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THE EFFECTS OF MONEY SUPPLY ON ECONOMIC WELFARE IN THE STEADY STATE*

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The idea that money has something to do with the intermediation of intertemporal exchange between agents of successive generations was recognized by Samuelson (1958) in his article on consumption loans. In that paper it was demonstrated that by introducing a fixed nominal supply of fiat money the optimal steady state allocation could be supported in competitive equilibrium. All mutually advantageous exchanges could be affected by trades in an asset with the return characteristics of money. An implication of this is that any successful attempt on behalf of the monetary authority to make the return to money holding an object of deliberate decision would necessarily reduce economic welfare in the steady state.

As a theory of optimal money supply, the consumption loan model is incomplete. How can money compete with claims to productive assets which can also serve to transfer wealth over time? Does uncertainty, broadly defined, modify the conclusion that monetary policy which holds constant the nominal money supply yields an optimal allocation? This paper extends the familiar overlapping generations model to analyze the role of money and monetary policy in providing financial intermediation in light of these considerations.

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The view that money can intermediate trade is not inconsistant with the idea that real cash balances yield services to its holder, much like physical capital. Friedman (1969) and others have emphasized this aspect of money in their discussion at the optimal quantity of money. Under the assumption that the provision of monetary services requires no real resources, it is argued that an optimal policy will maximize this amenity flow eminating from real balances. To induce households to hold this satiety level of money requires that the financial return to money equal that available on other assets. If money has an own nominal rate of zero, this is attained when the rate of deflation equals the real rate of interest. Such a prescription will be shown to be correct only in a model of infinitely long lived, static individuals who are endowed with money and have no reason to alter their holdings of real balances. If, however, real balances are acquired through exchange with a view towards their eventual disposition, effects of a change in the level of real balances must include a specification of the adjustments in either consumption or real asset holdings necessary to finance their acquisition. Such considerations alter the characterization of the optimal rate of inflation.

The paper is organized into two sections. Although both parts are about the effects of money creation on economic welfare in environments of rational expectations, homogeneous information, and competitive equilibrium, each is addressed to different issues. The first section analyzes the effects of alternative anticipated, constant money growth rates upon equilibrium real balances and capital intensity in a model which treats real balances as a consumption good. The principal result in this section is that although the "optimal" level of real balances will be induced under a policy which holds constant nominal money supply, the maximum sustainable utility level will require positive growth in money supply. The effects of inflation upon capital accumulation are shown to be qualitatively more important than the misallocation in the supply of real balances, at least for "small" rates of money growth. The second section analyzes the complications introduced by uncertainty. The model implies that whenever the real return to money holding is uncertain under a constant nominal supply policy, some nontrivial active response will be necessary for the competitive allocation to yield the maximum possible expected utility. An example is presented in which active policy increases welfare. The third and concluding section contains some interpretations of the principal analytic results.

I. The Deterministic Model[1]

People live two periods, young and old, and are endowed with a unit of exogenously supplied labor when young. Each individual derives utility from consumption when young, c_1, consumption when old, c_2, and his holdings of real balances m, evaluated at acquisition cost. The preference of each individual is expressed by the concave, increasing, additive, separable function

(1) $$U(c_1, c_2, m) = U_1(c_1) + L(m) + U_2(c_2) .$$

The production technology is homogeneous of degree one, so output per unit of labor, y_t, depends only upon the capital labor ratio, k_t:

(2) $$y_t = f(k_t) .$$

Output may be consumed or invested. There is no depreciation so that the real rate of interest is equal to the marginal product of capital

(3) $$r = f'(k_t) \ .$$

The wage rate, w_t , is equal to the marginal product of labor

(4) $$w = f(k_t) - k_t f'(k_t) \ .$$

Labor supply (and population) grows at a rate g

(5) $$L_{t+1} = (1+g)L_t \ .$$

The nominal money supply, M_t , grows at a steady preannounced rate ρ ,

(6) $$M_{t+1} = (1+\rho)M_t \ .$$

Although other specifications could be employed, the proceeds from money creation are assumed to be distributed to old people before trade begins on an equal per capita basis, unaffected by initial holdings.[2] In this way money growth is like a tax on real balances, the proceeds from which accrue to those who bear the burden of the tax.

