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THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART 34

A MULTIPERIOD TRADING ECONOMY WITH FIAT MONEY,  
BANK MONEY AND AN OPTIMAL BANKRUPTCY RULE\*

by

Martin Shubik

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1. INTRODUCTION

In two previous papers<sup>1, 2</sup> it has been suggested that an economy which uses paper money will need a legal system to enforce the contracts implicit in the use of a paper money. This system must provide rules to define insolvency and methods to cure insolvency. In particular when an individual or corporation cannot meet its obligations a bankruptcy procedure must be available. In the previous papers a one period market model with an "outside bank" or "treasury" issuing money was studied. In the one period model the only purpose for the use of money was to cover the float created by the use of a simultaneous bidding mechanism.

An optimal bankruptcy rule was suggested as one which enabled us to construct a bidding game which when solved for its noncooperative equilibria would have a subset of noncooperative equilibria (N.E.) which in

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the replicated markets approach the competitive equilibria (C.E.).\*

The discussion of nonoptimal bankruptcy rules is not given here. Many of the problems with nonoptimal bankruptcy rules are covered in the papers referred to.<sup>1, 2</sup>

In this paper we examine a two period economy that uses money. In an economy with two or more periods money plays two roles. It both finances the float and it finances intertemporal trade. In the subsequent parts of this paper a specific model is studied in detail in order to illustrate the problems in modelling an economy with two types of paper money issued by a central bank and by a private banking system.

## 2. A SIMPLE GENERAL EQUILIBRIUM SYSTEM

We consider a trading economy with  $2n$  individuals, two periods and one commodity in each period. Suppose that there are  $n$  individuals each of one of two types. The types differ only in the endowments of the (non-inventoriable) commodity held each period and in their time preferences. In particular any trader  $i$  of type 1 has endowments of  $(A_1, B_1)$  and any trader  $j$  of type 2 has endowments of  $(A_2, B_2)$ . Where  $A_1$  and  $A_2$  are endowments in the first period and  $B_1, B_2$  are endowments in the second period. All endowments are nonnegative.

All individuals  $i$  of the same type are presumed to have the same utility function of the form:\*\*

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\*Or alternatively a nonatomic bidding game whose noncooperative equilibria give traders the same distributions as a C.E.

\*\*See 4.3 for the general case.

$$(1) \quad U_i = \log q_1^i + B_i \log q_2^i$$

where  $q_t^i$  = the amount consumed by  $i$  in period  $t$  and  $B_i$  = the time discount factor of  $i$ .

The competitive equilibrium for a market with one trader of each type is given by solving the following system of equations

$$(2) \quad \frac{1}{q_1} = \lambda_1$$

$$(3) \quad \frac{B_1}{q_2} = \lambda_1 p_2$$

$$(4) \quad A_1 + p_2 B_1 = q_1^1 + p_2 q_2^1$$

$$(5) \quad \frac{1}{q_1} = \lambda_2$$

$$(6) \quad \frac{B_2}{q_2} = \lambda_2 p_2$$

$$(7) \quad A_2 + p_2 B_2 = q_1^2 + p_2 q_2^2$$

$$(8) \quad q_1^1 + q_1^2 = A_1 + B_1$$

and

$$(9) \quad q_2^1 + q_2^2 = A_2 + B_2 .$$

The  $q_1^1$ ,  $q_1^2$  and  $q_2^1$ ,  $q_2^2$  are the amounts consumed by traders 1 and 2 in the first and second periods,  $\lambda_1$  and  $\lambda_2$  are Lagrangian multipliers. The first price  $p_1$  may be set to  $p_1 = 1$ . The second price is  $p_2$ .

From (2), (3) and (5), (6) we obtain

$$(10) \quad \frac{q_1^2}{q_2} = \left( \frac{B_1}{B_2} \right) \frac{q_1^1}{q_2} .$$

From (10), (8) and (9) we have:

$$(11) \quad \frac{A_1 + B_1 - q_1^1}{A_2 + B_2 - q_2^1} = \left( \frac{B_1}{B_2} \right) \frac{q_1^1}{q_2^1}$$

or

$$(12) \quad q_1^1 = \frac{(A_1 + B_1)q_2^1}{(A_2 + B_2)\left(\frac{B_1}{B_2}\right) + \left(1 - \frac{B_1}{B_2}\right)q_2^1} .$$

From (4) and (7), (8), (9) we obtain:

$$(13) \quad \frac{A_1 - q_1^1}{A_2 - A_1 - B_1 + q_1^1} = \frac{q_2^1 - B_1}{A_2 - q_2^1} .$$

Substituting (12) in (13) we may solve for  $q_2^1$ . There are many cases, but as our prime concern is with monetary mechanisms we select three contrasting cases which will require different banking needs.

Case 1.  $A_1 = A_2 = A$ ,  $B_1 = B_2 = \epsilon A$  and  $0 < B_1 = B_2 = B \leq 1$  and  $\epsilon > 0$ .

Case 2.  $A_1 = A$ ,  $A_2 = B > A$ ,  $B_1 = \epsilon B$ ,  $B_2 = \epsilon A$  and  $0 < B_1 = B_2 = B \leq 1$  and  $\epsilon > 0$ .

Case 3.  $A_1 = A_2 = A = B_1 = B_2$  and  $B_2 < B_1$ .

	Case 1	Case 2	Case 3
$\lambda_1^{q_1}$	A	$A + B e^B \left( \frac{A+eB}{B+eA} \right)$	$2A(1+B_2)$
$\lambda_1^{q_2}$	$eA$	$\frac{1+B}{A \left( \frac{A+eB}{B+eA} \right) + BeA}$	$2AB_1(1+B_2)$
$\lambda_2^{q_1}$	A	$\frac{BA}{1+B} + eB \left[ 1 - \frac{B}{1+B} \left( \frac{A+eB}{B+eA} \right) \right]$	$2A(1+B_1)$
$\lambda_2^{q_2}$	$eA$	$\frac{eA}{1+B} + A \left[ e - \frac{1+B}{1+B} \left( \frac{A+eB}{B+eA} \right) \right]$	$\frac{2AB_2(1+B_1)}{[B_1(1+B_2) + B_2(1+B_1)]}$
$\lambda_1^{p_2}$	$B/e$	$B \left( \frac{A+eB}{B+eA} \right)$	$\frac{[B_1(1+B_2) + B_2(1+B_1)]}{[2+B_1+B_2]}$
$\lambda_2^{p_1}$	$\frac{1}{q_1}$	$\frac{1}{q_1}$	$\frac{1}{q_1}$
$\lambda_1^{p_2}$	$\frac{1}{q_2}$	$\frac{1}{q_2}$	$\frac{1}{q_2}$

TABLE I

Case 1 has identical traders with more or less to trade in the second period than in the first. In the general equilibrium solution there is no trade. Each consumes his own resources.

