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A CLOSED ECONOMIC SYSTEM WITH PRODUCTION AND EXCHANGE MODELLED AS A GAME OF STRATEGY

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APPENDIX B..

A CLOSED ECONOMIC SYSTEM WITH PRODUCTION AND EXCHANGE MODELLED AS A GAME OF STRATEGY*

by

Pradeep Dubey and Martin Shubik

1. INTRODUCTION

Models of a closed economic system with production and exchange have been studied both by methods of general equilibrium analysis and cooperative game theory. The conditions for the existence of a price system which clears all markets and is efficient have been investigated in recent years by Debreu[1] and many others.

A completely different approach to the study of the existence of a price system is to use a model of the economy as a game in coalitional form and to apply the cooperative game solution of the core to this game. The formulation of a market with trade alone as a game in coalitional form is relatively straightforward with or without sidepayments. The work of Shapley and Shubik,[2] Shubik,[3] Debreu and Scarf,[4] Aumann,[5] and many others attests to this. The formulation of a game with trade and production is more difficult but has been done by Hildenbrant,[6] Dubey,[7] and others.[6] The difficulties lie more with the basic economic modelling

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rather than the mathematics. Given decreasing returns to scale it becomes necessary to consider individually owned technologies. It is not the purpose of this paper to enter into a critique of the modelling of production in economies studied as games in cooperative or coalitional form. Our stress is upon the description of an economy with trade and production as a game in strategic form.

There have been no models constructed of an economy with trade and production as a game in strategic form. We construct one and analyze the resultant game for its noncooperative equilibria.

2. A STRATEGIC MODEL OF EXCHANGE AND PRODUCTION

In order to model a closed economic system as a game of strategy it is necessary to be completely specific about the market mechanism leading to the formation of prices; the details of information and the rules of production and trade.

The sequence of trade and production must be spelled out. There are many choices in modelling in fine detail where different variations each have their counterparts in actual methods of trade and production. In our first model our stress is on simplicity rather than on realism.

(1) Actors

We consider two types of economic actors in the economy. They are consumers, who are also owners of all resources and managers who run the (consumer-owned) firms.

Let there be n consumers and s firms.

(2) Commodities

We assume that there are m commodities and a distinguished $m+1^{st}$ commodity called "money." A consumer is called "monied" if he has some money. The money may be of intrinsic worth as a commodity, in which case it is a commodity money. Otherwise it is a fiat money.

We denote a quantity of the k^{th} commodity purchased during the first market by consumer (firm) i by x_k^i (y_k^i) . The quantity of commodity k held at the end of the second market is denoted by $\hat{x}_k^i$ $(\hat{y}_k^i)$.

(3) Ownership and Shares

At the start of the economy all goods are owned by the consumers. Let individual i , $i = 1, \ldots, n$ have an initial endowment of $(a_1^i, a_2^i, \ldots, a_{m+1}^i)$. The money is held initially by individuals and firms. A firm $k \in I_s$ has an initial holding of M^k of money.

At the start of the economy ownership claims to all firms are held by the consumers. Managers are not permitted to own or trade in shares of the companies. Let the initial endowment of individual i of the shares of firm k by η_k^i where $i \in I_n$, $k \in I_s$. We assume that the shares are nonvoting.

Without loss of generality (abbreviated as "wlog") we may set

$$\sum_{i=1}^{n} \eta_k^i = 1 \quad \text{for} \quad k \in I_s \ .$$

From the above description we can specify the total initial endowment of consumer i by:

$$(a_1^i, \ldots, a_m^i, a_{m+1}^i; \eta_1^i, \ldots, \eta_k^i) \quad \text{for} \quad i \in I_n , \quad k \in I_s$$

and the initial endowment of firm k by

$$(0, \ldots, 0; M^k) \text{ where } k \in I_s .$$

(4) Sequence of Trade and Production

The economy begins with a market in which all consumers and firms trade. The consumers sell and buy goods* and the firms buy goods. After the first market has ended the firms take delivery of the goods they have bought. These goods now serve as inputs for the production of the firms. All firms produce and then offer all of their final product for sale. In the second market only the consumers buy and sell.

At the end of the second market the consumers receive delivery of all goods obtained in the first and second markets and consume them. Furthermore at the end of the second market a final financial settlement is made. The financial settlement is discussed in (10) below.

At this point we have a minor choice to make in modelling. We may assume either that the consumers receive the income from their sales in the first market at the end of the first or at the end of the second. To begin with for simplicity in presentation we assume that they are paid at the end of the second market, however we sketch the other case in 5.5 .

(5) Preference Conditions

We assume that each consumer i has his preferences described by a concave nondecreasing utility function defined on the m goods and the money or some nonempty subset thereof. This in particular covers the

*In this model we assume shares are not traded. We hope to relax this condition later.

possibility that the money could be a fiat money and that some of the goods might have economic value only in production.

In this model we make the somewhat strong and unsatisfactory assumption that the manager of each firm acting as a perfect fiduciary for his stockholders attempts to maximize profits. This assumption (which is the usual assumption made in general equilibrium theory) is somewhat unsatisfactory for three reasons

(1) Managers are not modelled as having their own goals which could not be at variance with stockholders.*

(2) Stockholders might have different risk preferences for money.

(3) The interpretation of the definition of profits in a single period model may not be totally satisfactory especially when there are durable production goods.

In this model we may denote the final bundle of goods purchased by each firm i by $\hat{y}^i = (\hat{y}^i_1, \ldots, \hat{y}^i_m)$ and the prices which are formed in the second market by $\hat{p}_1, \ldots, \hat{p}_m$. The firm i's profits will be given by:

$$\Pi^i = \sum_{j=1}^{m} \hat{y}^i_j \hat{p}_j - \Sigma e^i_j$$

where e^i_j is the bid of firm i for commodity j. (This implies that unused inputs are of no final worth to the firm.)

(6) <u>Moves, Markets and Price Formation</u>

In the first market a move by a consumer i is a vector of $2m$ dimensions of the form

$$(b^i_1, \ldots, b^i_m; q^i_1, \ldots, q^i_m) \quad \text{for} \quad i = 1, \ldots, n$$

*We hope to be able to relax this condition, give the manager a utility function and prove that profit maximization comes about when there are many firms with each manager owning few shares in his firm.

where b_j^i is the amount of money offered for commodity j where $j = 1, \ldots, m$ and q_j^i for $j = 1, \ldots, m$ is the amount of commodity j offered for sale by consumer i.

We require that:

$$\sum_{j=1}^{m} b_j^i \leq M^i , \quad b_j^i \geq 0 \quad \text{and} \quad 0 \leq q_j^i \leq a_j^i \quad \text{for} \quad j = 1, \ldots, m .$$

A move by a firm i in the first market is a vector of bids for inputs $(e_1^i, \ldots, e_m^i)$ where $i \in I_s$ and where

$$\sum_{j=1}^{m} e_j^i \leq M^i \quad \text{and} \quad b_j^i \geq 0 .$$

In the second market a move by a consumer i is a vector of $2m$ dimensions. It can be expressed as $(\hat{b}_1^i, \ldots, \hat{b}_m^i; \hat{q}_1^i, \ldots, \hat{q}_m^i)$ where $\hat{b}_j^i$ is a bid by i for commodity j and $\hat{q}_j^i$ is an offer by i for the sale of j.

$$\sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} \hat{b}_j^i \leq a_{m+1}^i$$

also $$q_j^i + \hat{q}_j^i \leq a_j^i .$$

This condition, which will be changed in 5.5 implies that consumer i does not obtain the income from the sale of his goods in the first market until after he has bid in the second market.*

---

*A distinction between the cases in which payments from markets are made in parallel or sequentially can be made in terms of spacially or temporally separated markets. In the first instance we could imagine several separate but coexistant markets for the same goods, in the second there is only one market but it meets several times. As our concern here is with a sequential market we do not pursue the distinction in the interpretation of the models further.

In the second market a move by a firm i is a vector of m dimensions which can be denoted by $(\hat{y}_1^i, \ldots, \hat{y}_m^i)$ where $\hat{y} \in (y^i + Y^i) \cap \Omega^{m+1}$. We make the extra simplifying assumption to start with that a firm will offer all of its final product for sale, hence the offer can be denoted as $(\hat{y}_1^i, \ldots, \hat{y}_m^i)$.

Price formation in the first and second markets is caused by the volume of bids and offers. Thus

$$p_j = \frac{\sum_{i=1}^{n} b_j^i + \sum_{i=1}^{s} e_j^i}{\sum_{i=1}^{n} q_j^i}$$

and

$$\hat{p}_j = \frac{\sum_{i=1}^{n} \hat{b}_j^i}{\sum_{i=1}^{n} \hat{q}_j^i + \sum_{i=1}^{s} \hat{y}_j^i} .$$

An individual i who has bid b_j^i for item j will purchase an amount b_j^i/p_j .

(7) <u>Production</u>

Individual firms are each assumed to own a particular production technology. The specific description of the production technology is given in detail in 3 . Essentially we may consider either a production cone (constant returns to scale) or a convex production set (decreasing returns to scale). These are illustrated below for a two commodity example.

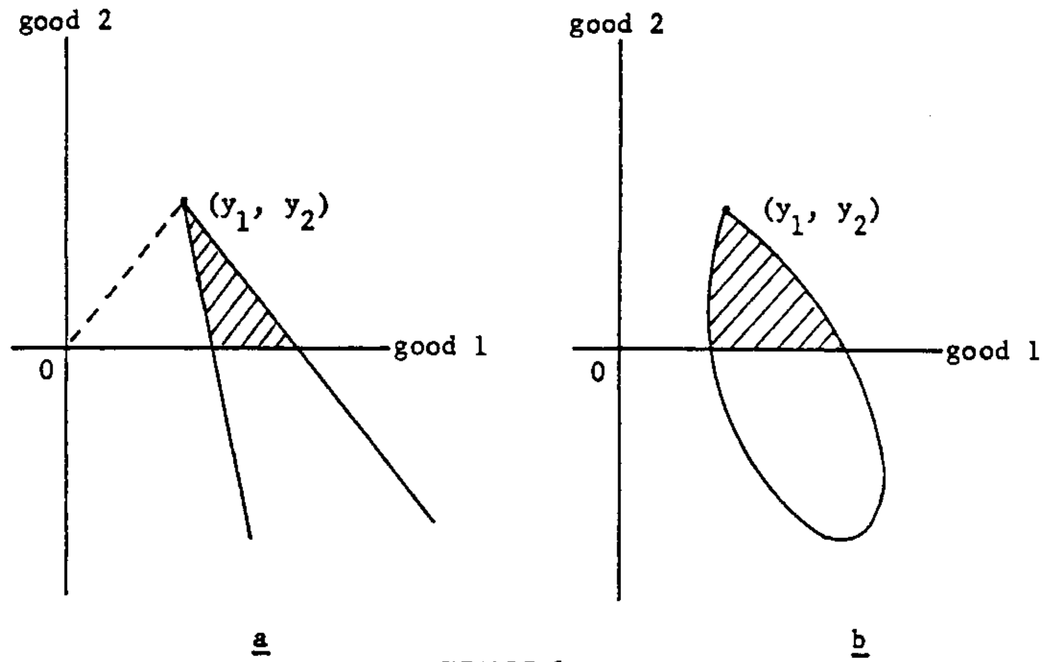

FIGURE 1

The firm begins with a bundle of inputs (y_1, y_2) and in Figure 1a is able to transform them into any bundle of outputs which lies within the production cone and is in R^2_+. Similarly in Figure 1b (decreasing returns) the outputs must be within the production set and in R^2_+.

(8) Information

A critical and subtle aspect of the description of the process of trade and production is the specification of information. In particular the availability of information and the use made of it are key elements in the description of the strategies of the individuals, in contrast with their moves. The availability of information enables an individual to plan at a more complex level than if he were unable to distinguish among different contingencies.

We suggest that, at least in mass markets, both simplicity and realism call for assumptions which give the actors as little information as possible. For example we may assume that each consumer knows his endowments and the aggregate information on the endowment of all others before he bids in the first market. After the first market is over he finds out only what the prices were and what he bought.

The producers at the start of the first market have only aggregate information beyond each knowing his own money holding. After the first market each producer knows the prices in the first market and his supply of inputs.

After the second market has taken place the consumers and producers find out about final prices, deliveries, profits and dividends.

(9) Strategies

In a multistage game with any information at all the proliferation of strategies is enormous. Although it is natural to be concerned with the existence of equilibria, in general as the strategy sets expand so do the sets of equilibria. The problem is frequently that there are too many equilibria rather than in proving existence.

When information becomes available it may happen that strategies become complex functions of functions and that desirable properties such as convexity are lost. If one is concerned with pure strategy equilibria it may be possible to lose existence with complicated strategies, as has been noted by Roberts and Sonnenschien.[8] By restricting ourselves to simple subsets of strategies characterized by limited contingency planning we may be able to show the existence of equilibria using only pure strategies.

In particular here we may consider the set of strategies for the consumers based upon the assumption that the prices known to exist in market 1 have an influence upon the bids made in market 2. Thus a strategy for a consumer i might be a vector of 2m components and a vector function where the first 2m components are bids and offers for goods in market 1 and the vector function determines bids and offers for the firm without questioning how or why the inputs are obtained.

A more complex strategy more in keeping with extremely high information conditions and close oligopolistic competition is one in which the production decision is based not only upon what is the vector of resources available for the production process, but how it was obtained. For example there might be a contingency plan of firm C which calls for one level of production if firm A had made a large bid for steel and firm B had made a low bid, and a different level of production if the bids of A and B were reversed even though those bids summed to the same amount and both led to firm C obtaining the same input of steel.

In Section 3 below we show that equilibria in pure strategies exist for games where strategies are of the simpler variety as described above.

It is important to note that as numbers increase the potential set of strategies for an individual increases astronomically. There are two natural ways in which this set can be cut down. One is by aggregating or limiting the information available to the agents of a player and for the m goods in market 2 given information of the results in market 1. Such a strategy for consumer i can be described as:

$$(b_1^i, \ldots, b_m^i; q_1^i, \ldots, q_m^i; \varphi_i(\cdot))$$

where φ is defined over all information states of i .

We assume that the producer knows only his own resources and the aggregate resources in the economy when he bids in the first market. We may formulate the strategy of a producer in terms of imagining the firm to have a buyer and a manager of manufacturing. Two versions are suggested. In the first version the buyer selects his bids, and the manager of manufacturing is told to prepare an overall production plan contingent on any vector of inputs obtained from the market. Thus a strategy consists of a vector and a transformation contingent upon inputs received.

$$(b_1^i, \ldots, b_m^i; T_i)$$

where T_i is defined over the information states of the manager of manufacturing.

This strategy complicated as it might seem is really both behavioristically relatively easy to interpret and implement and is by no means as complicated as a strategy could become, as will be shown below. In particular the interpretation here is of decentralization. The manager of production accepts the inputs as given and tried to optimize the other is trying to impose some reasonable limitation on the level of complexity of planning permitted by a player. The first approach is implicit in macroeconomic theorizing and the second approach is in keeping with a behavioral view of decisionmaking in the firm. These approaches are complementary with not opposed to the game theoretic models of competition.

