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COWLES FOUNDATION DISCUSSION PAPER NO. 466

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THE LONG RUN IMPLICATIONS OF A TWO SECTOR MODEL  
WITH IMMOBILE CAPITAL

by

Gary Smith

October 10, 1977

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Elsewhere, Smith and Starnes [1977] describe and analyze the short run properties of a discrete period two sector model with an imperfect market for physical capital. Used capital was assumed to be completely immobile, with a Keynesian investment demand for new capital arising from discrepancies between the commodity market price and the equity market valuation of capital.

In the fixed wage case, we found that a bond financed government expenditure increases interest rates and the production of the commodity which is purchased by the government but has an uncertain effect on production in the other sector. The net effect on national income and employment is also ambiguous. An expansionary open market operation lowers interest rates and stimulates production in both sectors. A money financed government expenditure is also unambiguously expansionary, when the period is assumed sufficiently long so that an increase in income has a positive effect on the demand for interest bearing financial assets. A money financed purchase of consumption goods lowers interest rates while a purchase of investment goods raises them.

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\*The research described in this paper was undertaken by grants from the National Science Foundation and the Ford Foundation.

In the flexible wage full employment version of this model, a demand induced expansion of one sector necessarily reduces employment and production in the other sector, even with an elastic supply of labor. This is because investment production in the short run is a positive function of the relative price of investment to consumption goods, while consumption production is a negative function of this same price ratio. Bond financed government purchases increase interest rates and tilt production toward the commodity purchased. An expansionary open market operation reduces interest rates and slants production toward investment goods. A money financed purchase of either commodity lowers interest rates and tilts production toward investment goods.

In the present paper I analyze the stationary state equilibria for these models and the long run implications of monetary and fiscal policies. This exercise is in the spirit of the Blinder-Solow [1973, 1974] and Tobin-Butler [1976] analyses of the one sector IS-IM model. For comparative purpose I will here use a continuous version of the Smith-Starnes two sector model.

In the Blinder-Solow and Tobin-Butler unemployment models, the levels of both wages and prices are fixed. Since fixed prices imply a constant relative price ratio, the two and one commodity models are identical. I have consequently followed Smith-Starnes in analyzing an unemployment model in which nominal wages are fixed but prices are flexible.

For both the unemployment and full employment models I find, in contrast to the Smith-Starnes short run analysis, that the signs of the long run multipliers in the one and two commodity models are identical. The stability results do however diverge somewhat. The one commodity model tends to be unstable with static price expectations while the sta-

bility of the two commodity model is formidably ambiguous.

The notation used in this paper is described below:

$N_i$  = employment in the  $i^{\text{th}}$  sector;  $i = C$  (consumption goods), or

$i = I$  (investment goods),

$K_i$  = capital stock in the  $i^{\text{th}}$  sector,

$\delta_i$  = rate of depreciation of  $K_i$ ,

$Q_i$  = output of the  $i^{\text{th}}$  sector,

$P_i$  = price of the  $i^{\text{th}}$  commodity,

$\pi_i$  = rate of change of  $P_i$ ,

$\pi_i^e$  = anticipated rate of change of  $P_i$ ,

$\rho = P_I/P_C$ ,

$w$  = wage rate,

$Y$  = real factor income, measured in terms of the consumption good,

$W$  = real private wealth, measured in terms of the consumption good,

$M$  = nominal monetary government debt,

$B$  = nominal interest-bearing government debt,

$r$  = anticipated real rate of return on debt,

$G_C$  = government purchases of consumption goods,

$G_C^1 = G_C$  plus net interest,  $(1-t)(r + \pi_C^e)B/P_C$ ,

$G_I$  = government purchases of investment goods,

$t$  = tax rate,

$\dot{x}$  = time derivative of  $x$ .

With the exception of the closing of the labor market, the dynamic short run model is described by the following equations:

- (1)  $P_C F_{CN}^{+ -}[K_C, N_C] = w = P_I F_{IN}^{+ -}[K_I, N_I]$  (demands for labor)
- (2)  $Q_C = F_C^{+ +}[K_C, N_C], Q_I = F_I^{+ +}[K_I, N_I]$  (production)
- (3)  $Y = Q_C + \rho(Q_I - \delta_C K_C - \delta_I K_I)$  (income)
- (4)  $W = \frac{M+B}{P_C} + \frac{(F_{CK}^{+ -}[K_C, N_C] - \rho\delta_C)}{r} K_C + \rho \frac{(F_{IK}^{+ -}[K_I, N_I] - \delta_I)}{r} K_I$  (wealth)
- (5)  $C[r, Y, W] + G_C^{+ +} = Q_C$  (consumption equilibrium)
- (6)  $\delta_C K_C + \delta_I K_I + K_C + K_I + G_I = Q_I$  (investment equilibrium)
- (7)  $L[r + \pi_C^e, Y, W] = M/P_C$  (money equilibrium)
- (8)  $\dot{K}_C = I_C[P_C F_{CK}^{+ -}[K_C, N_C] - P_I \delta_C - P_I r]$  (consumption sector investment)
- (9)  $\dot{K}_I = I_I[F_{IK}^{+ -}[K_I, N_I] - \delta_I - r]$  (investment sector investment)
- (10)  $\dot{B} = P_C G_C' + P_I G_I - t P_C Y$  (government dissaving)
- (11)  $\dot{\pi}_C^e = \beta(\pi_C - \pi_C^e)$  (commodity price expectations)

In the unemployment version, the model is closed by the assumption that nominal wages are exogenously given

(12)  $w = \bar{w}$ .