In the steady state, factor supplies, including aggregate real balances, grow at a common rate g . Letting m^* be real balances per young person, the aggregate real value of money at time t is equal to $L_0(1+g)^t m^*$ and the real value per unit is thus $m^*[L_0(1+g)^t/M_0(1+\rho)^t]$. The real return to holding money between two periods (equal to minus the inflation rate) is given by

(7) $$\frac{L_0(1+g)^{t+1}}{M_0(1+\rho)^{t+1}} \bigg/ \frac{L_0(1+g)^t}{M_0(1+\rho)^{t+1}} - 1 = \frac{1+g}{1+\rho} - 1 = \frac{g-\rho}{1+\rho}$$

At time $t+1$ there are $\rho(1+\rho)^t M_0$ new dollars distributed among the $L_0(1+g)^t$ old people, so that the proceeds from new money creation in real terms is, for each old person, equal to

$$(8) \qquad m^*\left[\frac{L_0(1+g)^{t+1}}{M_0(1+\rho)^{t+1}}\right]\left[\frac{\rho M_0(1+\rho)^t}{L_0(1+g)^t}\right] = m^*\frac{\rho(1+g)}{1+\rho} \ .$$

Each agent when young decides how to allocate his wealth among consumption, money holding, and capital holding to maximize his utility. He takes factor rewards and the proceeds from the creation of money (possibly negative) as given. If k^* is capital acquired when young, the capital labor ratio is $k^*/(1+g)$. In the steady state, the problem of each agent is to

$$\operatorname*{Max}_{m,k}\ U_1(c_1) + L(m) + U_2(c_2)$$

subject to

$$c_1 = w - m - k$$

$$c_2 = m\left[1 + \frac{g-\rho}{1+\rho}\right] + k(1+r) + m^*\frac{\rho(1+g)}{1+\rho}$$

where

$$w = f(k^*/(1+g)) - (k^*/(1+g))f'(k^*/(1+g))$$

$$r = f'(k^*/(1+g)) \ .$$

The necessary first order conditions for a steady state allocation to be a competitive equilibrium are

$$(9) \qquad -U_1'(w - m^* - k^*) + L'(m^*) + \left(\frac{1+g}{1+\rho}\right) U_2'(m^*(1+g) + k^*(1+r)) = 0$$

$$(10) \qquad -U_1'(w - m^* - k^*) + (1+r)U_2'(m^*(1+g) + k^*(1+r)) = 0 \ .$$

Equation (9) implies that money holdings are optimal given a real yield of $(g-\rho)/(1+\rho)$ and equation (10) implies that capital holdings are individually optimal given a real yield of r.

An implication of this formulation is that the competitive equilibrium forthcoming under a monetary policy which holds constant nominal money supply $(\rho \equiv 0)$ is characterized by a real rate of interest which must exceed the growth rate since

$$(11) \qquad r - g = \frac{L'(m)}{U_2'(c_2)} > 0 \ .$$

The availability of an asset with a real return of g, in addition to a positive amenity return, precludes the possibility of dynamic inefficiency associated with overcapitalization. If, however, the amenity return to money holding admitted the possibility of satiation $(L' \leq 0)$ this result would no longer hold.

Any discussion of the welfare aspects of alternative monetary policies must address both the criterion of optimality employed and strategies available to the monetary authority. In this paper, emphasis is on the effects of altering the real return to money holding. Thus the policy authority is assumed to control only the size of the nominal transfer to each old person. Particularly, the authority is denied the possibility of acquiring real capital.[3] The criterion of optimality is the welfare of an individual in the steady state.

Policy has real effects upon welfare only by altering factor supplies. An optimal policy will induce the optimal mix of real balances and real capital holdings. Neglecting, in the first instance, the implications of policy upon real capital formation, what determines the optimal quantity of real balances? Suppose a social planner could by legislative fiat fix the terms on which the young acquired the money balances of the old. Since there are $(1+g)$ old people per young person, the social planner could give each old person $1+g$ units of consumption for each unit taken from a young person. Holding constant factor supplies and rewards, he would seek to find that level of real balances, m, which maximizes the utility of a typical individual. The necessary first order condition for this maximum is

$$(12) \qquad -U_1'(w-m-k^*) + L'(m) + (1+g)U_2'(m(1+g)+k^*(1+r)) = 0$$

which is precisely condition (9) which determines the competitive value of real balances under a constant nominal supply policy.

