as the most favored player. The paradox is in the treatment of threats. The calculation of the characteristic function for Player 1 does not take into account the high cost to Player 2 incurred if he employs his strategy 2. A more detailed discussion of the problem of threats in the evaluation of the characteristic function is given elsewhere [1].

The possibility for cooperation is changed but not eliminated if comparison of utility and sidepayments are not permitted. We may define a generalized characteristic function or a "characterizing function" V(S) which maps every set of players S onto a set of optimal achievable payoffs.

A way of defining the generalized characteristic function is illustrated by Figure 6 which shows a three person example. We treat each

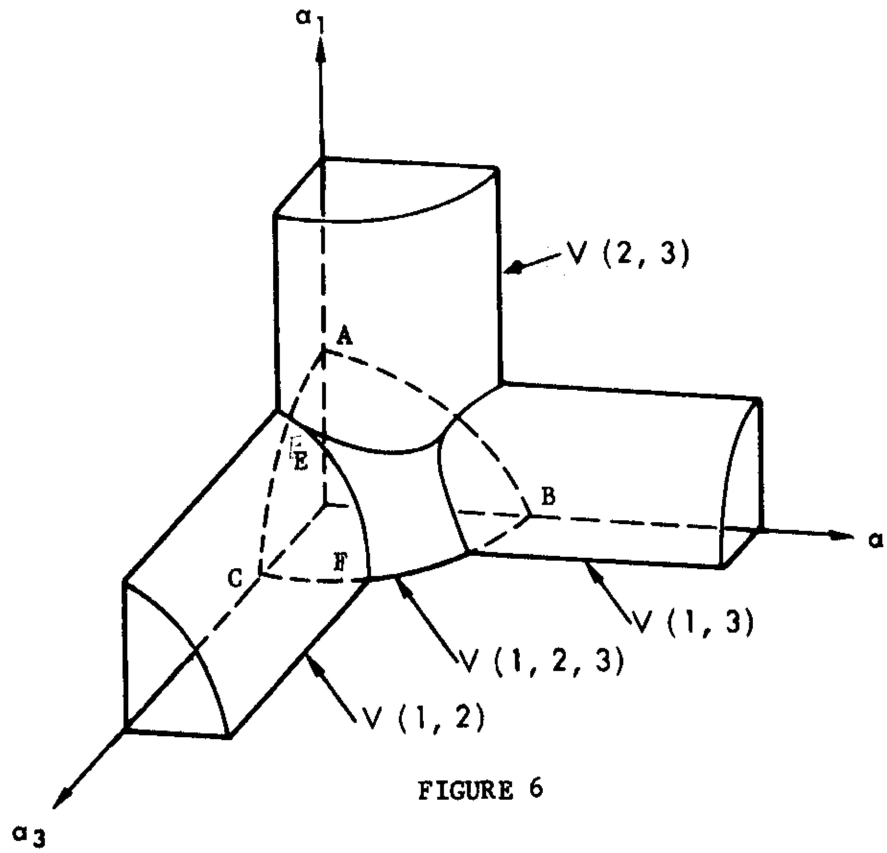

FIGURE 6

V(S) as though it were a cylinder which will punch out a part of the Pareto optimal surface of the n person game as a whole. For example the coalition $\overline{1,2}$ can obtain at least as much as they are offered at any point on that part of Pareto optimal surface delineated by EFC. The reader is referred to Aumann [14], Billera [15] and Scarf and Shapley [16].

Starting from the game in strategic form there are two ways of defining the effectiveness of a coalition in a game without sidepayments. We may specify either that which a <u>coalition can achieve</u> or that which it cannot be prevented from achieving. A simple example of this distinction first made by Aumann and Peleg [4] is provided by Shapley and Shubik [1].

A fairly natural restriction to place upon a characteristic function (with or without sidepayments) is that of superadditivity, i.e.

$$v(S \cup T) \geq V(S) + V(T) , \quad \text{where} \quad S \cap T = \theta \quad [2].$$

The argument for this condition is at the center of economic modelling and the modelling of any societal behavior. The presumption is that trade, exchange and social interaction takes place when all parties have something to gain from cooperation in contrast with opting for no cooperation.

When one attempts to obtain a more process oriented view of cooperation than is provided by the characteristic function this assumption is by no means as natural as it may seem. The costs of organization appear to be critical in determining the formation of coalitions, groups and institutions. Game theory techniques per se do not provide us with an adequate way to make these distinctions.

In spite of the many difficulties and limitations in the defining of a characteristic function, a game cast in this form is suited for the answering of several questions of interest concerning the power of individuals and coalitions, methods of fair division and patterns of social stability.

In recent years there has been a growth in experimentation with games in coalitional form. A survey of much of this work is given elsewhere [17].

1.5. Continua of Strategies, Time, Players and Goods

The first development of the theory of games concentrated upon situations with fixed numbers of players making choices from finite sets of alternatives in games with a finite end. Most of human affairs can be modelled to a good approximation by these conditions. But both for reasons of obtaining better approximations and to exploit deeper and more appropriate mathematical methods other assumptions are of use.

In particular there are many games in which continuous strategies and continuous time appear to be called for. The Cournot duopoly model already noted provides an example where even if mass production occurs in integral outputs, we may be able to construct models of greater mathematical tractability by assuming continuous and differentiable production functions.

Duals and pursuit problems provide examples of games where it is natural to think of events occurring in continuous time. There is a large literature on games on the unit square [18], duels, pursuit and differential games in general [19], [20], [21]. The application of these to economic problems has been relatively small [22].

Possibly the most important simplification to game theoretic theorizing in application to economics has come about in the development of games with a continuum of players. One of the key underlying assumptions in the study of mass market economies, polities and societies is the idea that although the individual may have freedom of choice, for many purposes his influence on the economy or society as a whole is negligible.

Many of the most paradoxical features in the understanding of the relationship between micro and macroeconomics rest upon the fallacy of composition that distinguishes individual from mass behavior.

The first attempts to mathematize the relationship between the influence of the single individual and the number of individuals in the market were made by Cournot [13] and Edgeworth [23]. The method of replication of players was essentially clearly spelled out by them. In the context of game theory applied to economics Shubik [24], [25], Shapley [26] and Debreu and Scarf [27] formulated and developed the method of replication.

Aumann first treated the set of traders in a closed economy as a continuum with individual nonatomic traders [28] thus capturing mathematically the meaning associated with a small trader in the market. Milnor and Shapley [29] and Shapley [30] had previously applied the concept of a continuum of small traders to voting processes in their treatment of oceanic games.

Most of the applications of game theory to economics to date have utilized a description of trade and production with a finite set of commodities. The classification and taxonomy of commodities is somewhat arbitrary. For some purposes two items may be regarded as perfect substitutes whereas for other purposes they may differ. It is clear thus that any result in economic theory which appears to depend in any critical manner upon the relative number of commodities and traders must be suspect. Although this problem is by no means one confined alone to game theory applications to economics it is nevertheless of importance in understanding monopolistic competition [31], [32], [33].

2. SOLUTIONS

2.1. Pre Solutions

The mathematical representation of a game in and of itself is a step towards answering the question that is being posed. Thus the descriptions of a game in extensive, strategic or cooperative form can be regarded as presolutions in the sense that the labor involved in translating and modelling may yield all the insights that are required. For example the characteristic function provides an indication of the potential gains from cooperation of various groups. This information alone may be all that is needed to understand what is at stake in a negotiation.

A natural presolution is the Pareto optimal surface. Consider a payoff vector x in an n-person game. x is feasible if:

$$x \in V(N) .$$

It is Pareto optimal if

$$x \notin D(N)$$

where $D(S)$ is the interior of $V(s)$.

The Pareto optimal surface satisfies our concepts of efficiency and societal rationality, but it does not include any conditions on individual rationality. There may be points on the Pareto optimal surface where an individual obtains less than he could get by acting by himself.

How much an individual can maintain without the cooperation of others is a matter of modelling political, economic and social reality. In economic models of trade it is usually assumed that an individual can maintain ownership over his initial endowments.

If we add a condition of individual rationality to the conditions for Pareto optimality we restrict ourselves to payoffs in the <u>imputation set</u> which is part of the Pareto optimal set. The additional condition is

$$x \notin D(\{i\}) , \quad \text{all} \quad i \in N .$$

For games with sidepayments the imputation set can be represented by a simplex which provides a particularly convenient geometric representation. Figure 7 shows a diagram analogous to that of Figure 6 for a three person game with sidepayments represented by the characteristic function

$$v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0$$
$$v(\overline{12}) = 1 , \quad v(\overline{13}) = 2 , \quad v(\overline{23}) = 3$$
$$v(\overline{123}) = 4 .$$