In the full employment model the final equation is the labor supply,

$$(13) \quad N_C + N_I = N^S[w/P_C] .$$

In order to increase the comparability with Tobin-Buiter, I will adopt their specification of the private demands for consumption goods and money:

$$(14) \quad C = (1-t) \left( Y + \frac{(r + \pi_C^e)B}{P_C} \right) - \pi_C^e \frac{M+B}{P_C} - s(\hat{\mu}Y - W)$$

$$(15) \quad L = \ell[(1-t)(r + \pi_C^e), Y/W]W .$$

### I. Unemployment

The differentiation and solution of the supply side equations (1) and (2) with a fixed nominal wage (11) imposed shows that sectoral employment and output are positively related to the capital stock and price level:

$$\begin{bmatrix} P_C F_{CNN} & 0 & 0 & 0 \\ 0 & P_I F_{INN} & 0 & 0 \\ -F_{CN} & 0 & 1 & 0 \\ 0 & -F_{IN} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta N_C \\ \Delta N_I \\ \Delta Q_C \\ \Delta Q_I \end{bmatrix} = \begin{bmatrix} -F_{CN} & -P_C F_{CNK} & 0 \\ 0 & -F_{IN} & 0 & -P_I F_{INK} \\ 0 & 0 & F_{CK} & 0 \\ 0 & 0 & 0 & F_{IK} \end{bmatrix} \begin{bmatrix} \Delta P_C \\ \Delta P_I \\ \Delta K_C \\ \Delta K_I \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta N_C \\ \Delta N_I \\ \Delta Q_C \\ \Delta Q_I \end{bmatrix} = \begin{bmatrix} \frac{-F_{CN}}{P_C F_{CNN}} & 0 & -\frac{F_{CNK}}{F_{CNN}} & 0 \\ 0 & \frac{-F_{IN}}{P_I F_{INN}} & 0 & -\frac{F_{INK}}{F_{INN}} \\ \frac{-F_{CN}^2}{P_C F_{CNN}} & 0 & F_{CK} - \frac{F_{CN} F_{CNK}}{F_{CNN}} & 0 \\ 0 & \frac{-F_{IN}^2}{P_I F_{INN}} & 0 & F_{IK} - \frac{F_{IN} F_{INK}}{F_{INN}} \end{bmatrix} \begin{bmatrix} \Delta P_C \\ \Delta P_I \\ \Delta K_C \\ \Delta K_I \end{bmatrix}$$

Qualitatively,

$$\begin{aligned}
 N_C &= N_C^+ [P_C^+, K_C^+] \\
 N_I &= N_I^+ [P_I^+, K_I^+] \\
 Q_C &= Q_C^+ [P_C^+, K_C^+] \\
 Q_I &= Q_I^+ [P_I^+, K_I^+] .
 \end{aligned}
 \tag{16}$$

Total real factor income from (3) is then given by

$$\Delta Y = \left( F_{CK} - \rho \delta_C - \frac{F_{CNK} F_{CN}}{F_{CNN}} \right) \Delta K_C + \rho \left( F_{IK} - \delta_I - \frac{F_{INK} F_{IN}}{F_{INN}} \right) \Delta K_I \\ + \left( \frac{-F_{CN}^2}{F_{CNN}} - \rho(Q_I - \delta_C K_C - \delta_I K_I) \right) \frac{\Delta P_C}{P_C} + \left( \frac{-F_{IN}^2}{F_{INN}} + Q_I - \delta_C K_C - \delta_I K_I \right) \frac{\Delta P_I}{P_C} .$$

Near a long run equilibrium with  $\dot{K}_C = \dot{K}_I = 0$ , equations (8) and (9) imply  $F_{CK} - \rho \delta_C = \rho(F_{IK} - \delta_I) = r$ , and thus

$$(17) \quad Y = Y[P_C, P_I, K_C, K_I]$$

Smith-Starnes assume  $\partial Y / \partial P_C > 0$ , but that assumption is not needed here.

With constant returns to scale production functions, the marginal products of capital depend only upon the price levels:

$$\Delta F_{CK}[K_C, N_C] = \frac{F_{CKK} F_{CNN} - F_{CKN}^2}{F_{CNN}} \Delta K_C - \frac{F_{CN} F_{CKN}}{P_C F_{CNN}} \Delta P_C \\ \Delta F_{IK}[K_I, N_I] = \frac{F_{IKK} F_{INN} - F_{IKN}^2}{F_{INN}} \Delta K_I - \frac{F_{IN} F_{IKN}}{P_I F_{INN}} \Delta P_I$$

so that

$$(18) \quad F_{CK} = F_{CK}^+[P_C] \\ F_{IK} = F_{IK}^+[P_I] .$$

Steady states are characterized by constant real asset stocks. When the government fixes the nominal stock of either asset, then the



steady states are stationary with the price level and nominal stocks of other assets also constant. Thus, using the behavioral equations (14) and (15) and the supply side results (16) and (18), the short run equilibria (5), (6), and (7) and the stationary conditions