In Case 2 as there is the possibility for one trader to have relatively more goods than the other in one period and vice versa in the second period there is a need for intertemporal trade.

In the first two cases each trader was assumed to have the same "natural" discount rate. In Case 3 their discount rates differ, hence even with the same resources they have reason for intertemporal trade.

### 3. A TRADING MARKET WITH AN OUTSIDE AND INSIDE BANK

In this section the same basic trading model as specified in Section 2 is reconsidered as an economy with a specific market mechanism with trade in two types of paper money and in bank shares. Before we specify the monetary structure the nature of the market mechanism is noted. Although several other market mechanisms have been considered elsewhere,<sup>3, 4, 5, 6</sup> the simplest is chosen here. It is assumed that at each period all traders are required to offer all of their goods for sale. They make their market move by offering an amount of money to the market.

As the model is presented in complete detail, before the notation is given, a flow diagram is given of the steps needed to issue government money, create the banks, issue bank shares, make bank loans, accept deposits, bid in the markets, refinance and finally liquidate the banking system and have a final settlement.



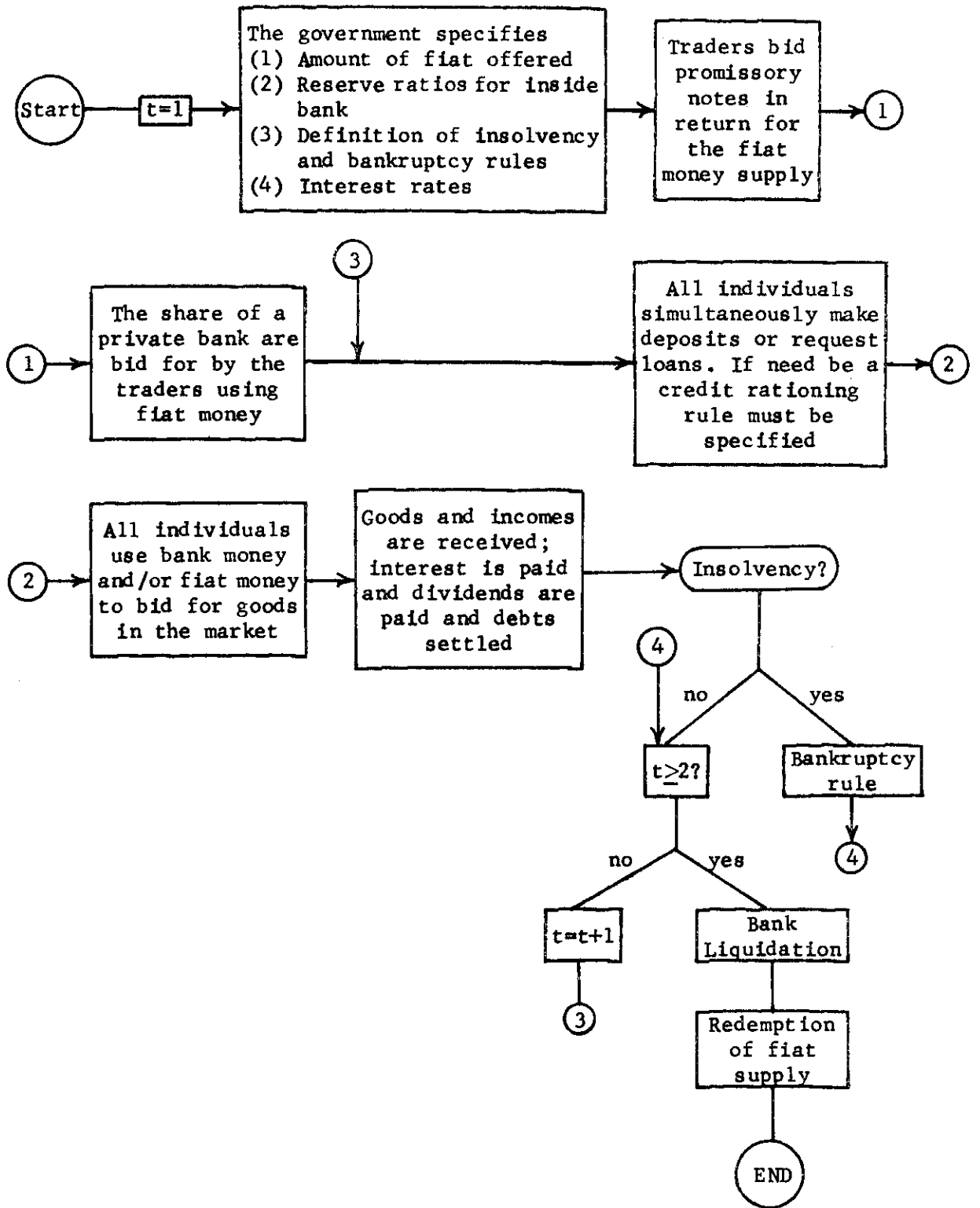


FIGURE 1

In Figure 1 we observe that we begin with several preliminary acts by a central government as the conditions needed to set the stage for an issue of fiat money. These involve deciding upon the amount of fiat to be issued, the method of issue, the rules for the creation of banks, the control of bank money, the rate of interest and the methods for handling insolvency and bankruptcy as well as credit rationing and bank failure must be given.

The government auctions off its fiat money in return for promissory notes to be redeemed in fiat money at the end of the economy.

After the issue of fiat a private bank is formed whose shares are sold to the public for fiat. The fiat money taken in provides the capital basis of the bank and this determines the size of loans that a bank can make.

Two (or more) banks are required for competition to be potentially present. If the government controls the interest rate one bank will do. There are several variants in the mechanisms for the issue of shares and for the extension of the ability of a bank to make loans based on its assets and deposits. No attempt is made here to cover them all or to be particularly realistic. The mechanism used here is not meant to resemble any specific mechanism closely. The main concern here is to specify completely some mechanism that works.

It is assumed that the bank is run by automata who attempt to maximize bank profits. In this simple model the conditions on the paying out of profits as dividends must be specified. Furthermore care must be taken in defining short run profits in a multistage model.

In this simple version it is assumed that the loan and deposit rates are identical. The bank is essentially a dummy. It has no strategic freedom.

After individuals have obtained their loans they bid in the markets for goods. After this they receive their incomes from the sale of goods and go to the bank to settle accounts. Rollovers are permitted thus all accounts need not be fully settled before new lending takes place.

In the second period borrowing and depositing is followed by trade. For simplicity a secondary market in bank shares has been left out of this version. The addition of trade in shares is noted in 5.2 below.

After the settlement of all debt at the banks in the second period the banks liquidate and pay out all capital then at the end of all economic activity the promissory notes issued against the fiat money supply are redeemed.