The distinction between this result and the conclusion of Friedman arises from the assumption that changes in the aggregate value of money must be accompanied by non-trivial changes in the real allocation of consumption over time. The rate of interest available on sustainable real balances is g. If this is less than the available yield on capital, an increase in real money holdings implies a loss of real income. Evaluated at the equilibrium under a constant nominal supply, this loss is just equal to the increase in utility afforded by access to higher real balances. Since the utility function is concave, any finite movement away from this position must decrease welfare. <u>When the real return to money is</u> g, <u>the competitively determined level of real balances is optimal</u>.

However, this is not sufficient to guarantee that the maximal sustainable utility will be attained at a zero growth rate of nominal money supply. Lowering the return to money will induce a substitution in favor of capital accumulation. In general, this will raise wages and lower the return to capital. If the initial situation is characterized by a rate of interest which exceeds the growth rate, the changes in factor payments induced by changes in factor supplies will have a first order effect on the welfare of the typical individual. The utility loss arising from a misallocation in real balances induced by inflation is a second order consideration, much as the "dead weight" loss from taxation is on the second order of smalls.

This may be derived rigorously by differentiating equations (9) and (10) with respect to ρ in the usual Hicksian manner. It may be shown that:

$$(13) \qquad \left.\frac{\partial k^*}{\partial \rho}\right|_{\rho=0} = \frac{-(1+g)U_1'[U_1''+(1+r)(1+g)U_2'']}{\Delta}$$

where

$$\Delta = U_1''U_2''(r-g)^2 + L''(U_1'' + (1+r)^2U_2'' + kf''(U_1''/(1+g)^2 + \frac{1+r}{1+g}U_2''))$$
$$+ \frac{U_2'}{1+g}f''(U_1'' + L'' + (1+g)^2U_2'')$$

is the determinant of the Jacobian of second derivatives. If the equilibrium is stable, this will be positive, so that equilibrium capital intensity will increase with money growth.

The change in the utility of an individual across steady states is given by

$$(14) \qquad \left.\frac{\partial U}{\partial \rho}\right|_{\rho=0} = \left(\frac{\partial w}{\partial k^*}U_1' + k^*\frac{\partial r}{k^*}U_2'\right)\frac{\partial k^*}{\partial \rho} + \frac{\partial m^*}{\partial \rho}\{-U_1' + L' + (1+g)U_2'\}$$

$$+ \frac{\partial k^*}{\partial \rho}\{-U_1' + (1+r)U_2'\}$$

which, after substituting

$$\frac{\partial r}{\partial k^*} = \frac{f''}{1+g}, \quad \frac{\partial w}{\partial k^*} = -\frac{k^* f''}{(1+g)^2}$$

$$\text{equals} = -k^* f'' U_1' \left(\frac{r-g}{(1+r)(1+g)^2}\right)\frac{\partial k^*}{\partial \rho} > 0 \ .$$

Thus so long as factor rewards are influenced by factor supplies ($f'' < 0$) steady state welfare increases with money growth, evaluated at a position of zero money growth. By the envelope theorem, there is no direct change in welfare induced by changes in the endogenous variables, irrespective of the magnitudes of these changes.[4]

II. Modifications Introduced by Uncertainty

Uncertainty is thought to motivate money holding. Tobin (1956) showed that money might be held as part of a risk avoidance strategy even if other assets offering higher expected returns are available. His analysis was partial equilibrium; there was no explanation of how money enlarged the opportunities for aggregate savings or the role of money in the allocation of social risk. These issues are of primary importance for assessing the merits of alternative monetary policies in environments of continuous capacity utilization.

One way to introduce uncertainty is to simply assume that the physical return accruing to aggregate capital is random, and that investment must be made before the realization of this uncertainty. To avoid dealing with problems of optimal capital accumulation, the population will be assumed stationary and the production function will be made additive and separable

(15) $$Y_t = wL_t + \tilde{\alpha}_t K_t$$

where $\tilde{\alpha}_t$ is random, distributed identically and independently over time. This implies that the real wage per young worker is w, constant over time. The <u>ex ante</u> yield available to real capital accumulation is also time independent. Thus the values of per capita real balances, m, and capital holdings per person, k, will be constant over time. The competitive solution under a constant nominal supply policy is determined by

(16) $$-U_1'(w-m-k) + L'(m) + E_{\tilde{\alpha}} U_2'(m+k\tilde{\alpha}) = 0$$

(17) $$-U_1'(w-m-k) + E_{\tilde{\alpha}} \tilde{\alpha} U_2'(m+k\tilde{\alpha}) = 0 \ .$$

Analogous to the preceeding discussion, the competitively determined value of real balances is such that it maximizes the expected utility of a typical individual relative to any other constant value.[5]

This model makes explicit the idea that money can serve to make available a safe return to aggregate savings which could not be duplicated by claims to physical capital. Furthermore, this property does not alter the proposition that a constant nominal stock of money is optimal relative to other constant growth rates of money supply, when factor rewards are independent of factor supplies.