$\dot{K}_C = \dot{K}_I = \dot{B} = \dot{\pi}^e = 0$  for (8), (9), (10), and (11) determine the long run stationary state values of  $P_C$ ,  $P_I$ ,  $r$ ,  $K_C$ ,  $K_I$ , and  $B$ :

$$(19) \quad F_{IK}^+[P_I] = \delta_I + r$$

$$(20) \quad P_C F_{CK}^+[P_C] = P_I(\delta_C + r)$$

$$(21) \quad \mu[(1-t)r, 1/\hat{u}] \hat{u}(P_C G'_C + P_I G'_I) = tM$$

$$(22) \quad G'_C + \rho(1-t)G'_I = tQ_C[P_C, K_C]$$

$$(23) \quad G_I + \delta_C K_C + \delta_I K_I = Q_I[P_I, K_I]$$

$$(24) \quad \hat{u}(G'_C + \rho G'_I) = \frac{M+B}{P_C} + P_I \frac{K_C + K_I}{P_C}.$$

Equations (19) and (20) describe a positive relationship between  $P_C$ ,  $P_I$ , and  $r$  which is unaffected by monetary and fiscal policies:

$$\frac{dP_I}{dr} = \frac{1}{F'_{CK}} > 0$$

$$\frac{dP_C}{dr} = \frac{P_I F'_{IK} + \delta_C + r}{F_{CK} F'_{IK} + P_C F'_{CK} F'_{IK}} > 0.$$

An increase in  $r$  requires an increase in  $P_I$  to raise employment and the marginal productivity of capital in the investment goods sector. Higher values for  $r$  and  $P_I$  mean higher marginal capital costs in the consumer goods sector;  $P_C$  must then rise to increase its employment and marginal revenue product. The relationship between  $r$  and the relative price ratio  $\rho$  depends upon the relative capital-labor intensities\* for the two sectors:

$$P_C^2 F'_{IK} (F_{CK} + P_C F'_{CK}) \frac{d\rho}{dr} = w \left( \frac{F_{CKN}}{-F_{CNN}} - \frac{F_{IKN}}{-F_{INN}} \right) = w \left( \frac{N_C}{K_C} - \frac{N_I}{K_I} \right).$$

Collecting these results,

$$(25) \quad \begin{aligned} P_C &= P_C[r] \\ P_I &= P_I[r] \end{aligned}$$

$$\rho = \rho[r], \quad \rho > 0 \text{ as } N_C/K_C > N_I/K_I.$$

The longrun LM curve (21) also describes a relationship between  $P_C$ ,  $P_I$ , and  $r$ . Using (25) to make this two-dimensional,

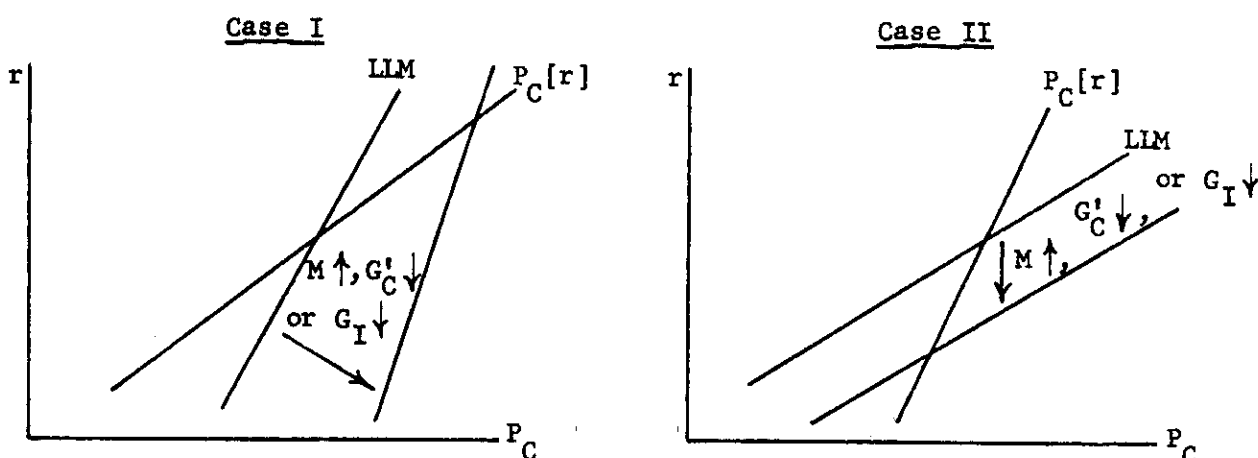
$$\begin{aligned} t\Delta M - l\hat{u}_C \Delta G'_C - l\hat{u}_I \Delta G'_I &= l_1(1-t)W\Delta r + l\hat{u}_C G'_C \Delta P_C + l\hat{u}_I G'_I \Delta P_I \\ &= l_1(1-t)W\Delta r + l\hat{u}_I \left( G'_C + G_I \frac{dP_I/dr}{dP_C/dr} \right) \Delta P_C. \end{aligned}$$

An increase in  $P_C$  (and from (25)  $P_I$ ) raises income and the demand for

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\*Writing the production functions in intensive form with  $F_i[K_i, N_i] = N_i f_i(K_i/N_i)$  gives  $F_{iKK} = f''/N_i$  and  $F_{iNK} = -K_i f''/N_i^2$  so that  $F_{iKN}/F_{iNN} = F_{iKK}/F_{iNK} = -N_i/K_i$ .

money, requiring an offsetting rise in  $r$ . This longrun LM curve (LLM) thus describes a positive relationship between  $r$  and  $P_C$  which shifts downward as  $M$  increases,  $G'_C$  falls, or  $G_I$  falls:



This curve may be either steeper or flatter than the relationship  $P_C[r]$  in (25) that results from the marginal productivity conditions. As the interest elasticity of the demand for money approaches zero the LLM curve becomes vertical. When the LM curve is steeper than  $P_C[r]$ , an increase in the money supply increases the steady state values of  $r$ ,  $P_C$  and, from (25),  $P_I$ ; an increase in  $G'_C$  or  $G_I$  lowers  $r$ ,  $P_C$ , and  $P_I$ . The opposite is true when the LM curve is flatter than  $P_C[r]$ .

