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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART 30 (REVISED)

THE OPTIMAL BANKRUPTCY RULE IN A TRADING ECONOMY USING FIAT MONEY

Martin Shubik and Charles Wilson

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by

Martin Shubik** and Charles Wilson

1. INTRODUCTION

In several previous papers^{1, 2, 3, 4, 5, 6, 12} models of a monetary economy have been solved as a noncooperative game. The problem of granting credit and the possibility of bankruptcy was avoided by the artifact of considering that all traders were supplied with "enough" of a commodity serving as a "money" or means of payment so that there was no need to borrow.

In this paper an outside bank, and borrowing are considered explicitly and the meaning of an optimal bankruptcy rule are considered. This paper deals primarily with problems in modelling and interpretation. Thus the argument is carried out primarily in terms of a specific relatively

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**The authors wish to thank Pradeep Dubey and Donald Brown for helpful conversations. This work builds not only on one of the authors previous work but on joint work with Shapley, Dubey, Whitt and Evers.

simple example to illustrate the market, banking and bankruptcy mechanism. Several general proofs pertaining to a broad class of trading models are given in a separate paper.¹⁵

2. THE MODEL

The model is a variant of the model originally suggested by Shubik¹ and investigated by Shapley,³ Shapley and Shubik,⁷ Shubik^{8, 9} and Dubey and Shubik.^{5, 6} The paper here however is self contained inasmuch as a complete model is built, although the references noted provide detailed discussion of some aspects of the model and proofs which are not supplied here.

2.1. A Trading Economy without Uncertainty

The procedure adopted here is to begin by taking a simple model of trade. This is formulated and solved for the standard competitive equilibrium solution. We then take the same economic background and model trade as a noncooperative game with a bank issuing loans to finance trade.

We solve the game for a type symmetric noncooperative equilibrium point (T.S.N.E.). This is an equilibrium point at which traders of the same type obtain equal treatment. Equal treatment is a property of the competitive equilibrium, but not necessarily of the noncooperative equilibrium. We then study the conditions under which the T.S.N.E. coincides or fails to coincide with the C.E. in terms of market prices and distribution of resources.

Consider n traders with endowments of $(A,0)$ and n with endowments of $(0,B)$. Traders of the first type have utility functions of the form

$$(1) \quad u^1 = \log x^\alpha y^{1-\alpha}$$

and the second type

$$(2) \quad u^2 = \log x^\beta y^{1-\beta} .$$

2.2. The Competitive Equilibrium and Pareto Optimal Surface

In order to solve for the competitive equilibrium we may consider two functions of the following form:

$$(3) \quad G_1 = \log(A-x)^\alpha y^{1-\alpha} + \lambda_1 [p_1 x - p_2 y]$$

$$(4) \quad G_2 = \log x^\beta (B-y)^{1-\beta} + \lambda_2 [p_2 y - p_1 x]$$

where λ_1 and λ_2 are Lagrangian multipliers. Taking derivatives wrt. x , y , λ_1 and λ_2 we obtain:

$$(5) \quad \frac{\alpha}{A-x} = \lambda_1 p_1$$

$$(6) \quad \frac{1-\alpha}{y} = \lambda_1 p_2$$

$$(7) \quad p_1 x = p_2 y$$

$$(8) \quad \frac{\beta}{x} = \lambda_2 p_1$$

and

$$(9) \quad \frac{1-\beta}{B-y} = \lambda_2 p_2 .$$

We may set $\boxed{p_1 = 1}$.

From (5) and (6) we obtain:

$$(10) \quad p_2 = \left(\frac{1-\alpha}{\alpha}\right) \left(\frac{A-x}{y}\right) .$$

From (7) and (10)

$$x = \left(\frac{1-\alpha}{\alpha}\right)(A-x)$$

or

$$(11) \quad \boxed{x = (1-\alpha)A} .$$

From (8) and (11)

$$(12) \quad \boxed{\lambda_2 = \frac{\beta}{A(1-\alpha)}} .$$

From (5) and (11)

$$(13) \quad \boxed{\lambda_1 = 1/A} .$$

From (9), (10) and (12)

$$(14) \quad \boxed{y = B\beta} ,$$

and from (10), (11) and (14)

$$(15) \quad \boxed{p_2 = \left(\frac{1-\alpha}{\beta}\right) \left(\frac{A}{B}\right)} .$$

In this extremely simple model it is easy to calculate the Pareto optimal surface. It is given by:

$$(16) \quad \begin{vmatrix} \frac{\partial u^1}{\partial x} & \frac{\partial u^1}{\partial y} \\ \frac{\partial u^2}{\partial x} & \frac{\partial u^2}{\partial y} \end{vmatrix} = 0$$

or $\frac{\alpha(1-\beta)}{(A-x)(B-y)} = \frac{\beta(1-\alpha)}{xy}$, which gives:

$$(17) \quad \left\{ \frac{\alpha - \beta}{\beta(1-\alpha)} \right\} xy = AB - Bx - Ay .$$

A check shows that (11) and (14) satisfy (17).

2.3. The Money Game

2.3.1. The Trading Mechanism

Let the amount bid by a trader i of type 1 be b^i and by a trader j of type 2 be d^j . As traders of type 1 only have the first commodity we may assume that they can offer only it for sale. Let the amount of the first commodity offered for sale by a trader i of the first type be x^i . A trader j of the second type offers an amount y^j of the second commodity.

Let the symbol $b = \sum_{i=1}^n b^i$ and similarly for the others.

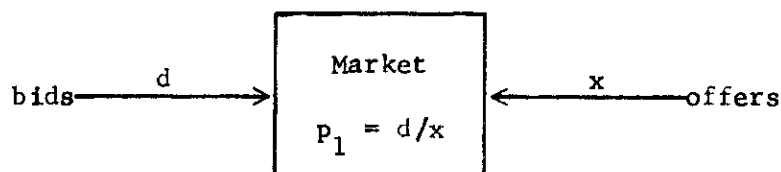


FIGURE 1

We may assume in this simple market that traders of type 1 bid only for the second good and traders of type 2 bid only for the first good. It has been shown elsewhere⁶ that there is no loss of generality

in making this assumption in large markets.*

The markets for the goods are extremely simple. Figure 1 illustrates the market for the first good. All bids are aggregated (d) and all offers are aggregated (x) and the price is fixed by dividing the amount of money bid by the amount of good offered.

There are several other price formation mechanisms which could have been selected. These are discussed elsewhere.⁹ This was selected because of its basic simplicity.

When the price has been formed, individual j who has bid d^j for the first good will obtain

$$(18) \quad z^j = \frac{d^j}{p_1} = \frac{d^j}{d} x \quad \text{where}$$

$$(19) \quad p_1 = \frac{d}{x}$$

similarly trader i of the first type obtains

$$(20) \quad w^i = \frac{b^i}{p_2} = \frac{b^i}{b} y \quad \text{where}$$

$$(21) \quad p_2 = \frac{b}{y} .$$

2.3.2. Banking: Credit and Promissory Notes

In 2.3.1 we did not specify the currency in which bids are to be made. We now assume that all bids are made using a fiat or bank money which must be obtained from an "outside bank" which is modelled as a

*When numbers are few "wash sales" may be of importance, but we leave this problem aside at this time.

mechanism or "dummy" in the sense that it is given a fixed strategy. In particular the bank fixes an amount of money it will issue. In an economy with $2n$ traders let this amount be nK .

The loan mechanism to lend out the nK units will be a simple "money market." Each trader i of type 1 is permitted to create a financial instrument of his own, to wit, a nonnegotiable promissory note of size or denomination u^i . Similarly a trader j of type 2 bids v^j .

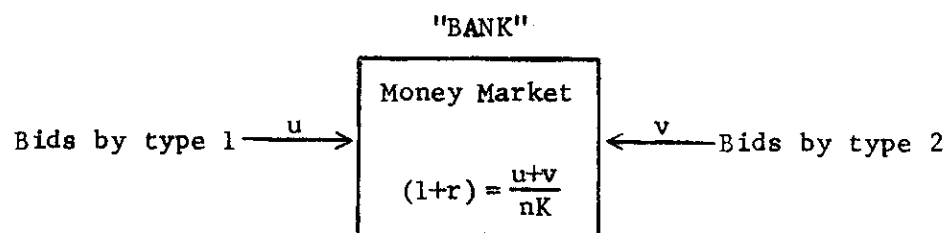


FIGURE 2

The traders use the promissory notes to bid for the money supply. The meaning of a promissory note is that it is a contract between an individual i and the bank which promises to return to the bank an amount of bank money u^i at the end of the period in return for an amount obtained at the start of the period. This amount is determined by the supply of loanable funds nK and the offers of promissory notes, i.e. if s^i and t^j are the loans obtained by i and j they are respectively:

$$(22) \quad s^i = \frac{u^i}{u+v}(nK)$$

and

$$(23) \quad t^j = \frac{v^j}{u+v}(nK) .$$

The model is naively simple. Banks in general would be suspicious of borrowers wishing to promise to pay back enormous sums for loans whose

size they do not know in advance. However the mechanism is well defined and in a mass market where individuals have estimates of the aggregate money supply and demand this can be interpreted more reasonably, as a single individual will not expect the price of money to move very much in response to his actions.