### 3.1. Traders, Endowments and Utility Functions

There are  $2n$  traders;  $n$  each of two types. Each trader  $i$  ( $j$ ) of type 1 (2) has endowments of  $(A_1, B_1)$  (or  $(A_2, B_2)$ ) where the consumer good is not durable. It is supplied at the start of any period and lasts only for that period. (Assume  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  are all positive.)

A trader  $i$  of type 1 has a utility function of the form:

$$(14) \quad U_1 = \log q_1^{i1} + \beta_1 \log q_2^{i1}$$

and a trader  $j$  of type 2:

$$(15) \quad U_2 = \log q_1^{j2} + \beta_2 \log q_2^{j2} ,$$

where  $q_1^{i1}$ ,  $q_2^{i1}$ ,  $q_1^{j2}$ ,  $q_2^{j2}$  are the amounts consumed in the first and second periods by traders  $i$  and  $j$  of type 1 and 2 respectively.

$$(16) \quad q_1^1 + q_1^2 = (A_1 + A_2), \quad q_2^1 + q_2^2 = (B_1 + B_2)$$

where 
$$q_1^1 = \sum_{k=1}^n q_1^{k1}.$$

### 3.2. The Issue of Fiat Money

The government begins the economy by offering  $2nM$  units of fiat money for sale by simultaneous sealed bid. A bid by a trader is made by creating his own promissory note or I.O.U.

$$\text{Trader } i \text{ of type 1 bids } u^{i1} \geq 0$$

$$\text{Trader } j \text{ of type 2 bids } u^{j2} \geq 0.$$

The amount of fiat money obtained by  $i$  ( $j$ ) is:

$$(17) \quad r^{i1} = \frac{2nu^{i1}M}{u^1 + u^2} \quad \left( \text{or } r^{j2} = \frac{2nu^{j2}M}{u^1 + u^2} \right).$$

We must specify the redemption conditions on the promissory notes and what happens if there is a failure to redeem.

A simple convention concerning the call conditions on the promissory notes is that they are to be redeemed after the close of all economic activity. In this simple example after the financial settlement with the banks at the end of the second period.

Because in actuality no economy has a finite termination date after which all institutions are to be dismantled it is not particularly important to specify precisely what are the offsetting assets against the fiat

money supply. In this simple model they are personal promissory notes.

Failure to redeem the promissory notes will be discussed in 4.2 and 4.3 which are devoted to insolvency and bankruptcy.

### 3.3. The Sale of Bank Shares and Formation of a Bank

In an economy run using spot markets and trading in money it may be necessary to vary the money supply and to provide a system for loans and credits to take advantage of intertemporal trade. This calls for a banking function.

There are many different ways in which "inside banks" can be modelled. For example if we were to limit our concern to loans (leaving out expansion or contraction of the money supply) then a simple "money market" where available fiat is offered and promissory notes are bid in a double auction would suffice. Here we select a model in which the bank can actually create money.

Before the mechanism of the bank is described we must note that in order to have competition in the banking sector we need at least two banks. If we assume that the banks are noncooperative then it can be argued that under certain circumstances the presence of two banks will be sufficient to do away with oligopolistic effects. These circumstances depend upon the choice of strategic variable (rate of interest, or "price" rather than credit rationing or "quantity"); information conditions and capacity constraints. It has been shown elsewhere<sup>7, 8</sup> that when the banks can be set up in a manner analogous to a Bertrand duopoly<sup>9</sup> then the oligopolistic effect of few banks is attenuated.

An alternative to modeling a competitive banking system is to assume that the inside banking system acts as a strategic dummy, i.e. the central

government sets the rates of interest and these can be treated as parameters of the system. In this instance the bank managers have no strategic freedom if they are also required to make any loan that is feasible, accept deposits and pay out short term profits.

Let there be a single bank with  $nS$  shares for sale. A trader  $i$  of type 1 bids a number  $0 \leq v^{i1} \leq 1$  which represents the percentage of the fiat money he has obtained from the central bank that he wishes to spend on buying (the inside) bank shares. Thus the amount that he actually spends will be  $v^{i1} r^{i1}$ . Similarly a trader  $j$  of type 2 bids  $0 \leq v^{j2} \leq 1$  and hence spends  $v^{j2} r^{j2}$ . The reason for having the traders bid percentages rather than monetary sums is that it enables us to define relatively simple strategies under conditions of extremely low information where simple rationing conventions guarantee feasibility. This point is developed in detail elsewhere.<sup>10</sup>

A trader  $i$  of type 1 will obtain a number of shares:

$$(18) \quad s^{i1} = \frac{nv^{i1}r^{i1}S}{\sum_{i=1}^n v^{i1}r^{i1} + \sum_{j=1}^n v^{j2}r^{j2}},$$

and similarly for  $s^{j2}$ . The rules we require for the purchase of bank shares is that payments are made in fiat money. This amounts to suggesting that personal promissory notes will not be regarded as adequate capital for an inside bank.

### 3.4. Running the Banks: Reserve Requirements and Setting the Interest Rates

The fiat money taken in by the sale of the banks' shares constitutes its assets. We may construct a flexible banking system by having laws

which set a "gearing ratio" for bank loans related to the size of the reserves. In actuality the loan limits may depend upon several items such as deposits as well as capital. Here, the simplest scheme appears to be the introduction of a number  $k$  by the central bank such that the bank may lend up to:

$$(19) \quad k(v^1 + v^2) .$$

In this model there is no further issue of bank shares. If we also require that banks pay out their profits to stockholders then a distinction is made between the fiat serving as bank capital and other money, in the sense that it is the only "high-power" money in the system, i.e. it can serve as a basis for a money expansion geared at a level of up to  $k$  to 1 .

As part of the rules we must specify not only the lending role of the banks, but how deposits are to be treated. In actuality, given transactions costs there is a spread in the interest charged on loans and paid on deposits. Furthermore deposits may serve as a basis for the expansion of loans. In attempting to construct the simplest inside banking system we assume here that a bank is required to pay the same rate on deposits as it charges for loans. Furthermore in this simple system deposits do not serve as a basis for further loan expansion. The central bank sets the interest rates.

### 3.5. Loans and Deposits

We distinguish two types of loans and two types of deposit at the bank. They are designated as a "float loan," a "time loan" and a call or sight or float deposit and a "time deposit." Figure 2 shows the structure of trade through time.