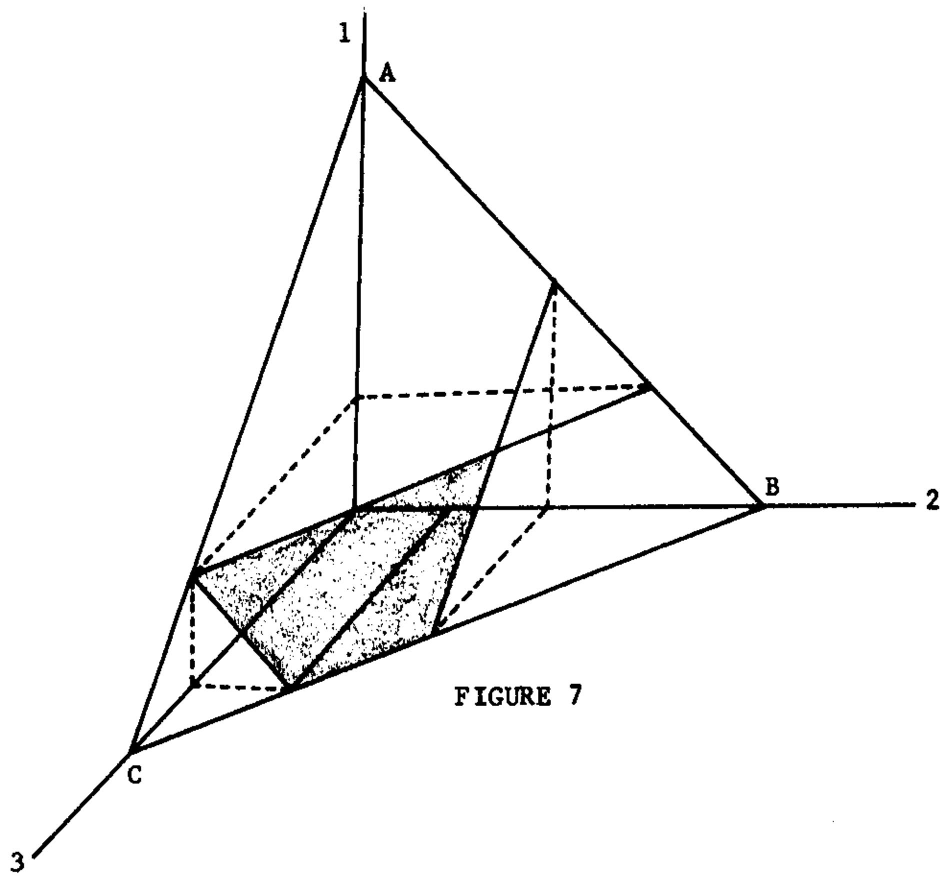

FIGURE 7

ABC in Figures 6 and 7 denote the imputation sets in the three person sidepayment and nosidepayment games respectively. Figure 8 shows the sidepayment game imputations as a simplex; dispensing with the higher dimensional representation in Figure 7. The lines $v(\overline{12}) = 1$, $v(\overline{13}) = 2$ and $v(\overline{23}) = 3$ are drawn on the simplex. An imputation $x = (x_1, x_2, x_3)$

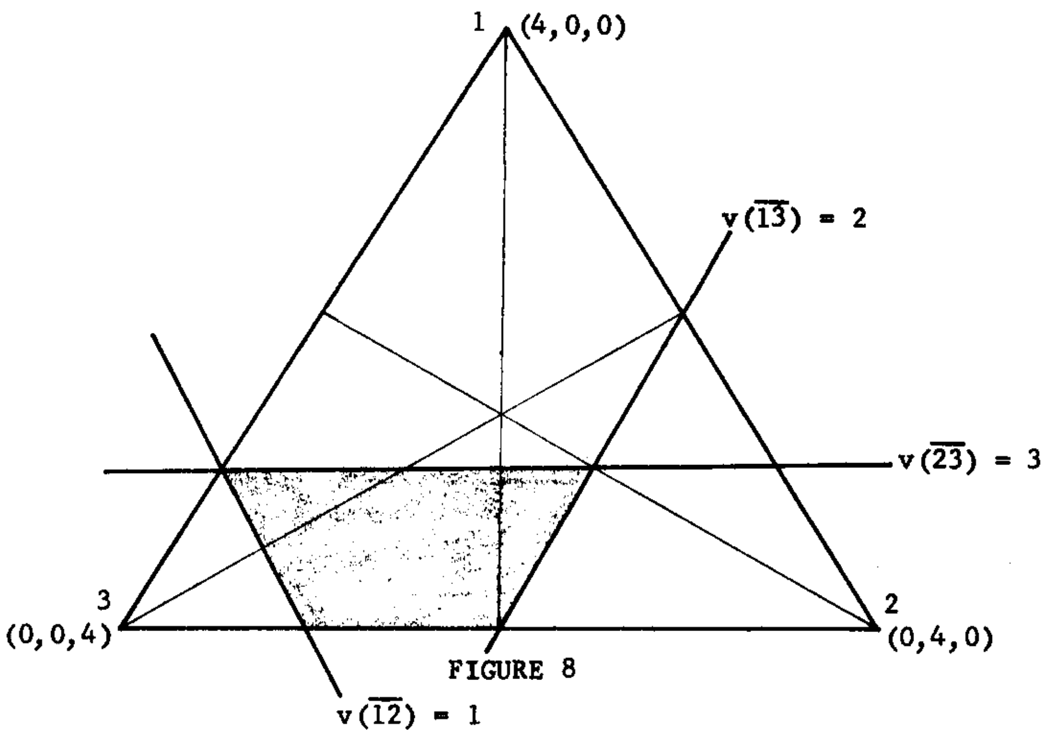

FIGURE 8

that can be obtained by the coalition $\overline{12}$ acting alone has to satisfy

$$x_1 \geq 0 , \quad x_2 \geq 0 , \quad x_3 \geq 0 \quad \text{and} \quad x_1 + x_2 \leq 1 \quad \text{and} \quad x_1 + x_2 + x_3 = 4 .$$

One additional kind of presolution has been suggested by Milnor [34]. A payoff vector $x = (x_i : i \in N)$ is defined as <u>reasonable</u> iff it satisfies

$$x_i \leq \max_{i \in S}[v(S) - v(S - \{i\})] , \quad \forall i \in N .$$

This condition states that no individual should ever obtain more than the most he contributes to any coalition.

A Caveat

The idea of presolution is a link between modelling and analysis; i.e. certain reasonable conditions are loaded onto the model before the heavier analysis begins. In particular it cannot be overstressed that the dangers in blindly accepting the characteristic function and derivative concepts are extremely large. Shapley and Shubik [1] have suggested the term c-game to stand for a game whose characteristic function adequately reflects the underlying structure of the behavioral situation.

2.2. Cooperative Solutions

A basic dichotomy has been made in the development of static solution concepts. This is the dichotomy between cooperative and noncooperative solutions. When we contemplate dynamics this facile dichotomy breaks down. We return to this point in 2.4 .

Cooperative solution theories in general use as their basis the characteristic function for sidepayment games or the extended characteristic function for nosidepayment games. The descriptions and definitions given below are for sidepayment games but subsequent comments note the differences of importance between sidepayment and nosidepayment solutions.

The solution concepts we consider are:

the core,
the value,
the von Neumann-Morgenstern stable set,
the bargaining set,
the kernel,
the nucleolus,
the ϵ-core,
and the inner-core.

Others have been suggested but this list certainly includes the major cooperative solutions.

The Core

The core was originally defined by Gillies [35], and suggested as an independent solution concept by Shapley [36]. Essentially it consists of the set of imputations which leave no coalition in a position to improve the payoffs to all of its members.

Formally the core consists of all imputations x such that:

$$\sum_{i \in S} x_i \geq v(S) , \quad S \subseteq N .$$

It is easy to observe that many games may have no core. In Figures 6 and 7 the cores of the three-person sidepayment and nosidepayment games are indicated by the shaded parts of the imputation sets.

A key link between game theory and economics comes in the defining of a class of games known as <u>market games</u> [5] originally considered by Shapley and Shubik in 1953; and in the recognition of the important link between the price system and the existence of cores in a game and all of its subgames.

A class of games more suited to the analysis of voting problems known as <u>simple games</u> has the property that the values of the characteristic function are only 0 or 1 or "lose" and "win." Most of the games of this variety have no core. Intuitively it appears that as one moves from nicely structured economic markets to markets with externalities to political means for distributing resources the chances for conditions for the existence of a price system and then for a core diminish.

Balanced Games

From the superadditivity property of a characteristic function we know that for every family $\{S_j\}$ of coalitions which forms a partition of S that

$$v(S_1) + \ldots + v(S_m) \leq v(S) .$$

In a market game we consider the possibility that S is broken up into groups which may overlap but for which each set S_j uses only a fraction f_j of the resources (or time) of each of its members. If it is possible to select the f_j to sum to 1 such that

$$f_1 v(S_1) + \ldots + f_m v(S_m) \leq v(S)$$

then the $\{S_j\}$ is said to be a balanced family of subsets. A game in characteristic function form is totally balanced if for every S it is possible to satisfy the balancing conditions.

It has been shown by Shapley and Shubik [5] that every market game is totally balanced, and for sidepayment games, viceversa.
Shapley [37] and Billera and Bixby [38] have considered the nosidepayment games.

The intuitive appeal of the core as a possible solution to problems in political economy is that if it exists it implies that there are ways of imputing wealth which not only satisfy individual and total group rationality but also satisfy all subgroup rationality, i.e. no subgroup is offered less than it could obtain by itself.

The Value

The core picks up the claims of groups, but offers no fair or equitable manner for resolving these claims. A completely different approach to a solution is offered by the value. Here a direct attempt is made to characterize or axiomatize a concept of fair division. Paradoxically these attempts not only succeeded in producing several fair division schemes, but they also showed the intimate relationship between considerations of fair division and power. In particular a key element where these considerations come together is in the definition of the *status quo* point needed to fix the initial conditions from where the fair division is to take place.

Using essentially three axioms, (1) efficiency, (2) symmetry and (3) additivity Shapley [39] was able to deduce a unique value for a side-payment game. The first two axioms are fairly evident; the third axiom is that if we consider two strategically independent games played by the same players the value calculated by considering the games as one will be the same as that calculated by assigning values to each and then adding them. Specifically the value to player i is given by:

$$\varphi_i = \sum_{\substack{S \subset N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S - \{i\})] .$$

There is a simple economic interpretation for this value. Each individual is assumed to enter every possible coalition in every way randomly he is then assigned the expected value of the incremental gain he brings to all. The value provides a combinatoric marginal evaluation.