Equations (22), (23), and (24) now recursively determine  $K_C$ ,  $K_I$ , and  $B$ :

$$(22') \quad t \frac{\partial Q_C}{\partial K_C} \Delta K_C = \Delta G'_C + \rho(1-t) \Delta G'_C - t \frac{\partial Q_C}{\partial P_C} \Delta P_C + (1-t) G_I \Delta \rho$$

$$(23') \quad \left( \frac{\partial Q_I}{\partial K_I} - \delta \right) \Delta K_I = \Delta G_I + \delta_C \Delta K_C - \frac{\partial Q_I}{\partial P_I} \Delta P_I$$

$$(24') \quad \frac{1}{P_C} \Delta B = \frac{1}{P_C} \Delta M - \rho \Delta K_C - \rho \Delta K_I - (K_C + K_I - \hat{u}G_I) \Delta \rho \\ + \frac{\hat{u}}{t} (\Delta G'_C + \rho \Delta G_I) + (M+B) \frac{\Delta P_C}{P_C^2} .$$

If  $\Delta \rho / \Delta P_C > 0$ , then all of these remaining multipliers are ambiguous.

If  $N_I/K_I \geq N_C/K_C$  so that  $\partial \rho / \partial P_C < 0$ , then by inspection\* the multipliers are as displayed in Table 1.

TABLE 1

## Stationary State Multipliers with Fixed Nominal Wages

	<u>Two Commodity Model</u>				
	<u>CASE I</u>		$P_C[r]$	<u>CASE II</u>	
	(IM steeper than $P_C[r]$ )		$\Delta M$	(IM flatter than $P_C[r]$ )	
	$\Delta G'_C, \Delta G_I$			$\Delta G'_C, \Delta G_I$	$\Delta M$
$\Delta r$	-	+	+	+	-
$\Delta P_C$	-	+	+	+	-
$\Delta P_I$	-	+	+	+	-
$\Delta K_C$	+	-	-	?	+
$\Delta K_I$	+	-	-	?	+
$\Delta B$	?	?	?	?	-
	<u>One Commodity Model</u>				
	$\Delta G$	$\Delta M$		$\Delta G$	$\Delta M$
$\Delta r$	-	+	+	+	-
$\Delta P$	-	+	+	+	-
$\Delta K$	+	-	-	?	+
$\Delta B$	?	?	?	?	-

\*To sign  $\Delta B$ , I also need the reasonable assumption that  $K_C + K_I > \hat{u}G_I$ , which states that capital is a larger fraction of wealth than net investment is of income.

The stability of these long run equilibria is quite uncertain. The characteristic equation of the system (5)-(11) is a 7x7 determinant which becomes a very long and complicated fourth order equation. Despite many hours of struggling, I have been unable to find sufficient conditions which were not so specific and severe as to be uninteresting.

The character of this ambiguity can be seen in a two step stability analysis for the special case of static expectations ( $\beta \rightarrow 0$ ). The three short run equilibrium equations (5), (6), and (7) can be used to determine  $r$ ,  $P_C$ , and  $P_I$  as functions of  $K_C$ ,  $K_I$ , and  $B$ . These can then be substituted into the dynamic equations (8), (9), (10) to give  $\dot{K}_C$ ,  $\dot{K}_I$ , and  $\dot{B}$  as functions of  $K_C$ ,  $K_I$ , and  $B$ . For the first step, substituting the supply conditions (16) and (18) into (5), (6), and (7) and differentiating gives

$$\begin{bmatrix} -a_{11} & a_{12} & -a_{13} \\ a_{21} & -a_{22} & -a_{23} \\ a_{31} & a_{32} & -a_{33} \end{bmatrix} \begin{bmatrix} \Delta P_C \\ \Delta P_I \\ \Delta r \end{bmatrix} = \begin{bmatrix} a_{14} & -a_{15} & -\frac{1}{P_C} C_W \\ -\delta_C & a_{25} & 0 \\ -a_{34} & -a_{35} & -\frac{1}{P_C} I_W \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \\ \Delta B \end{bmatrix}$$

where each  $a_{ij}$  on the left hand side is a positive parameter defined in Smith-Starnes, and

$$a_{14} = (1 - C_Y)(F_{CK} - \rho\delta_C - F_{CN}^F F_{CNK}/F_{CNN}) + \rho(\delta_C - C_W) ,$$

$$a_{15} = \rho C_W + \rho C_Y(F_{IK} - \delta_I - F_{IN}^F F_{INK}/F_{INN}) > 0 ,$$

$$a_{25} = F_{IK} - \delta_I - F_{IN}^F F_{INK}/F_{INN} > 0 ,$$

$$a_{34} = L_Y[F_{CK} - \rho\delta_C - F_{CN}^F F_{CNK}/F_{CNN}] + \rho L_W > 0 , \text{ and}$$

$$a_{35} = \rho L_W + \rho L_Y(F_{IK} - \delta_I - F_{IN}^F F_{INK}/F_{INN}) > 0 .$$

Although  $a_{14}$  is formally ambiguous, it can be reasonably assumed positive.