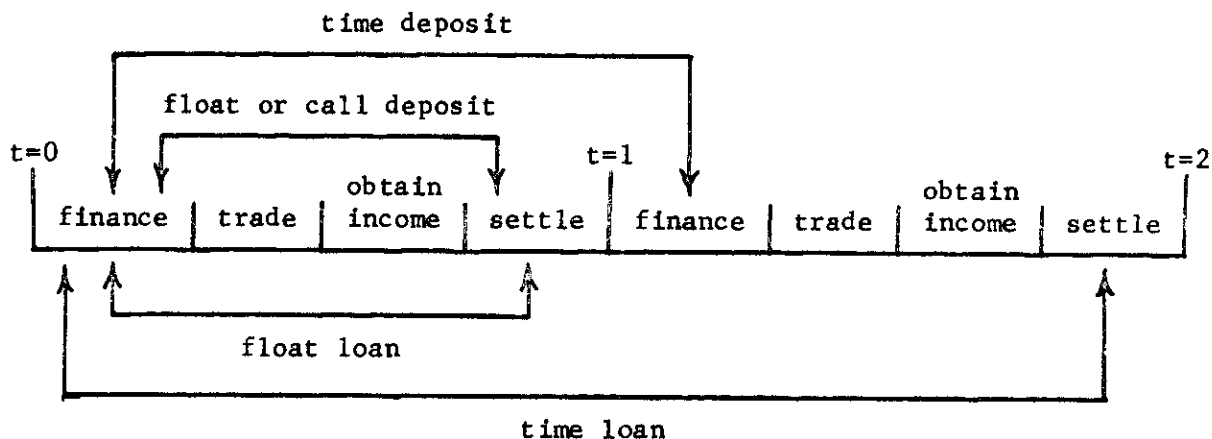


FIGURE 2

A float loan denoted by  ${}^+z_t^{i,j}$ , for trader  $i$  of type  $j$  at the start of period  $t$  is a loan which must be paid back before period  $t+1$ . If it is not repaid it is automatically converted into a time loan with a different interest rate

$\rho_1$  = the float rate of interest

$\rho_2$  = the time rate of interest.

A time loan or borrowing by a trader  $i$  of type  $j$  at the start of period  $t$  at the bank is denoted by  ${}^+y_t^{i,j}$ . It must be repaid or refinanced at the settlement date at  $t+1$ . A time deposit at the start of period  $t$  is denoted by  ${}^-y_t^{i,j}$ ; it cannot be withdrawn or otherwise utilized until the financing stage at the start of period  $t+1$ .

We adopt the convention that deposits are negative and borrowings (or the taking of loans) are positive.

A float deposit denoted by  ${}^-z_t^{i,j}$  could also be defined. It can be interpreted as a checking account.

The convention adopted here for the sake of simplicity is that



an individual will not both borrow and deposit simultaneously at the same rate of interest. It is stressed that at least for some markets the assumption that an individual will not enter on both sides is not necessarily true. A counterexample is provided elsewhere.<sup>5</sup>

$$(20) \quad |y_1^{-11}| \leq \frac{2nu^1 M}{u^1 + u} - v^{1,1}.$$

This merely states that a time deposit in period 1 must be less than or equal to cash on hand.

If a trader  $i$  applies for a time loan  $y_t^{+11}$  from the bank this means that he offers the amount of  $(1 + \rho_2) y_t^{+11}$  of promissory notes to be redeemed in money one period later (or rolled over or refinanced) in return for an amount  $y_t^{+11}$  of bank money now.\*

At this point we must specify how loans and deposits are made under credit rationing, i.e. suppose that the demand for loans exceeds the available loan limits of the bank, how is credit rationed?

One of the simplest of rationing conventions is that all individuals have their loans scaled back in proportion to their applications. When there are few individuals in a tight market this can lead to tactics such as borrowing heavily to deny the use of funds to others.

We view the bank loan and deposit activity as a clearinghouse activity hence although all borrowers and depositors act simultaneously the bank is first able to aggregate deposits so that it could still calculate its total lending ability if deposits influenced it.

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\*It is possible that a bank might "rediscount a loan," i.e. it might sell the I.O.U. notes to another financial institution. This is explicitly ruled out here. We regard the I.O.U. notes as "sterilized" or nontradable.

The actual loan obtained by  $i$  in period 1 is given by

$$(21) \quad {}^*y_1^{i1} = \min \left[ y_1^{i1}, \frac{{}^+y_1^{ij}}{{}^+y_1^1 + {}^+y_1^2} \{k(v^1 + v^2)\} \right].$$

The loans obtained in the second period may be specified similarly.

$$\text{The symbol } {}^+y_1^1 = \sum_{i=1}^n \max[0, y_1^{i1}].$$

### 3.6. Bidding in the Market for Goods

In this simple economy we assume that all goods are offered for sale. Thus in the first period there are  $A_1 + A_2$  and in the second period  $B_1 + B_2$  units for sale. (A model where not all goods go through the markets has already been examined.<sup>5</sup>)

A trader  $i$  of type 1 bids a number  $0 \leq b_t^{i1} \leq 1$  which is the percentage of the money he has available at time  $t$  (after financing but before the market for goods) that he wants to spend on the good. The money he has on hand in time 1 is  $r^{i1}(1 - v^{i1}) + {}^*y_1^{i1} + {}^*z_1^{i1}$  thus his bid is:

$$(22) \quad {}^*b_1^{i1} = b_1^{i1} \{r^{i1}(1 - v^{i1}) + {}^*y_1^{i1} + {}^*z_1^{i1}\}.$$

Similarly for a trader  $j$  of type 2.

The amount of the good obtained by  $i$  of type 1 is:

$$(23) \quad q_1^{i1} = \frac{{}^*b_1^{i1}}{{}^*b_1^1 + {}^*b_1^2} (A_1 + A_2) \quad \text{and by } j \text{ is } q_1^{j2} = \frac{b_1^{j2}}{b_1^1 + b_1^2} (A_1 + A_2).$$

The price in the first period is:

$$(24) \quad p_1 = \frac{b_1^* + b_2^*}{A_1 + A_2} .$$

### 3.7. Bank Settlement

It is trivially easy to construct examples where one individual must increase his debt without making repayments for an arbitrarily long time, if optimal trade is to be achieved. In practice one's local friendly bank does not relish permitting an individual to rollover and enlarge a loan for twenty or thirty years without at least making interest payments. In static models of the economy such as the Walrasian system this causes no problems as the books are only balanced at the end of time.

In a multistage noncooperative game model of exchange through time the conditions on repayment and the servicing of loans are of critical importance in several ways. In particular if an individual is permitted to roll over loans he can always avoid bankruptcy until the last period.

If individuals can roll over loans without making any payments then the definition of the bank's short term profits becomes an exercise in accounting conventions. In particular if dividend payments to stockholders are based upon short term earnings, but earnings are not actually collected but are merely accrued then if a bank is permitted to pay out booked rather than collected profits the preconditions for a Ponzi game or chain-letter financing have been met. This is in the sense that the bank is issuing credit to itself in the form of dividends to stockholders paid on profits which are banked but which may turn into bad loans and hence will not be profits.