However if the real return to money holding under a constant nominal supply policy is random, then the competitively determined value of real balances need not be optimal. This will arise, for example, in a model where the expectations of each generation regarding the return to investment is, itself, a source of uncertainty. Keynes (1936) wrote that the state of long term expectations (those concerning the marginal efficiency of investment) was "liable to sudden revision" and made this point central to his theory of income determination.

To capture the idea that people alter their expectations of the profitability of new investment, it will be assumed that the return to capital is announced with certainty one period ahead,[6] before investment is undertaken. The gross return, α, is random from period to period, distributed independently and identically in each period. The equilibrium values of money and per capita holdings of real capital will, in general, depend upon the particular announcement of α. An equilibrium is defined as two functions, $k(\alpha)$ and $m(\alpha)$ which are competitively sustainable given that individuals have rational expectations of future values. If the population and nominal money stock is constant over time this requires that for each α:

$$(18) \qquad -m(\alpha)U_1'(w - m(\alpha) - k(\alpha)) + m(\alpha)L'(m(\alpha)) + E_{\alpha'}m(\alpha')U_2'(m(\alpha') + \alpha k(\alpha)) = 0$$

$$(19) \qquad -U_1' + E_{\alpha'}\alpha U_2'(m(\alpha') + \alpha k(\alpha)) = 0$$

where α' is the expectation of the young next period of the yield to their real investments. In Appendix [A] suitable conditions on preferences and the distribution of α' are found to assure that such an equilibrium exists.

Is the competitively determined function $m(\alpha)$ optimal?[7] Suppose a social planner could choose this function arbitrarily and sought to maximize the expected utility of a typical agent, holding constant the function

$k(\alpha)$. The necessary first order condition for an optimal function $m(\alpha)$ is, for each α :

$$(20) \qquad \frac{\partial E[U]}{\partial m(\alpha)} = -U_1'(w - m(\alpha) - k(\alpha)) + L'(m(\alpha)) + E_{\alpha'}U_2'(\alpha' k(\alpha') + m(\alpha)) = 0 \ .$$

This criterion is different from the competitive solution since, in general

$$E_{\alpha'} m(\alpha')U_2'(m(\alpha') + \alpha k(\alpha)) \neq E_{\alpha'} m(\alpha)U_2'(m(\alpha) + \alpha' k(\alpha'))$$

unless the distribution of α' is degenerate, or $m(\alpha)$ and are $k(\alpha)$ are both constant over their domain.

The allocation of available consumption between young and old which maximizes the expected utility of each agent differs from the competitive solution because of the limited transaction opportunities for achieving intergenerational trade. There is no prehistoric meeting whereby agents can insure the value of their real balances. The uncertainty associated with the return to money holding does not stem from random components in the aggregate endowment of resources; it is an example of a private risk which is not a social risk.

The Role of Policy

Monetary policy can influence the state dependent equilibrium level of real balances. Thus, it influences the division of available consumption between young and old and alters opportunities for intergenerational trades. The role of policy is to duplicate that allocation which would have attained had agents access to complete contingent markets.

An example is presented to show that policy can improve welfare.