The solution is then given by

$$-|A| \begin{bmatrix} \Delta P_C \\ \Delta P_I \\ \Delta r \end{bmatrix} \begin{bmatrix} -a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} + a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & -a_{11}a_{33} - a_{13}a_{31} & a_{11}a_{23} + a_{14}a_{21} \\ -a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} - a_{12}a_{31} & a_{12}a_{21} - a_{11}a_{22} \end{bmatrix} \begin{bmatrix} a_{14} & -a_{15} & -\frac{1}{P}C_W \\ -\delta_C & a_{25} & 0 \\ -a_{34} & -a_{35} & -\frac{1}{P}L_W \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \\ \Delta B \end{bmatrix}$$

The only one of the nine complicated elements in this matrix product which can be signed is  $\Delta r/\Delta B > 0$ , with the Smith-Starnes assumption that

$a_{11}a_{22} - a_{12}a_{21} > 0$ . With so little information about the other effects of stock changes on the equilibrating variables, very little can be said about the stability of the system.

The analysis for the one commodity model is similar though simpler. (Since the long run implications of this familiar model have not been previously analyzed, they are themselves of interest.) The dynamic IS-LM model with a fixed nominal wage can be summarized by

$$s(\hat{W}Q[P, K] - W) + \pi^e \left( \frac{M+B}{P} \right) = \dot{K} + G' - tQ[P, K]$$

$$\ell[(1-t)(r + \pi^e), Q[P, K]/W]W = M/P$$

$$(26) \quad \dot{K} = I[F_K[P] - r]$$

$$\dot{B} = G' - tQ[P, K]$$

$$\dot{\pi}^e = \beta(\pi - \pi^e) .$$

The stationary state values of  $r$ ,  $P$ ,  $K$ , and  $B$  are then given by

$$r = F_K[P]$$

$$P \ell[(1-t)r, 1/\hat{W}] \hat{W} G' = tM$$

(27)

$$G' = tQ[P, K]$$

$$\hat{W} G' = tK + t(M+B)/P .$$

The first two equations in (27) gives two positive relationships between  $P$  and  $r$ , with the relative slopes ambiguous

$$\left( \frac{\partial r}{\partial P} \right)_{F_K(P)} = F'_K$$

(28)

$$\left( \frac{\partial r}{\partial P} \right)_{LLM} = \frac{M/P^2}{-(1-t)\ell_1 W} .$$

As  $\partial L/\partial r$  approaches zero, the LIM curve becomes vertical. The remaining two equations in (27) then recursively determine  $K$  and  $B$ . The signs of the multipliers are displayed in Table 1. It is striking, and surprising, that they do not differ qualitatively from the long run multipliers for the two sector model.

For the stability analysis, I will now define

$$b_{11} = (t + s\hat{u})\frac{\partial Q}{\partial P} + I'F'_K + s\left(\frac{M+B}{P^2} - \frac{KF'_K}{r}\right)$$

$$b_{12} = I' + sK/r > 0$$

$$b_{13} = (t + s\hat{u})\frac{\partial Q}{\partial K} - s$$

$$b_{21} = \frac{M}{P^2} + \ell_2 \frac{\partial Q}{\partial P} - L_W\left(\frac{M+B}{P^2} - \frac{F'_K K}{r}\right)$$

$$b_{23} = L_W + \ell_2 \frac{\partial Q}{\partial K} > 0 .$$

The conventional assumptions that  $b_{11}$  and  $b_{21}$  are positive state respectively that an increase in the price level raises saving relative to investment and raises the demand for money relative to supply. A positive sign for  $b_{13}$  states that an increase in the capital stock raises saving relative to investment; a sufficient condition for this is that the marginal propensity to consume out of income is less than one.

The characteristic equation for (26) for  $P$ ,  $r$ ,  $K$ ,  $B$ , and  $\pi^e$  can now be written as



$$\begin{aligned}
0 &= \begin{vmatrix} -b_{11} & -b_{12} & -b_{13} & s/P & -\frac{M+B}{P} \\ -b_{21} & b_{22} & -b_{23} & -L_W/P & -(1-t)\ell_1 W \\ I'F'_K & -I' & -\lambda & 0 & 0 \\ -t\frac{\partial Q}{\partial P} & 0 & -t\frac{\partial Q}{\partial K} & -\lambda & 0 \\ \lambda/P & 0 & 0 & 0 & -1 - \lambda/\beta \end{vmatrix} \\
&= \lambda^3 [(1-t)\ell_1 W/P + b_{22}(M+B)/P^2 + (b_{11}b_{22} + b_{12}b_{21})/\beta] \\
&\quad + \lambda^2 [b_{11}b_{22} + b_{12}b_{21} + I'(b_{23}(M+B)/P^2 - b_{13}(1-t)\ell_1 W/P) \\
&\quad\quad + I'[b_{13}(b_{22}F'_K - b_{21}) + b_{23}(b_{12}F'_K + a_{11})]/\beta] \\
&\quad + \lambda I'[b_{11}b_{23} - b_{21}b_{23} + b_{22}b_{13}] - \frac{t}{P^2} \frac{\partial Q}{\partial K} (L_W(M+B) + (1-t)s\ell_1 W) \\
&\quad\quad - \frac{t}{\beta P} \frac{\partial Q}{\partial K} (L_W(b_{11} + b_{12}F'_K) + s(b_{21} - b_{22}F'_K))] \\
&\quad - (tI'\frac{\partial Q}{\partial K}/P)[L_W(b_{11} + b_{12}F'_K) + s(b_{21} - b_{22}F'_K)] .
\end{aligned}$$