If we limit the period of financing obtainable from the bank and require, for example a balancing of the books at the end of each period

before new financing is permitted then the short term profits of the bank are well defined but Pareto optimality may be lost and furthermore the insolvency of one individual can cut the bank's short term profits which in turn can cut its dividends which in turn can force the insolvency of other individuals. It is possible to construct a "domino model" in which one insolvency forces a sequential insolvency of all others until finally the bank may fail.

#### Limited Liability and Bank Failure

If we model the system with only short term lending and limit the payment of dividends to actually obtained profits it is possible that with the insolvency of debtors a bank has negative profits. These negative profits could exceed paid in capital if  $k > 1$ . At this point a modelling decision concerning the liability of stockholders must be made. If unlimited liability is assumed the bank could flow through a negative dividend or an assessment. Otherwise the bank is constrained to pay a zero dividend but may not have the reserves to meet its depositors.

#### Individual Insolvency and Bankruptcy

Suppose that an individual is not in a position to pay the bank the amount it is owed. The individual is insolvent and we need to specify the manner in which the insolvency is to be cured. One way has been suggested in a previous paper.<sup>3</sup> A certain amount of goods are confiscated from the debtor. This particular "cure" is based upon the assumption that the goods can be claimed before the debtor has the opportunity to consume them.

If a debtor cannot pay the amount he owes and if the settlement process involves writing off the difference between what he pays and owes

then bank profits (and hence dividends) must be reduced accordingly. If the settlement involves the confiscation of goods which are not sufficiently durable that they can be sold in the next market then either the settlement must be regarded as essentially only punitive and the confiscated goods are written off or if value is to be derived the goods should be distributed to the stockholders as part of their returns. For simplicity we assume that if the goods are not durable they are destroyed. If durable, they are sold in a subsequent market, if one exists.\*

A simple 3 person example shows the chain effect possible with one insolvency that is cured by a bankruptcy procedure which writes off the difference between the monetary claim and collection. Suppose 3 individuals all own equal shares of a bank. They all have borrowed one unit at a rate of interest  $\rho$ . Furthermore all have outside incomes of 1. Table 2 below shows the simple balancing of the banks at settlement:

TABLE 2

	<u>Trader 1</u>	<u>Trader 2</u>	<u>Trader 3</u>
Amount owed	$1+\rho$	$1+\rho$	$1+\rho$
Outside income	1	1	1
Bank profit share	$\rho$	$\rho$	$\rho$
Net	0	0	0

---

\*Once durability is considered the meaning of short in contrast with long term profit maximization must be specified with care as a value must be imputed to the inventory of reclaimed goods.

Now suppose that trader 1 had a drop in outside income to say  $1-\rho$ . He becomes insolvent for an amount  $\rho$ , bank profits must be written down but the reduction in the size of the dividend forces traders 2 and 3 into insolvency.

### Bank Failure

It is also possible that the bankruptcy of a borrower could cause the insolvency or complete failure of a bank. A simple example shows this:

Bank Credit	M
Loans due to be collected	$N(1+\rho)$
Deposits called	$S(1+\rho)$

Suppose  $N > S$  then the bank runs at a profit if all loans are paid. If however there is a shortfall in repayments of say  $D$  then if

$$(25) \quad N(1+\rho) - D - S(1+\rho) < 0$$

the bank will have to use capital for repayments. If however:

$$(26) \quad N(1+\rho) + M - D - S(1+\rho) < 0$$

then the bank cannot meet its obligations and a bank failure rule is required.

We note that if there is unlimited liability of stockholders the bank failure conditions will be different from where there is limited liability.

There are several reasonable conventions such as creditors of the bank are paid in proportion to their claims and the bank is liquidated

after all assets are distributed. In order to fully define a model with bank failure we must specify if a new bank can be organized given the failure of an old bank.

Another convention might be to avoid liquidation by having the government issue extra fiat to pay all creditors in full and to cease the payment of dividends until the bank has recouped its losses.

As the prime concern of this paper is with the illustration of some relatively simple equilibrium conditions in an economy with fiat and bank money and no futures markets we select the simplest and to some extent less realistic assumptions in order to give us a well defined non-cooperative game. The unsatisfactory aspects of the model tend to highlight where the difficulties lie and help to illustrate the gap between a satisfactory microeconomic theory of finance and general equilibrium theory.

Before we turn to the full specification of the model to be analyzed some comments on the relationship between insolvency and inflation are needed.

### 3.8. Inflation and Insolvency

When an individual debt is created an individual issues promissory notes in return for goods or services or another financial instrument. A merchant, for example, who extends a ninety day credit to a customer takes a promissory note in exchange for goods. If, for simplicity we require that all trade be carried out in bank or fiat money then we rule out the credit creation by the firms and we limit ourselves to a system in which bank money and promissory notes jointly increase or decrease.

When an individual is declared bankrupt and the procedure obliterates

his promissory notes outstanding it does not simultaneously reduce the supply of bank notes outstanding hence the laws of conservation of claims are broken. In particular some individuals may now hold bank money against which the bank has no offsetting financial asset.

Suppose A has borrowed \$100 from the bank B which is now held by C. If A is bankrupt and the bank's claims against him are wiped out then the \$100 held by C is a claim against the bank without a counterbalancing asset.

When an individual holding bank money presents it to the bank for payment we must specify what is meant by payment. It usually means that the bank money is used to cancel personal I.O.U. notes outstanding or otherwise that the bank money is to be exchanged for fiat money.

When a bank fails then there can be a unilateral destruction of bank money thus a bank failure after individual bankruptcy could restore a balance between bank money and promissory notes. The lack of balance however does not imply that a bank has to fail. If its notes are not called the bank need not fail and the presence of the excess notes may lead to an increase in prices.

The full importance of the difficulty in balancing the books appears more strikingly when we consider multistage models with exogenous uncertainty. This is discussed elsewhere.<sup>11</sup>



#### 4. THE FORMAL MODEL: RULES, PLAYERS, STRATEGIES AND PAYOFFS

##### 4.1. The Rules

In the light of the discussion in Section 3 we make the following modelling choices:

- (1) The government and bank managers are strategic dummies. They are not modelled as active players, but are part of the rules of the game with given strategies.
- (2) The rate of interest on time loans or deposits in inside money  $p_2$  is given as a parameter of the system. The "gearing ratio" or the ratio between capital and loanable funds,  $k$  is also a parameter and is set to be sufficiently high that if the price system is normalized so that  $nM$  units of money are spent in period 1 then no competitive equilibrium will require more than  $nMk$  to be spent in any subsequent period.
- (3) Banks are required to accept all deposits and to grant all loans. If loan requests exceed available loanable funds then all loans are prorated.
- (4) Banks are required to pay out all short term profits as dividends.
- (5) Profits may be unrealized rather than realization profits, i.e. debtors are permitted to roll over principal and interest. Thus the banks are in a position to "create their profits" by crediting their equity with profits which may not be paid in.
- (6) All debts must be settled at the end of the last period. Thus bankruptcies will take place then. Prior to then an insolvency can be cured by a roll over of the debt and the bank can be protected by allowing it to bank unrealized profits as though they were an actuality.