Assume that the von Neumann-Morgenstern utility function for each agent is given by

(21) $$\log c_1 + \log m + \log c_2 \ .$$

For a constant population and nominal money supply, the equilibrium conditions [equations (18) and (19)] are, for each α :

(22) $$-\frac{m(\alpha)}{w-m(\alpha)-k(\alpha)} + 1 + E_{\alpha'} \frac{m(\alpha')}{\alpha k(\alpha)+m(\alpha')} = 0$$

(23) $$-\frac{1}{w-m(\alpha)-k(\alpha)} + E_{\alpha'} \frac{\alpha}{\alpha k(\alpha)+m(\alpha')} = 0 \ .$$

Multiplying equation (23) by $k(\alpha)$ and adding to equation (22) shows that $[m(\alpha)+k(\alpha)]/[w-m(\alpha)-k(\alpha)] = 2$, independent of α . Since w is constant, by assumption, the consumption of the young $w-m(\alpha)-k(\alpha)$ is also constant. From equation (23) this implies that $k(\alpha)$ is increasing in α ,[8] so that $m(\alpha)$ is decreasing. Favorable expectations of the profitability of new investment induce a substitution out of money holdings into real capital. This implies that money holding is risky, since the return depends upon the state of future expectations.

The proper role of policy in the model is to make the return to money holding safer than would occur under constant policy. Suppose that the policy authority announces that it will expand the nominal money supply by $\rho(\alpha)$% next period when the productivity of capital is α . The proceeds accrue to old people in an amount independent of initial holdings. Defining $\phi(\alpha) = \rho(\alpha)/[1+\rho(\alpha)]$, the real return to money holding between two consecutive periods is $(1-\phi(\alpha))[m(\alpha')/m(\alpha)]$ minus one, where $m(\alpha)$ and $m(\alpha')$ are the per capita values in the first and second periods. For a particular function $\phi(\alpha)$, and a scalar, a , the competitive equilibrium is characterized by

(24) $$\frac{-m(\alpha)}{w - m(\alpha) - k(\alpha)} + 1 + E_{\alpha'}(1 - a\phi(\alpha))\frac{m(\alpha')}{\alpha k(\alpha) + m(\alpha')} = 0$$

(25) $$\frac{-1}{w - m(\alpha) - k(\alpha)} + E_{\alpha'} \frac{\alpha}{\alpha k(\alpha) + m(\alpha')} = 0 \ .$$

In appendix [B] it is demonstrated that there exists a function $\phi^*(\alpha)$ such that

(26) $$\left.\frac{\partial m(\alpha)}{\partial a}\right|_{a=0} = \alpha - \bar{\alpha} \ .$$

The function $\phi^*(\alpha)$ when implimented on small scale will make the equilibrium value of real balances more nearly constant. The welfare effects of this policy may be evaluated by

(27) $$\left.\frac{\partial EU}{\partial a}\right|_{a=0} = E_\alpha \frac{\partial m(\alpha)}{\partial a} \frac{1}{w - m(\alpha) - k(\alpha)} + E_\alpha \frac{\partial m(\alpha)}{\partial a} \frac{1}{m(\alpha)}$$

$$+ E_{\alpha'} \frac{\partial m(\alpha)}{\partial a} \frac{1}{\alpha k(\alpha) + m(\alpha')}$$

$$= E_\alpha (\alpha - \bar{\alpha})\frac{3}{w} + E_\alpha \frac{\alpha - \bar{\alpha}}{m(\alpha)} + E_{\alpha'} \frac{\alpha' - \bar{\alpha}}{\alpha k(\alpha) + m(\alpha')} \ .$$

The first expression is zero. Since $m(\alpha)$ is decreasing, $1/m(\alpha)$ is increasing so that the second and third expressions are necessarily positive. The effects upon welfare of altering the equilibrium capital stock may be neglected by appealing to the envelope theorem. <u>By making the value of aggregate real balance more nearly constant, policy can increase the expected utility of each individual</u>.

III. Concluding Remarks

This paper has emphasized the role of money as a store of value which can intermediate intertemporal trade. This function is consistent with the idea that there is also an amenity, or service return, attached to holding real balances. The latter function is not meant as a literal description of reality, but as a way of summarizing the complex and esoteric function of money as economizing on transaction costs.

In the absence of uncertainty, the role of policy is to influence the division of savings between real balances and physical capital accumulation. This view of policy was first expressed by Tobin (1965) in his article on money and economic growth. However, because that model did not contain an explicit description of individuals' preferences no welfare comparisons of alternative policies was possible. The present analysis demonstrates that under plausible conditions the maximum sustainable utility will be obtained under positive money growth rates.

Endogenous, uncertain changes in the equilibrium value of money alters the role of policy. By influencing the return to money holding in each state, policy can alter the range of available transactions. In a sequence economy characterized by uncertainty there is no reason to suspect that all mutually advantageous intergenerational trades will be exhausted in competitive equilibrium. The "efficient market hypothesis" as put forth by Fama and others denies the possibility of arbitrage profits. In the absence of complete contingent markets, this does not imply Pareto efficiency. The role of policy is to create an asset which can overcome the failure of markets to exist.