A Note on Consumer Strategies

In order to give some insight into the nature of the strategy sets of an individual and their relationship to final outcomes we consider a market with one commodity being bought or sold with trade in a commodity money.*

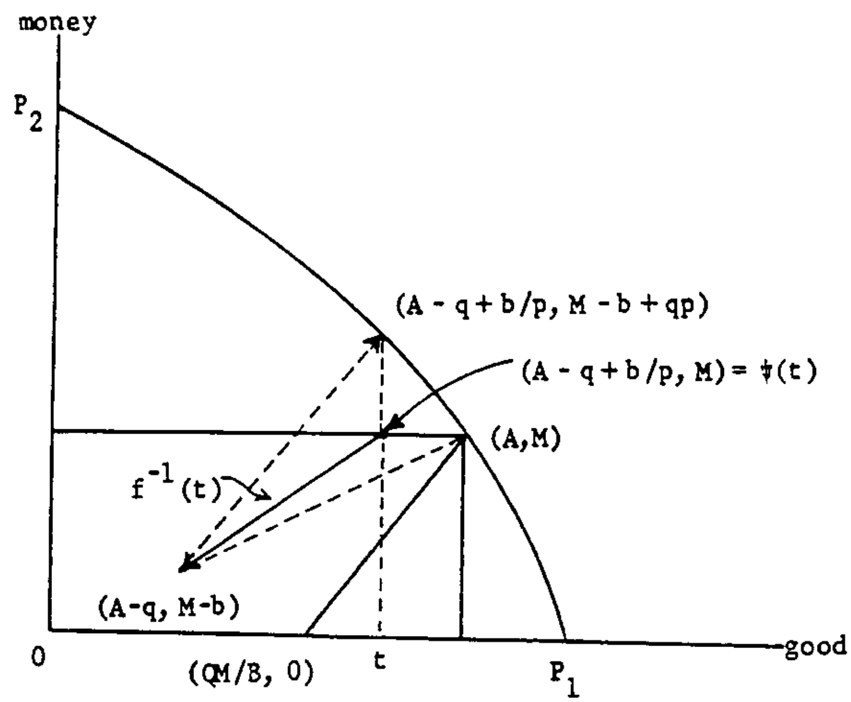

FIGURE 2

In Figure 2 the point with coordinates (A,M) indicates the initial endowment of an individual. Consider the rectangle given by $0 \leq q \leq A$ and $0 \leq b \leq M$. Any point in this rectangle is a bid and an offer.** Consider the strategy*** (q,b) ; we wish to study how it transforms the initial point (A,M) . Suppose that all other traders have offered Q units of the good and a total of B units of money. Then a bid of (q,b) takes the initial holding of (A,M) to a final holding of:

---

*This note has been given in a previous paper,[9] but as it is directly relevant to proofs in Section 3, for convenience it is given directly here.

**For simplicity in describing the strategies of a single individual in a single market the superscript i and subscript j has been dropped as the meaning should be clear.

***As our comments are limited to a single market we refer to the pair (q,b) as a strategy, although in the larger game it would only be a move.

$$(A - q + b/p,\ M - b + qp)$$

where

$$p = \frac{B+b}{Q+q} \quad (Q > 0)\ .$$

We may observe that had the individual bid $(q - b/p,\ 0)$ this too maps (A,M) into

$$(A - q + b/p,\ M - b + qp)\ .$$

The bids which map (A,M) into itself are given by

$$A = A - q + b/p \quad \text{and} \quad M = M - b + qp$$

or

$$b = q\left(\frac{B+b}{Q+q}\right) \quad \text{or} \quad Qb = Bq\ .$$

All bids on the line connecting $(QM/B,\ 0)$ to (A,M) map (A,M) onto itself. Any strategy to the right of this line maps AM onto a point on the curve P_1P_2 to the right.

All points on the curve P_1P_2 can be obtained by bids on the "L-shaped" boundary of the strategy set, i.e. bids of the firm $(q,0)$ or $(0,b)$ where $0 \leq q \leq A$ and $0 \leq b \leq M$.

Let (x,m) be a point on the curve P_1P_2 then it can be checked that the equation for this surface is given by:

$$m = M + \frac{(A-x)B}{Q+A-x}\ .$$

This is a concave curve.

(10) Payoff Functions

An individual consumer i has a utility function of the firm

$$u^i = u^i(\hat{x}^i_1, \ldots, \hat{x}^i_{m+1})$$

where $\hat{x}^i_j$ $(j = 1, \ldots, m+1)$ is his final endowment of goods (including money). The payoff function for consumer i is given by:

$$\Pi^i = \Pi^i(t^1, \ldots, t^n, \ldots, t^{n+s})$$

where t^j is a strategy for player j (a consumer for $j \leq n$ and a firm if $j > n$). The strategies are of the form indicated previously. We can express the payoff function in terms of utility functions with outcome variables transformed into strategy variables as follows:

$$\Pi^i = u^i\Bigg(a^i_1 - q^i_1 + \frac{b^i_1}{b_1}q_1 - \hat{q}^i_1 + \frac{\hat{b}^i_1}{\hat{b}_1}\hat{q}_1;\ \ldots;\ a^i_m - q^i_m + \frac{b^i_m}{b_m}q_m - \hat{q}^i_m + \frac{\hat{b}^i_m}{\hat{b}_m}\hat{q}_m;\ a^i_{m+1} - \sum_{j=1}^{m} b^i_j - \sum_{j=1}^{m} \hat{b}^i_j + \sum_{j=1}^{m} q^i_j p_j + \sum_{j=1}^{m} \hat{q}^i_j \hat{p}_j + \sum_{j=n+1}^{n+s} \eta^i_j \pi_j\Bigg)$$

where, in words, the following relationships hold:

Final holding of commodity j $(j = 1, \ldots, m)$ =

Initial amount - amount sold in market 1 + amount bought in market 1 - amount sold in market 2 + amount bought in market 2.

The final holding of money =

Initial amount - total bid in market 1 - total bid in market 2 + income from market 1 + income from market 2 + dividends.

The payoff function of a firm i has been given in (5) above.

3. THE EXISTENCE OF NONCOOPERATIVE EQUILIBRIUM POINTS

For a positive integer r we shall denote by I_r the set $\{1, \ldots, r\}$, by E^r the Euclidean space of dimension r, and by Ω^r the non-negative orthant of E^r.

Let

$$I_n = \text{the set of traders}$$

$$I_{m+1} = \text{the set of commodities}$$

$$I_{n+s} \setminus I_n = \text{the set of firms.}$$

The <u>initial allocation</u> of trader i is a vector $a^i \in \Omega^{m+1}$, where a^i_j is the amount of commodity j available to i (for $j \in I_m$), and a^i_{m+1} represents the money held by i. In addition, trader i has <u>shares</u> in the profits of the s firms. This we will denote by $\eta^i \in \Omega^s$ where η^i_ℓ is i's share in firm ℓ, $n+1 \leq \ell \leq n+s$. The traders' utility functions are real-valued:

$$u^i : \Omega^{m+1} \to \Omega^1 , \quad i \in I_n$$

and are assumed to be concave, continuous and non-decreasing. We shall say that trader i "desires" good j if $u^i(x^i)$ is an increasing function of the variable x^i_j, for any fixed choice of the other variables x^i_ℓ. A trader i for whom $a^i_{m+1} > 0$ will be called "monied"; and when $a^i_j > 0$ we will say that he is "j-furnished."

When we drop an index and use a bar it will indicate summation over the indexing set. Thus for $x^i \in \Omega^m$, $\bar{x}^i$ means $\sum_{j \in I_m} x^i_j$, etc.

We assume that $\bar{a}_j > 0$ for all $j \in I_m$, and $\bar{\eta}_k = 1$ for all $k \in I_s$.

Each firm ℓ is endowed with positive money M^ℓ and with a <u>production set</u> $Y^\ell \subset E^{m+1}$. Any $y \in Y^\ell$ represents a possible production that k may engage in: the negative (positive) components of y stand for the inputs (outputs) of production. We make the standard assumptions that Y^ℓ is convex, and that $Y^\ell \cap \Omega^{m+1} = \{0\}$.*

The production economy is modelled as a non-cooperative game that consists of two stages. There are m trading-posts, one for every $j \in I_m$. In the first stage trader i offers a quantity q_j^i of j for sale in the j^{th} trading-post, and bids b_j^i amount of money on it; firm ℓ merely bids e_j^ℓ. We require, obviously, that

$$0 \le q_j^i \le a_j^i ,$$

$$b_j^i \ge 0 , \quad \sum_{j \in I_m} b_j^i \le a_{m+1}^i ,$$

$$\sum_{j \in I_m} e_j^\ell \le M^\ell .$$

The price p_j of $j \in I_m$ is obtained by dividing the amount bid on j by the total supply of j :

$$p_j = \begin{cases} \dfrac{\bar{b}_j + \bar{e}_j}{\bar{q}_j} & \text{if } \bar{q}_j > 0 \\ 0 & \text{if } \bar{q}_j = 0 . \end{cases}$$

Next we calculate the allocation $x^i \in \Omega^{m+1}$ ($y^\ell \in \Omega^{m+1}$) that results for trader i (firm ℓ):

---

* 0 denotes the origin of Ω^{m+1}.

$$x^i_j = a^i_j - q^i_j + \frac{b^i_j}{p_j},^* \quad \text{for} \quad j \in I_m$$

$$x^i_{m+1} = a^i_{m+1} - \sum_{j \in I_m} b_j + \sum_{j \in I_m} p_j q^i_j$$

$$y^\ell_j = \frac{e^\ell_j}{p_j}$$

$$y^\ell_{m+1} = M^\ell - \sum_{j \in I_m} e^\ell_j .$$

In the second stage, firms produce with the resources they have purchased; thus firm ℓ selects a vector $\hat{y}^\ell \in (y^\ell + Y^\ell) \cap \Omega^{m+1}$. Firms then send all their products to the trading-posts for sale. Traders bid and offer as in the first stage. The market gets cleared as before. Let us denote by $(\hat{b}^i, \hat{q}^i)$ the bids and offers of trader i in the second stage market, and by $\hat{p}_j$ the price of $j \in I_m$. Then $\hat{q}^i_j \leq a^i_j - q^i_j$, and $\sum_{j \in I_m} \hat{b}^i_j \leq a^i_{m+1} \sum_{j \in I_m} b^i_j \hat{p}_j$ is given by:

$$\hat{p}_j = \frac{\bar{\hat{b}}_j}{\bar{\hat{q}}_j + \bar{\hat{y}}_j} .$$

The final allocation $\hat{x}^i$ to trader i is:

$$\hat{x}^i_j = x^i_j - \hat{q}^i_j + \frac{\hat{b}^i_j}{\hat{p}_j}, \quad \text{for} \quad j \in I_m$$

$$\hat{x}^i_{m+1} = x^i_{m+1} - \sum_{j \in I_m} \hat{b}^i_j + \sum_{j \in I_m} \hat{p}_j \hat{q}_j + \sum_{\ell \in I_{n+s} \setminus I_n} \Pi^i_\ell \pi^\ell$$

---

*We set $b^i_j / p_j = 0$ if $p_j = 0$.

where Π^{ℓ} is the profit* made by firm ℓ :

$$\Pi^{\ell} = \hat{y}^{\ell}_{m+1} + \sum_{j \in I_m} \hat{y}^{\ell}_j \hat{p}_j \ .$$

The payoff to trader i is the utility of his final bundle = $u^i(\hat{x}^i)$; that of firm ℓ is Π^{ℓ} .

To consider this as a non-cooperative game we must define the strategy sets of the players. We assume that at the end of stage 1, the players are informed about the aggregate bid and offer due to all others in each trading-post. Their second move is based on this information. We think of a strategy of a player as consisting of a move in the first stage, along with a set of moves in the second stage planned in advance for every possible market situation that may arise in the first stage. To state this precisely let $r = \max\{\bar{a}_1, \ldots, \bar{a}_m, \bar{a}_{m+1} + \bar{M}\} + K$ where K is some positive constant. r serves as an upper bound on the aggregate bids or offers that can be made in any trading post. The set of all possible aggregate bids and offers may be represented as a $2m$ dimensional set

$$\tilde{D} = \{(\alpha_1, \alpha_1', \ldots, \alpha_m, \alpha_m') \in \Omega^{2m} : |\alpha_j| \leq r \text{ and } |\alpha_j'| \leq r \text{ for } j \in I_m\}$$

where α_j (α_j') is thought of as the aggregate bid (offer) in the j^{th} trading-post. Then we can describe a plan of trader i for second stage moves by a function

*We assume that the firms pay out not only their profits but also they distribute all of their assets. Thus we do not subtract the M^k (even if we did the theorems would hold).

$$\tilde{D} \xrightarrow{\psi_i} \Omega^{2m} ,$$

where $\psi_i(\alpha) = [\hat{b}_1(\alpha), \hat{q}_1(\alpha), \ldots, \hat{b}_m(\alpha), \hat{q}_m(\alpha)]$ corresponds to bids and offers of i in the second stage, made with the information (given by α) about the first stage. Thus a strategy of i is a pair $\{(b,q), \psi_i\}$ such that

$$b \in \Omega^m , \quad q \in \Omega^m , \quad \tilde{D} \xrightarrow{\psi_i} \Omega^{2m} , \quad \text{and}$$

$$\bar{b} + \bar{\hat{b}}(\alpha) \le a^i_{m+1}$$

$$q_j + \bar{\hat{q}}_j(\alpha) \le a^i_j .$$

We will denote by $\tilde{S}^i$ the set of all strategies of trader i.

A plan for firm ℓ is again aptly described by a function

$$\tilde{D} \xrightarrow{\psi_\ell} \Omega^{m+1}$$

where $\psi_\ell(\alpha)$ is the output produced by ℓ for sale in stage 2. But $\psi_\ell(\alpha)$ must be feasible. To state this requirement precisely let $R^\ell = \{e = (e_1, \ldots, e_m) \in \Omega^m : \bar{e} \le M^\ell\}$ be the set of all bids that ℓ can make in stage 1. Let $y(e,\alpha) \in \Omega^{m+1}$ denote the input that accrues to ℓ at the end of stage 1 if he bids $e \in R^\ell$, and the aggregate bids and offers are given by $\alpha \in D$. Then a strategy for firm k is a pair $\{e, \psi_\ell\}$, $e \in R^\ell$, $D \xrightarrow{\psi_\ell} \Omega^{m+1}$, with the constraint on ψ_ℓ:

$$\psi_\ell(\alpha) \in [y(e,\alpha) + Y^\ell] \cap \Omega^{m+1}$$

for all $\alpha \in \tilde{D}$. The set of all strategies of firm ℓ (for $n+1 \le \ell \le n+s$)

will be denoted by $\tilde{S}^{\ell}$. Let $\tilde{S} = \tilde{S}^1 \times \ldots \times \tilde{S}^{n+k}$. The payoff functions $\tilde{S} \xrightarrow{\tilde{\Pi}^i} R$ were defined before for $i \in I_{n+s}$. Thus a non-cooperative game $\tilde{\Gamma}$ has been well defined.

A "Nash Equilibrium" (N.E.) of $\tilde{\Gamma}$ is a collection of strategies $(\tilde{s}^1, \ldots, \tilde{s}^{n+k}) = \tilde{s}$ (where $\tilde{s}^i \in \tilde{S}^i$) such that

$$\Pi^i(\tilde{s}) = \max_{t^i \in \tilde{S}^i} \Pi^i(\tilde{s}|t^i) \quad \text{for all } i .$$

[$(\tilde{s}|t^i)$ stands for $(\tilde{s}^1, \ldots, \tilde{s}^{i-1}, t^i, \tilde{s}^{i+1}, \ldots, \tilde{s}^{n+k})$.]

N.E.'s for this game exist trivially. For example, take those strategies where bids and offers of all traders are 0 in stages 1 and 2, and the bids and production of the firms are also 0 . This choice of strategies clearly constitutes a N.E.

We will be interested in establishing the existence of "nice" N.E.'s which converge to the competitive equilibrium (C.E.) under replication.

First let us reformulate the N.E. problem in a way that makes it mathematically more tractable. The transformed problem that we will look at was studied by Debreu in [10].

By ${}^{\epsilon}\tilde{\Gamma}$ $(\epsilon > 0)$ we will mean the modification of $\tilde{\Gamma}$ that results if an external agency places a bid and an offer of ϵ in each of the m trading-posts in both stages. This does not change the strategy sets of the players but it does of course change their payoffs.

We now define a "generalized game" (see [9]) ${}^{\epsilon}\Gamma$ associated with ${}^{\epsilon}\tilde{\Gamma}$. Let

$$D = \{x \in \Omega^{m+1} : |x_j| \leq r, \text{ for } 1 \leq j \leq m+1\} .$$

Thus D contains any possible holding that a player may acquire at the end of the first stage in ${}^{\epsilon}\tilde{\Gamma}$, provided ϵ is less than K (which we henceforth assume it is). Define

$$S^i = \{(q^i, b^i, \hat{q}^i, \hat{b}^i) : q^i \in \Omega^m, b^i \in \Omega^m, \hat{q}^i \in \Omega^m, \hat{b}^i \in \Omega^m,$$
$$\bar{b}^i + \bar{\hat{b}}^i \leq a^i_{m+1}, q^i_j + q_j \leq a^i_j \text{ for } j \in I_m\}$$

for $1 \leq i \leq n$. We will interpret $s^i \in S^i$ as a strategy of trader i in ${}^{\epsilon}\Gamma$ $(\epsilon \geq 0)$ where q^i $(\hat{q}^i)$ and b^i $(\hat{b}^i)$ denote his bids and offers in the first (second) stage. For $n+1 \leq \ell \leq n+s$, define

$$S^{\ell} = R^{\ell} \times E^{\ell}$$

where

$$E^{\ell} = (D + Y^{\ell}) \cap \Omega^{m+1} .$$

An element $s^{\ell} = (e^{\ell}, \hat{y}^{\ell})$ [where $e^{\ell} \in R^{\ell}$, $\hat{y}^{\ell} \in E^{\ell}$] of S^{ℓ} similarly represents a strategy of ℓ in ${}^{\epsilon}\Gamma$ $(\epsilon \geq 0)$: e^{ℓ} is the vector of his bids in stage 1, and $\hat{y}$ his output produced for sale in stage 2.