This is in general ambiguous. With static expectations ( $\beta \rightarrow 0$ ), however, the characteristic equation is dominated by

$$\begin{aligned}
0 &= \lambda^3 [b_{11}b_{22} + b_{12}b_{21}] + \lambda^2 I'[(s\hat{u}\ell + tL_W)\left(\frac{\partial Q}{\partial P}\right)\left(\frac{\partial Q}{\partial K} \frac{M+B}{P^2}\right) \\
&\quad - b_{13}(M/P^2 + (1-t)\ell_1 W F'_K)] + \lambda \left(tI'\frac{\partial Q}{\partial K}/P\right) \left[s\ell_2\left(\frac{\partial Q}{\partial P}\right) \right. \\
&\quad \left. + L_W\left((t+s\hat{u})\frac{\partial Q}{\partial P} - I'F'_K\right) + s\left(\frac{M}{P^2} + (1-t)\ell_1 W F'_K\right)\right] .
\end{aligned}$$

The expression  $(t+s\hat{u})\frac{\partial Q}{\partial P} - I'F'_K$  is the part of  $b_{11}$  that describes, neglecting wealth effects, the effect of a price increase on saving relative to

investment. This will usually be positive or small, if negative. From (28) the expression  $M/P^2 + (1-t)L_1 W F'_K$  is positive if the LIM curve is steeper than  $r = F'_K[P]$ , and negative if it is flatter. Thus, when  $\partial L/\partial r$  is relatively small and the LIM curve relatively steep, the characteristic function will be of the form  $0 = a\lambda^3 + b\lambda^2 - c\lambda$ , where  $a$  and  $c$  are positive and the system is therefore unstable.

## II. Full Employment

The differentiation of the labor marginal productivity conditions (1) and the labor supply (13) gives

$$\begin{bmatrix} 1 & -F_{CNN} & 0 \\ 1 & 0 & -\rho F_{INN} \\ -N^{S'} & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta(w/P_C) \\ \Delta N_C \\ \Delta N_I \end{bmatrix} = \begin{bmatrix} F_{CNK} & 0 & 0 \\ 0 & \rho F_{INK} & F_{IN} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \\ \Delta \phi \end{bmatrix},$$

with the solution to this system given by

$$(29) \quad \begin{bmatrix} \Delta(w/P_C) \\ \Delta N_C \\ \Delta N_I \end{bmatrix} = \frac{1}{(-\alpha)} \begin{bmatrix} -\rho F_{INN} F_{CNK} & -\rho F_{INK} F_{CNN} & -F_{IN} F_{CNN} \\ F_{CNK}(1 - \rho F_{INN} N^{S'}) & -\rho F_{INK} & -F_{IN} \\ -F_{CNK} & \rho F_{INK}(1 - F_{CNN} N^{S'}) & F_{IN}(1 - F_{CNN} N^{S'}) \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \\ \Delta \phi \end{bmatrix}$$

where  $\alpha = F_{CNN} + \rho F_{INN}(1 - F_{CNN} N^{S'}) < 0$ . Qualitatively,

$$(30) \quad w/P_C = w/P_C \left[ \overset{+}{\rho}, \overset{+}{K_C}, \overset{+}{K_I} \right]$$

$$(31) \quad N_C = N_C \left[ \overset{-}{\rho}, \overset{+}{K_C}, \overset{-}{K_I} \right]$$

$$(32) \quad N_I = N_I \left[ \overset{+}{\rho}, \overset{-}{K_C}, \overset{+}{K_I} \right] .$$

An increase in  $\rho = P_I/P_C$  increases  $N_I$ , decreases  $N_C$ , and raises aggregate employment by increasing  $w/P_C$ . An increase in either industry's capital stock will increase that industry's demand for labor, the real wage and aggregate employment while reducing the other industry's demand for labor.

In addition to these labor market conditions and the behavioral assumptions (14) and (15), the long run equilibrium is characterized by the short run equilibria (5), (6), and (7) and the stationary conditions for (8), (9), (10), and (11):

$$(33) \quad F_{IK}[P_I] = \delta_I + r$$

$$(34) \quad P_C F_{CK}[P_C] = P_I (\delta_C + r)$$

$$(35) \quad G_I + \delta_C K_C + \delta_I K_I = Q_I$$

$$(36) \quad tY = G_C' + \rho G_I$$

$$(37) \quad \hat{L}[(1-t)r, 1/\hat{U}] \hat{U}Y = M/P_C$$

$$(38) \quad \hat{U}Y = \frac{M+B}{P_C} + \rho(K_C + K_I) .$$

Differentiating the capital marginal productivity conditions (33) and (34) gives

$$\begin{bmatrix} \rho^2 & F_{CK} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta p \end{bmatrix} = \begin{bmatrix} \rho F_{CKK} & 0 \\ 0 & F_{IKK} \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \end{bmatrix} + \begin{bmatrix} \rho F_{CKN} & 0 \\ 0 & F_{IKN} \end{bmatrix} \begin{bmatrix} \Delta N_C \\ \Delta N_I \end{bmatrix}$$