As can be seen from Figure 2 the price of money is given by:

$$(24) \quad 1+r = \frac{u+v}{nK}$$

where r can be interpreted as a money rate of interest.*

2.3.3. Fiat or Bank Money?

Before we continue with the model a minor digression on terminology is made. There are strong reasons to distinguish government issued fiat money from bank issued money. In particular it is argued elsewhere that efficiency calls for an issue of noninterest bearing fiat money to cover the float,^{10, 11} whereas interest bearing bank money is called for for intertemporal trade. These distinctions are obliterated in a one period model.

Furthermore one might wish to consider different banking institutions or credit operations such as private inside banks or finance companies. At our level of modelling here no attempt is made to go beyond a most rudimentary bank which is really only being used to finance the float.

In essence in the model here individual traders will not accept

*In a one period model the concept of an interest rate seems somewhat strange. A different way of phrasing the role of r is that it is the payment made for having the bank finance trade or cover the float.

each others' promissory notes. They begin with no money, but there exists a bank which will accept promissory notes against its note issue and the markets as a whole will accept the bank's notes.

As there is only one outside bank issuing notes we do not attempt to make any distinction between the terms "fiat money" or "bank money." This money is accepted in trade, but is intrinsically worthless.

2.3.4. Settlement and Bankruptcy

After the market is over individuals obtain their final allocations of goods, obtain their money incomes and must make their final settlements at the bank.

If an individual ends up with a positive amount of money after having paid the bank, this has no positive value to him. If on the other hand he is unable to honor his debts in full, a penalty is leveled against him. This penalty does not necessarily have to be the same for all individuals. In this model it is proposed to study it parametrically. Two parameters μ_1 and μ_2 are introduced associated with a linear term which measures the "punishment" leveled against anyone who is unable to repay his debts. The specific structure of the payoff functions is shown in 2.3.6 below.

We may consider the bankruptcy act as economic or not directly so. An example of the former might involve a sale of assets of the debtor, an example of the latter would be a prison sentence or death. In a one period model we may imagine that assets which are confiscated are sold off in a subsequent period.

Regardless of how we regard the punishment, its presence is needed if we wish to influence individuals against making exorbitant repayment

commitments to increase the current loans, knowing that they will be unable to meet their obligations.

2.3.5. The Payoff Functions

We have seen that the utility functions of the traders (shown in (1) and (2)) involve only the two consumer commodities. We define the payoff functions in terms of strategies and outcomes including the final disposition of money. We may write the payoff to a trader i of type 1 as:

$$(25) \quad \Pi_1^i = \log(A - x^i) \left(\frac{b^i y}{b} \right)^{1-\alpha} + \mu_1 \min \left[0, \left\{ \frac{d}{x} x^i - b^i + \frac{u^i nK}{u+v} - u^i \right\} \right]$$

and the payoff to a trader j of type 2 is:

$$(26) \quad \Pi_2^j = \log \left(\frac{d^j x}{d} \right)^\beta (B - y^j)^{1-\beta} + \mu_2 \min \left[0, \left\{ \frac{b}{y} y^j - d^j + \frac{v^j nK}{u+v} - v^j \right\} \right]$$

where the x^i and y^j are considered as functions of the u^i and v^j , and where $0 \leq b^i \leq u^i nK/u+v$ and $0 \leq d^j \leq v^j nK/u+v$.

$$(27) \quad p_1 = d/x \quad \text{and} \quad p_2 = b/y .$$

We note that the bankruptcy conditions enter as linear terms, or as zero.

We assume $0 < \alpha$, $\beta < 1$ to ensure a solution with some trade. The preferences of the individuals are represented by concave utility functions. The introduction of the bankruptcy condition can be considered as though the utility function for an individual had been defined to include money holdings where nothing is added for positive holdings but debt

(which may be interpreted as negative holdings) has a negative worth.

Limiting our illustration to one real commodity and money and debt Figure 3 shows the modified utility function extending into a negative money or debt zone.

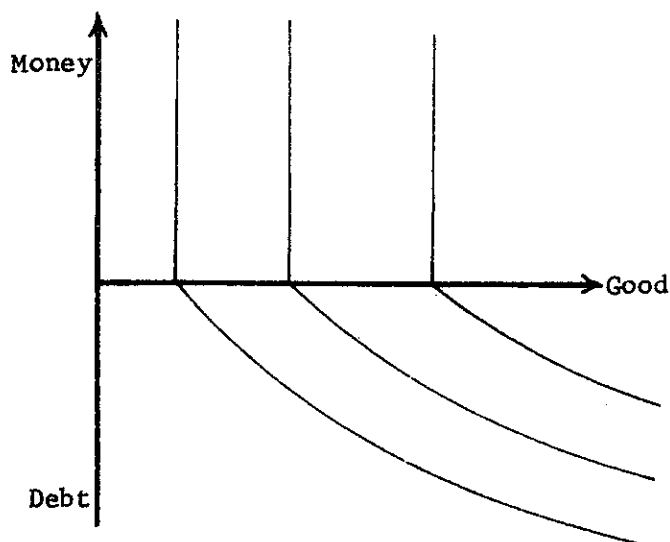


FIGURE 3

2.4. The Noncooperative Equilibrium with No Information

Equilibria may be extremely sensitive to information conditions. In multistage games such as this, the less information the traders have the easier it may be to examine the equilibrium points of the game.*

The simplest situation is where all traders are required to announce simultaneously not only u^i and v^j but** also b^i , d^j and x^i ,

* In the analysis which follows, only the first order conditions for utility maximization are used to describe the equilibrium. In particular, we have not provided a complete proof that any of our solutions are in fact non-cooperative equilibria. The conscientious reader is encouraged to provide his own proofs where necessary.

**As an individual may not be sure of his money supply before his bid b^i (d^j) is fixed, we require a feasibility or acceptability condition in the game. For example a nonfeasible bid is replaced by a zero bid or some convention is given which transforms the bid into a feasible act.

y^j .*

We may consider that a trader i of the first type attempts to maximize:

$$(28) \quad G_1^i(u^i, b^i, x^i) = \alpha \log(A - x^i) + (1-\alpha) \log(yb^i/b) \\ + \lambda_1^i \{ dx^i/x - b^i + (M-1)u^i \} + \gamma_1^i (Mu^i - b^i) .$$

There will be a similar expression for a trader j of the second type. We define:

$$(29) \quad M = \frac{nK}{u+v} \geq 0, \quad u = \sum u^i \quad (\text{and similarly for the others})$$

where λ_1^i and γ_1^i can be regarded as Lagrangian multipliers with the following conditions:

$$(30) \quad \begin{aligned} \text{if } dx^i/x - b^i + (M-1)u^i > 0 \text{ then } \lambda_1^i &= 0 \\ &= 0 \text{ then } 0 \leq \lambda_1^i \leq \mu_1 \\ < 0 \text{ then } \lambda_1^i &= \mu_1 \end{aligned}$$

$$(31) \quad \begin{aligned} \text{if } (Mu^i - b^i) > 0 \text{ then } \gamma_1^i &= 0 \\ &= 0 \text{ then } \gamma_1^i \geq 0 . \end{aligned}$$

First order maximization conditions give:

$$(32) \quad \frac{\partial G_1^i}{\partial x^i} = \frac{-\alpha}{A - x^i} + \lambda_1^i d \left\{ \frac{1}{x} - \frac{x^i}{x^2} \right\} = 0$$

*It is straightforward to check that degenerate equilibria always exist --that is, equilibria with $x^i, y^i = 0$ and no bankruptcy. In what follows, we will be restricting our attention to non-degenerate equilibria.

and from symmetry:

$$(33) \quad \frac{\alpha}{A - x^i} = \lambda_1^i \left(\frac{n-1}{n} \right) \frac{d}{x} .$$

Similarly from $\partial G_1^i / \partial b^i = 0$ and $\partial G_1^i / \partial u^i = 0$ we obtain

$$(34) \quad (1-\alpha) \left(\frac{n-1}{n} \right) = b^i (\lambda_1^i + \gamma_1^i)$$

and

$$(35) \quad \lambda_1^i = M(\lambda_1^i + \gamma_1^i) \left\{ \frac{v + \left(\frac{n-1}{n} \right) u}{u+v} \right\} .$$

From (33) and $x = nx^i$

$$x\alpha = \lambda_1^i \left(\frac{n-1}{n} \right) d(A - x^i) \quad \text{gives}$$

$$(36) \quad x^i = \frac{\lambda_1^i d A \left(\frac{n-1}{n} \right)}{n\alpha + \lambda_1^i d \left(\frac{n-1}{n} \right)}, \quad \text{for } \lambda_1^i > 0 .$$

If $\lambda_1^i = 0$, the only solution consistent with equation (36) is $x^i = 0$.