Let $S = S^1 \times \ldots \times S^{n+s}$. For any $\epsilon \geq 0$, we can define

$${}^{\epsilon}p^i : S \to R$$

exactly as before, given the interpretation of S^i, and assuming an external bid and offer of ϵ in each trading post. For

$$s = (s^1, \ldots, s^{n+s}) \in S ,$$

let

$$\bar{s}^i = (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^{n+s}) ;$$

and

$$\bar{S}^i = S^1 \times \ldots \times S^{i-1} \times S^{i+1} \times \ldots \times S^{n+s} .$$

Consider any $\bar{s}^i \in \bar{S}^i$. Define*

$$Q_j(\bar{s}^i) = \sum_{t \in I_n \setminus \{i\}} q_j^t$$

$$B_j(\bar{s}^i) = \sum_{t \in I_n \setminus \{i\}} b_j^t + \sum_{t \in I_{n+s} \setminus I_n \cup \{i\}} e_j^t$$

For $n+1 \leq i \leq n+s$, define ${}^{\varepsilon}\xi_{\bar{s}^i} : R^i \to \Omega^{m+1}$ by

$$[{}^{\varepsilon}\xi_{\bar{s}^i}(e^i)]_j = e_j^i \frac{[Q_j(\bar{s}^i) + \varepsilon]}{[B_j(\bar{s}^i) + \varepsilon + e_j^i]} , \quad \text{for } j \in I_m ,$$

$$[{}^{\varepsilon}\xi_{\bar{s}^i}(e^i)]_{m+1} = M^i - \sum_{j \in I_m} e_j^i ,$$

for any $e^i \in R^i$. Now we define a mapping ${}^{\varepsilon}\gamma^i$ ** from $\bar{S}^i$ into 2^{S^i} for all $1 \leq i \leq n+s$. If $i \leq n$,

$${}^{\varepsilon}\gamma^i(\bar{s}^i) = S^i .$$

If $i > n$,

$${}^{\varepsilon}\gamma^i(\bar{s}^i) = \{(e^i, \hat{y}^i) : e^i \in R^i, \hat{y}^i \in E^i, \hat{y}^i \in [{}^{\varepsilon}\xi_{\bar{s}^i}(e^i) + Y^i] \cap \Omega^{m+1}\} .$$

Following Debreu [10], we will define a Nash equilibrium of ${}^{\varepsilon}\Gamma$ to be

*Throughout we will use the notation $s^i = (q^i, b^i, \hat{q}^i, \hat{b}^i)$ for $1 \leq i \leq n$; and $s^i = (e^i, \hat{y}^i)$ for $n+1 \leq i \leq n+s$.

** $\gamma_{\varepsilon}^i(\bar{s}^i)$ is the subset of S^i to which i is restricted when the other players choose strategies $\bar{s}^i$.

a collection $s = (s^1, \ldots, s^{n+s}) \in S$ such that

(1) $s^i \in \gamma^i_\epsilon(\bar{s}^i)$

(2) $P^i_\epsilon(s) = \max_{v \in \gamma^i_\epsilon(\bar{s}^i)} P^i_\epsilon(s|v)$.

From a N.E. for ${}^\epsilon\Gamma$ we can extract an N.E. for ${}^\epsilon\tilde{\Gamma}$. Let $s = (s^1, \ldots, s^{n+s})$, $s^i \in S^i$, be a N.E. for ${}^\epsilon\Gamma$. Construct $\psi_i : \tilde{D} \to \Omega^{2m}$ as follows. For $1 \leq i \leq n$

$$\psi_i(\alpha) = \begin{cases} (\hat{b}^i, \hat{q}^i) \ \underline{\text{if,}} \text{ for all } j \in I_m , \ \alpha_j = (\sum_{i \in I_n \setminus \{i\}} b^i_j + \sum_{i \in I_{n+s} \setminus I_n} e^i_j) + \epsilon \\ \qquad \alpha'_j = (\sum_{i \in I_n \setminus \{i\}} q^i_j) + \epsilon \\ 0 \ \text{otherwise} \end{cases}$$

For $n+1 \leq i \leq n+s$

$$\psi_i(\alpha) = \begin{cases} \hat{y}^i \ \underline{\text{if,}} \text{ for all } j \in I_m , \ \alpha_j = \epsilon + (\sum_{i \in I_n} b^i_j + \sum_{i \in I_{n+s} \setminus I_n \cup \{i\}} e^i_j) \\ \qquad \alpha'_j = \sum_{j \in I_n} q^i_j \\ 0 \ \text{otherwise} \end{cases}$$

Then it is clear that if we put

$$\tilde{s}^i = \begin{cases} [(q^i, b^i), \psi_i] & \text{for } 1 \leq i \leq n \\ [e^i, \psi_i] & \text{for } n+1 \leq i \leq n+s \end{cases}$$

then $(\tilde{s}^1, \ldots, \tilde{s}^{n+s})$ is an N.E. for ${}^\epsilon\tilde{\Gamma}$.

[Note also that if ${}^\epsilon\tilde{\Gamma}$ were to be defined as a game with perfect information (i.e., each player's second move is made after knowing the detailed individual bids and offers due to all others in the first stage), even then a N.E. for ${}^{\epsilon}\Gamma$ can be converted in a similar manner to an N.E. of ${}^{\epsilon}\tilde{\Gamma}$.]

Lemma 1. A N.E. exists for every ${}^{\epsilon}\Gamma$, $\epsilon > 0$.

Proof. Observe that the sets S^i are convex, and that ${}^{\epsilon}\gamma^i$ is clearly continuous for every i and maps $\bar{S}^i$ into compact subsets of S^i . In Lemma 2 we verify that the sets

$$ {}^{\epsilon}\bar{\gamma}^i(\bar{s}^i) = \{v^* \in \gamma^i_\epsilon(\bar{s}^i) : P^i_\epsilon(s|v^*) = \max_{v \in \gamma^i_\epsilon(\bar{s}^i)} P^i_\epsilon(s|v)\} $$

are contractible.* Then, by the theorem in [10], a N.E. for ${}^{\epsilon}\Gamma$ exists.

Q.E.D.

Lemma 2. The sets ${}^{\epsilon}\bar{\gamma}^i(\bar{s}^i)$ are contractible for any $1 \leq i \leq n+s$, and any $\bar{s}^i \in \bar{S}^i$.

(We will say that a set S in E^ℓ is contractible to $S' \subset S$ if there exists a continuous function

$$ F : [0,1] \times S \to S $$

such that

$$ F(0,s) = s , \quad \text{for all } s \in S , $$

and

$$ \{F(1,s) : s \in S\} = S' . $$

Recall that [9] S is said to be contractible if it is contractible to

*For the definition of "contractible" see Lemma 2.

some single point in S .)

Before the proof of Lemma 2 we will need some other results. Let for $i \leq n$,

$$g_j^i = a_j^i - q_j^i + \frac{b_j^i}{p_j}$$

$$h_j^i = a_j^i - \hat{q}_j^i + \frac{\hat{b}_j^i}{\hat{p}_j} .$$

If the offers and bids due to all the others except i (including the external agency) are kept fixed at (Q_j, B_j) in the first stage, and at $(\hat{Q}_j, \hat{B}_j)$ in the second stage, then g_j * and h_j depend only on $s^i \in S^i$. Denoting by g $[h]$ the vector $(g_1, \ldots, g_m)$ $[(h_1, \ldots, h_m)]$, we will think of (g,h) as a (vector-valued) function $f(s^i)$ of $s^i \in S^i$.

Claim 1. The set $T = \{f(s^i) : s^i \in S^i\}$ is convex.

Proof. Let $(g', h') \in T$ and $(g'', h'') \in T$; and let $(g^*, h^*) = \lambda(g', h') + \mu(g'', h'')$ where $\lambda \geq 0$, $\mu \geq 0$, $\lambda+\mu = 1$. We want to show that $(g^*, h^*) \in T$.

Suppose $f((b', q', \hat{b}', \hat{q}')) = (g', h')$ and $f((b'', q'', \hat{b}'', \hat{q}'')) = (g'', h'')$. Consider $(\bar{q}, \bar{b}, \bar{\hat{b}}, \bar{\hat{q}})$ defined as follows:

$$\bar{q} = \min\{q', q''\} \text{ **}$$

$$\bar{\hat{q}} = \min\{\hat{q}', \hat{q}''\}$$

$$\bar{b} = \lambda b' + \mu b''$$

$$\bar{\hat{b}} = \lambda\hat{b}' + \mu\hat{b}''$$

*We drop the superscript i since no confusion will result.

**I.e., $\bar{q}_j = \min\{q_j', q_j''\}$ for each $j \in I_m$.

It is easily checked that $(\bar{q}, \bar{b}, \bar{\hat{q}}, \bar{\hat{b}}) \in S^i$. Let $(\bar{g}, \bar{h}) = f((\bar{q}, \bar{b}, \bar{\hat{q}}, \bar{\hat{b}}))$. We assert that $\bar{g} \geq g^*$, and $\bar{h} \geq h^*$. To see this first observe that, if we keep b_j $[\hat{b}_j]$ fixed, then g_j $[h_j]$ is a decreasing function of q_j $[\hat{q}_j]$. Hence

$$(\bar{g}', \bar{h}') = f((\bar{q}, b', \bar{\hat{q}}, \hat{b}')) \geq f((q', b', \hat{q}', \hat{b}'))$$

$$= (g', h')$$

and

$$(\bar{g}'', \bar{h}'') = f((\bar{q}, b'', \bar{\hat{q}}, \bar{\hat{b}}'')) \geq f((q'', b'', \hat{q}'', \hat{b}''))$$

$$= (\bar{g}'', \bar{h}'') .$$

But if we keep q_j $[\hat{q}_j]$ fixed then g_j $[h_j]$ is a concave function of b_j $[\hat{b}_j]$. Hence

$$\begin{aligned}(\bar{g}, \bar{h}) &= f((\bar{q}, \bar{b}, \bar{\hat{q}}, \bar{\hat{b}})) \\ &\geq \lambda f((\bar{q}, b', \bar{\hat{q}}, \bar{\hat{b}}')) + \mu f((\bar{q}, b'', \bar{\hat{q}}, \bar{\hat{b}}'')) \\ &\geq \lambda f((q', b', \hat{q}', \hat{b}')) + \mu f((q'', b'', \hat{q}'', \hat{b}'')) \\ &= \lambda(g', h') + \mu(g'', h'') = (g^*, h^*) .\end{aligned}$$

Put

$$\tilde{q} = \lambda q' + \mu q''$$

$$\tilde{\hat{q}} = \lambda \hat{q}' + \mu \hat{q}'' .$$

Note that $(\tilde{q}, 0, \tilde{\hat{q}}, 0) \in S^i$. Let $(\tilde{g}, \tilde{h}) = f((\tilde{q}, 0, \tilde{\hat{q}}, 0))$. Then

$$\tilde{g}_j = a_j - \tilde{q}_j$$

$$= a_j - (\lambda q_j' + \mu q_j'')$$

$$= "(a_j - q_j') + \mu(a_j - q_j'')$$

$$\leq \lambda g_j' + \mu g_j'' = g_j^* \ .$$

Similarly

$$\tilde{h}_j \leq h_j^* \ .$$

Now g_j $[h_j]$ is a continuous function of (q_j, b_j) $[(\hat{q}_j, \hat{b}_j)]$ for which $g_j(\overline{q}_j, \overline{b}_j) \geq g_j^*$, $g_j(\tilde{q}_j, 0) \leq g_j^*$. By the intermediate value theorem in calculus there is some $\overline{q}_j \leq q_j^* \leq \tilde{q}_j$ and $0 \leq b_j^* \leq \overline{b}_j$ for which

$$g_j(q_j^*, b_j^*) = g_j^* \ .$$

Similarly, there is a $\overline{\hat{q}}_j \leq \hat{q}_j^* \leq \tilde{\hat{q}}_j$ and $0 < \hat{b}_j^* \leq \overline{\hat{b}}_j$ for which $h_j(\hat{q}_j^*, \hat{b}_j^*) = h_j^*$. Hence $f(q^*, b^*, \hat{q}^*, \hat{b}^*) = (g^*, h^*)$. It is obvious that $(q^*, b^*, \hat{q}^*, \hat{b}^*) \in S^1$. Therefore $(g^*, h^*) \in T$.

Q.E.D.

Claim 2. The set $f^{-1}(t)$ is convex for any $t \in T$.

Proof. Consider g_j as a function of the variables q_j and b_j

$$g_j(q_j, b_j) = a_j - q_j + b_j\left(\frac{Q_j + q_j}{B_j + b_j}\right)$$

where $0 \leq q_j \leq a_j$, and $0 \leq b_j \leq a_{m+1}$. It was shown in Section 3.1 of [9] and noted earlier in this paper that for any t in the range of g_j , $g_j^{-1}(t)$ is a convex set of dimension 1. (So is $h_j^{-1}(t)$ where we define h_j with $\hat{Q}_j$ and $\hat{B}_j$ in place of Q_j and B_j .) Then

$$f^{-1}(t) = S^i \cap [g_1^{-1}(t_1) \times \dots \times g_m^{-1}(t_m) \times h_1^{-1}(t_{m+1}) \times \dots \times h_m^{-1}(t_{2m})]$$

where $t = (t_1, \dots, t_{2m})$. This is clearly a convex set.

Q.E.D.

Remarks

(1) $f^{-1}(t)$ is obviously not empty for $t \in T$.

(2) For any $t \in T$, we will denote by $\psi(t)$ the unique point in $f^{-1}(t)$ for which $q_j b_j = 0$ and $\hat{q}_j \hat{b}_j = 0$ for every $j \in I_m$. [See Figure 1.] It is easy to verify that ψ is a homeomorphism between T and $\psi(T)$.

Let $\hat{x}_j(s)$ be the final holding of $j \in I_{m+1}$ by $i \in I_n$ when he uses his strategy s (other strategies being fixed as before).

Claim 3. Denote $\sum_{k \in I_s} \eta_k^{i,k} \hat{y}_j$ by c_j . Then

$$\hat{x}_{m+1}(s) = a_{m+1} + \sum_{j \in I_m} \frac{B_j(a_j - g_j(s))}{Q_j + a_j - g_j(s)} + \sum \frac{\hat{B}_j(a_j + c_j - h_j(s))}{\hat{Q}_j + a_j - h_j(s)} .$$

Proof. L.S.H. - $a_{m+1} = \sum_{j \in I_m} (q_j p_j - b_j) + \sum_{j \in I_m} [(\hat{q}_j + c_j)\hat{p}_j - \hat{b}_j]$

R.H.S. - a_{m+1}

$$= \sum_{j \in I_m} \frac{B_j(q_j - b_j/p_j)}{Q_j + q_j - b_j/p_j} + \sum_{j \in I_m} \frac{\hat{B}_j(\hat{q}_j + c_j - \hat{b}_j/\hat{p}_j)}{\hat{Q}_j + \hat{q}_j - \hat{b}_j/\hat{p}_j} .$$

But

$$\frac{B_j(q_j - b_j/p_j)}{Q_j + q_j - b_j/p_j} = \frac{B_j/p_j(p_jq_j - b_j)}{[(Q_j + q_j)(B_j + b_j) - b_j(Q_j + q_j)]/(B_j + b_j)}$$

$$= \frac{B_j/p_j(p_jq_j - b_j)}{B_j/p_j} = p_jq_j - b_j \ .$$

Similarly

$$\frac{\hat{B}_j(\hat{q}_j + c_j - \hat{b}_j/\hat{p}_j)}{\hat{Q}_j - \hat{q}_j + \hat{b}_j/\hat{p}_j} = \hat{p}_j(\hat{q}_j + c_j) - \hat{b}_j \ .$$

This establishes the claim.