- (7) At the end of the last period the sequence of final settlement is as follows:
- (a) (hypothetical) profits are paid out to stockholders
  - (b) loans are settled and insolvencies are cured by the bankruptcy procedure
  - (c) creditors (depositors) are paid in full if possible--otherwise in prorated payment
  - (d) the bank is liquidated with any remaining capital paid out in proportion to shares held. The minimum payment is zero.
  - (e) The government calls for the redemption of the promissory notes issued against the fiat money supply. It assesses its bankruptcy penalty against those unable to redeem their notes.
  - (f) The bankruptcy penalties for failure to redeem promissory notes with the inside bank and the outside bank (or government) are parameters of the system and could be different, but will be assumed to be the same.

#### 4.2. The Players

The government and outside bank is a mechanism without specified preferences and with a fixed strategy. In particular the government announces at the start the following parameters:

- $nM$  = the fiat money supply to be auctioned  
 $nS$  = the number of inside bank shares to be sold  
 $k$  = the inside bank loan ratio against assets  
 $\rho_1$  = the float interest rate  
 $\rho_2$  = the short term trade interest rate  
 $\mu_1$  = bankruptcy penalty for failure to meet the claims of  
the inside bank  
 $\mu_2$  = bankruptcy penalty for failure to meet the claims of  
the outside bank (we assume  $\mu_1 = \mu_2 = \mu$  ).

The inside bank is a mechanism without specified preferences and with a fixed strategy. Each period it accepts all deposits, makes loans, then at the end of each period calculates pro forma profits and pays them out as dividends. If profits are less than or equal to zero then zero dividends are paid.

The pro forma profit of the bank at the end of time  $t$  is given by:

$$(27) \quad \pi_t = (z_t^{+*} - z_t^{-*}) \rho_1 + (y_{t-1}^{+*} - y_{t-1}^{-*}) \rho_2 .$$

If  $\pi_t \geq 0$  dividends are paid. If  $\pi_t \leq 0$  dividends are zero. If  $\pi_t$  is sufficiently negative the bank could be forced into insolvency. In this rather simplified model this possibility can be averted by modelling the simple equivalent of a "central bank bailout," i.e. the central bank is willing to exchange debt with the inside bank sufficiently to prevent bank failure. We may specify an interest rate of zero for this exchange.

As the solutions we are looking for will lie away from bank failure we can sidestep this problem in this paper. But full rigor would call

for this mechanism to be completely spelled out. This cannot be avoided in any richer model which includes the possibility of large perceptual errors in lending policy.

The traders all have preferences and strategic freedom. Their strategies are described in 4.3. In general, if there were  $m$  consumer goods available each period, we assume the preferences of individuals could be described by a concave utility function:

$$(28) \quad U_i = \sum_{t=1}^T \beta_i^{t-1} \phi_i(q_{1t}^i, \dots, q_{mt}^i) .$$

We introduce the bankruptcy penalties as directly influencing the utility functions of each trader. Thus we replace (28) by:

$$(29) \quad \tilde{U}_i = \sum_{t=1}^T \beta_i^{t-1} \phi_i(q_{1t}^i, \dots, q_{mt}^i) + \mu_i \min[0, w^{i1}]$$

where net liquid assets for trader  $i$  of type 1 are given by:

$$(30) \quad \begin{aligned} w^{i1} = & r^{i1}(1 - v^{i1}) + z_t^{i1} + y_t^{i1} - \sum_j b_{jt}^{i1} + \sum_j p_{jt} A_{jt}^i - (1 + \rho_1) z_t^{i1} \\ & - (1 + \rho_2) y_t^{i1} + \frac{s^{i1}}{S} \pi_1 + z_{t+1}^{i1} + y_{t+1}^{i1} + \sum_j p_{j,t+1} A_{j,t+1}^i \\ & - (1 + \rho_1) z_{t+1}^{i1} - (1 + \rho_2) y_{t+1}^{i1} + \frac{s^{i1}}{S} \pi_2 . \end{aligned}$$

This is an arbitrary and unsatisfactory way of treating bankruptcy. The implicit assumption is that settlement takes place at the end of the economy and that a not necessarily economic penalty is applied to each individual. This mere subtraction from "score" is a crude approximation of a more satisfactory model involving the sale of repossessed assets each period.

The employment of the penalty at the end avoids all of the difficulties with the definition of short term profits which have been noted above.

#### 4.3. An Aside on Why Separability is not Needed

In (29) the utility function of an individual was described by a set of separable additive terms, one for each time period, followed by another separable and linear term representing the bankruptcy penalty. These forms are the simplest and easiest for modelling, exposition and the exploring of an explicit example. It is important to stress that neither separability nor linearity of any of the terms is required for the general results based on this type of model of an economy with a bankruptcy penalty (see Dubey and Shubik<sup>10</sup>).

Suppose that originally we begin with a trading economy where individuals trade for  $k$  time periods in  $m$  commodities. We can specify the utility function of trader  $i$  to be:

$$(31) \quad \varphi_i(q_{11}^i, \dots, q_{m1}^i; \dots; q_{1k}^i, \dots, q_{mk}^i)$$

where  $q_{jt}^i$  = the amount of  $j$  consumed by  $i$  during  $t$ .

We may introduce the bankruptcy penalty in a nonseparable, nonlinear manner as follows. The new utility function is:

$$(32) \quad \varphi_i(q_{11}^i, \dots, q_{m1}^i; \dots; q_{1k}^i, \dots, q_{mk}^i; f(\min[0, w^i]))$$

where  $w^i$  = the net liquid assets of trader  $i$  at the end of trade.

These can be positive or negative or zero.

$$(33) \quad \text{For } w^i \geq 0, \quad \vartheta_i = \varphi_i$$

$$\text{For } w^i < 0, \quad \vartheta_i \leq \hat{\vartheta}_i \quad \text{where}$$

$$\hat{\vartheta}_i = \varphi_i + \mu_i \min[0, w^i] .$$

#### 4.4. Strategies

A trader must decide how big a promissory note he should bid to obtain fiat money. Given his supply of fiat, how much he should bid for bank shares. Then he must decide upon loans, deposits and trade. Specifically we specify his strategy for the simplest possible instance where he must make all of his moves without gathering information until the end. This appears to be unrealistic, but as is argued elsewhere,<sup>10</sup> can be interpreted more realistically when the number of traders is large.