But this means that each type i trader receives 0 rather than d^i units of money from type j traders. As a consequence, it is easy to verify that no solution exists for $\lambda_1^i = 0$. Therefore, (30) implies:

$$(37) \quad d - b + (M-1)u \leq 0 \quad \text{and} \quad b - d + (M-1)v \leq 0$$

$$\text{or} \quad (M-1)(u+v) \leq 0 .$$

Hence if $u+v > 0$ then

$$(38) \quad M \leq 1 .$$

From (35) and (38) and symmetry then $\gamma_1 > 0$, and similarly $\gamma_2 > 0$; hence

$$(39) \quad Mu = b \quad \text{and} \quad Mv = d .$$

From (34), (35) and (39) and symmetry:

$$(40) \quad u^i = \frac{1-\alpha}{\lambda_1} \left(\frac{n-1}{n} \right) \left(\frac{v + \left(\frac{n-1}{n} \right) u}{u+v} \right)$$

$$v^i = \frac{\beta}{\lambda_2} \left(\frac{n-1}{n} \right) \left(\frac{u + \left(\frac{n-1}{n} \right) v}{u+v} \right) .$$

From (30) and (39):

$$(41) \quad \lambda_1 < \mu_1 \implies Mv = u \quad \text{and} \quad \lambda_2 < \mu_2 \implies Mu = v$$

and

$$(42) \quad Mv < u \implies \lambda_1 = \mu_1, \quad Mu < v \implies \lambda_2 = \mu_2 .$$

We now confine our attention to the limiting behavior in markets as $n \rightarrow \infty$. From (40) we obtain:

$$(43) \quad u^i \rightarrow \frac{1-\alpha}{\lambda_1} \quad \text{and} \quad v^j \rightarrow \frac{\beta}{\lambda_2} .$$

We wish to consider all λ_1 and λ_2 consistent with (41) or (42).

From (29) and (43)

$$M = \frac{\lambda_1 \lambda_2 K}{\lambda_2 (1-\alpha) + \lambda_1 \beta}, \quad \text{hence}$$

$$(44) \quad \text{Mu}^i = \frac{K \frac{1-\alpha}{\lambda_1}}{\frac{1-\alpha}{\lambda_1} + \frac{\beta}{\lambda_2}} \quad \text{and} \quad \text{Mv}^j = \frac{K \frac{\beta}{\lambda_2}}{\frac{1-\alpha}{\lambda_1} + \frac{\beta}{\lambda_2}} .$$

There are four cases arising from (41) and (42). They are as follows:

$$(A) \quad u^i = \text{Mv}^j \quad \text{and} \quad v^j = \text{Mu}^i$$

$$(B) \quad u^i = \text{Mv}^j \quad \text{and} \quad v^j < \text{Mu}^i$$

$$(C) \quad u^i < \text{Mv}^j \quad \text{and} \quad v^j = \text{Mu}^i$$

$$\text{and } (D) \quad u^i < \text{Mv}^j \quad \text{and} \quad v^j < \text{Mu}^i .$$

Figure 4 shows the four cases as μ_1 and μ_2 are varied. The calculations of the values are given in Appendix A.

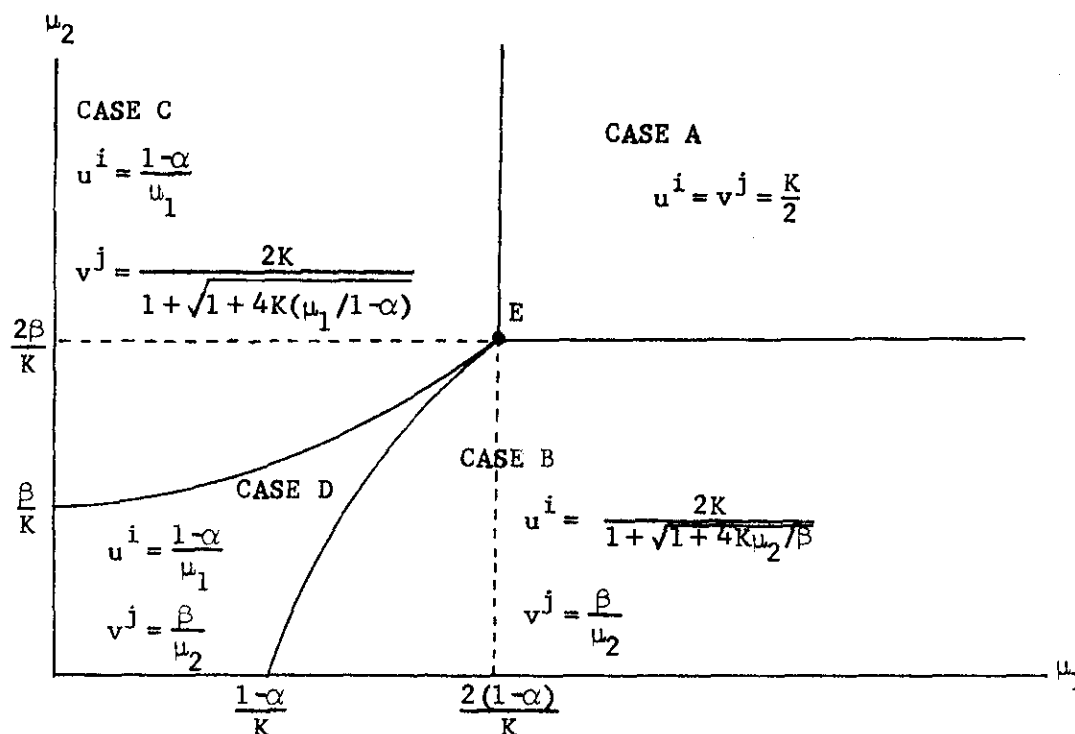


FIGURE 4

In zones A and B traders of type 1 stay solvent; in zones A and C traders of type 2 stay solvent.

We observe that at the point E, $\mu_1 = 2(1-\alpha)/K$ and $\mu_2 = 2\beta/K$ where $K = 2A(1-\alpha)$. Hence $\mu_1 = 1/A = \lambda_1$ and $\mu_2 = \beta/A(1-\alpha) = \lambda_2$ which are the same as the Lagrangian multipliers obtained from solving the competitive equilibrium. Thus for these bankruptcy penalties the limit noncooperative equilibrium coincides with the competitive equilibrium.

For other bankruptcy penalties we see that if the μ_1 and/or the μ_2 are set to be less than λ_1 or λ_2 that one or both types will elect bankruptcy and trade will not be efficient.

2.4.1. A Finite, Infinite Game Solution Distinction

We can see from (35) and (38) that for a finite n hoarding cannot take place, i.e. $\gamma_1, \gamma_2 > 0$. This is not so if there were a continuum of nonatomic traders. If each trader felt that his influence on the aggregates were negligible a new solution appears.

Suppose each trader takes x , y , b , d , and M as given, and chooses his optimal x^i , b^i and u^i accordingly. Then the first order conditions for a maximum (given by equations (32), (34), and (35)) simplify to:

$$(45) \quad x^i = \frac{A\lambda_1^i d - \alpha x}{\lambda_1^i d}$$

$$(46) \quad b^i = \frac{1-\alpha}{\lambda_1^i + \gamma_1^i}$$

$$(47) \quad \gamma_1^i = \frac{1-M}{M} \gamma_1^i .$$

Using (47), (46) becomes:

$$(48) \quad b^i = M \frac{1-\alpha}{\lambda_1^i} .$$

Adding (45) n times and simplifying yields:

$$x^i = \frac{A\lambda_1^i d}{n\alpha + \lambda_1^i d} .$$

From equation (28), $b^i > 0$ implies $\lambda_1^i > 0$, and therefore (47) requires $M \leq 1$ to keep $\gamma_1^i \geq 0$. Note that $\gamma_1^i = 0$ if and only if $M = 1$.