Q.E.D.

Remark. (4) $\hat{x}_{m+1}$ is a concave function of $(g, h) \in T$. To show this, note that the second derivative (w.r.t. h_j) of

$$F(h_j) = \frac{\hat{B}_j(a_j + c_j - h_j)}{\hat{Q}_j + a_j - h_j} \quad \text{is} \quad \frac{2\hat{B}_j(c_j - \hat{Q}_j)}{(\hat{Q}_j + a_j - h_j)^3} \ .$$

But $h_j \leq \hat{Q}_j + a_j$ since $\hat{Q}_j + a_j$ is the total amount of j available; also $\hat{Q}_j = \sum_{i\in I_n} a_j^i + \sum_{k\in I_s} \hat{y}_j^k \geq \sum_{k\in I_s} \eta_k^i \hat{y}_j^k = c_j$, since $\eta_k^i \leq 1$. Therefore $F''(h_j) \leq 0$, and F is a concave function of h_j, hence of (g,h) as well. Similarly $[B_j(a_j - g_j)]/[Q_j + a_j - g_j]$ is a concave function of (g,h). Thus $\hat{x}_{m+1}$ being a sum of concave functions of $(g,h) \in T$ is itself concave.

Note that for any $(g,h) \in T$, the final allocation is given by

$$\hat{x}_j(g,h) = -a_j + g_j + h_j \ , \quad \text{for } j \in I_m$$

$\hat{x}_{m+1}$ is given by the function in Claim 3. So the payoff to player i at $(g,h) \epsilon T$ is $u^i(\hat{x})$.

Claim 5. Let $\hat{T}$ be the subset of T on which u^i is maximized. Then $\hat{T}$ is convex.

Proof. Suppose $(g', h') \epsilon \hat{T}$, $(g'', h'') \epsilon \hat{T}$. Let $(g,h) = \lambda(g', h') + \mu(g'', h'')$ for $\lambda \geq 0$, $\mu \geq 0$, $\lambda+\mu = 1$. By Claim 1, $(g,h) \epsilon T$. Moreover, by Claim 3, $\hat{x}_{m+1}(g,h) \geq \lambda\hat{x}(g', h') + \mu\hat{x}(g'', h'')$. Finally, for $j \epsilon I_m$,

$$
\begin{aligned}
\hat{x}_j(g,h) &= -a_j + g + h \\
&= -a_j + (\lambda g' + \mu g'') + (\lambda h' + \mu h'') \\
&= \lambda(-a_j + g' + h') + \mu(-a_j + g'' + h'') \\
&= \lambda\hat{x}_j(g', h') + \mu\hat{x}_j(g'', h'') .
\end{aligned}
$$

Therefore

$$\hat{x}(g,h) \geq \lambda\hat{x}(g', h') + \mu\hat{x}(g'', h'') .$$

Since u^i is concave, this implies

$$u^i[\hat{x}(g,h)] \geq \lambda u^i[\hat{x}(g', h')] + \mu u^i[\hat{x}(g'', h'')] .$$

The result easily follows.

Q.E.D.

Proof of Lemma 1. Since $f^{-1}(t)$ is convex for every $t \epsilon \hat{T}$, and contains $\psi(t)$, there is a continuous mapping

$$H^t : [0,1] \times f^{-1}(t) \to f^{-1}(t)$$

such that

$$H^t(0,y) = y ,$$

$$H^t(1,y) = \psi(t)$$

for every $y \in f^{-1}(t)$. Define

$$H : [0,1] \times f^{-1}(\hat{T}) \to f^{-1}(\hat{T})$$

by

$$H(x,y) = H^t(x,y) \quad \text{if} \quad y \in f^{-1}(t) .$$

[Since $f^{-1}(t) \cap f^{-1}(t') = \emptyset$ if $t \neq t'$ this is well-defined.] H contracts $f^{-1}(\hat{T})$ to $\psi(\hat{T})$. But $\psi(\hat{T})$ is homeomorphic to $\hat{T}$, and $\hat{T}$ is convex and thus contractible. It can now be easily shown that $\psi(\hat{T})$ is contractible; hence so is $f^{-1}(\hat{T})$. This proves the lemma for $i \leq n$.

To prove it for $i > n$, let us first consider the set y of initial resources that i can purchase. Denote by Q_j $(\hat{Q}_j)$ and B_j $(\hat{B}_j)$ the aggregate supply and bid due to the others. From the equations

$$y_j = \frac{e_j Q_j}{B_j + e_j} , \quad \text{for} \quad j \in I_m ;$$

$$y_{m+1} = M - \sum_{j \in I_m} e_j$$

we deduce, as in the proof of Claim 3, that

$$y_{m+1} = M^i - \sum_{j \in I_m} \frac{B_j y_j}{Q_j - y_j} = h(y_1, \ldots, y_m) , \quad \text{say}$$

which is clearly a concave function of $(y_1, \ldots, y_m)$. Moreover it is easily verified that the set

$$S = \left\{(y_1, \ldots, y_m) \in \Omega^m : y_j = \frac{e_j Q_j}{B_j + e_j}, \sum_{j \in I_m} e_j \leq M\right\}$$

is convex. Thus the set H of initial holdings that the firm can purchase before production is a surface defined by the concave function h on the convex domain S.

Define $F : \Omega^{m+1} \to R$ by

$$F(x) = \max\left\{y_{m+1} + \sum_{j \in I_m}\left(\frac{B_j}{Q_j + y_j}\right) y_j \ : y \in (x + Y^i) \cap \Omega^{m+1}\right\}.$$

We check that F is concave. Let $f(y) = y_{m+1} + \sum_{j \in I_m} \frac{B_j y_j}{(Q_j + y_j)}$ and note that it is concave. Suppose $z = \lambda x + \mu y$, where $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = 1$, $x \in \Omega^{m+1}$, $y \in \Omega^{m+1}$. Suppose $F(x) = f(x')$ where $x' \in (x + Y^i) \cap \Omega^{m+1}$; and $F(y) = f(y')$ where $y' \in (y + Y^i) \cap \Omega^{m+1}$. Put $z' = \lambda x' + \mu y'$. Since Y^i is convex, $z' \in (z + Y^i) \cap \Omega^{m+1}$. But

$$F(z') \geq f(z') \geq \lambda f(x') + \mu f(y')$$

$$= \lambda F(x) + \mu F(y).$$

Let $\overline{H} = \{x^* \in H : F(x^*) = \max_{x \in H} F(x)\}$. We claim that $\overline{H}$ is convex and in fact consists of a single point. For let $y' \in \overline{H}$, $y'' \in \overline{H}$, $y'' \neq y'$. Consider $y = \lambda y' + \mu y''$, for $\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$. Then since S is convex, and* $y'|_{\Omega^m}$ and $y''|_{\Omega^m}$ are in S, $y|_{\Omega^m} \in S$. Hence

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* $x|_{\Omega^m}$ denotes $(x_1, \ldots, x_m) \in \Omega^m$, for any $x \in \Omega^{m+1}$.

$\hat{y} = (y|_{\Omega^m}, h(y)) \in H$. Moreover $\hat{y}_{m+1} = h(y|_{\Omega^m}) > \lambda(y'|_{\Omega^m}) + \mu(y''|_{\Omega^m}) = y_{m+1}$, since h is strictly concave. But then F is strictly increasing in the $m+1^{st}$ variable, therefore

$$F(\hat{y}) > F(y) \geq \lambda F(y') + \mu F(y'') ,$$

a contradiction.

Let A be the set

$$\{y^* \in (\bar{H} + Y^i) \cap \Omega^{m+1} : f(y^*) = \underset{y \in (\bar{H}+Y^i) \cap \Omega^{m+1}}{\text{maximum}} f(y)\} .$$

A is convex because $(\bar{H} + Y^i) \cap \Omega^{m+1}$ is convex, and because f is concave. It now follows that

$$\gamma^i_\varepsilon(\bar{s}^{-i}) = \bar{H} \times A$$

which is also convex.

Q.E.D.

To obtain a "nice" N.E. of $\tilde{\Gamma}$, we will establish the existence of a "nice" N.E. of Γ . We consider a sequence of N.E.'s of ${}^\varepsilon\Gamma$, $\varepsilon \to 0$, and show that any cluster point of this sequence is a N.E. of Γ . A N.E. of $\tilde{\Gamma}$ is then derived from this N.E. as described earlier. To distinguish such a N.E. from other kinds of N.E.'s we will call it an E.P. (Equilibrium Point) of $\tilde{\Gamma}$.

Our first step in this direction is

Lemma 3. For each $j \in I_m$ suppose that there are at least 3 monied traders who desire j, and at least 3 j-furnished traders who desire money. Moreover suppose that $\eta_\ell^i < \frac{1}{2}$ $(\ell \in I_{n+s} \setminus I_s)$ if i is a trader of the latter type. Let ${}^\epsilon p_j$, ${}^\epsilon \hat{p}_j$ be the prices at a N.E. of ${}^\epsilon\Gamma$. There exist positive constants K, C_j and D_j (for $j \in I_m$) such that

$$C_j \leq {}^\epsilon p_j \leq D_j$$

$$C_j \leq {}^\epsilon \hat{p}_j \leq D_j$$

for any $\epsilon \leq K$.

Proof.* W.l.o.g. let $j = m$, and assume that 1, 2 and 3 are monied and desire m; and that 4, 5, and 6 are m-furnished and desire money. Put

$$H^k = \max\{|x| : x \in (D + Y^k) \cap \Omega^{m+1}\}$$

$$H = \sum_{k \in I_s} H^k + K, \quad \text{for some } K > 0.$$

(Thus H is an upper bound on the amount of any commodity or money that a player may hold at the end of the game Γ_ϵ, if $\epsilon < K$. We will assume in what follows that $\epsilon < K$.)

Also put

$$h = \min[h(u^i, m, H) : i = 1, 2, 3]^{**}$$

$$A = \frac{1}{m+1}\min[a^i_{m+1} : i = 1, 2, 3]$$

---

*This proof takes its cue from Shapley's ingenious proof of Lemma 4 in [11].

**See Appendix A for the definition of $h(u^i, m, H)$, and the statement of Lemma 4 in [11], which we use in this proof.

$$\hat{h} = \min[h(u^i, m+1, H) : i = 4, 5, 6]$$

$$\hat{A} = \frac{1}{2}\min[a_m^i : i = 4, 5, 6]$$

$$\beta = \max[\eta_\ell^i : i = 4, 5, 6, \ell \in I_s] .$$

Note that these are all positive under our assumptions (except for β which may be 0), and independent of ϵ . Also note that $\beta < 1/2$, i.e. $1/2 - \beta > 0$. First we establish the existence of C_m . Put

$$\delta = {}^{\epsilon}p_m = \frac{\bar{b}_m + \bar{e}_m + \epsilon}{\bar{q}_m + \epsilon}$$

and

$$\hat{\delta} = {}^{\epsilon}\hat{p}_m = \frac{\bar{\hat{b}}_m + \epsilon}{\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon} .$$

Suppose first that the condition

$$\text{(1)} \qquad b_m^i \leq \frac{\bar{b}_m}{2} , \quad \bar{\hat{b}}_m^i \leq \frac{\bar{\hat{b}}_m}{2} \quad \text{and} \quad a_{m+1}^i - \sum_{j \in I_m} b_j^i - \sum_{j \in I_m} \hat{b}_j^i \geq A$$

holds for at least one of $i = 1, 2, 3$; say for $i = 1$. Then an increase Δ in 1's bid for m in stage 1 would be feasible if $0 < \Delta \leq \min(\epsilon, A)$, and would have the following incremental effect on his final holding:

$$\hat{x}'_j(\Delta) - \hat{x}'_j = 0 \quad \text{for} \quad j \in I_{m-1} ;$$

$$\hat{x}'_m(\Delta) - \hat{x}'_m = \frac{(\bar{q}_m+\varepsilon)(b'_m+\Delta)}{\bar{b}_m+\bar{e}_m+\varepsilon+\Delta} - \frac{(\bar{q}_m+\varepsilon)b'_m}{\bar{b}_m+\bar{e}_m+\varepsilon}$$

$$= (\bar{q}_m+\varepsilon)\Delta\left[\frac{\bar{b}_m+\bar{e}_m+\varepsilon-b'_m}{(\bar{b}_m+\bar{e}_m+\varepsilon)(\bar{b}_m+\bar{e}_m+\varepsilon+\Delta)}\right]$$

$$\geq (\bar{q}_m+\varepsilon)\Delta\left[\frac{\bar{b}_m/2+\bar{e}_m/2+\varepsilon/2+\Delta/2}{(\bar{b}_m+\bar{e}_m+\varepsilon)(\bar{b}_m+\bar{e}_m+\varepsilon+\Delta)}\right]$$

$$= \frac{(\bar{q}_m+\varepsilon)\Delta}{2(\bar{b}_m+\bar{e}_m+\varepsilon)} = \frac{\Delta}{2\delta} .$$

(The "$\geq$" above follows from:

$$\bar{b}_m + \bar{e}_m + \varepsilon - b'_m \geq \frac{\bar{b}_m+\bar{e}_m}{2} + \varepsilon$$

$$\geq \bar{b}_m/2 + \bar{e}_m/2 + \varepsilon/2 + \Delta/2 \quad) ;$$

$$\hat{x}'_{m+1} - \hat{x}'_{m+1} = \left(\frac{q'_m}{\bar{q}_m+\varepsilon} - 1\right)\Delta \geq -\Delta .$$

Define

$$z = -2\delta e^{m+1}$$

and note that we have the vector inequality

$$(2) \qquad x^1(\Delta) \geq x^1 + \frac{\Delta}{2\delta}(z+e^m) .$$

We are now in a position to apply Lemma 4 in [11] taking $f = u^1$, $j = m$, and $y = x^1 + z$. We have $x^1 \in \Omega^{m+1}$ and $\|x^1\| \leq H$. So, by the lemma, if both $x^1 + z \geq 0$ and $z \leq h$, then

$$u^1(x^1 + z + e^m) > u^1(x^1) .$$

Since u^1 is concave, this implies that

$$u^1(x^1 + \frac{\Delta}{2\delta}(z + e^m)) > u^1(x^1)$$

holds for sufficiently small Δ , and hence, by (32) and the monotonicity of u^1 , that

$$u^1(x^1(\Delta)) > u^1(x^1)$$

for such Δ . But this means that trader 1 could have improved, contradicting (29). Hence either $x^1 + z < 0$, or $\|z\| > h$. If the former, we have

$$\hat{x}^1_{m+1} - 2\delta < 0 .$$

But $\hat{x}^1_{m+1} \geq A/2$; hence

(3) $\boxed{2\delta > A/2}$

If the latter, we have

(4) $\boxed{2\delta > h}$.

Similarly, an increase in Δ in 1's bid for m in stage 2 would produce the following incremental effect:

$$\hat{x}_j^1(\Delta) - \hat{x}_j^1 = 0 \quad \text{for} \quad j \in I_{m-1} \ ;$$

$$\hat{x}_m^1(\Delta) - \hat{x}_m^1 \geq \frac{\Delta}{2\hat{\delta}} \ ;$$

$$\hat{x}_{m+1}^1(\Delta) - \hat{x}_m^1 = \left(\frac{\hat{q}_m^1}{\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon} - 1\right)\Delta + \sum_{j \in I_m} \left\{ \sum_{k \in I_{n+s} \setminus I_n} \left(\pi_k^1 \hat{y}_j^k \frac{\Delta}{\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon} \right) \right\}$$

$$\geq -\Delta \ .$$

Defining $z = -2\hat{\delta}e^{m+1}$, (2) holds as before; and, arguing as before, we get that either

(5) $\boxed{2\hat{\delta} > A/2}$

or

(6) $\boxed{2\hat{\delta} > h}$.