Nash developed a two person bargaining scheme for nosidepayment games [40] which was generalized with some difficulties remaining by Harsanyi [41]. Shapley has suggested a value solution for n-person nosidepayment games which differs somewhat from that of Harsanyi [42].

The fundamental difficulties to be overcome in the development of the value were how to treat variable threats to fix the status quo point and how to cope with the nosidepayment game.

Owen has suggested a natural extension of Shapley's model which reflects the possibility that the likelihood of players joining coalitions may be biased [43]. Aumann and Shapley [44] and Dubey [45] have considered values and generalized values for games with a continuum of players.

The Stable Set Solution

In their book von Neumann and Morgenstern [2] offered a rather sophisticated concept of solution which turned out to be not as fruitful or general as had been originally hoped.

The essential idea behind the stable set solution is that all imputations belonging to a stable set must exhibit the properties of internal stability and external stability.

In order to illustrate these it is first necessary to define domination and effective set.

An imputation x dominates y if there exists a coalition S such that

$$x_i > y_i \qquad \text{for all} \quad i \in S$$

and

$$\sum_{i \in S} x_i \leq v(S) \; .$$

This last condition states that the coalition S is an effective set for x ; i.e. by itself it could obtain what its members obtain in x .

A set of imputations is <u>internally stable</u> if no member of the set is dominated by another member of the set.

A set of imputations is <u>externally stable</u> if any imputation not in the set is always dominated by some imputation in the set.

A set of imputations is a stable set solution if it is both internally and externally stable.

A large bibliography on stable set solutions is provided elsewhere [1]. Originally it had been conjectured by von Neumann that all side-payment games had stable set solutions. Lucas was able to prove that this was false, giving a 10 person game counterexample [46]. Shapley and Shubik were able to show that the Lucas counterexample could be regarded as a market game [5] hence there would exist economies without stable set solutions.

<u>The Bargaining Set</u>

This is a solution concept originally due to Aumann and Maschler [47]. It has been defined in several slightly different ways [48]. Its genesis was inspired by observing players in an experimental bargaining game.

A <u>bargaining point</u> of the game (N,v) has the property that for each pair $i, j \in N$ any objection of i against j can be met by a counterobjection by j against i .

An objection consists of a coalition S containing i but not j together with an imputation for which S is effective which is preferred to the given imputation by all members of S .

A counterobjection consists of a different coalition T containing j but not i together with an imputation for which T is effective that is (weakly) preferred to the objection by every member of $T \cap S$ and is (weakly) preferred to the original imputation by every member of $T-S$.

If x is the original imputation and (S,y) the objection and (T,z) the counterobjection then

$$\sum_{k \in S} y_k \leq v(S) \quad \text{and} \quad y_k > x_k\,, \quad \text{for all} \quad k \in S$$

and

$$\sum_{k \in T} z_k \leq v(T) \qquad \begin{array}{ll} z_k \geq y_k\,, & \text{for all} \quad k \in T \cap S \\ z_k \geq x_k\,, & \text{for all} \quad k \in T-S\,. \end{array}$$

The bargaining set is the set of all bargaining points.

Although some use of the bargaining set has been made in experimental studies, little application of the bargaining set has been made to economics. The computational difficulties for games larger than 3 or 4 make it somewhat unattractive.

The Kernel

Davis and Maschler [49] have suggested a solution which is contained within the bargaining set.

In order to define the kernel it is convenient to first define the excess and the surplus.

By the excess of a coalition S at an imputation x we mean:

$$e(S,x) = v(S) - \sum_{i \in S} x_i \;;$$

i.e. it is the amount by which the worth of the coalition exceeds its proferred payoff.

The surplus of a player i against another player j with respect to a given imputation is the largest excess of any coalition that contains i and not j.

The surplus basically measures a potential bargaining pressure of i against j. The kernel solution consists of all imputations x such that for any two players i and j

$$\max_{\substack{i \in S \\ j \notin S}} e(S,x) = \max_{\substack{j \in T \\ i \notin T}} e(T,x) .$$

It picks up the idea of symmetry or equalization of bargaining pressure.

The Nucleolus

The nucleolus is a solution concept introduced by Schmeidler [50] which is a unique outcome in the kernel of a sidepayment game. Although it should be noted that no totally satisfactory definition of the nucleolus for a nosidepayment game exists.*

The nucleolus is the imputation for which the maximal excess is minimal. Intuitively it is the point that minimizes dissatisfaction. It seems that it should have a natural application in the design of taxation and subsidies, yet although there have been some applications of the nucleolus [51] to operations research, the lack of an adequate nosidepayment nucleolus has limited its application. For that matter the bargaining set, kernel and nucleolus all appear to have been underemployed in the context of economic models.

*One could use a λ-transfer approach, but it does not seem to be satisfactory.

The ϵ-Core

The core of a game can be "fat" or for that matter nonexistent. We may consider a way to uniformly tax or subsidize the players so that cores can be made to appear or shrink. Two ways of doing this are suggested here.

The strong ϵ-core [52] consists of the set of Pareto optimal outcomes x such that

$$\sum_{i \in S} x_i \geq v(S) - \epsilon , \quad \text{for all} \quad S \subseteq N .$$

The weak ϵ-core consists of the set of Pareto optimal outcomes x such that

$$\sum_{i \in S} x_i \geq v(S) - s\epsilon , \quad \text{for all} \quad S \subseteq N .$$

We may regard the ϵ as an overall cost or a per capita cost to the formation of coalitions or a frictional threshold below which it is not worth acting.

By increasing the size of ϵ we can eventually produce a core in any sidepayment game without a core. We can define the least core or near core to be the smallest strong ϵ-core. It is evident that this is close to, but not the same as the idea underlying the nucleolus.

The Inner Core

Although many results which hold true for sidepayment games also have their analogues for nosidepayment games, this is by no means always the case. Among the important problems in game theory and its applications to political economy is the characterization of the differences. For example a brief contemplation of Figures 6 and 7 should suffice to indicate that although the core of a sidepayment game will always remain simply connected this is not true for a nosidepayment game and a counter-example can be produced for $n = 3$.

A difference between sidepayment and nosidepayment games leads us to the definition of the inner core of a game. Consider the core of the nosidepayment game illustrated in Figure 6. Suppose that at any point in the core we constructed a tangent hyperplane and used the direction cosines of this hyperplane to define an intrinsic comparison of utility among the players in an associated sidepayment game which has only the point of tangency in common with the nosidepayment game.

Using the comparison of utility we can describe the feasible sets of all coalitions in terms of hyperplanes. A natural question to ask is, is the point of tangency which is a point in the core of the nosidepayment game also a point in the core of the associated sidepayment game? The answer is not necessarily.

We define the inner core of a nosideplayment game to be those imputations in the core which are also in the core of the associated sidepayment games.

The construction for obtaining the inner core is essentially cardinal. Yet it is of interest to note that the inner core is contained within the core of a nosidepayment game defined ordinally. Furthermore for a market game the core shrinks under replication and the inner core is nonempty hence the core and inner core approach the same limit, the competitive equilibria.

2.3. Noncooperative Solutions

The cooperative solutions dealt with games in characteristic function form, the noncooperative solutions are basically applied to games in strategic form. In fact much of the basic interest in game models in oligopoly and other aspects of economics is focussed on games which are played many times over. A discussion of solutions to such games is deferred to 2.4. It is precisely here that the nice distinctions between cooperative and noncooperative solution concepts start to evaporate.

Two Person Constant Sum Games

Two person zero sum games and their strategically equivalent constant sum games are games of pure opposition. The goals of the players are diametrically opposed.

Both two person zero sum games and the minimax theorem are well known [2], and need not be presented again here. It is important however to note that two person zero sum game theory, although of considerable importance in the study of military tactical problems is of extremely limited value to the study of political economy. Extremely few situations in political economy, if any meet the conditions of pure opposition.

Noncooperative Equilibrium Points

Consider an n person game in strategic form where each player i has a set of strategies S_i , $i = 1, \ldots, n$.

Let $P_i(s_1, s_2, \ldots, s_n)$ be the payoff function to player i then an equilibrium point is a vector of strategies $(s_1^*, s_2^*, \ldots, s_n^*)$ such that for each $i = 1, \ldots, n$

$$P_i(s_1^*, \ldots, s_n^*) = \max_{s_i \in S_i} (s_1^*, \ldots, s_i, \ldots, s_n^*) .$$

The general concept of a noncooperative equilibrium and its existence for matrix games was given by Nash [53], although the basic idea and its relevance to economics was given by Cournot [13].

It is possible to show the existence of noncooperative equilibria for games with a continuum of players [54]. This is of direct relevance in attempts to model the nuances of meaning in the concept of competitive markets.

In general the difficulties with the noncooperative equilibrium solution concept come far less in problems of existence, than in the multiplicity of equilibria. Furthermore it appears to be relatively easy to produce models where many of the equilibrium points appear to be quite unreasonable. A brief rogue's gallery of 2 x 2 matrix games is given in Figure 9. In the first there is a single equilibrium yielding (0,0) . In the second there are two pure strategy equilibria, one favoring the first and the other the second player; there is also a bad mixed strategy equilibrium. In the third all outcomes are equilibrium points.