Suppose $M = 1$, then $u^i + v^j = K$. From (48) and an equivalent expression for type 2 traders, $b^i \leq \mu_1^i$ and $\lambda_1^i \leq \mu_1$ then imply:

$$(49) \quad \frac{1-\alpha}{\mu_1} = \frac{\beta}{\mu_2} \leq K .$$

Furthermore, (37) implies that $b^i = d^j$. Therefore,

$$(50) \quad \frac{1-\alpha}{\lambda_1} = \frac{\beta}{\lambda_2} ,$$

which does not violate (49) only if:

$$(51) \quad \frac{1-\alpha}{\mu_1}, \frac{\beta}{\mu_2} \leq \frac{K}{2} .$$

Following the argument in Appendix A, one can verify that $M = 1$ if and only if (51) holds. If (51) does not hold, then the limiting results of the previous section hold. However if (51) is satisfied, not only do both types stay solvent, but the possibility of hoarding occurs. In

this case the following values of x^i , y^j , b^i , d^j , u^i , v^j satisfy the conditions for an equilibrium.

$$(52) \quad x^i = A(1-\alpha) ; \quad y^j = \beta B$$

$$(53) \quad b^i = d^j \geq \max\left(\frac{1-\alpha}{\mu_1}, \frac{\beta}{\mu_2}\right)$$

$$(54) \quad u^i + v^j = K ; \quad u^i, v^j \geq b^i = d^j .$$

Not only may there be an indeterminate solution for u and v , but even when u and v is specified any values of b and d satisfying (53) and (54) will generate an equilibrium.

3. TWO STAGE NONCOOPERATIVE EQUILIBRIA

3.1. The Extensive Form, Information and Strategies

The game is played as follows: First, all individuals simultaneously bid for loans. They obtain no information beyond the size of the loan they have secured. After they have obtained their loans they then all bid simultaneously in the markets for goods.

If the individuals know the size of their own loans, as they know their own bids they can calculate the price of money. And if they all know the size of the total money supply they can calculate the aggregate amount of loans obtained by their competitors.

Let \bar{u}^i signify $u+v-u^i$ and \bar{v}^j signify $u+v-v^j$. Then a strategy for a trader i of type 1 is a number u^i and two functions $x^i = \varphi_1^i(u^i, \bar{u}^i)$, $b^i = \varphi_2^i(u^i, \bar{u}^i)$ and similarly for a trader j of type 2.

If we wished we might assume that trader i is given complete

information as to who has borrowed how much. In this case a strategy would be a number u^i and two functions

$$x^i = \varphi_1^i(u^1, u^2, \dots, u^n; v^1, v^2, \dots, v^n) \text{ and}$$

$$b^i = \varphi_2^i(u^1, u^2, \dots, u^n; v^1, v^2, \dots, v^n) .$$

In the first instance the trader bases his plan on macroeconomic statistics, in the second, on microeconomic detail. The refinement of information in a general noncooperative game could easily create many new equilibrium points associated with a noncooperative solution. We will not consider this possibility in this paper. It is conjectured that in general the competitive equilibrium points of the trading economy associated with the noncooperative game described here can be attained as limit noncooperative equilibrium points making either the macroeconomic or microeconomic assumption concerning information. In the subsequent calculations, the microeconomic assumption is made and it will be seen that (at least without communication) it does not appear to be critical to the analysis.

Fortunately there is a simple way to pick these functions. Assume that u^i and v^j , $i = 1, \dots, n$, $j = 1, \dots, n$ are given. Any individual i will select the pair (x^i, b^i) by maximizing his payoffs in the one stage game. Any individual j will select the pair (y^j, d^j) in the same manner. We may solve for x^i , y^j , b^i , d^j as functions of the u^i and v^j then solve the bidding-for-loans stage of the market by maximizing with respect to (wrt) u^i and v^j .

3.2. Threats and Two Stage Equilibria

In 3.1 we have noted that a strategy in a two stage game can consist of a number in the first period and a function depending upon the information concerning the moves of all others in the first period. The extreme generality of this function enables traders to convey highly implausible threats which may nevertheless give rise to new (and improbable) equilibrium points. Thus in a two stage game, in general, the problem is not with existence of noncooperative equilibria but with a surfeit of them.

Is it possible to distinguish "plausible" or nice equilibria from the others? As yet there does not seem to be a completely general satisfactory way to do so. However, we could use the type of backward solution used in dynamic programming. Unfortunately there are two basic difficulties in doing so. One concerns the information conditions. We need perfect information between the stages so that subgames can be well defined. It is likely that by making use of the special structure of these economic games where it is possible to aggregate moves we could weaken the information requirements, however this is not explored further here. The second difficulty concerns uniqueness. If we wish to replace the second stage games by the values of their noncooperative equilibria we need them either to be unique, or at least we require some sort of indexing scheme which enables us to associate the equilibrium points we select by some natural property such as continuity. Fortunately in the simple example considered here by appealing to continuity we can choose a unique value of the payoff function so that a backward solution is well defined. A further direction for research is to investigate how the properties of the bankruptcy condition might help in generalizing the class of trading games with unique values of the second stage equilibria.

3.3. Symmetric Perfect Equilibria--The Second Stage

In this section we assume that individuals enter the second stage with perfect information. For ease we number equations from (1) on. We fix the u^i and v^j , define $M = nK/u+v$ as before and solve the second stage for a type symmetric noncooperative equilibrium.

The objective of a trader of type 1 is given by:

$$(1) \quad \max U(x^i, b^i) = \alpha \log(A - x^i) + (1-\alpha) \log \frac{b^i y}{b} \\ + \mu_1 \min \left[0, \frac{d}{x} x^i - b^i + (M-1)u^i \right] \\ \text{subject to } b^i \leq Mu^i$$

or:

$$(2) \quad \max_{b^i, x^i} \alpha \log(A - x^i) + (1-\alpha) \log \frac{b^i y}{b} + \lambda_1 \left(\frac{d}{x} x^i - b^i + (M-1)u^i \right) \\ + \gamma_1 (Mu^i - b^i)$$

where γ_1 is a Lagrangian multiplier and λ_1 is a Lagrangian multiplier if $\lambda_1 < \mu_1$. If we assume that $x^i, b^i, d, y > 0$, then the first order conditions are:

$$(3) \quad \frac{\alpha}{A - x^i} = \lambda_1 d \left(\frac{x - x^i}{x^2} \right)$$

$$(4) \quad (1-\alpha) \frac{b - b^i}{bb^i} = \lambda_1 + \gamma_1$$

$$(5) \quad Mu^i \geq b^i ; \quad \gamma_1 \geq 0 ; \quad \gamma_1 (Mu^i - b^i) = 0$$

$$(6) \quad \lambda_1 \geq 0 ; \quad \text{and}$$

$$\frac{d}{x} x^i - b^i + (M-1)u^i > 0 \text{ implies } \lambda_1 = 0 ;$$

$$0 < \lambda_1 < \mu_1 \text{ implies } \frac{d}{x} x^i - b^i + (M-1)u^i = 0 ;$$

$$\frac{d}{x} x^i - b^i + (M-1)u^i < 0 \text{ implies } \lambda_1 = \mu_1 .$$

Using the fact that $x = nx^i$ (3) can be solved for:

$$\alpha nx^i = \lambda_1 d \left(\frac{n-1}{n} \right) (A - x^i)$$

or

$$(7) \quad x^i = \frac{\alpha \lambda_1 d \left(\frac{n-1}{n} \right)}{\alpha n + \lambda_1 d \left(\frac{n-1}{n} \right)} .$$

Using $nb^i = b$, (4) becomes:

$$(8) \quad b^i = \left(\frac{n-1}{n} \right) (1-\alpha) / (\lambda_1 + \gamma_1)$$

or

$$(\lambda_1 + \gamma_1) = \left(\frac{n-1}{n} \right) \frac{1-\alpha}{b^i} .$$

Since $b^i \leq \mu u^i$ (5) and (8) imply that

$$(9) \quad b^i = \min \left(\frac{n-1}{n} \frac{1-\alpha}{\lambda_1}, \mu u^i \right)$$

or

$$b^i = \frac{n-1}{n} \frac{1-\alpha}{\lambda_1} \text{ if } \lambda_1 \geq \left(\frac{n-1}{n} \right) \frac{1-\alpha}{\mu u^i} .$$

We now determine b^i and x^i as functions of d . There are two cases to consider:

Case 1: $\mu_1 \leq \frac{n-1}{n} \frac{1-\alpha}{\text{Mu}^i}$. By equation (9)

$$(10) \quad \boxed{b^i(d) = \text{Mu}^i}$$

for all $u \geq d > 0$ since $\lambda_1 \leq \mu_1$.