Now consider the case where (1) fails for $i = 1, 2, 3$. W.l.o.g. we may assume that

$$b_m^1 \leq \frac{\bar{b}_m}{2} \ , \quad \hat{b}_m^1 \leq \frac{\bar{\hat{b}}_m}{2} \ .$$

From the failure of (1) for $i = 1$ we therefore have

$$\sum_{j \in I_m} b_j^1 + \sum_{j \in I_m} \hat{b}_j^1 \geq a_{m+1}^1 - A \geq mA \ .$$

Hence either

(7) $b_j^1 > A/2$ for some $j \in I_m$,

or

(8) $\hat{b}_j^1 > A/2$ for some $j \in I_m$.

First suppose that (7) occurs. If $j = m$, then $b^1_m > A$ and so <u>a fortiori</u>

(9) $\boxed{\delta \geq A/2H}$.

If $j \neq m$, trader 1 could then decrease b^1_j by a small $\Delta > 0$ and increase b^1_m by the same amount with the incremental effect:

$$\hat{x}^1_\mu(\Delta) - \hat{x}^1_\mu = 0 \quad \text{for} \quad \mu \in I_m \setminus \{m,j\} ;$$

$$\hat{x}^1_j(\Delta) - \hat{x}^1_j = \frac{(b^1_j-\Delta)(\bar{q}_j+\varepsilon)}{(\bar{b}_j+\bar{e}_j+\varepsilon-\Delta)} - \frac{b^1_j(\bar{q}_j+\varepsilon)}{(\bar{b}_j+\bar{e}_j+\varepsilon)}$$

$$\geq \frac{(b^1_j-\Delta)(\bar{q}_j+\varepsilon)}{(\bar{b}_j+\bar{e}_j+\varepsilon)} - \frac{b^1_j(\bar{q}_j+\varepsilon)}{(\bar{b}_j+\bar{e}_j+\varepsilon)}$$

$$= -\frac{\Delta(\bar{q}_j+\varepsilon)}{(\bar{b}_j+\bar{e}_j+\varepsilon)} ;$$

$$\hat{x}^1_{m+1}(\Delta) - \hat{x}^1_{m+1} = [p_j(\Delta) - p_j]q^1_j + [p_m(\Delta) - p_m]q'_m$$

$$\geq [p_j(\Delta) - p_j]q^1_j$$

$$= -\left(\frac{q^1_j}{\bar{q}_j + \varepsilon}\right)\Delta .$$

(The "$\geq$" above follows because $p_m(\Delta) > p_m$.) *

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*Here $p_m(\Delta) = \dfrac{\bar{b}_m + \bar{e}_m + \varepsilon + \Delta}{\bar{q}_j + \varepsilon}$, $p_j(\Delta) = \dfrac{\bar{b}_j + \bar{e}_j + \varepsilon - \Delta}{\bar{q}_j + \varepsilon}$, etc.

$$\hat{x}^1_m(\Delta) - \hat{x}^1_m \geq \frac{\Delta}{2\delta},$$

by the same calculation as earlier. If we define

$$z = \left[\frac{-2\delta(\bar{q}_j + \epsilon)}{(\bar{b}_j + \bar{e}_j + \epsilon)}\right] e^j + \left[\frac{-2\delta q^1_j}{\bar{q}_j + \epsilon}\right] e^{m+1}$$

then (1) is satisfied. Hence, arguing as before, either $\hat{x}^1 + z < 0$ or $\|z\| > h$. If $\hat{x}^1 + z < 0$, then either $|z_j| > \hat{x}^1_j$ or $|z_{m+1}| > \hat{x}^1_{m+1}$. But we have

$$\hat{x}^1_j \geq \frac{(\bar{q}_j + \epsilon) b^1_j}{(\bar{b}_j + \bar{e}_j + \epsilon)} \geq \frac{(\bar{q}_j + \epsilon) A}{2(\bar{b}_j + \bar{e}_j + \epsilon)}$$

and

$$\hat{x}^1_{m+1} \geq \frac{b^1_j q^1_j}{\bar{q}_j + \epsilon} \geq \frac{q^1_j A}{2(\bar{q}_j + \epsilon)},$$

which gives

(10) $\boxed{2\delta > A/2}$.

If $\|z\| > h$, then either $|z_j| > h$, or $|z_{m+1}| > h$. In the first case the inequalities $\bar{e}_j + \bar{b}_j + \epsilon > b^1_j > A/2$, and $\bar{q}_j + \epsilon < H$ yield

(11) $\boxed{2\delta > hA/2H}$;

in the second, $q^1_j \leq \bar{q}_j$ yields

(12) $\boxed{2\delta > h}$.

The bid Δ could, however, also have been transferred from b_j^1 (even when $j = m$) to $\hat{b}_m^1$ with the incremental effect:

$$\hat{x}_j^1(\Delta) - \hat{x}_j^{1*} \geq -\left(\frac{\bar{q}_j + \varepsilon}{\bar{e}_j + \bar{b}_j + \varepsilon}\right)\Delta ;$$

$$\hat{x}_\mu^1(\Delta) - \hat{x}_\mu^1 = 0 \quad \text{for} \quad \mu \in I_m \setminus \{j,m\}$$

$$\hat{x}_m^1(\Delta) - \hat{x}_m^{1*} \geq \frac{\Delta}{2\hat{\delta}} ;$$

$$\hat{x}_{m+1}^1(\Delta) - \hat{x}_{m+1}^1 \geq \left(\frac{\hat{q}_m^1}{\bar{\hat{q}}_m + \bar{\hat{y}}_m + \varepsilon} - \frac{q_j^1}{\bar{q}_j + \varepsilon}\right)\Delta + \sum_{k \in I_{n+s} \setminus I_n} \eta_m^k \hat{y}_m^k [\hat{p}_m(\Delta) - \hat{p}_m] .$$

But $\hat{p}_m(\Delta) - \hat{p}_m \geq 0$ since an increase of the bid on m only increases the price of m. Hence

$$\hat{x}_{m+1}^1(\Delta) - \hat{x}_{m+1}^1 \geq -\left(\frac{q_j^1}{\bar{q}_j + \varepsilon}\right)\Delta .$$

Then again we derive conditions bounding $\hat{\delta}$ from below:

(13) $\boxed{2\hat{\delta} > A/2}$;

or

(14) $\boxed{2\hat{\delta} > hA/2H}$;

or

(15) $\boxed{2\hat{\delta} > h}$.

Next suppose that (8) occurs. If $j = m$, clearly we have*

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*If $j = m$, then these terms are the changes in the holdings of j in the first and the second stages. But the argument goes through in this case too.

(16) $\boxed{\hat{\delta} \geq A/2H}$.

Otherwise, a transfer of a small Δ from $\hat{b}_j^1$ to $\hat{b}_m^1$ leads to:

$$\hat{x}_j^1(\Delta) - \hat{x}_j^1 \geq -\left(\frac{\bar{\hat{q}}_j + \bar{\hat{y}}_j + \epsilon}{\bar{\hat{b}}_j + \epsilon}\right)\Delta ;$$

$$\hat{x}_\mu^1(\Delta) - \hat{x}_\mu^1 = 0 \quad \text{for} \quad \mu \in I_m\setminus\{j,m\} ;$$

$$\hat{x}_m^1(\Delta) - \hat{x}_m^1 \geq \Delta/2\hat{\delta} ;$$

$$\hat{x}_{m+1}^1(\Delta) - \hat{x}_{m+1}^1 \geq \left(\frac{\hat{q}_m^1}{\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon} - \frac{\hat{q}_j^1}{\bar{\hat{q}}_j + \bar{\hat{y}}_j + \epsilon}\right)\Delta - \left[\sum_{k\in I_{n+s}\setminus I_n} \eta_k^1 \hat{y}_j^k \left(\frac{\Delta}{\bar{\hat{q}}_j + \bar{\hat{y}} + \epsilon}\right)\right]$$

$$\geq -\left(\frac{\hat{q}_j^1 + \sum_{k\in I_{n+s}\setminus I_n} \eta_k^1 \hat{y}_j^k}{\bar{\hat{q}}_j + \bar{\hat{y}}_j + \epsilon}\right)\Delta$$

$$= -\alpha\Delta , \quad \text{say.}$$

(Note that $\alpha \leq 1$, since $\eta_k^1 \leq 1$ for $k \in I_s$.) Put

$$z = \left[-2\hat{\delta}\left(\frac{\bar{\hat{q}}_j + \bar{\hat{y}}_j + \epsilon}{\bar{\hat{b}}_j + \epsilon}\right)\right]e^j + [-2\alpha\hat{\delta}]e^{m+1} .$$

Then

(17) $$\hat{x}^1(\Delta) \geq \hat{x}^1 + \frac{\Delta}{2\hat{\delta}}(z + e^m) ;$$

and arguing as before we get that either $\hat{x}^1 + z < 0$ or $\|z\| > h$. If $\hat{x}^1 + z < 0$, then either $|z_j| > \hat{x}_j^1$ or $|z_{m+1}| > \hat{x}_{m+1}^1$. But

$$\hat{x}_j^1 \geq \left(\frac{\bar{\bar{q}}_j + \bar{\bar{y}}_j + \varepsilon}{\bar{\bar{b}}_j + \varepsilon}\right)\hat{b}_j^1 \geq \frac{(\bar{\bar{q}}_j + \bar{\bar{y}}_j + \varepsilon)A}{2(\bar{\bar{b}}_j + \varepsilon)}$$

and

$$\hat{x}_{m+1}^1 \geq \frac{\hat{b}_j^1}{\bar{\bar{q}}_j + \bar{\bar{y}}_j + \varepsilon}\left[\hat{q}_j^1 + \sum_{k \varepsilon I_{n+s} \setminus I_n} \eta_k^1 \hat{y}_j^k\right]$$

$$\geq \frac{A\alpha}{2} .$$

Therefore

(18) $\boxed{2\hat{\delta} > A/2}$.

Of $\|z\| > h$, then either $|z_j| > h$, or $|z|_{m+1} > h$. In the first case, the inequalities $\bar{\bar{b}}_j + \varepsilon > \hat{b}_j^1 > A/2$ and $\bar{\bar{q}}_j + \bar{\bar{y}}_j + \varepsilon < H$ yield

(19) $\boxed{2\hat{\delta} > hA/2H}$;

in the second, $\alpha \leq 1$ yields

(20) $\boxed{2\hat{\delta} > h}$.

Finally, a transfer of Δ from $\hat{b}_j^1$ to b_m^1 (the argument holds also when $j = m$) leads to the incremental effect:

$$\hat{x}_\mu^1(\Delta) - \hat{x}_\mu^1 = 0 , \quad \text{for } \mu \in I_m \setminus \{j, m\} ;$$

$$\hat{x}_j^1(\Delta) - \hat{x}_j^1 \geq -\left(\frac{\bar{\bar{q}}_j + \bar{\bar{y}}_j + \varepsilon}{\bar{\bar{b}}_j + \varepsilon}\right)\Delta ;$$

$$\hat{x}^1_m(\Delta) - \hat{x}^1_m \geq \frac{\Delta}{2\delta}$$

$$\hat{x}^1_{m+1}(\Delta) - \hat{x}^1_{m+1} \geq \left(\frac{-\hat{q}^1_j}{\bar{\bar{q}}_j + \bar{\bar{y}}_j + \epsilon}\right)\Delta - \left[\sum_{k\epsilon I_{n+s}\setminus I_n} \eta^i_k \hat{y}^k_j\left(\frac{\Delta}{\hat{\bar{q}}_j + \hat{y}_j + \epsilon}\right)\right]$$

$$= -\alpha\Delta \ .$$

Put $$z = \left[-2\delta\left(\frac{\bar{\bar{q}}_j + \bar{\bar{y}}_j + \epsilon}{\bar{\bar{b}}_j + \epsilon}\right)\right]e^j + [-2\delta\alpha]e^{m+1} \ .$$

Then (2) holds, and as in the previous case, we derive lower bounds for δ :

(21) $\boxed{2\delta > hA/2H}$

(22) $\boxed{2\delta > h}$

(23) $\boxed{2\delta > A/2}$.

Pick C_m to be the minimum of $A/4$, $h/2$, $A/2H$, $hA/4H$.

The existence of D_m is established in a similar manner. For the sake of completeness, we will grind through the details. Suppose first that the condition

$$q^i_m \leq \bar{q}_m/2 \ , \quad \hat{q}^i_m \leq \bar{\bar{q}}_m/2 \ , \quad a^i_m - q^i_m - \hat{q}^i_m \geq \hat{A}$$

holds for at least one of $i = 4, 5, 6$; say for $i = 4$. Then an increase of Δ in q^4_m would be feasible if $0 < \Delta \leq \min(\epsilon, \hat{A})$, and would have the following incremental effect on his final holding:

$$\hat{x}^4_j(\Delta) - \hat{x}^4_j = 0 \quad \text{for} \quad j \in I_{m-1} \ ;$$

$$\hat{x}^4_m(\Delta) - \hat{x}^4_m = \left(\frac{b^4_m}{\bar{b}_m + \bar{e}_m + \varepsilon} - 1\right)\Delta$$

$$\geq -\Delta$$

$$\hat{x}^4_{m+1}(\Delta) - \hat{x}^4_{m+1} = \left[\left(\frac{\bar{b}_m + \bar{e}_m + \varepsilon}{\bar{q}_m + \varepsilon + \Delta}\right)(q^4_m + \Delta)\right] - \left[\left(\frac{\bar{b}_m + \bar{e}_m + \varepsilon}{\bar{q}_m + \varepsilon}\right)q^4_m\right]$$

$$= \frac{\Delta(\bar{b}_m + \bar{e}_m + \varepsilon)(\bar{q}_m + \varepsilon - q^4_m)}{(\bar{q}_m + \varepsilon + \Delta)(\bar{q}_m + \varepsilon)}$$

$$\geq \frac{\Delta(\bar{b}_m + \bar{e}_m + \varepsilon)}{(\bar{q}_m + \varepsilon)} \cdot \frac{1}{2}$$

$$= \frac{\Delta\delta}{2} ,$$

ince $\bar{q}_m + \varepsilon - q^4_m \geq \bar{q}_m/2 + \varepsilon/2 + \Delta/2$.

Define $z = -\frac{2}{\delta} e^m$, and note that then

(24) $$\hat{x}^4(\Delta) \geq \hat{x}^4 + \frac{\Delta\delta}{2}[z + e^{m+1}] .$$

Arguing as before we get that either $\hat{x}^4 + z < 0$ or $\|z\| > \hat{h}$. If the former,

$$\frac{2}{\delta} > \hat{x}^4_m > \hat{A} ,$$

hence

(25) $$\boxed{\delta < 2/\hat{A}} ;$$

if the latter,

$$\frac{2}{\delta} > \hat{h} ,$$

hence

(26) $\boxed{\delta < 2/\hat{h}}$.