	1	2
1	5,5	-1,10
2	10,-1	0,0

	1	2
1	10,1	-20,-20
2	-20,-20	1,10

	1	2
1	1,6	10,6
2	1,3	10,3

FIGURE 9

When we contemplate a game in strategic form the unsatisfactory static feature of the equilibrium point becomes clear. There is undoubtedly a circular stability to an equilibrium: "if A knew what B was doing then he would do such-and-such and vice-versa!" Unfortunately there is no indication of how or why the players will generate expectations to bring about an equilibrium. In short the noncooperative equilibrium theory is static, does not indicate how communication is to be handled and the equilibrium points are frequently not unique. These comments apply not only to n-person noncooperative games in general but to economic markets viewed as noncooperative games.

2.4. Other Solutions

So much of economic analysis in one form or the other has apparently depended upon solution concepts which implicitly or explicitly amount to noncooperative equilibria that a full understanding of what knowledge is conveyed by the solution is critical.

In particular our problem is more in modelling and in the understanding of behavior than it is mathematics. The definition of an equilibrium point is purely static. The mathematics tells us what the equilibrium point is not how it came about. Dissatisfaction with this has forced many game theorists to consider the extensive form of the game as key to the understanding of process.

In trying to model in the extensive form an immediate difficulty is encountered. Are communications, negotiations and messages modelled as a part of the game? If so, how are messages, language and other forms of communication put into the extensive form? at this time there is no adequate answer to this basic coding problem. It is however clear that if everything is in the game then the distinction between cooperative and noncooperative theories becomes blurred. Binding commitments and coalitions outside of the game are ruled out and commitments within the game become the art of the possible. It can be shown that in games of indeterminate length virtually every outcome in a sub-game can be converted into an equilibrium even using behavior strategies. It is easy to see from the example in Figure 10 that the outcomes (5,5) and (0,0) and (-11,-11) can all be enforced as equilibria yet it is hard to believe that they are all equally plausible.

	1	2	3
1	5,5	-1,8	-30,-12
2	8,-1	0,0	-30,-12
3	-12,-30	-12,-30	-11,-11

FIGURE 10

Selten has introduced and refined [55], [56] the concept of a perfect equilibrium point for games in extensive form. In his original definition a perfect equilibrium point had the property that the players are in equilibrium in each subgame attained. A quick example helps to illustrate a perfect and a not perfect equilibrium. Consider the game shown in Figure 10 played twice. The strategy pairs (some strategy for each player)

> "I play 1 to start, if he plays 1 then I play 2 next; otherwise I play 3 ."
>
> and "I play 2 on both occasions regardless of what he does."

both give equilibria; the first is not perfect because in the last play even if the other player has failed to play 1 to begin with there is no ex post motivation beyond "desire to punish" to play 3 . In contrast the second is perfect.

The original definition of a subgame perfect equilibrium leaves problems with that part of the game tree not attained in the equilibrium path. An example of a 3-person game provided by Selten illustrates this.

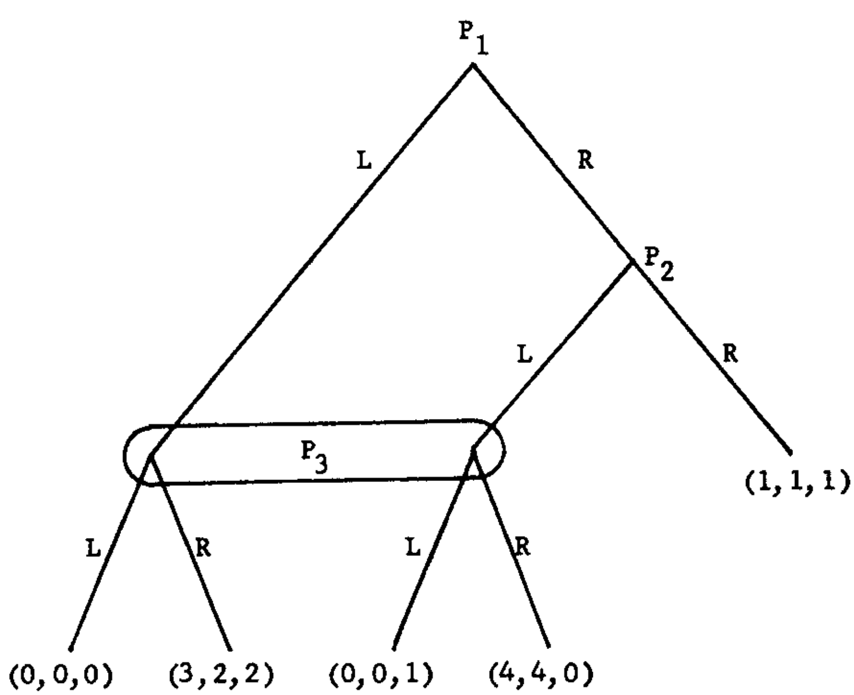

FIGURE 11

Each player has but two choices L or R hence a behavior strategy for i can be characterized by a probability p_i for selecting R . The two types of equilibria are:

$$\text{Type 1:} \quad p_1 = 1 \,, \quad p_2 = 1 \,, \quad 0 \leq p_3 \leq \frac{1}{4}$$

$$\text{Type 2:} \quad p_1 = 0 \,, \quad \frac{1}{3} \leq p_2 \leq 1 \,, \quad p_3 = 1 \,.$$

Note that in the equilibrium points of type 2 players 2's information set is not reached hence his expected payoff is independent of his choice. In particular consider the equilibrium point (0,1,1) which is of type 2. If by "accident," say a random pertubation player 2 were reached is it reasonable for him to play 1 given he believes that player 3 will also play 1 giving him 4 if he switches to 0? Clearly type 2 equilibria are unreasonable, a player's choices should be conditional upon the information set reached. Selten develops a model with a perturbed game where with some very small probability a player will make a mistake. In this manner all information sets may be reached.

Harsanyi [57] has been concerned with the possibility of selecting a single preferred noncooperative equilibrium point out of the myriads which may exist.

In general games of interest to political economy have a surfeit of equilibrium points. Dubey and Shubik [58] have shown that for any game representable by a finite game tree if player information sets are refined the new game will contain as equilibrium points those of the old game and probably more. In particular this means that if the general equilibrium trading model is not interpreted as single simultaneous move game then it has more noncooperative equilibria than the competitive equilibria.

Many economic models can be interpreted as a game that is more or less repeated many times or indefinitely through time. Thus it is attractive to contemplate some form of dynamic programming approach as suggested in several of the papers in Blaquiere [2]. This super-rationalistic approach which can be characterized by a backward induction for finite length games yielding perfect equilibrium points can be contrasted with the type of expectation updating devices which move forward in time in macroeconomic behavioral models. The work of Sobel [59, 60] is devoted to considering the latter type of equilibrium.

The discipline of game theory has answered few questions concerning dynamics, information and communication, but at least it provides a way to formulate precisely many of the critical unsolved problems.

Under the catchall of "other solutions" there still are more cooperative and noncooperative solutions that we have not dealt with mainly because space limitations and lack of many economic applications force this economizing. These solutions include the disruption nucleolus, the p-center; the competitive equilibrium of a game, the Banzhaf and other values and game payoff transformations such as maxmin the difference in a two person nonconstant sum game, beat the average in certain n person games and games of status.

3. APPLICATIONS

This brief review is offered as a somewhat short and possibly too compact survey and reference guide to the development of the applications of game theory to economics. It is not meant to be complete nor completely selfcontained.

3.1. Oligopolistic Markets

The major applications of game theory to date have been to the study of various aspects of oligopolistic competition and collusion and to the study of the emergence of price in a closed economic system. The discussion of the latter is given in Section 3.2.

A reasonable division of the study of oligopolistic markets is into:

1. Duopoly
2. Noncooperative and "quasi-cooperative" oligopoly
3. Bilateral monopoly and bargaining
4. Experimental gaming
5. Auctions and bidding

Duopoly

Models of duopoly have always held a fascination for mathematically inclined economists. The literature is so vast that it merits a separate study. A good set of references have been provided by Chamberlin [31]. The first well-known model which can be clearly identified as a well defined mathematical description of competition in a duopolistic market is that of Cournot [13]. It is a game in strategic form where the competitors, selling identical products, are assumed to select a level of production each in ignorance of the other's action. The solution suggested is that of a noncooperative equilibrium. Bertrand's [6] critique of Cournot was primarily directed towards the former's selection of production as the strategic variable. He suggested that price would be a more natural variable. Edgeworth's [62] model of duopoly introduced rising costs, or the equivalent of a capacity restraint; Hotelling [63], also working with a noncooperative model, introduced transportation costs as a form of product differentiation. Chamberlin's [31] model introduced product differentiation in a more general manner.

A variety of duopoly models have been proposed and analyzed by authors such as Coase, Harrod, Kahn, Nichol, Stigler and many others. These and many subsequent works cannot be properly regarded as explicitly game theory models inasmuch as the authors do not concern themselves with the specification in detail of the strategy spaces of the two firms. Indeed the careful mathematical statement of a duopoly model calls for considerable care and detail as is shown by Wald's [64] analysis of equilibrium.

Beckmann, Levitan and Shubik, Mayberry, Nash and Shubik, Shapley and Shubik, and Shubik have presented a series of explicitly game theoretic models of duopoly studying the effects of inventory carrying costs, fluctuations in demand, capacity constraints and simultaneous decisions on both price and production. Other models have compared noncooperative solutions with other types of solution; stockout conditions and variation in information conditions have also been considered.

There is a growing literature on dynamic models of duopoly. Frequently, in teaching even the simple Cournot duopoly model, a dynamic process entailing action and reaction is sketched. An important early work was that of von Stackelberg, in which a series of quasi-dynamic models were suggested. For example, the papers of Smithies and Savage [65], Shubik and Thompson [66] and Cyert and De Groot [67] provide more mathematical, game theoretical and behavioral models of duopolistic behavior.