A strategy for trader  $i$  of type 1 consists (for the general case with  $m$  commodities) of  $2m+6$  numbers, of the form:

$$(34) \quad x^{i1} = (u^{i1}, v^{i1}; y_1^{i1}, z_1^{i1}, b_{11}^{i1}, \dots, b_{m1}^{i1}; y_2^{i1}, z_2^{i1}, b_{12}^{i1}, \dots, b_{m1}^{i1})$$

where  $u^{i1}, v^{i1} \geq 0$ ,  $b_{jt}^{i1} \geq 0$  and  $\sum_j b_{jt}^{i1} \leq 1$ .

#### 4.5. Payoffs

Let  $P_1(x^{11}, \dots, x^{n1}; x^{12}, \dots, x^{n2})$  be the payoff to a trader of type 1 (and  $P_2$  for a trader of type 2). Then each trader  $i$  selects a strategy  $x^{i1}$  (or  $x^{j2}$ ) and the payoff is given by:

$$(35) \quad P_1(x^{11}, \dots, x^{n1}; x^{12}, \dots, x^{n2}) = U_1(q_{11}^{i1}, \dots, q_{m1}^{i1})$$

$$+ \beta_1 U_1(q_{12}^{i1}, \dots, q_{m2}^{i1}) + \mu_1 \min[0, w^{i1}] .$$

## 5. AN EXAMPLE CONSIDERED

For the relatively simple example sketched in Section 2 a strategy consists of 8 numbers for trader  $i$  of type 1:

$$(u^{i1}, v^{i1}, y_1^{i1}, z_1^{i1}, b_1^{i1}, y_2^{i1}, z_2^{i1}, b_2^{i1})$$

with a similar expression for a trader  $j$  of type 2.

We may express the payoff functions directly in terms of these variables as follows:

$$(36) \quad P_1 = \log \left\{ \frac{b_1^{i1} n(A_1 + A_2)}{b_1^1 + b_1^2} \right\} + \beta_1 \log \left\{ \frac{b_2^{i1} n(B_1 + B_2)}{b_1^1 + b_1^2} \right\} \\ + \mu_1 \min \left[ 0, \frac{u^{i1} nM}{u^1 + u} (1 - v^{i1}) + *y_1^{i1} + *z_1^{i1} - b_1^{i1} + \left\{ \frac{b_1^1 + b_1^2}{A_1 + A_2} \right\} A_1 \right. \\ \left. - (1 + \rho_1) z_1^{i1} - (1 + \rho_2) *y_1^{i1} + \frac{s^{i1}}{s} \pi_1 + *y_2^{i1} + z_2^{i1} - b_2^{i1} \right. \\ \left. + \left\{ \frac{b_2^1 + b_2^2}{B_1 + B_2} \right\} B_1 - (1 + \rho_1) z_2^{i1} - (1 + \rho_2) y_2^{i1} + \frac{s^{i1}}{s} \pi_2 \right]$$

where  $0 \leq v^{i1} \leq 1$ ,  $s^{i1} = \frac{nSv^{i1}}{v^1 + v}$ ,  $b_t^{i1} \leq 1$  for  $t = 1, 2$ .

Before we can define bank profits we must specify interest rates.

We define two rates:

$\rho_1$  = the float rate

$\rho_2$  = the short term trade rate.

The float rate is for money to be returned within the same period.

The short term trade rate is for money paid one period later. Thus:

$$\begin{aligned}\pi_1 &= \rho_1 |z_1| \\ \pi_2 &= \rho_1 |z_2| + \rho_2 |y_1| .\end{aligned}$$

We may assume that as this economy lasts for only two periods there is no market for new time loans or deposits in the second period.

We consider Case 1 of Section 2 and use the following parameters:

$$\rho_2 = \frac{1-\beta}{\beta}, \quad \lambda_1 = \frac{1}{A} \quad \text{and} \quad \lambda_2 = \frac{1}{A} .$$

With these settings the assertion is that the limit N.E. of the noncooperative game will approach the C.E. of the Walrasian economy.

Even for as basically simply an example as the one presented here the calculations for finite  $n$  and the investigations of behavior for large  $n$  are laborious and tedious in the extreme. It is shown elsewhere by Dubey and Shubik<sup>2</sup> that we may replace the game with a finite number of traders by one with a continuum of traders each with essentially no individual market power. If we do this for this model we may rewrite (36) as follows:

$$\begin{aligned}(37) \quad P_1 &= \log\{b_1^{i1}/p_1\} + \beta_1 \log\{b_2^{i1}/p_2\} + \mu_1 \min\{0, \{u^{i1}/p_3 - v^{i1}/p_4 \\ &\quad + y_1^{i1} + z_1^{i1} - b_1^{i1} + p_1 A_1 - (1 + \rho_1)z_1^{i1} - (1 + \rho_2)y_1^{i1} \\ &\quad + (s^{i1}/S)\pi_1 + z_2^{i1} - b_2^{i1} + p_2 B_1 - (1 + \rho_1)z_2^{i1} + (s^{i1}/S)\pi_2\}\end{aligned}$$

where

$$(38) \quad v^{i1} \leq u^{i1}/p_3$$

$$(39) \quad b_1^{i1} \leq y_1^{i1} + z_1^{i1} + \{u^{i1}/p_3 - v^{i1}\}$$



$$(40) \quad b_2^{i1} \leq \frac{u^{i1}}{p_3} - \frac{v^{i1}}{p_4} + y_1^{i1} + z_1^{i1} - b_1^{i1} + p_1 A_1 - (1 + \rho_1) z_1^{i1} \\ - (1 + \rho_2) y_1^{i1} + \frac{s^{i1}}{S} \pi_1 + z_2^{i1}$$

where  $p_1$  = price of commodity in period 1 in money terms  
 $p_2$  = price of commodity in period 2 in money terms  
 $p_3$  = price of outside money in terms of I.O.U. notes  
 $p_4$  = price of inside bank shares.

The referee or "government" supplies as parameters:

$M$  = the outside money supply

$S$  = the number of inside bank shares for sale

$\rho_1$  = interest rate on float inside money

$\rho_2$  = interest rate on time inside money

$\mu_1$  = bankruptcy penalty on traders of type 1

$\mu_2$  = bankruptcy penalty on traders of type 2.

For simplicity and without any further loss of generality in this example we may fix  $M = S = A_1 + A_2$ .

We solve the first example noted in Section 2, where  $B_1 = B_2 = \epsilon A$ . It is straightforward to check that a solution for an N.E. exists when the inequalities (38), (39) and (40) are satisfied as equalities and the additional conditions:

$$(41) \quad \frac{1}{b_1^{i1}} = \mu_1$$

$$(42) \quad \frac{\beta_1}{b_2^{i1}} = \mu_1$$

and

(43) the second term in the min function of (39) is equal to zero.