Similarly, an increase of $\Delta \leq \min(\epsilon,\hat{A})$ in $\hat{q}_m^4$ results in:

$$\hat{x}_j^4(\Delta) - \hat{x}_j^4 = 0 \quad \text{for} \quad j \in I_{m-1} ;$$

$$\hat{x}_m^4(\Delta) - \hat{x}_m^4 \geq -\Delta ;$$

$$\hat{x}_{m+1}^4(\Delta) - \hat{x}_{m+1}^4 = \left[\frac{(\bar{\hat{b}}_m + \epsilon)(\hat{q}_m^4 + \Delta)}{(\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon + \Delta)}\right] - \left[\frac{(\bar{\hat{b}}_m + \epsilon)\hat{q}_m^4}{(\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon)}\right]$$

$$\sum_{k\in I_{n+s}\setminus I_n} \eta_k^4 \hat{y}_m^k \left[\frac{(\bar{\hat{b}}_m + \epsilon)}{(\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon + \Delta)} - \frac{(\bar{\hat{b}}_m + \epsilon)}{(\bar{\hat{q}}_m + \bar{\hat{y}}_m + \epsilon)}\right]$$

$$\geq \frac{\Delta\hat{\delta}}{2} - \left[\frac{\sum_{k\in I_{n+s}\setminus I_n} \eta_k^4 \hat{y}_m^k}{\hat{q}_m + \hat{y}_m + \epsilon + \Delta}\right]\Delta\hat{\delta}$$

But $$\sum_{k\in I_{n+s}\setminus I_n} \eta_k^4 \hat{y}_m^4 \leq \sum_{k\in I_s} \beta \hat{y}_m^k$$

$$= \beta\bar{\hat{y}}_m .$$

Hence

$$\hat{x}_m^4(\Delta) - \hat{x}_m^4 \geq (1/2 - \beta)\Delta\hat{\delta} .$$

Define $$z = -\frac{1}{\hat{\delta}(\frac{1}{2} - \beta)} e^m .$$

Then

(27) $$\hat{x}^4(\Delta) \geq \hat{x}^4 + \Delta\hat{\delta}[\tfrac{1}{2} - \beta](z + e^{m+1}) \ .$$

Arguing as before, we get that either $x^4 + z < 0$ or $\|z\| > \hat{h}$. If the former

$$\frac{1}{\hat{\delta}(\frac{1}{2} - \beta)} > \hat{x}^4_m > \hat{A} \ ,$$

hence

(28) $$\boxed{\hat{\delta} < \frac{1}{\hat{A}(\frac{1}{2} - \beta)}} \ .$$

If the latter,

$$\frac{1}{\hat{\delta}(\frac{1}{2} - \beta)} > \hat{h}$$

hence

(29) $$\boxed{\hat{\delta} < \frac{1}{\hat{h}(\frac{1}{2} - \beta)}} \ .$$

Next, consider the case where (24) fails for $i = 4, 5, 6$. W.l.o.g. let

$$\hat{q}^4_m \leq \bar{\hat{q}}_m/2 \ , \quad q^4_m \leq \bar{q}_m/2 \ .$$

From the failure of (24) for $i = 4$, we have

$$q^4_m + \hat{q}^4_m > a^4_m - \hat{A} \geq \hat{A}/2 \ .$$

Let $q^4_m > \hat{A}/4$. Then it is immediate that

(30) $\boxed{\delta < 4H/\hat{A}}$.

Now 4 may decrease q_m^4 by a small $\Delta > 0$, and increase his supply of m in the 2nd stage by the same amount with the incremental effect:

$$\hat{x}_j^4(\Delta) - \hat{x}_j^4 = 0 \quad \text{for} \quad j \in I_{m-1} ;$$

$$\hat{x}_m^4(\Delta) - \hat{x}_m^4 = \left(\frac{\hat{b}_m^4}{\bar{b}_m + \epsilon} - \frac{b_m^4}{\bar{b}_m + \bar{e}_m + \epsilon}\right)\Delta ;$$

$$\hat{x}_{m+1}^4(\Delta) - \hat{x}_{m+1}^4 \geq \left(\frac{\bar{b}_m + \bar{e}_m + \epsilon}{\bar{q}_m + \epsilon - \Delta}\right)(q_m^4 - \Delta) - \left(\frac{\bar{b}_m + \bar{e}_m + \epsilon}{\bar{q}_m + \epsilon}\right)q_m^4 + (\tfrac{1}{2} - \beta)\Delta\hat{\delta}$$

$$\geq - \frac{(\bar{b}_m + \bar{e}_m + \epsilon)\Delta}{(\bar{q}_m + \epsilon)} + (\tfrac{1}{2} - \beta)\Delta\hat{\delta} .$$

Put
$$z = -\left[\frac{1}{\hat{\delta}(\frac{1}{2} - \beta)} \frac{(\bar{b}_m + \bar{e}_m + \epsilon)}{(\bar{q}_m + \epsilon)}\right]e^{m+1} - \left[\frac{1}{\hat{\delta}(\frac{1}{2} - \beta)} \frac{b_m^4}{(\bar{b}_m + \bar{e}_m + \epsilon)}\right]e^m .$$

Then (27) holds; and we get that either $\hat{x}^4 + z < 0$ or $\|z\| > \hat{h}$. If the former, either $z_{m+1} > \hat{x}_{m+1}^4$ or $z_m > \hat{x}_m^4$. But

$$\hat{x}_m^4 \geq \frac{b_m^4 q_m^4}{\bar{b}_m + \bar{e}_m + \epsilon} \geq \frac{\hat{A} b_m^4}{4(\bar{b}_m + \bar{e}_m + \epsilon)}$$

and

$$\hat{x}_{m+1}^4 \geq \frac{(\bar{b}_m + \bar{e}_m + \epsilon) q_m^4}{(\bar{q}_m + \epsilon)} \geq \frac{\hat{A}(\bar{b}_m + \bar{e}_m + \epsilon)}{4(\bar{q}_m + \epsilon)} .$$

Hence

$$\frac{1}{\hat{\delta}(\frac{1}{2}-\beta)} > \frac{\hat{A}}{4}$$

or

(31) $$\boxed{\hat{\delta} < \frac{4}{\hat{A}(\frac{1}{2}-\beta)}} \ .$$

If the latter, then the inequalities $\bar{b}_m + \bar{e}_m + \epsilon < H$, $\bar{q}_m \geq q_m^4 > \hat{A}/2$, and $b_m^4 < \bar{b}_m + \bar{e}_m + \epsilon$, imply that either

$$\frac{1}{\hat{\delta}(\frac{1}{2}-\beta)} \cdot \frac{4H}{\hat{A}} > \hat{h}$$

i.e.

(32) $$\boxed{\hat{\delta} < \frac{4H}{\hat{A}\hat{h}(\frac{1}{2}-\beta)}}$$

or

(33) $$\boxed{\hat{\delta} < \frac{1}{\hat{h}(\frac{1}{2}-\beta)}} \ .$$

If $q_m^4 > \hat{A}/4$, then clearly

(34) $$\boxed{\hat{\delta} < 4H/\hat{A}} \ .$$

Again we can consider the effect of a transfer of a small Δ from $\tilde{q}_m^4$ to q_m^4. A similar argument as before yields that either

(35) $\boxed{\delta < 8/\hat{A}}$ or

(36) $\boxed{\delta < 8H/\hat{A}\hat{h}}$ or

(37) $\boxed{\delta < 2/\hat{h}}$.

Pick D_m to be the maximum of

$$\left\{\frac{1}{\hat{A}(\frac{1}{2}-\beta)}, \frac{1}{\hat{h}(\frac{1}{2}-\beta)}, \frac{4H}{\hat{A}}, \frac{4}{\hat{A}(\frac{1}{2}-\beta)}, \frac{4H}{\hat{A}\hat{h}(\frac{1}{2}-\beta)}\right\} .$$

Q.E.D.

Remark (5). Note that if, for each $j \in I_m$, there is at least one monied trader in Γ_1 who desires j, and one j-furnished trader who desires money, then the hypotheses of Lemma 2 are satisfied for ${}^{\epsilon}_{k}\Gamma$, $k \geq 3$.

Let s'_ϵ be a N.E. of ${}^{\epsilon}\Gamma$. Consider the sequence $\{s'_{\epsilon_i}\}_{i=1}^{\infty}$ where $\epsilon_i \to 0$. Lemma 3 enables us to pick a subsequence $\{s_{\epsilon_i}\}$ such that $p_j^{\epsilon}(s_{\epsilon_i}) \to p_j$ where $c_j \leq p_j \leq D_j$. Let s be a cluster point of this subsequence. Then s is a N.E. of Γ since it is a point of continuity of the payoff functions P^i. (If the total bid and supply at s^* are both 0 in market j, i.e. if the market is inactive, there is no problem because a trader will lose j if he only supplies j, will lose money if he only bids on j, and will merely retain what he has if he both bids and supplies.)

The N.E. s of Γ corresponds to an N.E. of $\tilde{\Gamma}$. We have proved

Theorem 1. An E.P. of $\tilde{\Gamma}$ exists.

With this E.P. we will associate the prices $(p_1, \ldots, p_m, 1, \hat{p}_1, \ldots, \hat{p}_m, 1)$ * obtained for s as a limit of ${}^{\epsilon_i}p_j$ $({}^{\epsilon_i}\hat{p}_j)$ at the N.E. s_{ϵ_i} of ${}^{\epsilon}\Gamma$.

---

*We take the price of money to be 1 in both stages.

4. LIMIT EQUILIBRIUM POINTS AND COMPETITIVE EQUILIBRIA

4.1. The Convergence Proof

Let us consider a "replication sequence" of economies $\Gamma_1, \ldots, \Gamma_k, \ldots$ there are a fixed number t_1 of types of traders, characterized by their utilities u^i, endowments a^i, and shares π^i. Also there are a fixed number t_2 of firms characterized by their production sets Y^i and money M^i. The economy Γ_k has kt_1 traders and kt_2 firms, k of each type. (The shares of traders of any fixed type are divided equally. Thus in Γ_k the share of a trader of type i in a firm of type ℓ is ${}_k\pi^i_\ell$ π^i_ℓ/k.) An E.P. in which traders and firms of the same type choose the same strategies is called a type-symmetric E.P. (denoted T.S.E.P.)* and such a T.S.E.P. ν can be represented as a vector in $\displaystyle\mathop{X}_{i=1}^{t_1+t_2} S^i$ where S^i is the strategy set of trader i (firm i) in Γ_1 for $1 \le i \le t_1$ (for $t_1 + 1 \le i \le t_2$). Thus, for each k, a T.S.E.P. ${}_k\nu$ in Γ_k gives us a price ${}_kp$ and an allocation ${}_k\hat{x}$ in Γ_1 via ${}_k\bar{\nu}$ in S. We will say that ${}_k\nu$ is an interior T.S.E.P. if, for ${}_k\bar{\nu} = (s^1, \ldots, s^{t_1+t_2})$ and $s^i = (q^i, b^i, \hat{q}^i, \hat{b}^i)$ for $1 \le i \le t_1$.

Claim 6. Let C_j and D_j be as in Lemma 2. Then, for all k,

$$C_j \le {}_kp_j \le D_j$$

and

$$C_j \le {}_k\hat{p}_j \le D_j .$$

---

*That such a T.S.E.P. always exists is shown in Appendix B.

<u>Proof</u>. First note that if $\epsilon < K$, then in ${}^{\epsilon}_{k}\Gamma$ the maximum amount any commodity or money that a player can hold is bounded above by H. Suppose not. Let ${}^{\epsilon}_{k}x^{i}_{j} > H$. Now there are k players of the same type as i, hence at the T.S.E.P. ${}^{\epsilon}_{k}\nu$ they have identical final holdings. Then the total amount of j in ${}^{\epsilon}_{k}\Gamma$ is greater than kH. This is clearly impossible.

Note that all the other constants in Lemma 2 do not change with replication. Therefore the bounds in (3)-(6), (13)-(15), (18)-(23), (25), (26), (28), (29), (31)-(33), and (35), (36) hold for the prices ${}^{\epsilon}_{k}p$, ${}^{\epsilon}_{k}\hat{p}$. It is trivial to verify that (9), (12), (16), (30), and (34) also hold. Consequently $C_j \leq {}^{\epsilon}_{k}p_j \leq D_j$ and $C_j \leq {}^{\epsilon}_{k}\hat{p}_j \leq D_j$ for all k, if $\epsilon \leq K$. But ${}_{k}p_j = \lim_{\epsilon \to 0} {}^{\epsilon}_{k}p_j$, and ${}_{k}\hat{p}_j = {}_{\epsilon \to 0}\, {}^{\epsilon}_{k}\hat{p}_j$. The lemma follows by taking limits.

Q.E.D.

<u>Remark</u>. If ${}_{k}\hat{\nu}$ is a sequence of T.S.E.P. such that $\lim_{k \to \infty} {}_{k}p_j$ exists and is p_j, say, then the claim shows that $C_j \leq p_j \leq D_j$. (So also for $\lim_{k \to \infty} {}_{k}\hat{p}_j$.)

<u>Theorem 2</u>. Suppose we have a symmetric, interior* sequence ${}_{k}\nu$ of T.S.E.P., with associated prices $({}_{k}p_1, \ldots, {}_{k}p_m, 1, {}_{k}\hat{p}_1, \ldots, {}_{k}\hat{p}_m, 1)$, such that $({}_{k}p_1, \ldots, {}_{k}p_m, 1, {}_{k}\hat{p}_1, \ldots, {}_{k}\hat{p}_m, 1)$ converges to $(p_1, \ldots, p_m, 1, \hat{p}_1, \ldots, \hat{p}_m, 1) > 0$ as $k \to \infty$. Then $(p_1, \ldots, p_m, 1, \hat{p}_1, \ldots, \hat{p}_m, 1)$ are competitive prices (see Appendix A for definition of a C.E.) for Γ_k, $k = 1, \ldots, n, \ldots$.

*Included in ${}^{\epsilon}_{k}\beta^{i}_{j}$ $({}^{\epsilon}_{k}\beta_j)$ is the external bid of ϵ, etc.

(That such a sequence ${}_k\nu$ can be chosen is justified by Claim 6.)

Before the proof of the theorem we will need some other results. Let ${}^{\epsilon}_{k}\Gamma$ stand for the ϵ-modification ($\epsilon \leq K$) of ${}_k\Gamma$; and let ${}^{\epsilon}_{k}\nu$ (${}^{\epsilon}_{k}p$, ${}^{\epsilon}_{k}\hat{p}$) , ${}^{\epsilon}_{k}x$ denote respectively the N.E. of ${}^{\epsilon}_{k}\Gamma$, and the prices and allocation at this N.E., so chosen that ${}^{\epsilon}_{k}p \to {}_k p$, ${}^{\epsilon}_{k}\hat{p} \to {}_k\hat{p}$ as $\epsilon \to 0$. Further, denote (for $1 \leq i \leq t_1$) by ${}^{\epsilon}_{k}\beta^i_j$ (${}^{\epsilon}_{k}\hat{\beta}^i_j$) and ${}^{\epsilon}_{k}Q^i_j$ (${}^{\epsilon}_{k}\hat{Q}^i_j$) the aggregate bid and supply at ${}^{\epsilon}_{k}\nu$ due to all the players except i in the j^{th} trading-post in stage 1 (stage 2). If we drop the superscript i we will mean the aggregate due to all the players. Finally put ${}^{\epsilon}_{k}C^i_j = \sum_{\ell} {}_k\Pi^i_{\ell}\, {}^{\epsilon}_{k}\hat{y}^{\ell}_j$, where ℓ varies over the kt_2 firms in Γ_k , and ${}^{\epsilon}_{k}\hat{y}^{\ell}_j$ is of course the produced output of ℓ at ${}^{\epsilon}_{k}\nu$.

Claim 7. At ${}^{\epsilon}_{k}\nu$ trader i maximizes his utility subject to the constraint that he buy and sell in the $2m$ markets at fixed prices ${}^{\epsilon}_{k}p^i_j$, ${}^{\epsilon}_{k}\hat{p}^i_j$ ($j \in I_m$) given by

$$ {}^{\epsilon}_{k}p^i_j = \frac{{}^{\epsilon}_{k}Q^i_j}{{}^{\epsilon}_{k}B^i_j} \cdot ({}^{\epsilon}_{k}p_j)^2 $$

$$ {}^{\epsilon}_{k}\hat{p}^i_j = \frac{{}^{\epsilon}_{k}\hat{Q}^i_j}{{}^{\epsilon}_{k}\hat{B}^i_j} \cdot ({}^{\epsilon}_{k}\hat{p}_j)^2 - \frac{{}^{\epsilon}_{k}C^i_j}{{}^{\epsilon}_{k}\hat{B}^i_j} \cdot (\hat{p}_j)^2 \ . $$

Proof. Consider the "holdings-surface" H of trader i when all others keep their strategies fixed at ${}^{\epsilon}_{k}\nu$. The equation of this surface $\hat{x}_{m+1} = F(g,h)$ is of the form given in Claim 3, from which we observe that it is strictly concave and differentiable. Let H' be the unique hyperplane through ${}^{\epsilon}_{k}x^i \in H$, such that H lies below it. H' is determined by the partial derivatives of F at ${}^{\epsilon}_{k}x^i$, and it can be easily checked that

$$\frac{\partial F}{\partial g_j} = {}_k^{\epsilon}p_j^i$$

and

$$\frac{\partial F}{\partial h_j} = {}_k^{\epsilon}\hat{p}_j^i \; .$$

To establish the claim, we must show that ${}_k^{\epsilon}x^i$ maximizes u^i on H'. Suppose not. Let $y' \epsilon H'$ (see Figure 3) be such that y' can be attained by i (if he sells and buys at the fixed prices ${}_k^{\epsilon}p_j^i$, ${}_k^{\epsilon}\hat{p}_j^i$) and $u^i(y') > u^i({}_k^{\epsilon}x^i)$. By the continuity of u^i we can pick a y'' below H' such that $u^i(y'') > u^i({}_k^{\epsilon}x^i)$. Consider the straight line joining y'' and ${}_k^{\epsilon}x^i$. Since H is concave and H' is the unique hyperplane that supports it at ${}_k^{\epsilon}x^i$, the straight line must intersect H at some point $z \epsilon H$. But then $u^i(z) > u^i({}_k^{\epsilon}x^i)$, because u^i is concave. This contradicts the fact that $u^i({}_k^{\epsilon}x^i) = \max_{y \epsilon H} u^i(y)$.