Consider then equation (6). Note that by symmetry $\frac{d}{x} x^i = d^j$.

Then

$$(11) \quad d^j > u^i \text{ implies } \lambda_1 = 0$$

$$0 < \lambda_1 < \mu_1 \text{ implies } d^j = u^i$$

$$d^j < u^i \text{ implies } \lambda_1 = \mu_1.$$

Therefore by equation (7)

$$(12) \quad \boxed{\begin{aligned} x^i(d) &= \frac{A\mu_1 d \left(\frac{n-1}{n}\right)}{\alpha n + \mu_1 d \left(\frac{n-1}{n}\right)} \text{ for } d < u \\ 0 < x^i(d) &\leq \frac{A\mu_1 d \left(\frac{n-1}{n}\right)}{\alpha n + \mu_1 d \left(\frac{n-1}{n}\right)} \text{ for* } d = u \\ x^i(d) &\text{ not defined for } d^j > u^i \text{ if } y > 0 \end{aligned}}$$

*The indeterminacy in (12) reflects the fact that as x falls the optimal x^i falls proportionately. This probably is the result of using the log linear utility function. Note that x^i must stay greater than zero. For when x actually reaches zero, the proceeds from sale of x drops from d^i to 0 discontinuously.

Case 2: $\mu_1 > \frac{n-1}{n} \frac{1-\alpha}{\mu_1}$. Let $d^* = (1-M)u^i + \frac{n-1}{n} \frac{1-\alpha}{\mu_1}$. Then it is easy to show that equations (9) and (6) can be satisfied only if:

$$(13) \quad \begin{cases} b^i(d) = \frac{1-\alpha}{\mu_1} \frac{n-1}{n} & \text{for } d \leq d^* \\ b^i(d) = d^j + (M-1)u^i & \text{for } d^* < d \leq u \end{cases}$$

Since b^i determines λ_1 if $b^i < \mu_1 u^i$, we have

$$(14) \quad \begin{cases} x^i(d) = \frac{A\mu_1 d \left(\frac{n-1}{n}\right)}{n\alpha + \mu_1 d \left(\frac{n-1}{n}\right)} & \text{for } d \leq d^* \\ x^i(d) = \frac{A(1-\alpha) \left(\frac{n-1}{n}\right)^2 d}{\alpha(M-1)u + d \left[\alpha + (1-\alpha) \left(\frac{n-1}{n}\right)^2 \right]} & \text{for } d^* < d < u \\ 0 \leq x^i(d) \leq \frac{A(1-\alpha) \left(\frac{n-1}{n}\right)^2}{\alpha M + (1-\alpha) \left(\frac{n-1}{n}\right)^2} & \text{for } d = u \\ x^i(d) \text{ not defined for } d > u \text{ if } y > 0 \end{cases}$$

Note that the second line of equation (14) becomes:

$$(15) \quad x^i(d) = \frac{A(1-\alpha) \left(\frac{n-1}{n}\right)^2}{\alpha + (1-\alpha) \left(\frac{n-1}{n}\right)^2} \text{ for } d^* < d < u \text{ and } M = 1.$$

Figures 5 and 6 describe the results for the two cases as $n \rightarrow \infty$. In Figure 5, we assume $\mu_1 u^i < (1-\alpha/\mu_1)$. Therefore, for every value of $d^j < u^i$ the optimal response for type i traders is to bid $\mu_1 u^i$.