Unfortunately, this does not yield any simple and general prescription of just how this ought to be accomplished. An example concluded that a policy which stabilized the equilibrium value of real balances would improve economic welfare. This rested crucially on the assumption that money was the only vehicle for affecting intergenerational transfers. Clearly, if capital lasts longer than a single period, claims to physical capital could also serve this function. If the technology for producing new capital goods was such that the price of capital in terms of consumption was influenced by demand for new capital, then the state of long term expectations would influence the price of existing capital goods. If the proper role of monetary policy is to stabilize the value of the portfolio of assets which affect intermediation, then it might not be optimal to stabilize the aggregate value of real balances, which is only one component of this portfolio.

The traders in this model have no bequest motive; they begin and end life without assets. The issues addressed would not, however, be irrelevant in a world of optimal bequests. If agents are actively engaged in intertemporal trades, and use money as a store of value, policy will influence the division of output between those who wish to dispose of real balances and those who wish to acquire additional real balances. Policy should aim to duplicate that allocation which would have obtained had agents been able to share risks optimally.

APPENDIX A

The existence of a pair of continuous functions $m(\alpha)$ and $k(\alpha)$ satisfying

$$(15) \qquad -m(\alpha)U_1'(w - m(\alpha) - k(\alpha)) + m(\alpha)L'(m(\alpha)) + E_{\alpha'}m(\alpha')U_2'(m(\alpha') + \alpha k(\alpha)) = 0$$

$$(16) \qquad -U_1'(w - m(\alpha) - k(\alpha)) + E_{\alpha'}\alpha U_2'(m(\alpha') + \alpha k(\alpha)) = 0$$

will be proved under the following assumptions

$$(A1) \qquad \lim_{x \to 0} U_1'(x) = \infty$$

$$(A2) \qquad \lim_{x \to 0} xL'(x) > 0$$

$$xL'(x) < k \quad \text{some } k, \text{ all } x$$

$$(A3) \qquad 0 < U_2'(x) < k \quad \text{some } k, \text{ all } x$$

$$(A4) \qquad \alpha' \in Z = \{\alpha_1, \ldots, \alpha_n\}, \quad 0 \le \alpha_i \le b \quad \text{all } i .$$

(A1) is sufficient to conclude that there is positive consumption when young. (A2) will insure that the value of money in equilibrium is alway positive.

Let

$$A = \{x | 0 \le x \le w\}$$

$$B = \{(x,y) \in A \times A | x+y \le w\} .$$

Define $\Psi_1 : B \to R^+$

by $\Psi_1(m,k) = mU_1'(w-m-k)$

$\Psi_2 : B \to R^+$

by $\Psi_2(m,k) = kU_1'(w-m-k)$.

Let $G : R^+ \times A \to A$ be such that

$$G(\Psi(x,y),\ x) = y \quad \text{for} \quad x < w$$

$$= 0 \quad \text{for} \quad x = w$$

G is continuous over its domain since $\lim_{x \to w} G(\cdot,x) = 0$.

For any pair of functions $f(\alpha)$, $h(\alpha)$, $\alpha \varepsilon Z$, define the mapping

$$T(f(\alpha),\ h(\alpha)) = G(f(\alpha)L'(f(\alpha)) + E_{\alpha'}f(\alpha')U_2'(f(\alpha') + \alpha h(\alpha)),\ h(\alpha))$$

$$G(E_{\alpha'}\alpha h(\alpha)U_2'(f(\alpha') + \alpha h(\alpha)),\ f(\alpha)) \ .$$

Let $C \subset R^N$ be the set of vectors x such that $0 \leq x_i \leq w$. The set C is compact and convex. $T(\ )$ is a continuous mapping of the set $C \times C$ into itself, so by Brouwers Fixed Point Theorem it has a fixed point. This fixed point satisfies equations (18) and (19).

APPENDIX B

The existence of a function $\phi^*(\alpha)$ with the properties described in the text [equation (21)] is proved by construction.