Substituting for  $\Delta N_C$  and  $\Delta N_I$  from (29) and solving gives

$$\begin{bmatrix} \Delta r \\ \Delta p \end{bmatrix} = \frac{-1}{\beta} \begin{bmatrix} -\{\rho^2 F_{IN}^F F_{IKN}^F F_{INN}^F F_{CKK} (1 - N^S F_{CNN})\} & -\{\rho^2 F_{IN}^F F_{IKN}^F F_{INK}^F F_{CKN} (1 - N^S F_{CNN})\} \\ +\{\rho F_{IN}^F F_{CKN}^F F_{IKN}^F F_{CNK} - \alpha F_{CK}^F F_{LJN}^F F_{CNK}\} & +\{\rho F_{IN}^F F_{CKN}^F F_{IKK}^F F_{CNN} - \alpha F_{CK}^F F_{IKK}^F F_{CNN}\} \\ -\alpha \rho^2 \{F_{IKN}^F F_{CNK} - F_{INN}^F F_{CKK}\} & \alpha \rho^2 \{F_{CKN}^F F_{INK} - F_{IKK}^F F_{CNN}\} \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \end{bmatrix}$$

where  $\beta = \alpha \rho^2 F_{IKN}^F F_{IN}^F (1 - N^S F_{CNN}) - \alpha^2 F_{CK}^F + \alpha \rho F_{IN}^F F_{CKN} < 0$ . Therefore,

$$(39) \quad r = r[K_C^-, K_I^-],$$

$$(40) \quad \rho = \rho[K_C^?, K_I^?], \quad \Delta \rho / \Delta K_I \geq 0 \text{ as } N_I / K_I \leq N_C / K_C.$$

The signs in (40) follow from the earlier footnote. The substitution of the actual equation for (40) into (29) now gives

$$\begin{aligned} w/p_C &= w/p_C [K_C^+, K_I^+] \\ (41) \quad N_C &= N_C [K_C^+, K_I^-] \\ N_I &= N_I [K_C^-, K_I^+], \end{aligned}$$

where the signs of the partial derivatives in (41) are determinate despite the ambiguity in (40). Thus, an increase in the capital stock of either industry will unambiguously increase real wages, aggregate employment, and employment in that sector and decrease employment in the other sector. It

follows directly from (41) and the production functions (2) that each industry's output is positively related to its own capital stock and negatively related to the other industry's capital stock,

$$Q_C = Q_C^+ [K_C, K_I^-]$$

$$Q_I = Q_I^- [K_C^-, K_I^+] .$$

As for aggregate income, from (3)

$$\begin{aligned} \frac{dY}{dK_C} &= F_{CK} - \rho \delta_C + F_{CN} \frac{\partial N_C}{\partial K_C} + \rho F_{IN} \frac{\partial N_I}{\partial K_C} + (Q_I - \delta_C K_C - \delta_I K_I) \frac{\partial \rho}{\partial K_C} \\ &= \rho r + \frac{w}{P_C} \frac{\partial N^S}{\partial K_C} + (Q_I - \delta_C K_C - \delta_I K_I) \frac{\partial \rho}{\partial K_I} \\ \frac{dY}{dK_I} &= \rho (F_{IK} - \delta_I) + F_{CN} \frac{\partial N_C}{\partial K_I} + \rho F_{IN} \frac{\partial N_I}{\partial K_I} + (Q_I - \delta_C K_C - \delta_I K_I) \frac{\partial \rho}{\partial K_I} \\ &= \rho r + \frac{w}{P_C} \frac{\partial N^S}{\partial K_I} + (Q_I - \delta_C K_C - \delta_I K_I) \frac{\partial \rho}{\partial K_I} . \end{aligned}$$

I will assume that these are positive, which is unambiguously so if there are few government purchases of investment goods ( $G_I = 0$ ) or if the investment goods industry is relatively labor intensive so that  $\partial \rho / \partial K_C, \partial \rho / \partial K_I > 0$ ,

$$(43) \quad Y = Y^+ [K_C^+, K_I^+] .$$

Now differentiating (35)-(38), with (39)-(43) substituted in gives

$$\begin{bmatrix} \delta_C - \frac{\partial Q_I}{\partial K_C} & \delta_I - \frac{\partial Q_I}{\partial K_I} & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & c_{32} & \frac{M}{P_C} & 0 \\ -c_{41} & -c_{42} & \frac{M+B}{P_C} & -1 \end{bmatrix} \begin{bmatrix} \Delta K_C \\ \Delta K_I \\ \Delta P_C \\ \Delta B \end{bmatrix} = \begin{bmatrix} -\Delta G_I \\ \Delta G'_C + \rho \Delta G_I \\ \Delta M \\ \Delta M \end{bmatrix}$$

where

$$c_{21} = t\pi + t\frac{w}{P_C} \frac{\partial N^S}{\partial K_C} - (1-t)G_I \frac{\partial \rho}{\partial K_C}$$

$$c_{22} = t\pi + t\frac{w}{P_C} \frac{\partial N^S}{\partial K_I} - (1-t)G_I \frac{\partial \rho}{\partial K_I}$$

$$c_{31} = P_C(1-t)\ell_1 w \frac{\partial r}{\partial K_C} + P_C \ell \hat{u} \frac{\partial Y}{\partial K_I} > 0$$

$$c_{32} = P_C(1-t)\ell_1 w \frac{\partial r}{\partial K_I} + P_C \ell \hat{u} \frac{\partial Y}{\partial K_I} > 0$$

$$c_{41} = \left( -\hat{u} \frac{\partial Y}{\partial K_C} + \rho + K_C \frac{\partial \rho}{\partial K_C} \right) P_C$$

$$c_{42} = \left( -\hat{u} \frac{\partial Y}{\partial K_I} + \rho + K_I \frac{\partial \rho}{\partial K_I} \right) P_C .$$