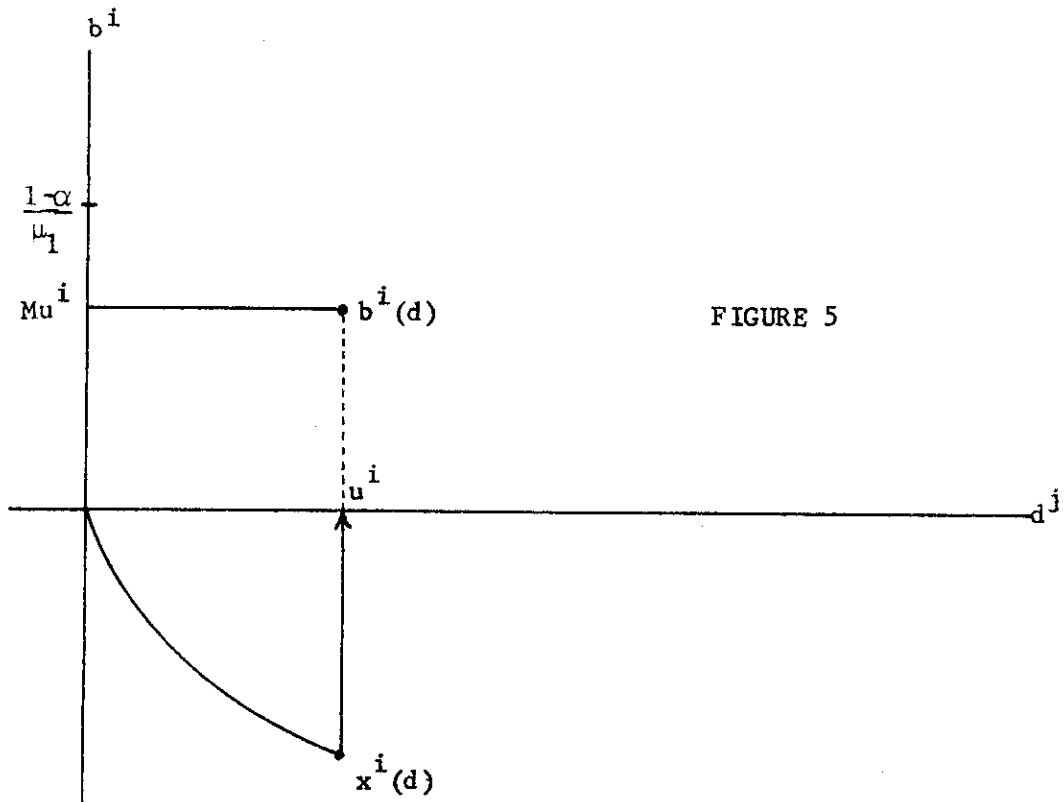


FIGURE 5

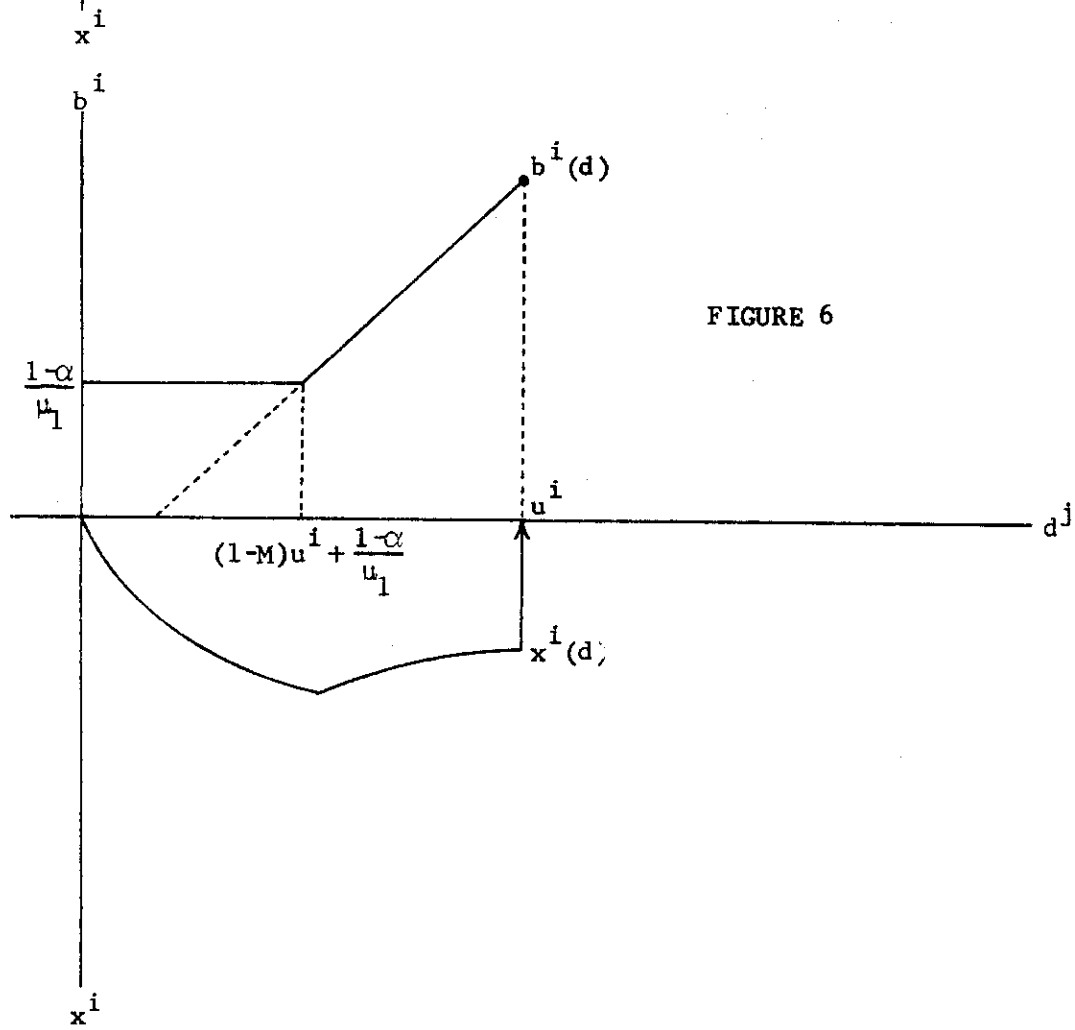


FIGURE 6

However, their optimal supply of x^i will rise with d until $d^j = u^i$. At that point there is a continuum of symmetric optimal responses, from $A\mu_1 u^i / (\alpha + \mu_1 u^i)$ to zero. For larger values of d , there is no optimal offer of x^i . For in that case, if x^i is positive, each consumer receives d^j , which is more money than he needs, and so x^i is reduced. But when $x^i = 0$ their receipts fall discontinuously from d to 0. Note that for all values of $d < u$, type i traders are going bankrupt.

In Figure 6, we assume that $Mu^i > 1 - \alpha / \mu_1$. In this case there are two intervals of d^j which are of interest. For $0 < d^j \leq u^i + (1 - \alpha / \mu_1 - Mu^i)$, the optimal x^i rises with d^j and the optimal $b^i = 1 - \alpha / \mu_1$. In this region, type i traders are going bankrupt, but hoarding occurs (i.e. $Mu^i > b^i$). For $u^i + (1 - \alpha / \mu_1 - Mu^i) < d^j \leq u^i$, x^i falls slightly as d rises when $M < 1$, but when $M = 1$, $x^i(d) = A(1 - \alpha)$ --the same result as under perfect competition. For all values of M , b^i increases at the same rate as d^j . In this region the traders stay solvent, but continue to hoard until $d^j = u^i$. In this case, we again get a continuum of optimal offers, x^i . There is no solution when $d^j > u^i$.

Similar results are obtained for type j traders.

For any given u and v , equilibrium can be found by superimposing corresponding graphs for type i and j traders as illustrated in Figures 7, 8 and 9. Equilibrium occurs where the $b^i(d^j)$ and $d^j(b^i)$ curves intersect. In Figure 7, a case has been constructed where both types are hoarding and both go bankrupt. The equilibrium values of x^i and y^j are given by x_e^i and y_e^j .

In Figure 8, a case has been constructed where type i consumers stay solvent but type j consumers go bankrupt. Both hoard.