Q.E.D.

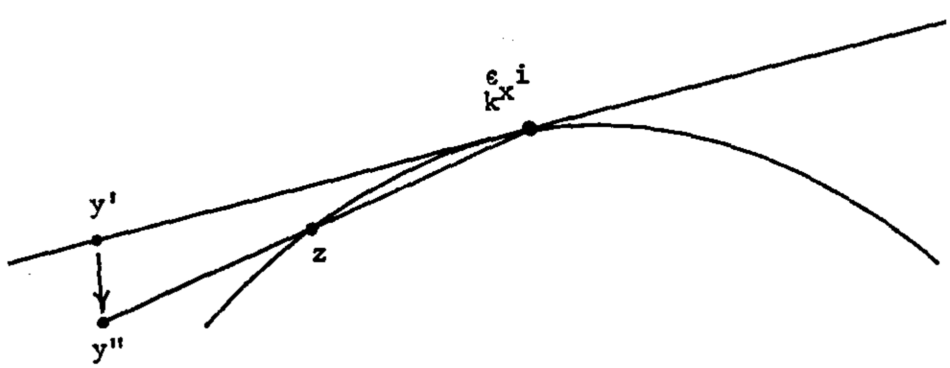

FIGURE 3

Claim 8. For any $\delta > 0$, there is a k_δ such that $\lim_{\epsilon\to 0} {}^{\epsilon}_{k}p^{i}_{j}$ and $\lim_{\epsilon\to 0} {}^{\epsilon}_{k}\hat{p}^{i}_{j}$ exist for all traders i and all $j \in I_m$, and

$${}_{k}p_{j} - \delta \le \lim_{\epsilon\to 0} {}^{\epsilon}_{k}p^{i}_{j} \le {}_{k}p_{j} + \delta , \quad \text{if} \quad k \ge k_\delta .$$

Proof. First observe that

$$1 \le \frac{{}^{\epsilon}_{k}Q_{j}}{{}^{\epsilon}_{k}Q^{i}_{j}} = \frac{Q + (k+1)\,{}^{\epsilon}_{k}q^{i}_{j}}{Q + k\,{}^{\epsilon}_{k}q^{i}_{j}} \quad \text{for some} \quad Q > 0$$

$$\le 1 + \frac{1}{k} .$$

Similarly,

$$1 \le \frac{{}^{\epsilon}_{k}B_{j}}{{}^{\epsilon}_{k}B^{i}_{j}} \le 1 + \frac{1}{k}$$

$$1 \le \frac{{}^{\epsilon}_{k}\hat{B}_{j}}{{}^{\epsilon}_{k}\hat{B}^{i}_{j}} \le 1 + \frac{1}{k}$$

$$1 \le \frac{{}^{\epsilon}_{k}\hat{Q}_{j}}{{}^{\epsilon}_{k}\hat{Q}^{i}_{j}} \le 1 + \frac{1}{k} .$$

Finally,

$$0 \le \frac{{}^{\epsilon}_{k}C^{i}_{j}}{{}^{\epsilon}_{k}\hat{Q}_{j}} \le \frac{1}{k}$$

since $$k \cdot {}^{\epsilon}_{k}C^{i}_{j} = k \cdot \sum_{\ell} {}_{k}\eta^{i}_{\ell} \, {}^{\epsilon}_{k}\hat{y}^{\ell}_{j}$$

$$= \sum_{\ell} \eta^{i}_{\ell} \, {}^{\epsilon}_{k}\hat{y}^{\ell}_{j}$$

$$\leq \sum_{\ell} {}^{\epsilon}_{k}\hat{y}^{\ell}_{j}$$

$$\leq {}^{\epsilon}_{k}\hat{Q}_{j} \; .$$

Now, recall that ${}^{\epsilon}_{k}p_{j} = {}^{\epsilon}_{k}B_{j}/{}^{\epsilon}_{k}Q_{j}$, ${}^{\epsilon}_{k}\hat{p}_{j} = {}^{\epsilon}_{k}\hat{B}_{j}/{}^{\epsilon}_{k}\hat{Q}_{j}$, and that ${}^{\epsilon}_{k}p_{j} \xrightarrow[\epsilon \to 0]{} {}_{k}p_{j}$, ${}^{\epsilon}_{k}\hat{p}_{j} \xrightarrow[\epsilon \to 0]{} {}_{k}\hat{p}_{j}$. The claim easily follows.

Q.E.D.

<u>Proof of Theorem 2</u>. By the continuity of the payoff functions, it follows that ${}_{k}x^{i} = \lim_{\epsilon \to 0} {}^{\epsilon}_{k}x^{i}$ is optimal for trader i if he trades at the fixed prices ${}_{k}p_{j}$, ${}_{k}\hat{p}_{j}$. But since he does not bid all his money, ${}_{k}x^{i}$ is optimal for i even if we were to relax his money constraint. For $({}_{k}b^{i}, {}_{k}\hat{b}^{i})$ is the solution to the following problem:

$$\text{Maximize } u^{i}(\theta(w, \hat{w}))$$

$$\text{subject to } a^{i}_{m+1} - \sum_{j \in I_{m}} (w_{j} + \hat{w}_{j}) \geq 0 \; , \quad w \geq 0 \; , \quad \hat{w} \geq 0$$

where $\theta(w,\hat{w})$ is the final bundle that accrues to i if he purchases commodities at fixed prices $({}_{k}p^{i}, {}_{k}\hat{p}^{i})$ with bids $(w,\hat{w})$, keeping his sales fixed at $({}_{k}q^{i}, {}_{k}\hat{q}^{i})$. Since u^{i} is clearly a concave function of $(w,\hat{w})$, by the Kuhn-Tucker theorem there exists a non-negative number λ such that the above problem is equivalent to

$$\text{Maximize} \quad u^i(\theta(w,\hat{w})) + \lambda[a^i_{m+1} - \sum_{j \in I_m} (w_j + \hat{w}_j)]$$

$$\text{Subject to} \quad w \geq 0 \ , \quad \hat{w} \geq 0 \ .$$

But $({}_k b^i, {}_k \hat{b}^i)$ is the solution, therefore

$$\lambda[a^i_{m+1} - \sum_{j \in I_m} ({}_k b^i_j + {}_k \hat{b}^i_j)] = 0$$

which implies $\lambda = 0$ since ${}_k\nu$ is an interior T.S.E.P. Thus in fact ${}_k x^i$ is optimal for i on $B^i({}_k p^i, {}_k \hat{p}^i)$.* Then from the fact that ${}_k x^i \xrightarrow[k\to\infty]{} x^i$ $({}_k p^i, {}_k \hat{p}^i) \xrightarrow[k\to\infty]{} (p,\hat{p})$, it is easy to deduce that x^i is optimal for i on $B^i(p,\hat{p})$.

In a similar manner we can show that firms maximize profits in the limit. Indeed, by a similar argument as was used for consumers, we find that a firm's input ${}_k y^i$ for production is optimal if he were to purchase inputs at fixed prices given by the partial derivatives of the function h (see page) evaluated at ${}_k y^i$. These turn out to be

$${}_k p^i_j + \lim_{\epsilon\to 0} {}^{\epsilon}_k p^i_j = \lim_{\epsilon\to 0} \frac{{}^{\epsilon}_k B^i_j \, {}^{\epsilon}_k Q^i_j}{({}^{\epsilon}_k Q^i_j - {}^{\epsilon}_k y^i_j)^2} \ .$$

Also ${}^{\epsilon}_k \hat{y}^i$ maximizes firm i's profit

$$\Pi^i(w) = w_{m+1} + \sum_{j \in I_m} \left(\frac{{}^{\epsilon}_k \hat{B}_j}{{}^{\epsilon}_k \hat{Q}_j + w_j} \right) w_j$$

---

*See Appendix C for the definition of $B^i(p^i, \hat{p}^i)$.

where $w \in ({}_k^{\epsilon}y^i + Y^i) \cap \Omega^{m+1}$, a convex set. Π^i is concave. Hence ${}_k^{\epsilon}\hat{y}^i$ is an optimal production for i if he were to sell at the fixed prices ${}_k\hat{p}_j^i = \lim {}_k^{\epsilon}\hat{p}_j^i$, where ${}_k^{\epsilon}\hat{p}_j^i$ are the partial derivatives of Π^i at ${}_k^{\epsilon}\hat{y}^i$. We can easily check as before that $({}_kp^i, {}_k\hat{p}^i) \underset{k\to\infty}{\longrightarrow} (p,\hat{p})$, and that therefore $\hat{y}^i$ is a profit-maximizing production for the firm i "in the limit." Finally, to ensure that the limiting prices and allocations is a C.E. for Γ_1, we need to check that the balance conditions on trade and production hold. This holds at each N.E. ${}_k\bar{\nu}$, and hence in the limit at $\bar{\nu}$.

Finally, it is trivial to show that a C.E. for Γ_1 (with allocations replicated) gives C.E.'s for Γ_k, $k \geq 2$.

Q.E.D.

4.2. Two Examples

In this section an example is provided to show that in this model with production even at the limit T.S.E.P. the price for a commodity in stage 1 will not necessarily be equal to the price in stage 2.

Furthermore, a second example is provided to illustrate the difference between totally competitive markets and a market with competitive traders but a monopolistic firm.

Example 1

In this two stage model of production and trade, even in the large economy, prices of the same commodity before and after production need not necessarily be the same. It can be shown easily (see Appendix A) that prices in the second stage cannot be higher than in the first, however it is possible that the reverse holds true. This state can be caused by a tightness in supply of an input commodity which, after production could be more abundant. An example with one type of consumer and two types of firms illustrates this.

Suppose that each consumer has an initial endowment of (25, 1, 2) of three commodities and each has an adequate supply of money so that no credit constraints are active. Let each have a utility function of the form

$$U = 8(\sqrt{x}+\sqrt{y}) + \frac{1}{2}z + m$$

where the x, y and z are the amounts of the three commodities and m is the amount of money.

Suppose that the firms of the first type control a production process

$$y = 2\min[x, 4z]$$

and those of the second type control a process of the form

$$z = 4\min[x,y] .$$

Let prices in the first period be p_1, p_2, p_3 and in the second $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$. The profits of firms of the first and second type are:

$$\Pi_1 = \hat{p}_2\hat{y}_1 - (p_1x_1 + p_3z_1)$$

and

$$\Pi_2 = \hat{p}_3\hat{z}_2 - (p_1x_2 + p_2y_2)$$

where x_1, z_1 and x_2, y_2 are inputs and $\hat{y}_1$, $\hat{z}_2$ are outputs. Suppose that the following prices are announced $(p_1, p_2, p_3) = (1, 1, 4)$ and $(\hat{p}_1, \hat{p}_2, \hat{p}_3) = (1, 1, 1/2)$. We may check that the following is a comtitive equilibrium: Firm 1 buys (8, 0, 2) and transforms it into (0, 16, 0) . Firm 2 buys (1, 1, 0) and transforms it into (0, 0, 4) .

$$\Pi_1 = 1(16) - (1(8)+4(2)) = 0$$

$$\Pi_2 = \frac{1}{2}(4) - (1(1)+1(1)) = 0 .$$

Consumers sell (9, 1, 2) and buy (0, 16, 4) .

In making the comparison with the Walrasian competitive equilibrium model we could consider a model with $2m$ rather than m commodities. The limit N.E. of this production and trading game with two goods markets will approach the C.E. of the $2m$ commodity Walrasian model in general, but when input constraints are not binding the C.E. of the m commodity model will be approached.

Example 2

In our models all participants are modelled as strategic players. The effect of a mass market can be studied as a whole or in part by replicating all or only some of the players. In the examples given here we show a (1) competitive market, i.e., one in which all economic units are small, and (2) a market with competitive consumers but a monopolistic firm. We do not calculate the case where there are many firms and few consumers although the conditions are given in the model.

Consider two types of trader, each with the same utility function.

$$U = \log x + \log y + m .$$

The first have an initial endowment of (1, 3, 100) and the second of (1, 35, 100) .

There is a single type of firm with a production process that converts $y \to x$, i.e. one unit of the second commodity as inputs yields one unit of the first as output. The firms have as initial resources (0, 0, 100) . The traders of the first type own all the shares of the firms equally.

We assert and then demonstrate that there is an equilibrium in which traders of both types buy the first commodity, traders of the first type and the firms buy the second commodity; the firms produce and sell the first commodity. If this were so we would write the payoffs as follows:

(37) $$\Pi_1^i = \log(1+\hat{b}_1^{1,i}/p_1) + \log(3+b_2^{1,i}/p_2) + 100 - \hat{b}_1^{1,i} - b_2^{1,i} + \frac{1}{n}\sum_{k=1}^{s} \Pi_3^k$$

(38) $$\Pi_2^j = \log(1+\hat{b}_1^{1,j}/p_1) + \log(35-q_2^{2,j}) + 100 - b_1^{2,j} + q_2^{2,j}p_2$$

(39) $$\Pi_3^k = \hat{b}_1^1 + \hat{b}_1^2 - e_2^3 + 100 = (\hat{p}_1 - p_2)^3 q_1 + 100$$

where $b_1^{1,i}$ is the bid of trader i of type 1 for the first commodity $q_2^{2,j}$ is the offer of trader j of type 2 for sale of the second commodity and $b_1^1 = \sum_{i=1}^{n} b_1^{1,i}$. $\hat{p}_1$ and p_2 are the prices of the first and second commodities.

Competition

Suppose that we assumed that there were so many traders and firms that all could treat p_1 and p_2 as fixed. We can take equations (1) to (3) and solve for the competitive equilibrium. We obtain:

(40) $$\frac{1}{\hat{p}_1 + \hat{b}_1^{1,i}} = 1 \quad \text{or} \quad \hat{b}_1^{1,i} = 1 - \hat{p}_1$$

(41) $$\frac{1}{3p_2 + b_2^{1,i}} = 1 \quad \text{or} \quad b_2^{1,i} = 1 \cdot 3p_2$$

(42) $$\frac{1}{\hat{p}_1 + \hat{b}_1^{2,j}} = 1 \quad \text{or} \quad \hat{b}_1^{2,j} = 1 - \hat{p}_1$$

(43) $$\frac{1}{35 - q_2^{2,j}} = p_2 \quad \text{or} \quad q_2^{2,j} = 35 - 1/p_2$$

(44) $$\hat{p}_1 = p_2 \ .$$

It can be checked that the following values provide a solution.

$$\hat{b}_1^{1,i} = \hat{b}_1^{2,j} = .9 \ , \quad \hat{p}_1 = p_2 = .1 \ , \quad q_2^{2,j} = 25 \ ,$$

$$b_2^{1,i} = .7 \ , \quad b_2^{3,k} = 1.8 \ , \quad \hat{y}^{3,1} = 18 \ .$$

These yield

$$\Pi_1^i = \log(10) + \log(10) + 100 - 1.6 + 0$$

$$\Pi_2^j = \log(10) + \log(10) + 100 - .9 + 2.5$$

$$\Pi_3^k = 0$$

Monopoly

Suppose that as we replicate the original market there is only one firm whose resources grow in proportion to the replication. Thus its endowment at replication n will be $(0, 0, 100n)$.