Oligopoly

A more or less standard way of considering oligopolistic markets is to construct a model which can be studied for duopoly and which can be compared with duopoly or with a competitive market as numbers are increased. Frequently the assumption of symmetrically related firms enables us to make comparisons among markets of different size. Thus, for example, Cournot proceeds from an analysis of two firms to many and Chamberlin considers small and large groups of competitors.

It is important to bear in mind that two, if not three different skills are called for in the investigation of oligopolistic markets. They are the skills of the economist at describing economic institutions and activities and selecting the relevant variables and relationships; the skills of the modeler in formulating a mathematical structure that reflects the pertinent aspects of the economic phenomenon; and the skills of the analyst in deducing the properties of the mathematical system that has been formulated. Thus, for example the work of Chamberlin may be regarded as a considerable step forward over that of Cournot in terms of its greater relevance and reality, however it was no advance at all (and possibly a retrogression) in terms of rigorous mathematical formulation and analysis when compared with Cournot. Both the Cournot and Chamberlinian large group analyses are based on a noncooperative equilibrium analysis.

All the mathematical rigor in the world cannot make up for lack of economic insight and understanding in the creation of the model to be analyzed, thus the development of an adequate theory depends heavily upon verbal description and less than fully formal models as suggested by Stackelberg and Fellner. Special variables must be considered. Brems introduces technological change, Bain considers entry, Marris, Baumol and Shubik stress managerial structure, Levitan and Shubik [68], Kirman and Sobel [69] consider the role of inventories and there are many other works dealing with other important and special variables such as transportation, advertising, production change costs, multiple products, financing and so forth.

Where does the theory of games fit into these considerations beyond being a mathematical tidying up device which merely translates the insights of others into a more heavily symbolic language? The answer to this can be best seen when it is understood that the discipline called for in specifying in detail the strategic options of the individual actors leads to the discovery of gaps in the logic of less formally defined models. Many of these models are "quasi-dynamic" in description. In other words they describe some sort of adjustment process in terms which gloss over the information conditions (precisely who knows what, when and how much does it matter?). In general in the description of action and reaction, time is not explicitly accounted for, not even by at least a rate of interest such as in the work of Cross [70].

The Chamberlinian analysis of large group behavior and the large literature by Sweezy, Stigler and many others on the "kinked oligopoly curve" provide important examples of both the power and danger of an informal mix of verbal and diagrammatic modeling. This is easily shown when one tries to formulate the structure of the market as a well defined model. The kinked oligopoly curve has no objective existence, it presupposes an extremely limited set of reactions by all competitors. It is obtained by implicitly assuming symmetry in strategic power, structure of moves and information conditions for all firms (or by not knowing enough to see that explicit or implicit assumptions must be made if the robustness of the conclusions is to be examined).

Furthermore, the arguments describing equilibrium or a tendency toward equilibrium using either the kinked oligopoly curve or Chamberlin's large group analysis depend only upon the local properties of these subjective curves. Edgeworth's [62] analysis of duopoly led him to conclude that no equilibrium need exist, but his results were obtained by considering the objective structure of oligopolistic demand over all regions of definition. In other words the analysis requires that we should be able to state what the two firms will find their demand to be, given every pair of prices (p_1, p_2).

Shubik [71] suggested the term "contingent demand" to describe the demand faced by an individual firm given the actions of the others as fixed. It is possible to show that the contingent demand structure may easily depend upon details of marketing involving the manner in which individual demands are aggregated. Levitan [72] showed the relationship between the description of oligopolistic demand and the theory of rationing. Based on the study of the shape of contingent demand curves Levitan and Shubik were able to show that the Chamberlin large group equilibrium may easily be destroyed for much the same reasons as indicated in Edgeworth's analysis.

A full understanding of the problems posed by oligopoly calls for a clear distinction to be made separating aspects of market structure, intent of the firms and behavior of the firms. Then a study of the interrelationships among these factors is called for.

Perhaps the most important aspect of game theoretic modeling for the study of oligopoly comes in describing information conditions and providing formal dynamic models which depend explicitly upon the information conditions. There is a growing interest in state strategy models, i.e., models in which the system dynamics are dependent upon only the state that the system is in currently. The work of Shubik and Thompson, Selten, Miyasawa and others working on sequential game models of economic processes provides examples.

The sensitivity of an oligopolistic market to changes in information has been studied [73]. When information is relatively high there is no strong reason to suspect that a few firms in an oligopolistic market will employ state strategies. Instead we may expect that they will use historical strategies where previous history, threats and counterthreats play an important role. Marschak and Selten [74], Selten [75] and Shubik [71] have considered this possibility.

Among the books directly devoted to a game theoretic investigation of oligopoly are those of Friedman [76], Jacot [77], Shubik [71] and Telser [78].

Bilateral Monopoly and Bargaining

Whereas most of the models of oligopolistic behavior have either offered solutions based on the noncooperative equilibrium or have sketched quasi-dynamic processes, the work on bilateral monopoly and bargaining has primarily stressed high levels of communication with a cooperative outcome, or a dynamic process which leads to an optimal outcome. A few of the models suggest the possibility of nonoptimal outcomes such as strikes which materialize after threats are ignored or rejected.

Many of the models of bargaining arise from highly different institutional backgrounds. The major ones are bilateral trade among individual traders as characterized by Böhm-Bawerk's horse market or Bowley's model. Frequently however, the model proposed refers to international trade or to labor and employer bargaining. Edgeworth's famous initial model was cast in terms of the latter, as was the work of Zeuthen.

The work of Edgeworth is clearly related to the game theory solution of the core, as has been noted by Shubik, Scarf, and Debreu and Scarf. Böhm-Bawerk's analysis may be regarded as an exercise in determining the core and price in a market with indivisibilities [79].

Zeuthen's analysis of bargaining is closely related to the various concepts of value as a solution. This includes the work of Nash, Harsanyi, Shapley and Selten.

Another area of importance in application which is possibly closer to political science than economics is international strategic bargaining. Reference to this type of application is made elsewhere [1].

A "solution" to an economic problem may attempt to do no more than cut down the feasible set of outcomes to a smaller set. No specific outcome is predicted. The solution narrows down the set of outcomes but does not tell us exactly what will or should happen. The contract curve of Edgeworth and the core are solutions in this sense.

Other solutions may be used in an attempt to single out one final outcome as that which should or which will emerge. In their static versions most of the various value solutions and other fair division solutions which have been proposed may be regarded as normative in their suggestions and abstracted from any particular institutional background in their presentation. These remarks hold for the works of Nash, Shapley, Harsanyi, Braithwaite, Kuhn, Knaster, Steinhaus and others.

Still other solutions which may be used to select a single outcome are phrased in terms of a dynamics of the bargaining process. These include the works of Cross, Harsanyi, Pen, Raiffa, Shubik, Zeuthen and others.

<u>Gaming</u>

One result of the development of the theory of games and the high speed digital computer has been a growth in interest in using formal mathematical models of markets for gaming for teaching and/or experimental purposes. The earliest published article on an informal economic game experiment was by Chamberlin. This however appears in isolation from the rest of the literature. The first "business game" built primarily for training purposes was constructed by Bellman, Clark and others several years later, this was followed by a deluge of large computerized business games. The use of these games has been broadly accepted in business schools and in some economics faculties. References on gaming are given elsewhere [1] [7].

Much of the earlier experimental work with games in economics did not use the computer. The games were frequently presented in the form of matrices or diagrams. Siegal and Fouraker, Fouraker and Siegal, Fouraker, Shubik and Siegal were concerned with bilateral monopoly under various information conditions and duopoly and triopoly. Stern, Dolbear and others investigated the effect of numbers of competitors in a market. Friedman has considered the effect of symmetry and lack of symmetry in duopoly as well as several other aspects of oligopolistic markets. Smith has considered the effect of market organization.

Experiments using computerized games have been run by Hoggatt, Hoggatt and Selten, Friedman and Hoggatt, Shubik, Wolf and Eisenberg, McKenney and several others. These games provide advantages in control and in ease of data processing that the noncomputer games do not offer.

Several of the games noted above can be and have been solved for various game theoretic and other solutions. This means that, for instance, in the duopoly games studied by Friedman it is possible to calculate the Pareto optimal surface and the noncooperative equilibrium points. Similarly in the duopoly or oligopoly investigations of Hoggatt, Stern, Fouraker, Siegal and Shubik and others it is usually possible to calculate, the joint optimum, the noncooperative equilibria and the competitive price system.

The game designed by Levitan and Shubik was specifically designed to be amenable to game theoretic analysis. Thus the joint maximum, the price noncooperative equilibrium, the quantity noncooperative equilibrium, the range of the Edgeworth cycle, the beat-the-average and several other solutions have been calculated for this game.

It is possibly too early to attempt a critical survey of the implications of all of the experimental work to oligopoly theory, however a general pattern does seem to be emerging. All other things being equal an increase in the number of competitors does appear to lower price, so does an increase in cross-elasticities between products. However, with few competitors information and communication conditions appear to be far more critical than a reading of oligopoly theory would indicate.

In all of the experiments and in games for teaching such as those of Jackson, the Carnegie Tech. and Harvard Business School games the importance of considering a richer behavioral model of the individual emerges when the way in which the players attempt to deal with their environment is observed. This observation is by no means counter to a game theory approach. It is complimentary with it. As yet there exists no satisfactory dynamic oligopoly solution provided by either standard economic theory or game theory. This appears to be due to the difficulties in describing the role of information processing and communication.