Totally differentiating the necessary first order conditions [equations (19) and (20)] with resepct to the scalar a for an arbitrary function $\phi(\alpha)$ yields the pair of equations:

$$\begin{bmatrix} -\frac{1}{w-m-k}-\frac{m}{(w-m-k)^2} & -\frac{m}{(w-m-k)^2}-\alpha E\frac{m'}{(\alpha k+m')^2} \\ -\frac{1}{(w-m-k)^2} & -\frac{1}{(w-m-k)^2}-\alpha^2 E\frac{1}{(\alpha k+m')^2} \end{bmatrix} \begin{bmatrix} \frac{\partial m(\alpha)}{\partial a} \\ \frac{\partial k(\alpha)}{\partial a} \end{bmatrix}$$

$$= \begin{bmatrix} \phi(\alpha)E\frac{m'}{\alpha k+m'} - E\frac{\partial m'}{\partial a}\frac{\alpha k}{(\alpha k+m')^2} \\ E\frac{\partial m'}{\partial a}\frac{\alpha}{(\alpha k+m')^2} \end{bmatrix}$$

where the functional dependence of m and k on α has been suppressed, and all expectations are taken over α', the expectations of the young next period and $m' \equiv m(\alpha')$.

The determinant of this matrix

$$\Delta = \frac{w}{(w-m-k)^2}\alpha^2 E\frac{1}{(\alpha k+m')^2} > 0$$

$$\left.\frac{\partial m(\alpha)}{\partial a}\right|_{a=0} = \frac{1}{\Delta}\left\{\frac{w}{(w-m-k)^2}E\frac{\partial m'}{\partial a}\frac{\alpha}{(\alpha k+m')^2}\right.$$

$$\left. + \phi(\alpha)\left(E\frac{m'}{\alpha k+m'}\right)\left(-\frac{1}{(w-m-k)^2} - E\frac{\alpha^2}{(\alpha k+m')^2}\right)\right\} .$$

We seek to find a function $\phi^*(\alpha)$ such that $\partial m(\alpha)/\partial a = \alpha - \bar{\alpha}$, which implies that $\partial m(\alpha')/\partial a = \alpha' - \bar{\alpha}$. This function must satisfy, for each α

$$\phi^*(\alpha) \equiv \frac{\Delta(\alpha - \bar{\alpha}) - \frac{w}{(w-m-k)^2} E\frac{\alpha(\alpha' - \bar{\alpha})}{(\alpha k+m')^2}}{\left(E\frac{m'}{\alpha k+m'}\right)\left(-\frac{1}{(w-m-k)^2} - E\frac{\alpha^2}{(\alpha k+m')^2}\right)} .$$

FOOTNOTES

[1]This model has been used in Stein (1969). Diamond (1965) originally analyzed the effect of government debt in a growing economy, but such debt was a perfect substitute for physical capital; there was no "amenity" return to holding this debt.

[2]If the proceeds from money creation were distributed equally to everybody (including the young) the qualitative results would not be altered. See note 4 below.

[3]Without this additional constraint, the model implies that the prescription for the optimum quantity of money calls for the government to monetize claims to real capital and then determine the optimal capital stock. This theoretical possibility was noticed by Barro and Fisher (1976).

[4]If additional money were distributed to the young also, there would be an additional first order welfare gain. Such a modification may be likened to a wealth transfer to the young from the old. Let x be the equilibrium value of this transfer per young, so that $(1+g)x$ is the payment from each old person. The welfare change of this is $xU_1' - (1+g)xU_2'$. Since $U_1' - (1+r)U_2'$ is zero and $r > g$ this expression is necessarily positive. This modification would not alter the proposition that money growth reduces the yield on money holding.

[5]If, however, a social planner wanted to maximize expected utility he could "insure" the old against the possibility of low returns on investments by increasing the value of money in this event. For this to be beneficial it is necessary that the aggregate return to capital be more risky than the return to labor.

[6]A random component to the return could be included, without effecting the argument.

[7]Analogous to the discussion of footnote 5 it would be preferable from the point of expected utility to make the value of m depend both upon α , the expectations of the young and $\alpha' k(\alpha')$, the capital income of the old.

[8]
$$k'(\alpha) = E_{\alpha'} \frac{m(\alpha')}{(\alpha k(\alpha) + m(\alpha'))^2} \Big/ E_{\alpha'} \frac{\alpha^2}{(\alpha k(\alpha) + m(\alpha'))^2} > 0 \ .$$

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