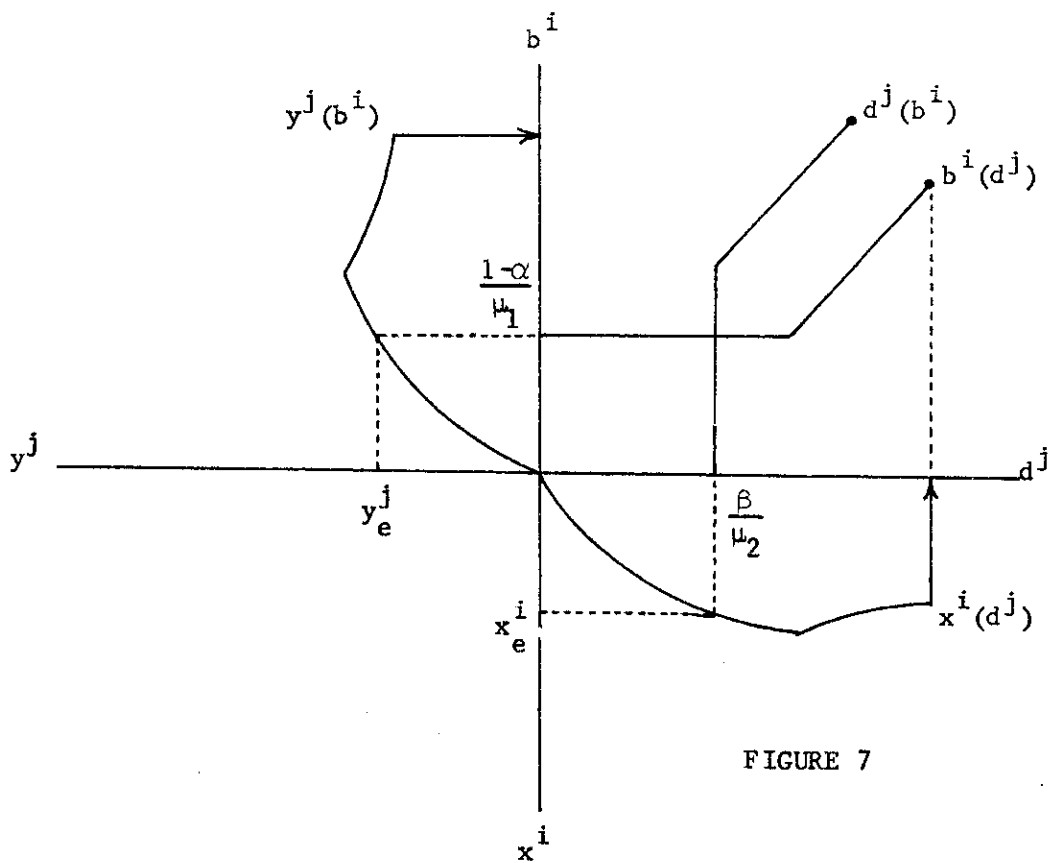


FIGURE 7

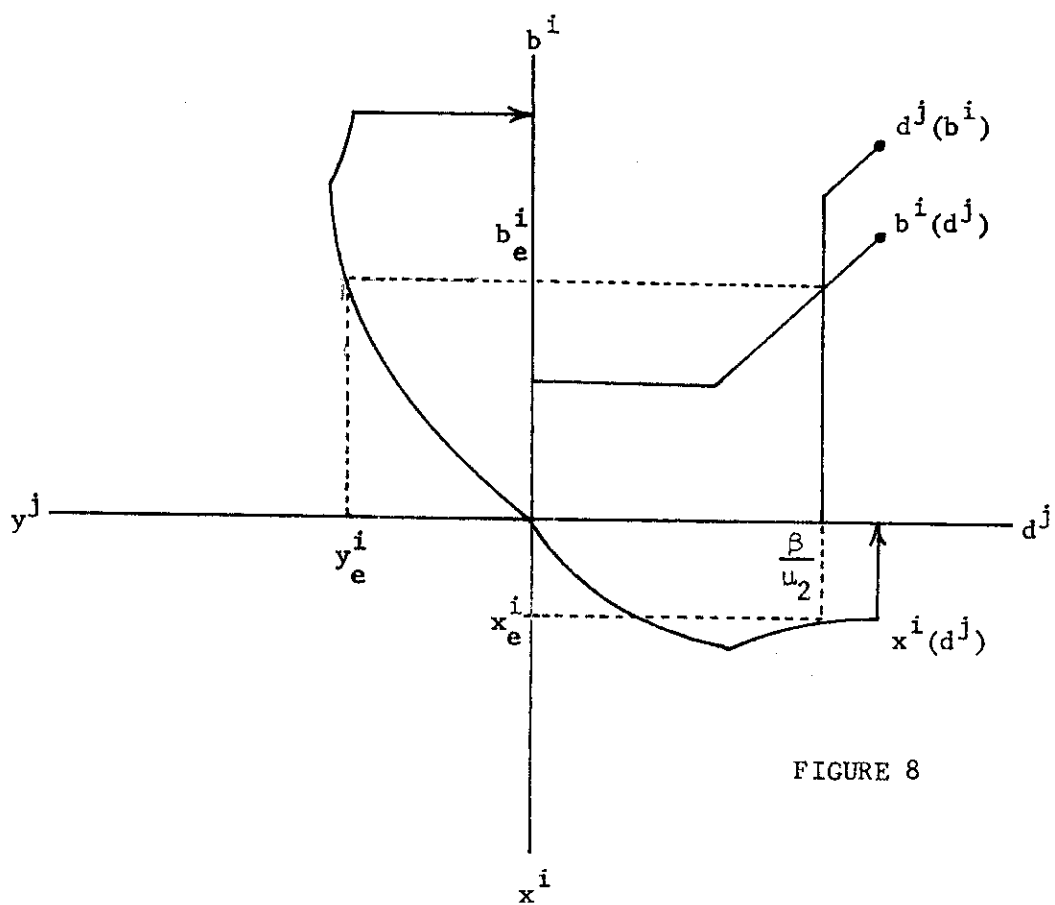


FIGURE 8

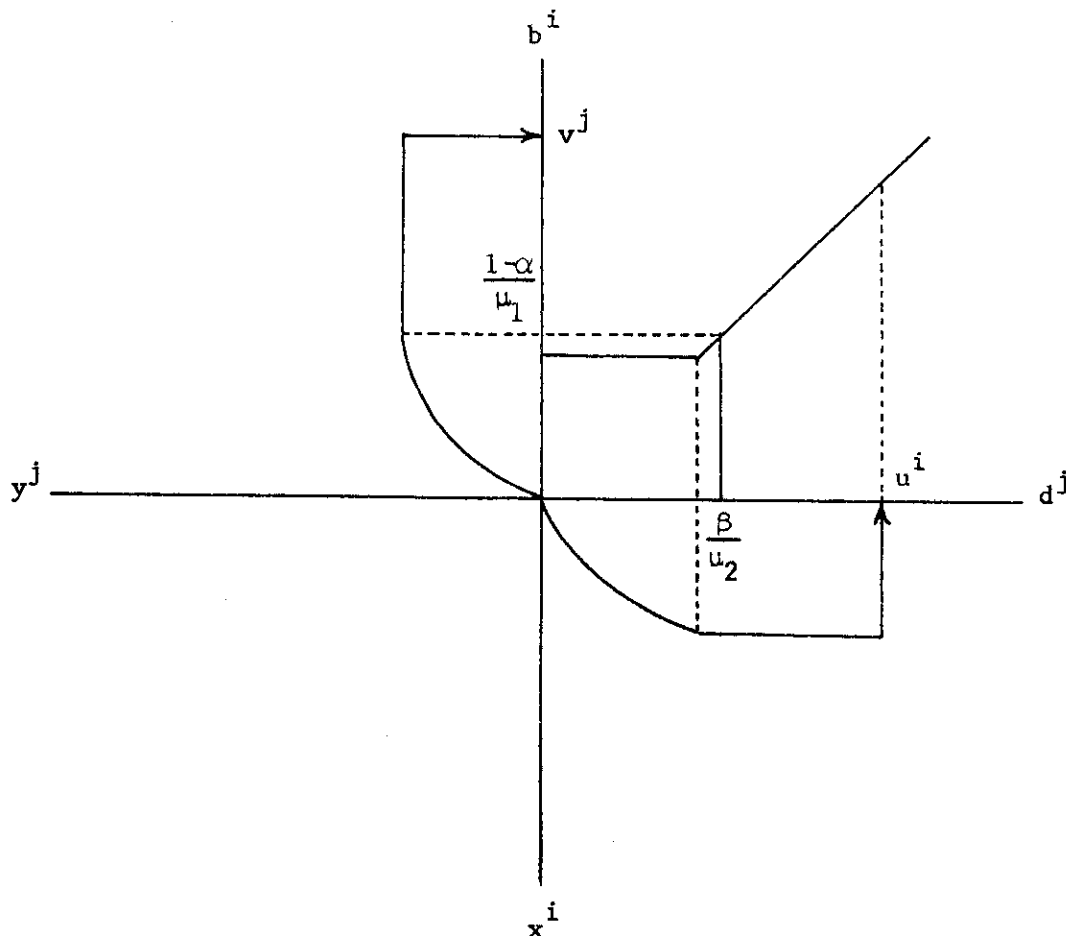


FIGURE 9

In Figure 9, a case is drawn where both types stay solvent. Note that this must imply that $M = 1$. As it has been constructed, a continuum of equilibria are possible since bankruptcy penalties of both types are sufficiently high. Note that as long as hoarding occurs, the equilibrium x^i and y^j is unique. But if, for example, $b^i = d^j = u^i$, then there is a range of equilibrium x^i . This yields an indeterminacy in the solution of the second period game, we will restrict attention only to the largest value of x^i or y^j consistent with equilibrium.

The reader may verify that no non-degenerate equilibrium is possible if $M > 1$.

3.4. The First Stage

When determining the equilibrium bid for money for each type of consumer, a new difficulty emerges. Our discussion of the second stage equilibrium was based on the assumption that traders of the same type start with the same amount of money. In the first stage, however, we want to explore the possibility that an incentive exists for a single trader to change his bid for money. This introduces an asymmetry in the second stage problem, which we have not yet examined. It is possible to compute the change in the solution of the second stage game when one trader changes his money demand in the first stage, but the process is very tedious and tends to be redundant. Therefore, we will not analyze the problem in great detail. Rather, we will sketch the basic results and present an intuitive argument for why they hold. Also, to simplify the exposition, we will only consider the results as $n \rightarrow \infty$.

In general, when trader 1 of type 1 considers changing u^1 , he must consider the effects on not only b^1 and x^1 , but also on d^j , y^j ; as well as x^i , b^i for $i \neq 1$, in order to compute the change in his utility. If all the derivatives exist, then:

$$(16) \quad \frac{dU}{du^1} = -\frac{\alpha}{A-x^1} \frac{\partial x^1}{\partial u^1} + (1-\alpha) \left[\frac{\partial y}{\partial u^1} / y - \frac{\partial b}{\partial u^1} / b + \frac{\partial b^1}{\partial u^1} / b^1 \right] \\ + u_1 \left[\frac{\partial d}{\partial u^1} / n - \frac{d}{nx} \frac{\partial x}{\partial u^1} + \frac{d}{x} \frac{\partial x^1}{\partial u^1} - \frac{\partial b^1}{\partial u^1} - \frac{M}{n} \frac{u}{u+v} + M - 1 \right].$$

Furthermore, it can be shown that $\partial y / \partial u^1 / y \rightarrow 0$, $\partial b / \partial u^1 / b \rightarrow 0$, $\partial d / \partial u^1 / n \rightarrow 0$, $\partial x / \partial u^1 d / xn \rightarrow 0$, and $M/n u / u+v \rightarrow 0$ as $n \rightarrow \infty$. Therefore, for n large (16) can be approximated by:

$$(17) \quad \frac{\partial u}{\partial u^1} = -\frac{\alpha}{A-x^1} \frac{\partial x^1}{\partial u^1} + (1-\alpha) \frac{\partial b^1}{\partial u^1} / b^1 + \mu_1 \left[\frac{d}{dx} \frac{\partial x^1}{\partial u^1} - \frac{\partial b^1}{\partial u^1} + M - 1 \right].$$

We can also show that as $n \rightarrow \infty$:

$$(a) \quad b^1 \leq \frac{1-\alpha}{\mu_1} \frac{n-1}{n} \text{ implies } \frac{\alpha}{A-x^1} + \mu_1 \left(\frac{d}{dx} \frac{\partial x^1}{\partial u^1} \right) \rightarrow 0.$$

This result follows from the fact that when $b^1 < \frac{1-\alpha}{\mu_1} \frac{n-1}{n}$, type i traders are going bankrupt. Hence consumers choose x^i to equate the gain in utility with the increase in the bankruptcy penalty from a fall in x^i .

$$(b) \quad b^1 > \frac{1-\alpha}{\mu_1} \frac{n-1}{n} \text{ implies } \frac{d}{dx} \frac{\partial x^1}{\partial u^1} - \frac{\partial b^1}{\partial u^1} + M - 1 \rightarrow 0$$

$$\text{and } -\frac{\alpha}{A-x^1} \frac{\partial x^1}{\partial u^1} + \frac{1-\alpha}{b^1} \frac{\partial b^1}{\partial u^1} \rightarrow \frac{1-\alpha}{b^1} (M-1).$$

The first result follows from that fact that since type i traders are staying solvent, x^i , b^i , and u^i must change so that the solvency condition is maintained. The second follows from utility maximization at the second stage of the game and the fact that $(1-M)$ additional units of money must be available at the end of period 2 when u^1 increases by one unit, if the trader is to stay solvent.