I will assume that  $c_{21}$  and  $c_{22}$  are positive, which states that an increase in either capital stock increases government tax revenue relative to expenditures. With these assumptions, the Jacobian determinant

$$|C| = -\frac{M}{P_C} \left( c_{22} \left( \delta_C - \frac{\partial Q_I}{\partial K_C} \right) + c_{21} \left( \frac{\partial Q_I}{\partial K_I} - \delta_I \right) \right) < 0$$

is negative and the long run multipliers are given by

$$-|C| \begin{bmatrix} \Delta K_C \\ \Delta K_I \\ \Delta P_C \\ \Delta B \end{bmatrix} = \begin{bmatrix} c_{22} \frac{M}{P_C} & \left( \frac{\partial Q_I}{\partial K_I} - \delta_I \frac{M}{P_C} \right) & 0 & 0 \\ -c_{21} \frac{M}{P_C} & \left( \delta_C - \frac{\partial Q_I}{\partial K_C} \frac{M}{P_C} \right) & 0 & 0 \\ c_{21} c_{32} - c_{22} c_{31} & c_{32} \left( \frac{\partial Q_I}{\partial K_C} - \delta_C \right) - c_{31} \left( \frac{\partial Q_I}{\partial K_I} - \delta_I \right) & -|C| \frac{P_C}{M} & 0 \\ \frac{M+B}{P_C} (c_{21} c_{32} - c_{31} c_{22}) & \left( \delta_I - \frac{\partial Q_I}{\partial K_I} \right) \left( c_{31} \frac{M+B}{P_C} + c_{41} \frac{M}{P_C} \right) & -|C| \frac{M+B}{M} & |C| \\ -\frac{M}{P_C} (c_{22} c_{41} - c_{21} c_{42}) & -\left( \delta_C - \frac{\partial Q_I}{\partial K_C} \right) \left( c_{32} \frac{M+B}{P_C} + c_{42} \frac{M}{P_C} \right) & -|C| \frac{M+B}{M} & |C| \end{bmatrix} \begin{bmatrix} -\Delta G_I \\ \Delta G'_C + \rho \Delta G_I \\ \Delta M \\ \Delta M \end{bmatrix}$$

The multiplier  $\Delta K_C / \Delta G_I$  is positive since

$$\begin{aligned} -|C| \frac{P_C^3}{M} \frac{\Delta K_C}{\Delta G_I} &= \rho \left( F_{IK} - \delta_I + F_{IN} \frac{\partial N_I}{\partial K_I} \right) - \left( t \sigma + t \frac{w}{P_C} \frac{\partial (N_I + N_C)}{\partial K_I} - (1-t) G_I \frac{\partial \rho}{\partial K_I} \right) \\ &= (1-t)r + (1-t) \frac{w}{P_C} \frac{\partial N_I}{\partial K_I} - t \frac{w}{P_C} \frac{\partial N_C}{\partial K_I} + (1-t) G_I \frac{\partial \rho}{\partial K_I} > 0. \end{aligned}$$

This result also directly yields  $\Delta P_C / \Delta G_I < 0$ . Some lengthier but straightforward manipulation shows that sufficient conditions for  $\Delta B / \Delta G'_C$  and  $\Delta B / \Delta G_I$  to be negative are that the supply elasticity of labor is small ( $N^S \rightarrow 0$ ) and either there are few government investment goods purchases ( $G_I \rightarrow 0$ ) or the capital goods sector is not labor intensive ( $N_C / K_C \geq N_I / K_I$ ). The signs of the remaining multipliers displayed in Table 2 follow by inspection.

TABLE 2

## Stationary State Multipliers with Flexible Wages

<u>Two Commodity Model</u>			
	$\Delta G'_C$	$\Delta G'_I$	$\Delta M$
$\Delta K_C$	+	+	0
$\Delta K_I$	+	+	0
$\Delta r$	-	-	0
$\Delta P_C$	-	-	$P_C/M$
$\Delta B$	-	-	$B/M$

<u>One Commodity Model</u>		
	$\Delta G'$	$\Delta M$
$\Delta K$	+	0
$\Delta r$	-	0
$\Delta P$	-	$P/M$
$\Delta B$	-	$B/M$

Monetary policy is neutral in the long run. Expansionary fiscal policies increase the capital stocks in both sectors, while reducing real interest rates, the price level, and government debt.

The stability of these multipliers is again quite uncertain. The characteristic equation of the system (5)-(11) using (1)-(4), (13)-(15), and (30)-(32) is a 7x7 determinant which becomes a very long and complicated fourth order equation. As with the unemployment model, it seems to take a number of very specific assumptions to make the system unambiguously stable or unstable.

The Tobin-Buiter stationary state multipliers for the one commodity



model are reproduced in Table 2. As with the unemployment scenario, the agreement between the one and two commodity long run multipliers is striking. The stability analysis for the one commodity model is not quite as tedious (a  $5 \times 5$  determinant which becomes a cubic equation), but the stability of the system is generally uncertain. However, in the case of completely static expectations ( $\beta \rightarrow 0$ ), the one commodity model is unambiguously unstable.

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