A different set of games have been used for experimentation with a certain amount of economic content, but far less identified with an economic market than the oligopoly games. These include simple bidding and bargaining exercises used by Flood, Kalish, Milnor, Nash and Nering, Stone, Maschler, Riker, Shubik and others. In all of these instances there was a direct interest in comparing the outcomes of the experiments with the predictions of various game theory solutions.

Auctions and Bidding

Auctions date back to at least Roman times. In many economies they still play an important role in financial markets and in commodity markets. Sealed bids are used in the letting of large systems contracts or in the sale of government property. Their history as economic market mechanisms is a fascinating subject by itself [80]. Furthermore, as an auction or a bidding process is usually quite well defined by a set of formal rules (together, on occasions, with customs or other informal rules) it lends itself naturally to formal mathematical modeling.

The mathematical models of auctions and bids fall into two major groups. Those for which the role of competition is modeled by assuming a Bayesian mechanism and those where the model is solved as a game of strategy using the solution concept of the noncooperative equilibrium or some other solution.

There is a considerable literature on problems encountered in different types of bidding and on features such as problems in evaluation, risk minimization and incentive systems.

The study of auctions and bidding lies heavily in the zone between theory and application as can be seen by observing the tendency of the publications to appear in journals such as the Operations Research Quarterly or Management Science. A useful bibliography on bidding has been supplied by Stark and Rothkopf [81].

In general game theoretic work on auctions and bidding has been useful in two ways: descriptive and analytic. The careful specification of the mathematical models has forced attention to be paid to understanding the actual mechanisms including informal rules and customs. The attempts at solution have shown that the models are extremely sensitive to information conditions and that many of the important features of auctions involve the individual's ability to evaluate what an object is worth to him and to others. This is a far cry from the economic models where the assumption is made that all have complete knowledge of all individual's preferences.

3.2. General Equilibrium

An important area of application of the theory of games to economic analysis has been to the closed general equilibrium model of the economy. The solution concepts which have been explored are primarily cooperative solutions. Noncooperative solutions appear to be intimately related with monetary economies and this work is discussed in Section 3.4.

In much economic literature it has been claimed that the study of economics requires only the assumption of a preference ordering over the prospects faced by an individual. Apart from the fact that such a strong assumption immediately rules out of economic consideration topics like bargaining and fair division where virtually no analysis can be made without stronger assumptions on the measurement of utility, even if we restrict our investigation to the free functioning of a price system the assumption of only a preference ordering is not sufficient. At least one must restrict transformations to those which preserve concavity of utility functions, otherwise markets involving gambles would emerge [82].

The investigations of game theoretic solutions in application to economic problems have been devoted primarily to the core and secondarily to other cooperative solutions. Because of the predominance of the former we deal with it separately.

The Core

The first economist to consider bargaining and market stability in terms of the power of all feasible coalitions was Edgeworth. He was dealing with a structure which can be described as a market game hence his solution could not be described as the core, in general. Shubik observed that the Edgeworth analysis was essentially an argument that could be described in terms of the core. He constructed a two-sided market model to demonstrate this and used a method of replication to illustrate the emergence of a price system. This was done for the sidepayment game. He conjectured this to be true in general for nosidepayments and proposed this problem to Scarf. The replication method of studying the limiting behavior of a many player game involves starting with a given set of different types of traders, where a type is defined by a utility function and endowment. Suppose we start with a game with k players, one of each of k types. The n^{th} replication of such a market game consists of a market with nk players with n of each type.

Essentially the replication method boils down to considering an economy with thousands of butchers, bakers and candlestick makers. Scarf and Debreu [27] and Scarf [83] using the method of replication were able to generalize the previous results considerably. They showed that under replication, in a market with any number of different traders the core "shrinks down" (under the appropriate definition which takes into account the increasing dimensions) until a set of imputations which can be interpreted as price systems emerge as the limit of the core. Hildenbrand has generalized the method of replication, doing away with the rigidity of maintaining identical types [84].

A different way of considering markets with many traders is by imagining that we can "chop up" traders into finer and finer pieces. Going directly to measure theory we may consider a continuum of traders where the individual trader whose strategic power is of no significance to the market is described as having a measure of zero. Aumann [28] first developed this approach.

In the past fifteen years there has been a proliferation of the literature on the core of a market. Although much of these writings have been devoted to the relationship between the core and the competitive equilibrium (for example, given a continuum of small traders of all types, the core and competitive equilibrium can be proved to be identical), some of the work has been directed towards other problems; thus Shapley and Shubik [52] considered the effect of nonconvex preference sets and Aumann [85], Shitovitz [86] and others have been concerned with the economics of imperfect competition. Caspi [87] and Shubik [82] also considered the effect of uncertainty on the core. Debreu [89] has considered the speed of convergence of the core.

A difficulty with the cooperative game formulation appears when one tries to model production. Are production sets jointly available or individually owned? Dubey [45] has considered the latter case for nonatomic sidepayment games and has established the coincidence of the core, value and competitive equilibrium. Hildenbrand [84] assuming a production possibility set for each coalition has extended the definition of a Walrasian equilibrium and proved the coincidence of the core for the nonatomic game with the extended Walrasian equilibrium.

In summary the import of this work to economic analysis is that it extends the concept of economic equilibrium and stability to many dimensions and it raises fundamental questions concerning the role and the nature of coalitions in bringing about economic stability.

Other Solutions

The core can be regarded as characterizing the role of countervailing power among groups. There are other solution concepts which reflect other views for the determination of the production and distribution of resources. In particular the family of solutions which can be described as the value of an n-person game, stress fair division where the division is based both upon the needs or wants of the individual and his basic productivity and ownership claims.

The Value

There are a variety of differences among the different value solutions which have been suggested, which depend upon three major factors. They are (1) whether there are two or more individuals, (2) whether or not a sidepayment mechanism is present and (3) whether threats play an important role and the status quo is difficult to determine.

Leaving aside the finer points, all of the value solutions are based in one form or the other upon a symmetry axiom and an efficiency axiom. Describing them loosely, if individuals have equal claims they should receive equal rewards and the outcome should be Pareto optimal.

These solutions clearly appear to have no immediate relationship with a price system, yet Shapley and Shubik using the method of replication were able to show that under the appropriate conditions as the number of individuals in a market increases the value approaches the imputation selected by the price system. Shapley and Aumann and Shapley have also considered other models. In the latter work markets with a continuum of traders have been investigated and the coincidence of the value with the competitive equilibrium has been established.

The Bargaining Set, Nucleolus and Kernel

Rather than appeal to countervailing power arguments or to considerations of fairness one might try to delimit the outcomes by bargaining considerations. Aumann and Maschler [47] suggested a bargaining set and Peleg established that such a set always exists [48]. Shapley and Shubik [91] were able to show that under the appropriate conditions the bargaining set lies within an arbitrarily small region of the imputation selected by the price system when the number of traders in an economy is large, however this was an extremely restricted result.

As has already been noted neither the nucleolus nor kernel appear to have yielded significant economic applications yet.

Solutions, Market Games and the Price System

The class of games known as market games provides a representation of a closed economic system for which a price system exists.

The study of market games with large numbers of participants has shown a remarkable relationship between the imputations selected by a price system and the core, value, bargaining set, kernel and nucleolus of large market games. Each solution concept models or picks up an extremely different aspect of trading. The price system may be regarded as stressing decentralization (with efficiency); the core shows the force of countervailing power; the value offers a "fairness" criterion; the bargaining set and kernel suggest how the solution might be delimited by bargaining conditions; and the nucleolus provides a way to select a point at which dissatisfaction with relative tax loads or subsidies is minimized.

If for a large market economy these many different approaches call for the same imputation of resources then we have what might be regarded as a nineteenth century laissez faire economist's dream. The imputation called for by the price system has virtues far beyond that of decentralization it cannot be challenged by countervailing power, it is fair, it satisfies certain bargaining conditions.

Unfortunately in most economies as we know them these euphoric conclusions do not hold for two important reasons. The first is that there is rarely if ever enough individuals of all types that oligopolistic elements are removed from all markets. The second is that the economies frequently contain elements that modify or destroy the conditions for the existence of an efficient price system. In particular these include external economies and diseconomies, indivisibilities and public goods.

3.3. Public Goods, Externalities and Welfare Economics

When we examine the literature on public goods it is difficult to make a completely clear distinction between economic analysis and political science studies. The basic nature of the problems is such that their investigation requires an approach based on political economy. More or less arbitrarily even though it is related to welfare economics we do not discuss the work on voting systems.

It is well known when externalities are present in an economy, an efficient price system may not exist. It might be that a tax and subsidy system can be designed to make it possible for an efficient price system to function. Shapley and Shubik [92] and Foley [93] have been able to show that a tax system, essentially the one suggested by Lindahl [94] serves the purpose. However, in the former work it is noted that such a system may not be effective when external diseconomies are present. Klevorick and Kramer [95] have worked on a specific taxation scheme for pollution management using prices. Aumann and Kurz [96] have applied a mixed model of threats, prices and the value solution to taxation.

The role of threats is of considerable importance when studying many of the problems posed by externalities and public goods. Features such as can you force an individual to share a public good, can you prevent him from using a good unless he pays his share lead to differentiating many types of public goods. Shubik [97] suggested a taxonomy of public goods based on these considerations.

Considerable application of game theory to tax problems and public finance has been made by Schleicher [98].