This firm will influence price in both the factor and product markets. Equation (44) will no longer hold. We need to consider its influence on price explicitly, hence we note

$$(45) \qquad p_1 = \frac{\hat{b}_1^1 + \hat{b}_1^2}{\hat{y}_1^3}$$

and

$$(46) \qquad p_2 = \frac{b_2^1 + e_2^3}{q_2^2}$$

where

$$(47) \qquad \hat{y}_1^3 = \frac{e_2^3 q_2^2}{b_2^1 + b_2^3} .$$

From (40), (42) and (45) we can obtain

$$(48) \qquad \hat{y}_1^3 = 2n \frac{(1 - \hat{p}_1)}{\hat{p}_1} .$$

From (41), (43) and (46) we obtain

$$(49) \qquad e_2^3 = 2n(19p_2 - 1) .$$

From (41), (43), (47), (48) and (49) we obtain

$$(50) \qquad \frac{2n(1 - p_1)}{p_1} = \frac{2n(19p_2 - 1)}{p_2}$$

or

$$(51) \qquad p_2 = \frac{p_1}{2p_1 - 1} .$$

We may rewrite $\Pi_3 = (p_1 - p_2)q_1^3$ as:

$$(52) \qquad \Pi_3 = 2n\left(p_1 - \frac{p_1}{2p_1 - 1}\right)\left(\frac{1 - p_1}{p_1}\right) .$$

Maximizing with respect to b_2^3 we obtain as a first order condition

$$(1 - p_1)\left[\frac{1}{(2p_1 - 1)^2}\right] - \left[\frac{19p_1 - 1}{2p_1 - 1}\right] = 0$$

giving

(53) $$380(\hat{p}_1)^2 - 38 = 0 \quad \text{or} \quad p_1 = .316 \ .$$

Table 1 presents a comparison of the full competition conditions with those of the monopolistic firm

TABLE 1

	$\hat{p}_1$	p_2	b_2^3	q_1^3	$b_1^{1,i}$	$b_1^{2,j}$	$b_2^{1,i}$	$q_2^{2,j}$
Competition	.1	.1	1.8	18	.9	.9	.7	25
Monopoly	.316	.0594	.2568n	4.3246n	.6838	.6838	.8218	18.165

The payoffs are:

$$\Pi_1^i = \log(3.16) + \log(16.84) + 100 - 1.5056 + 1.1107$$

$$\Pi_2^j = \log(3.16) + \log(16.84) + 100 - .6838 + 1.0790$$

$$\Pi_3 = .2568(4.3246)n + 100n = 1.1107n + 100n \ .$$

TABLE 2

	Π_1	Π_2	Π_3
Competition	103.00	106.20	100
Monopoly	103.58	104.37	101.11n

5. DISCUSSION AND FURTHER PROBLEMS

5.1. The Role of Credit

In this and a previous paper[12] we have avoided having to consider in detail problems involving the granting of credit. This was done by the device of assuming that all individuals have "enough money" where enough money requires special definition. In a separate paper[13] credit, insolvency and bankruptcy are all discussed.

We believe that at this stage in the development of these models it is desirable to separate phenomena as much as possible. Thus here although some of the implications of sequential trade caused by production have been studied the roles of credit, bridging finance, durable assets and the trading of stock have not been dealt with.

In particular the need for credit in this model depends not only upon the amount of money held by each individual or firm, but also upon the specific pattern of payments. If, for example, all payments are received after the inputs of first market has cleared, far less money may be required than if payments come in after the second market. Even at this level of detail we must distinguish between a payment that is not received because it is in the float during the span of two markets and one that is not received because the buyer delays payment and utilizes the credit he has created for himself.

In this model the firms have no debt. Their assets are money which are offset by the equity claim of the stockholders. A far more satisfactory model of the firms would be to have them own an array of durable assets, but this would raise further questions concerning corporate finance i.e. do we wish to assume that all assets are financed by equity or do we wish to introduce some debt?

5.2. Capital Goods

The introduction of producer durables poses several new problems and offers an opportunity for new solutions. In particular if the durables have lives of different lengths we may need to distinguish financing of different length. This cannot be done yet in these models. Exogenous or endogenous uncertainty must be present before we can operationally distinguish the need for and different costs of financing of different length.

In a finite horizon model with durables which outlast the time span being studied we face a problem in attaching an end state evaluation to these items. Furthermore with durable assets present, the definition of short term profits becomes somewhat arbitrary as is shown in many accounting texts. This in turn may effect dividend policy.

When there is an array of producer durables with different durability there will tend to be production time lags present which may result in systems whose dynamic behavior in disequilibrium is heavily governed by the specifics of these lags and the credit conditions.

The presence of producer (or consumer) durables produces a need for financing, but also provides a means for financing. In particular highly marketable durable assets can serve as "hostages" against default on loans made on them. Thus long term financing need not call for more trust or perception on the part of the lenders provided that the quality of the long term assets is sufficiently high. In terms of a noncooperative game theory this implies that claims by creditors to assets may be sufficient to turn long run credit arrangements which would otherwise be unstable, into stable arrangements.

5.3. The Trading of Shares

In the models we have examined we have not included trade in shares. Individuals were given fixed endowments of shares which they were required to hold. A more general treatment would be to permit them to trade shares. Although we do not carry through a formal analysis of the market with the trading of shares in this paper, several observations and conjectures are made on the effect of having shares traded.

As information conditions are critical in models of this type we must be explicit about the sequencing of markets and the availability of information. The simplest model might be to have shares traded before or after the first round of trade prior to production, but with no information to the traders. This is equivalent to having the goods market and stock market decisions simultaneous.

We must specify the relationship between the date of ownership and the dividend receipt as well as the sequence of transfer and payment for shares. Simple rules for this would be that there is one or two stock market trading sessions, say before the first and second goods markets. Dividends are paid to the final owners after the last stock trades.

We conjecture that for mass markets the price of the shares approaches the value of the dividends plus liquidation payment to be received from the shares at the limit N.E.

The major effect of the introduction of shares, if payment is received for selling them before all trade has taken place, is to increase the liquidity of some of the traders and hence to relax the conditions calling for "enough money" or for credit.

At the limit N.E. which approaches a competitive equilibrium in the models with trade in shares it is conjectured that a new indeterminacy (or multiplicity) appears in the existence of equilibrium points. In particular if cash flow constraints are not binding then individuals will be indifferent between holding shares and obtaining dividends or selling them for a price equal to the dividends plus liquidation payment.

In the modelling of shares here we assume that they are nonvoting and are of unlimited liability. Otherwise new analytical difficulties and different results are to be expected.

Even in the simplest of models with trade in nonvoting shares only once, as is shown in Figure 4, when numbers are few we may expect new

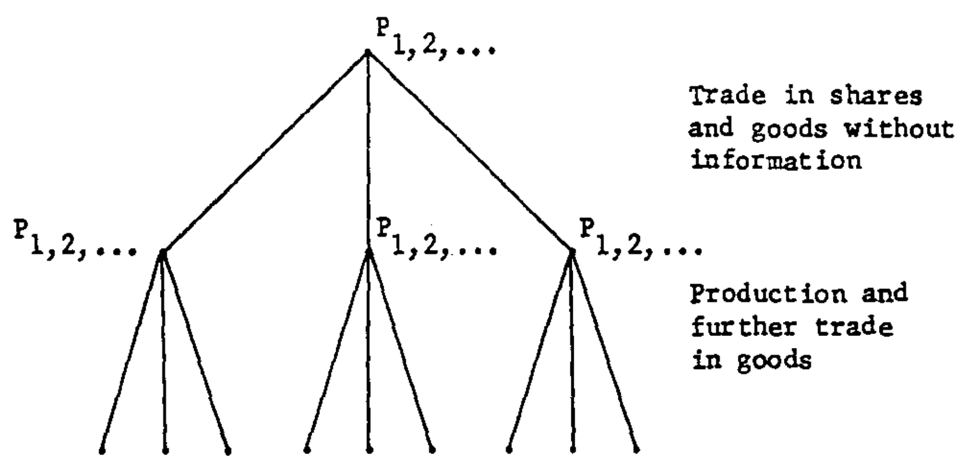

FIGURE 4

phenomena. In this model, if there are m goods and k firms a strategy by an individual trader consists of a vector of $2(m+k)$ numbers (a bid and offer for each good and each type of share) and a function for stage 2 which has as its argument which could be as high as any one of the vectors of $2(m+k)n + 2mk$ dimensions which could arise from the possible moves of all n traders and k firms in the first stage.

In this model we may expect new equilibria to appear for a finite number of traders. These equilibria will reflect the influence of "wash sales" on both the goods markets and stock markets. An example of the influence of wash sales has been given in a previous paper by Dubey and Shubik.[10]

Another set of problems is encountered if we permit the firms to hold and trade in shares. In particular we must either rule out the control aspects of the holding of shares or open a new Pandora's box. Even with this simplification a new dimension of oligopolistic manipulation is opened up. We must also decide whether treasury stock (i.e. stock of a company held by itself) obtains dividends. This raises a *pro forma* accounting problem in as much as the residual worth of the firm will be distributed on liquidation.

If a firm is permitted to own its own shares we must decide how to treat the possibility that it owns all of its shares. In this case we might require that the owner be a player with a utility function rather than a bloodless fiduciary maximizing profits for the share owners. Profit maximization in such a market cannot *a priori* be equated with utility maximization.

5.4. A Comment on Existence of Oligopolistic Equilibria

The model in this paper contrasts with the work of Arrow and Hahn,[14] Fitzroy,[15] Gabszewicz and Vial,[16] Laffont and Laroque,[17] Marschak and Selten,[18] Negishi,[19] and Roberts and Sonnenschein[8] in several ways. In particular all traders and firms are treated as strategic players; a different market mechanism is used; a means of payment or a "money" plays a crucial role and a two stage trade is used in order to be able to deal with the procurement of inputs while preserving laws of conservation.

In multistage models of economic activity it may be desirable to distinguish between "perfect equilibria" and more general noncooperative equilibria. The former might not exist in systems in which the latter nevertheless exist.

5.5. A Variation of the Model

A variant of the model given here would allow the traders to use the money and commodities acquired at the end of the first stage as bids and offers in the second stage. We chose the simpler model for ease of presentation. The existence and convergence results nevertheless hold for the variant model as well. The proofs are in essence identical to the ones given here.

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APPENDIX A

We quote, with proof, the following lemma from [11] that is invoked in the proof of Lemma 3 in Section 3.

LEMMA C. (Uniform monotonicity). Let $j \in I_m$ *, let* $f(x)$ *be a continuous, nondecreasing function from* Ω^m *to the reals that is actually increasing in the variable* x_j *, and let* H *be a positive constant. Then a positive number* $h = h(f,j,H)$ *exists such that for all* x *and* y *in* Ω^m *, if*

$$\|x\| \leqq H \quad \text{and} \quad \|y-x\| \leqq h ,$$

then

$$f(y+e^j) > f(x) .$$

Proof. Let $Q(\alpha)$ denote the cube $\{x \in \Omega^m : \|x\| \leqq \alpha\}$. The function $f(x+e^j) - f(x)$ is positive and continuous, and so has a positive minimum μ in $Q(H)$. The function f itself is uniformly continuous in $Q(H+1)$, so there exists a $0 < \delta < 1$ such that $x \in Q(H)$, $y \in \Omega^m$, and $\|y-x\| < \delta$ imply that $|f(y) - f(x)| < \mu$, and hence that $f(y+e^j) - f(x) > 0$. So we may take $h = \delta$. Q.E.D.

APPENDIX B*

We wish to prove** that there will always exist a T.S.N.E. of ${}^{\epsilon}\Gamma$. We will assume that everybody desires money. (This assumption is required--see [11]--to guarantee the interiority of the N.E.'s.) It can then easily be checked that $\hat{T}$ consists of a single point. Hence the sets ${}^{\epsilon}\gamma^{i}(\bar{s}^{i})$ are in fact convex. Define S^* to be the subset of S which consists of type-symmetric strategies, i.e.

$$S^* = \{s \in S : s^i = s^j \text{ if } i \text{ and } j \text{ are of the same type}\} .$$

S^* is clearly convex. Consider the map

$$S^* \xrightarrow{\phi} 2^{S^*}$$

given by

$$\phi(s) = \left[\mathop{X}_{i=1}^{n+s} {}^{\epsilon}\gamma^{i}(\bar{s}^{i}) \right] \cap S^*$$

$\phi(s)$ is always non-empty because at any $s \in S^*$ all players of any given type face the same optimization problem, therefore there will be identical strategies that lie in their "best-response sets" ${}^{\epsilon}\gamma^{i}(\bar{s}^{i})$. ϕ is moreover convex-valued and u.s.c. We appeal to the Kakutani fixed point theorem, and see that the fixed point of ϕ is a T.S.N.E. of ${}^{\epsilon}\Gamma$.

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*We adhere to the notation of Section 3.

**This proof is essentially the same as the one in [11].

APPENDIX C

The trading and production economy that corresponds to our model involves $2m+1$ commodities. The traders utility functions

$\Omega^{2m+1} \xrightarrow{\hat{u}^i} \Omega^1$ are given by

$$\hat{u}^i(x_1, \ldots, x_m, x_1^*, \ldots, x_m^*, x_{m+1}) = u^i(x_1 + x_1^*, \ldots, x_m + x_m^*, x_{m+1}) \ .$$

Their initial endowments are $A^i = (a_1^i, \ldots, a_m^i, 0, \ldots, 0, a_{m+1}^i)$. Firm ℓ has initial endowment $(0, \ldots, 0, M^\ell)$, and a convex production set $\hat{Y}^\ell < R^{2m+1}$ with the property:*

$$[(x, 0, x_{m+1}) + \hat{Y}^\ell \cap \tilde{\Omega}^m = \{0\}$$
$$\cap \Omega^{2m+1} = \{0\} \ .$$

The first condition states that the first m commodities are inputs for production. We denote the vector of prices by $(p_1, \ldots, p_m, \hat{p}_1, \ldots, \hat{p}_m, 1)$, the set of productions available to firm ℓ at the prices $(p,\hat{p})$ is

$$\tilde{Y}^\ell(p,\hat{p}) = \{[(x, 0, x_{m+1}) + Y^\ell] \cap \Omega^{2m+1} : x \in \tilde{\Omega}^m, \ x_{m+1} + p\cdot x \le M^\ell\} \ .$$

Let $\Pi^\ell(p,\hat{p})$ denote the maximum of $\hat{p}\cdot\hat{y} + y_{m+1}$ for $(0, \hat{y}, y_{m+1}) \in \tilde{Y}^\ell(p,\hat{p})$. Then the budget set of trader i at the prices $(p_1, \ldots, p_m, \hat{p}_1, \ldots, \hat{p}_m, 1)$ is

$$B^i(p,\hat{p}) = \{(x, x^*, x_{m+1}) \in \Omega^{2m+1} : p\cdot x + \hat{p}\cdot x^* + x_{m+1}$$
$$\le p\cdot a^i + a_{m+1}^i + \sum_{\ell \in I_{n+s} \setminus I_s} \eta_\ell^i \Pi^\ell(p,\hat{p})\}$$

---

* $\tilde{\Omega}^m = \{(x_1, \ldots, x_m, 0, \ldots, 0, 0) \in \Omega^{2m+1} : (x_1, \ldots, x_m) \in \Omega^m\}$.

A competitive equilibrium consists of prices $(p, \hat{p}, 1)$, allocations $x^i \in \Omega^{2m+1}$ for $i \in I_n$, and productions $(0, \hat{y}, \hat{y}^{\ell}_{m+1}) \in \Omega^{2m+1}$ for $\ell \in I_{n+s} \setminus I_s$ such that the balance conditions for trade and production are met and x^i maximizes $\hat{u}^i$ on $B^i(p,\hat{p})$ for $i \in I_n$, $\hat{p} \cdot \hat{y}^{\ell} + \hat{y}^{\ell}_{m+1} = \Pi^{\ell}(p,\hat{p})$ for $\ell \in I_{n+s} \setminus I_n$ with

$$x^i \in B^i(p,\hat{p})$$

$$(0, \hat{y}^{\ell}, \hat{y}^{\ell}_{m+1}) \in Y^{\ell}(p,\hat{p}) .$$

APPENDIX D

Suppose ${}^{\epsilon}p_j < {}^{\epsilon}\hat{p}_j$ at a N.E. of ${}^{\epsilon}\Gamma$. It follows immediately that $q_j^i = 0$ and $\hat{b}_j^i = 0$ for all traders i. But

$${}^{\epsilon}p_j = \frac{\epsilon + \bar{b}_j + \bar{e}_j}{\epsilon} \geq 1$$

and

$${}^{\epsilon}\hat{p}_j = \frac{\epsilon}{\epsilon + \bar{\hat{y}}_j + \bar{\hat{q}}_j} \leq 1 ,$$

which is a contradiction.