There are six similar conditions for traders of type 2.

An N.E. strategy for a trader  $i$  of type 1 can be characterized by:

$$(44) \quad (u^{i1}, v^{i1}, y_1^{i1}, z_1^{i1}, b_1^{i1}, y_2^{i1}, z_2^{i1}, b_2^{i1}) = (1, 1, 1, 0, 1, 0, 1, 1) \\ \text{or } (1, 1, 0, 1, 1, 0, 1, 1)$$

and similarly for traders of type 2; where as given parameters of the system we have

$$(45) \quad p_2 = \frac{1-\beta}{\beta}, \quad \mu_1 = \frac{1}{A}, \quad \mu_2 = \frac{1}{A}$$

prices are given by:

$$(46) \quad (p_1, p_2, p_3, p_4) = (1, 1/\epsilon, 1, 1), \quad \text{hence } p_1 = 0,$$

and the final distribution of goods are given by  $(A, \epsilon A)$  to each type of trader.

The other two cases require a similar analysis.

### 5.1. An Interpretation of the Results

This paper has provided a somewhat elaborate six stage model together with an example with a solution calculated for one simple case.

The generality of the observations made here will be established in a separate paper by Dubey and Shubik.<sup>10</sup> Essentially the general observations are as follows:

Leaving aside uncertainty a monetary system serves two purposes (a) the financing of the float and (b) the financing of intertemporal trade. It is possible to construct a system which has two types of money "outside" and "inside" money, or fiat and inside bank money. These monies may exchange on a 1:1 basis but the first will carry a zero rate of interest, while the second may carry a positive rate of interest when used for time loans or deposits even though float loans and deposits carry a zero interest rate.

The outside money serves to purchase the shares of an inside bank designed to issue inside money. As the shares of the bank are held within the system even with positive interest conservation is preserved as interest payments merely amount to a redistribution of wealth. If no one goes bankrupt then as  $M$  units of outside money must be returned the highest rate for outside money must be zero.

It is important to note that without the explicit introduction of uncertainty it becomes operationally meaningless to distinguish between debt and equity. Thus there is a tendency to talk about fiat money, or bonds or consols or stock where the key operational differences in terms of risk taking and other ownership functions are not really there in a model without uncertainty.

## 5.2. Further Problems

Even given a general proof of the items noted above; the models raise as many, if not more questions than they answer. A list of some of the new problems follows.

(a) In the model presented the inside money rate of interest is introduced as a parameter. Can we construct a model where competition

among inside banks produces this rate of interest endogenously? My conjecture is yes, with the following qualification. The inside money supply and the inside rate of interest are dual variables and one of them must be supplied before it is possible to calculate the other; i.e. the central bank must control either the supply of bank money or its rate of interest.

(b) There are only spot markets here. Furthermore all time interest rates are only for one period, after which all loans are refinanced. Can we introduce futures markets and long run interest rates? What effect do they have on the system? The answer appears to be that we can and that no new phenomena of basically important distinction occur without the introduction of uncertainty.

(c) In the model presented here shares in the inside bank are not traded. It appears to be straightforward, but onerous to add two extra strategic stages to the six stages required for the two period model in order to have trade in shares.

(d) As is shown in the example in Section 2 and in the example above the inside money rate of interest is related to the "natural time discount" which different individuals may have, but it is not necessarily a particularly simple function of them.

(e) The model presented here extended to any number of periods allows for unrestricted roll-over of loans from period to period until the end. Furthermore all bankruptcy penalties are exacted at the end. A more attractive model might take into account the possibility of insolvency and bankruptcy at any time. This calls for rules concerning the capturing and selling of the goods of the defaulter as well as the rules concerning the cancellation of debt other than by repayment.

(f) The models described here and elsewhere tend to have dealt with finite fixed time periods. It may be of interest to consider continuous time. In such a model, it is conjectured that the necessary size of the initial inside money supply approaches zero, however velocity increases so that the product provides a measure of the flow of trade. The presence of a zero rate of interest to finance the float becomes vanishingly small.

(g) The models have been ones with trade and no production and durable assets. Dubey and Shubik<sup>12</sup> have already cast production into this mold; however production and the presence of assets introduce new constraints on the dynamics and the need for long term financing which have not yet been considered.

(h) The key next step calls for the introduction of exogenous uncertainty. The presence of assets provides lenders with insurance against lending to borrowers whose debt would otherwise be "naked."

The stress in this series of papers has been to separate difficulties so that many different monetary and financial phenomena can be investigated separately prior to attempting to put them together to study the extra phenomena and interactions characteristic of a complex monetary and financial system.

## REFERENCES

1. Shubik, Martin and C. Wilson, TMFI, Part 30: The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money, CFDP 424, 2/9/76 (Revised with C. Wilson 6/7/76.)
2. Shubik, Martin, "Mathematical Models for a Theory of Money and Financial Institutions." ADAPTIVE ECONOMIC MODELS, ed. Richard H. Day and Theodore Groves, Academic
3. Shubik, Martin, "Commodity Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model," Western Economic Journal, X, 4(December 1972) pp. 24-38. (Also CFP 391, earlier versions CFDP 324, RAND p-4686.)
4. Shubik, Martin, Lloyd Shapley, "Trade Using One Commodity as a Means of Payment," The Journal of Political Economy, forthcoming 1977.
5. Shubik, Martin, P. Dubey, "TMFI, Part 28: The Noncooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies CFDP 422, 2/10/76.
6. Shubik, Martin, "A Trading Model to Avoid Tatonnement Metaphysics," STUDIES IN GAME THEORY AND MATHEMATICAL ECONOMICS, Auctioning for Procurement and Allocation, Y. Amihud (ed.), New York, NYU Press, 1976, 129-142.
7. Shubik, Martin, "A Theory of Money and Banking in a General Equilibrium System, RM 48 7/70. Institute for Advanced Study, Vienna, Austria.
8. Shubik, Martin, "A Noncooperative Model of a Closed Economy with Many Traders and Two Bankers," Zeitschrift fur Nationalokonomie, forthcoming, 1976.
9. Bertrand, J., "Theorie Mathematique de la richesse sociale" (review), Journal des Savants, Paris (September 1883), pp. 499-508.
10. Dubey, Pradeep and M. Shubik, "Bankruptcy in a multistage model with Inside and outside money," in process as a CFDP.
11. Shubik, Martin, Part 31: Banks, Insurance and Futures Markets CF60819, August 15, 1976.
12. Dubey, Pradeep and M. Shubik, "A Closed Economic System with Production and Exchange Modelled as a Game of Strategy, CFDP 429, 5/18/76.