Indivisibilities and other features which may cause nonconvexities to be present in the consumption or production possibility sets of the individual have been studied by Shapley and Scarf, Shapley and Shubik, Shubik and others.

Externalities caused by different ownership arrangements [99] as well as pecuniary externalities [100] caused by the presence of markets have been studied. Although not primarily game theoretic in content the work of Buchanan and Tulloch, Davis and Whinston, Zeckhauser and others is closely related to the game theory approach to public goods and welfare economics.

Another aspect of welfare economics where game theory analysis is of direct application involves the study of lump sum taxation, subsidies and compensation schemes. These depend delicately on assumptions made concerning the availability of a sidepayment mechanism and the relationship between social and economic prospects and the structure of individual preferences.

3.4. Money and Financial Institutions

In recent years a considerable interest has been evinced in the construction of an adequate microeconomic theory of money. In general this work has taken as its basis the general equilibrium nonstrategic model of the price system. The writings of Foley, Hahn, Starr and others serve as examples of the statics and Grandmont provides an excellent coverage of the dynamics [101].

In contrast with the nonstrategic approaches Dubey and Shubik [102], [103], Shapley [104], Shapley and Shubik [105], Shubik [106], [107], and others [108], [109] have considered noncooperative game models of trading economies using a commodity or a fiat money. The major thrust of this work has been to suggest that strategic modelling calls for the introduction of rudimentary structures and rules of the game which can be interpreted in terms of markets, financial institutions and laws. In specifying the use of money distinction must be made between money and credit. Bankruptcy laws must be specified [110]. The way money and credit enters the system must be noted. A decision must be made whether or not to model bankers as separate distinguished players in the "money game" [111].

Information conditions clearly are of considerable importance in a mass economy. Dubey and Shubik [112] have noted the sensitivity of market models to changes in information conditions and have obtained a noncooperative equilibrium in markets with nonsymmetric information conditions.

Some results have been obtained using cooperative game theoretic analysis. In particular it has been shown that if trade is assumed to take place via markets then pecuniary externalities are real [100].

Other game theoretic aspects of insurance have been considered by Borch [113].

3.5. Other Applications

There has only been slight application of game theory to macro-economic problems beyond the work of Nyblen [114] and Faxen [115]. These developments are nevertheless suggestive of the possibility of treating aggregated units as players in a game of strategy.

Beyond the applications noted above there have been some scattered papers in economics or topics closely allied to economics in the form of the work of Schleicher [98] on public finance, Shapley on bureaucracy and organization design [6], Shubik on the design of incentive systems [116], Littlechild [51] on operations research costing and pricing problems and Gately and Kyle on cartel problems [117].

REFERENCES

[1] Shapley, L. S. and M. Shubik. "Game Theory in Economics," Rand R-904/1, 2, 3, 4, 6-NSF, 1971-1974.

[2] Von Neumann, J. and O. Morgenstern. Theory of Games and Economic Behavior. Princeton: Princeton University Press, 1944.

[3] Shapley, L. S. and M. Shubik. "Solutions of n-person Games with Ordinal Utilities" (abstract), Econometrica, 21 (1953), 348-349.

[4] Aumann, R. J. and B. Peleg. "Von Neumann-Morgenstern Solutions to Cooperative Games without Sidepayments," Bulletin of American Mathematical Society, 66 (1960), 173-179.

[5] Shapley, L. S. and M. Shubik. "On Market Games," Journal of Economic Theory, 1 (1969), 9-25.

[6] Shapley, L. S. "Simple Games: An Outline of the Descriptive Theory," Behavioral Science, 7 (1962), 59-66.

[7] Kuhn, H. W. "Extensive Games and the Problem of Information," Annals of Mathematics Study, 28 (1953), 193-216.

[8] Rapoport, A. and A. M. Chammah. Prisoner's Dilemma. Ann Arbor: University of Michigan Press, 1965.

[9] Rapoport, A., M. J. Guyer and D. J. Gordon. The 2 x 2 Game. Ann Arbor: University of Michigan Press, 1975.

[10] Rapoport, A. and M. J. Guyer. "A Taxonomy of 2 x 2 Games," General Systems, 11 (1966), 203-214.

[11] Luce, R. D. and H. Raiffa. Games and Decisions. New York: John Wiley, 1957.

[12] Schelling, T. C. The Strategy of Conflict. Cambridge: Harvard University Press, 1960.

[13] Cournot, A. A. Researches into the Mathematical Principles of the Theory of Wealth. New York: Macmillan, 1897. (Translated from French, 1838.)

[14] Aumann, R. J. "The Case of a Cooperative Game without Sidepayment," Transactions of Am. Math. Society, 98 (1961), 539-552.

[15] Billera, L. J. "Some Theorems on the Core of an n-person Game without Sidepayments," SIAM Journal of Applied Math, 18 (1970), 567-579.

[16] Scarf, H. E. and L. S. Shapley. "On Cores and Indivisibility" in Studies in Optimization, ed. by G. Dantzig and C. Eaves. Washington, D.C.: Math. Assoc. of America, 1973.

[17] Shubik, M. Games for Society Business and War. Amsterdam: Elsevier, 1975.

[18] Dresher, M. Games of Strategy: Theory and Applications. Englewood Cliffs, N.J.: Prentice Hall, 1961.

[19] Berkovitz, L. D. and M. Dresher. "A Game Theory Analysis of Tactical Airwar," Operations Research, 7 (1959), 599-620.

[20] Fox, M. and G. S. Kimeldorf. "Noisy Duels," SIAM Journal of Applied Math, 17 (1969), 353-361.

[21] Friedman, A. Differential Games. New York: John Wiley, 1971.

[22] Blaquiere, A. Topics in Differential Games. Amsterdam: North Holland, 1973.

[23] Edgeworth, F. Y. Mathematical Psychics. London: Keagon Paul, 1881.

[24] Shubik, M. Edgeworth Market Games. Seminar Notes, Center for Advanced Study in Behavioral Sciences, Palo Alto, California, 1955.

[25] Shubik, M. "Edgeworth Market Games," Ann. Math. Study, 40 (1959), 267-278.

[26] Shapley, L. S. Unpublished notes and discussions with M. Shubik, 1954-1960.

[27] Debreu, G. and H. E. Scarf. "A Limit Theorem on the Core of an Economy," International Economic Review, 4 (1963), 235-246.

[28] Aumann, R. J. "Markets with a Continuum of Traders," Econometrica, 32 (1964), 39-50.

[29] Milnor, J. W. and L. S. Shapley. Values of Large Games II: Oceanic Games, RM-2649. RAND, Santa Monica, California, 1961.

[30] Shapley, L. S. "Values of Games with Infinitely Many Players," in Recent Advances in Game Theory. Princeton University Conference, Princeton, New Jersey, 1962, 113-118.

[31] Chamberlin, E. H. The Theory of Monopolistic Competition. Cambridge: Harvard University Press, 1950, 6th ed.

[32] Shapley, L. S. and M. Shubik. "Price Strategy Oligopoly with Production Variation," Kyklos, 22, 1 (1969), 30-44.

[33] Bewley, T. "Existence of Equilibria in Economies with Infinitely Many Commodities," Journal of Economic Theory, 4 (1972), 514-540.

[34] Milnor, J. W. Reasonable Outcomes for n-Person Games. RM-916 RAND, Santa Monica, California, 1952.

[35] Gillies, D. "Solutions to General Nonzero sum Games," Annals Math. Study, 40 (1959), 47-85.

[36] Shapley, L. S. Markets as Cooperative Games, P-B29 RAND, Santa Monica, California, 1955--but actually introduced earlier in unpublished lecture notes in 1953-54 seminar at Princeton.

[37] Shapley, L. S. "On Balanced Games without Sidepayments," Mathematical Programming, T. C. Hu and S. M. Robinson, eds. New York: Academic Press, 1973.

[38] Billera, L. J. and Bixby, R. E. "A Characterization of Polyhedral Market Games," International Journal of Game Theory, 2, 4 (1973), 252-61.

[39] Shapley, L. S. "A Value for N-Person Games," Ann. of Math. Study, 28 (1953), 307-317.

[40] Nash, J. F., Jr. "Two-person Cooperative Games," Econometrica, 21 (1953), 128-140.

[41] Harsanyi, J. C. "A Bargaining Model for the Cooperative n-Person Game," Ann. Math. Study, 40 (1959), 325-355.

[42] Shapley, L. S. "Values of Large Market Games: Status of the Problem," RAND RM-3957-PR, 1964.

[43] Owen, G. "Multilinear Extensions of n-Person Games," Management Science, 18, 5, Part II (1972), P-64-P-79.

[44] Aumann, R. J. and L. S. Shapley. Values of Nonatomic Games. Princeton: Princeton University Press, 1974.

[45] Dubey, P. "Some Results on Values of Finite and Infinite Games," Ph.D Thesis, Cornell University, 1975.

[46] Lucas, W. F. "The Proof That a Game May not Have a Solution," Trans. Amer. Math. Soc., 137 (1969), 219-229.

[47] Aumann, R. J. and M. Maschler. "The Bargaining Set of Cooperative Games," in Advances in Game Theory, ed. by M. Dresher, L. S. Shapley, and A. W. Tucker. Princeton: Princeton University Press, 1964, 443-447.

[48] Peleg, B. "Existence Theory for the Bargaining Set $M_1^{(i)}$," Essays in Mathematical Economics: In Honor of Oskar Morgenstern. Princeton: Princeton University Press, 1967, 53-56.

[49] Davis, M. and M. Maschler. "The Kernel of a Cooperative Games," Naval Research Logistics Quarterly, 12 (1965), 223-295.

[50] Schmeidler, D. "The Nucleolus of a Characteristic Function Game," SIAM Journal of App. Math, 17 (1969), 1163-1170.

[51] Littlechild, S. C. "A Simple Expression for the Nucleolus in a Special Case," International Journal of Game Theory, 3, 1 (1974), 21-29.

[52] Shapley, L. S. and M. Shubik. "Quasi-Cores in an Economy with Nonconvex Preferences," Econometrica, 34 (1966), 805-827.

[53] Nash, J. F., Jr. "Equilibrium Points in n-Person Games," Proc. Nat. Acad. Sci., U.S.A., 36 (1950), 48-49.

[54] Dubey, P. and L. S. Shapley. "Noncooperative Exchange with a Continuum of Traders," RAND, August 1977, P-5964.

[55] Selten, R. "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit," Zeitschrift für die gesamte Staatswissenschaft, 121, 301-324 and 667-689 (1965).

[56] Selten, R. "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory, 4 (1975), 25-56.

[57] Harsanyi, J. C. "The Tracing Procedure: A Bayesian Approach to Defining a Solution for n-Person Noncooperative Games," International Journal of Game Theory, 4 (1975), 61-94.

[58] Dubey, P. and M. Shubik. "Information Conditions, Communication and General Equilibrium," CFDP 467, October 1977.

[59] Sobel, M. J. "Non Cooperative Stochastic Games," Annals of Mathematical Statistics, 42 (1971), 1095-1100.

[60] Sobel, M. J. "Continuous Stochastic Games," Journal of Applied Probability, 10 (1973), 597-604.

[61] Bertrand, J. "Theorie Mathematique de la Richesse Sociale" (Review), Journal des Savants, Paris (1883), 499-508.

[62] Edgeworth, F. Y. Papers Relating to Political Economy I. London: MacMillan, 1925, 111-142.

[63] Hotelling, H. "Stability in Competition," Economic Journal (1929), 41-57.

[64] Wald, A. "On Some Systems of Equations of Mathematical Economics," Econometrica, 19 (1951), 368-403.

[65] Smithies, A. and L. J. Savage. "A Dynamic Problem in Duopoly," Econometrica, 8 (1940), 130.

[66] Shubik, M. and G. Thompson. "Games of Economic Survival," Naval Logistics Research Quarterly, 6 (1959), 111-123.

[67] Cyert, R. M. and H. M. De Groot. "An Analysis of Cooperation and Learning in a Duopoly Context," American Economic Review, 63 (1973), 24-37.

[68] Levitan, R. E. and M. Shubik. "Price Variation Duopoly with Differentiated Products and Random Demand," Journal of Economic Theory, 3 (1971), 23-39.

[69] Kirman, A. P. and M. J. Sobel. "Dynamic Oligopoly with Inventories," Econometrica, 42, 2 (1974), 279-287.

[70] Cross, J. G. The Economics of Bargaining. New York: Basic Books, 1969.

[71] Shubik, M. Strategy and Market Structure. New York: John Wiley and Sons, 1959.

[72] Levitan, R. E. "Oligopoly Demand," IBM Research, Yorktown Heights, RC 1239, July 23, 1964.

[73] Shubik, M. "Information Duopoly and Competitive Markets: A Sensitivity Analysis," Kyklos, 26 (1973), 736-761.

[74] Marschak, T. and R. Selten. General Equilibrium with Price Making Firms. Berlin: Springer, 1974.

[75] Selten, R. "A Simple Model of Imperfect Competition Where 4 Are Few and 6 Are Many," International Journal of Game Theory, 2, 3 (1973), 141-201.

[76] Friedman, J. W. Oligopoly and the Theory of Games. North Holland, Amsterdam, 1977.

[77] Jacot Simon-Pierre. Strategie et Concurrence. Paris: SEDES, 1963.

[78] Telser, L. Competition Collusion and Game Theory. Chicago: University of Chicago Press, 1972.

[79] Shapley, L. S. and M. Shubik. "The Assignment Game I: The Core," International Journal of Game Theory, 2 (1972), 111-130.

[80] Cassady, R. Auctions and Auctioneering. Berkeley: University of California Press, 1967.

[81] Stark, R. M. and M. H. Rothkopf. "Competitive Bidding: A Comprehensive Bibliography," University of Delaware, Mimeographed, April 1977.

[82] Shubik, M. "The Core of a Market with Exogenous Risk and Insurance," New Zealand Economic Papers, 7 (1973), 121-127.

[83] Scarf, H. E. "The Core of an N-Person Game," Econometrica, 35 (1967), 50-69.

[84] Hildenbrand, W. Core and Equilibria of a Large Economy. Princeton: Princeton University Press, 1974.

[85] Aumann, R. J. "Disadvantageous Monopolies," Journal of Economic Theory, 6 (1973), 1-11.

[86] Shitovitz, B. "Oligopoly in Markets with a Continuum of Traders," Econometrica, 41, 3 (1973), 467-501.

[87] Caspi, Y. "A Limit Theorem on the Core of an Economy Under Uncertainty," Hebrew University, RR 43, December 1972.

[88] Shubik, M. "The Case of a Market with Exogenous Risk and Insurance," New Zealand Economic Papers, 7 (1973), 121-127.

[88] Debreu, G. "The Rate of Convergence of the Core of an Economy," Journal of Mathematical Economics, 2 (1975), 1-7.

[89] Shapley, L. S. and M. Shubik. "Pure Competition, Coalitional Power and Fair Division," International Economic Review, 10 (1969), 337-362.

[90] Shapley, L. S. and M. Shubik. "Convergence of the Bargaining Set for Differentiable Market Games," working notes, mimeographed, 1972.

[91] Shapley, L. S. and M. Shubik. "On the Case of an Economic System with Externalities," American Economic Review, 59, 4 (1969), Part I, 678-684.

[92] Foley, D. K. "Lindahl's Solution and the Core of an Economy with Public Goods," Econometrica, 38, 1 (1970).

[93] Lindahl, E. "Just Taxation--A Positive Solution," reprinted in part in R. A. Musgrave and A. E. Peacock (eds.), Classics in the Theory of Public Finance. London: MacMillan, 168-176. Original in German, 1919.

[94] Klevorick, A. K. and G. H. Kramer. "Social Choice on Pollution Management: The Genossenschaften," Journal of Public Economics, 2 (1973), 101-146.

[95] Aumann, R. J. and M. Kurz. "Power and Taxes," Econometrica, 45 (1977), 1137-1163.

[96] Shubik, M. "Notes on the Taxonomy of Problems Concerning Public Goods," CFDP 208, April 19, 1966.

[97] Schleicher, H. Staatshaushalt und Strategie. Berlin: Dunker and Humblot, 1971.

[98] Shapley, L. S. and M. Shubik. "Ownership and the Production Function," Quarterly Journal of Economics, 81 (1967), 88-111.

[99] Shubik, M. "Pecuniary Externalities: A Game Theoretic Analysis," American Economic Review, 61, 4 (1971), 713-718.

[100] Grandmont, J. M. "Temporary General Equilibrium Theory," Econometrica, 45 (1977), 535-572.

[101] Dubey, P. and M. Shubik. "The Noncooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies," Journal of Economic Theory (forthcoming 1978).

[102] Dubey, P. and M. Shubik. "A Closed Economic System with Production and Exchange Modelled as a Game of Strategy," Journal of Mathematical Economics (forthcoming 1977).

[103] Shapley, L. S. "Noncooperative General Exchange," in Theory and Measurement of Economic Externalities, ed. by A. Y. Lin. New York: Academic Press, 1976.

[104] Shapley, L. S. and M. Shubik. "Trade Using One Commodity as a Means of Payment," Journal of Political Economy (1977), 937-968.

[105] Shubik, M. "Commodity Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model," Western Economic Journal, 10, 4 (1972), 24-38.

[106] Shubik, M. "The General Equilibrium Model Is Incomplete and Not Adequate for the Reconciliation of Micro and Macroeconomic Theory," Kyklos, 28 (1975), 545-573.

[107] Postlewaite, A. W. and D. Schmeidler. "Approximate Efficiency of Non Walrasian Nash Equilibria," Econometrica (forthcoming 1978).

[108] Shubik, M. and W. Whitt. "Fiat Money in an Economy with One Nondurable Good and No Credit (A Noncooperative Sequential Game)," in Topics in Differential Games, ed. by A. Blaquiere. Amsterdam: North Holland Publishing Co., 1973, 401-448.

[109] Shubik, M. and C. Wilson. "The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money," Zeitschrift fur Nationalokonomie (forthcoming 1977).

[110] Shubik, M. "A Noncooperative Model of a Closed Economy with Many Traders and Two Bankers," Zeitschrift fur Nationalokonomie (1976), 10-18.

[111] Dubey, P. and M. Shubik. "Trade and Prices in a Closed Economy with Exogenous Uncertainty Different Levels of Information, Money and Compound Futures Markets," Econometrica (forthcoming 1977).

[112] Borch, K. The Economics of Uncertainty. Princeton: Princeton University Press, 1968.

[113] Nyblen, G. The Problem of Summation in Economic Science. Lund Gleerup, 1951.

[114] Faxen, K. O. Monetary and Fiscal Policy under Uncertainty. Stockholm: Almqvist and Wiksell, 1957.

[115] Shubik, M. "The Assignment of Joint Costs and Internal Pricing," Management Science, 8, 2 (1962), 325-342.

[116] Gately, D. and J. F. Kyle. "Optimal Strategies for OPEC's Pricing Decisions," New York University, Center for Applied Economics Discussion Paper 76-13, November 1976.