Substituting these results into equation (17), and using the fact that $b^1 = \mu u^1$ when $\mu u^1 \leq \frac{1-\alpha}{\mu_1} \frac{n-1}{n}$, we get:

$$(18) \quad \left\{ \begin{array}{l} \frac{\partial U}{\partial u^i} \rightarrow \frac{1-\alpha}{u^i} - \mu_1 \quad \text{for } Mu^i \leq \frac{1-\alpha}{\mu_1} \\ \frac{\partial U}{\partial u^i} \rightarrow \frac{1-\alpha}{b^i} (M-1) \quad \text{for } Mu^i > \frac{1-\alpha}{\mu_1} . \end{array} \right.$$

Similar results hold for type 2 traders:

$$(19) \quad \left\{ \begin{array}{l} \frac{\partial V}{\partial v^j} \rightarrow \frac{\beta}{v^j} - \mu_2 \quad \text{for } Mv^j \leq \frac{\beta}{\mu_2} \\ \frac{\partial V}{\partial v^j} \rightarrow \frac{\beta}{d^j} (M-1) \quad \text{for } Mv^j > \frac{\beta}{\mu_2} . \end{array} \right.$$

The preceding results hold when the outcome of the second period game changes continuously with u^i or v^j . At certain boundary points, however, the outcome changes discontinuously.

We have already argued that $x^i, y^i > 0$ implies that $dx^i/x - b^i + (M-1)u^i \leq 0$. Summing over all i , we get:

$$(20) \quad d - b + (M-1)u \leq 0 .$$

Now suppose that $\beta/\mu_2 > Mv^j = u^i$. Then from our analysis in section 3.2, $d^i = Mv^j$. Suppose, then that either v^j rises or u^i falls. Then in either case, $d = Mv$ will rise, and u will either remain unchanged or fall. But this will imply that $d > u$, and hence $b \leq Mu$ implies that (20) will be violated.

Therefore, if $Mv^j = u^i < \beta/\mu_2$, there is a discontinuous fall in the payoffs to both players as u falls or as v rises. A similar result holds if $Mu^i = v^j < (1-\alpha)/\mu_1$.

Using the relations in (3) and (4) and the discussion which follows, it is possible to describe all of the equilibria associated with each

μ_1 , μ_2 . The first point to note is that $u^i = v^j = K/2 = b^i = d^j$ is always an equilibrium with no bankruptcy. However, the allocation of x^i and y^j may be smaller than that associated with the competitive equilibrium if μ_1 and μ_2 are sufficiently small. This equilibrium always arises because: (1) no trader has an incentive to change u^i or v^j ; or (2) any change in u^i or v^j by only one trader will necessarily result in a degenerate solution in the second stage. Secondly, unless both types go bankrupt, a continuum of equilibrium will generally result. Using the classification of cases described in section 2.4, the results are summarized in Figure 10.

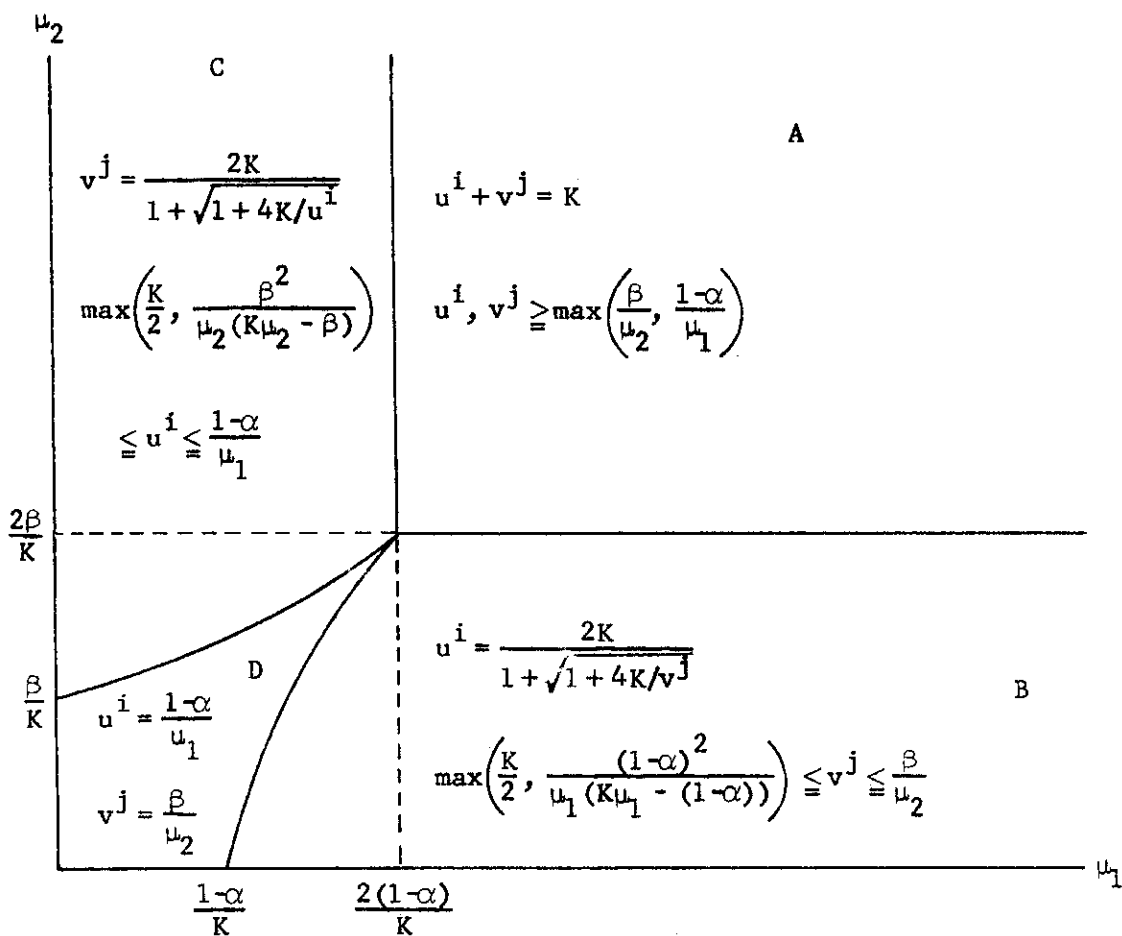


FIGURE 10

In case D, the result is the same as in section 2.4. In all other cases, there is generally a continuum of equilibria with the equilibrium of section 2.4 as a special case. In cases A, B and C, the multiplicity of equilibria reflects an awareness on the part of traders that under some circumstances a small change in u^i or v^j will induce a discontinuous jump to the degenerate equilibrium. That awareness was absent in the case of no information discussed in section 2.4. In case A, whenever u^i and v^j is greater than $\max(1-\alpha/\mu_1, \beta/\mu_2)$, there is the additional indeterminacy discussed in the previous section where the traders hoard, but both remain solvent. No hoarding ever occurs in cases B, C, or D.

4. SO WHAT?

In this paper we have labored through a large amount of tedious calculation on a special example in order to at least begin to disclose the fine structure of a general class of economic problems that we believe are of importance.

In particular a modern economy which uses a paper money or some other symbolic credit instrument¹⁴ for payments must have rules to specify what happens when debts cannot be repaid. Any positive penalty provides some support for the value of the original paper. However as we have shown here, and is generally true, penalties which are equal to the Lagrangian multipliers in the nonmonetary Walrasian model lead to limit noncooperative equilibria in the monetary model which converge to the same distribution and prices as the Walrasian model. Any other bankruptcy penalties may give different results--particularly if set too low. These different results are illustrated in the welter of cases we have worked out here in this example.

There are two items of note concerning this type of model of trade. The first is that the price level is not homogeneous of order zero with respect to the money supply. In order to leave the set of equilibria unchanged except for price level a covariant change in the money supply and the bankruptcy penalties must take place. For example, if the money supply is doubled the bankruptcy penalty must be halved.

If the bankruptcy penalties are picked so that they coincide with the Lagrangian multipliers associated with some particular equilibrium in the Walrasian model the limit equilibrium of the noncooperative game will be that particular equilibrium point. Thus it appears that the money supply and penalty settings provide a way to guide a market economy to a specific equilibrium. However when this is considered from the viewpoint of computation and information content this appears to indicate that a market economy with an outside bank requires as much calculation to guide it as does an economy with a central planning board using the Scarf¹⁶ or Smele methods for calculating a price system. The calculation of the appropriate μ 's appears as the dual of the calculation of the prices.

The model here appears strange to those used to thinking about the one period general equilibrium model in the sense that in that model a money or means of payment and a cost of money or interest rate do not seem to have any role. Here because of decentralized or simultaneous bidding⁹ a float must be financed.^{10, 11} We in fact obtain a "zero rate of interest" when the penalties are set appropriately.

A better model which will capture intertemporal trade and the role of money in financing this trade requires at least two periods, an outside bank and an inside bank. In this model it is conjectured that the outside bank money will still bear a zero rate of interest, but the inside

bank money may have a positive rate of interest corresponding to the appropriate rate to finance intertemporal trade efficiently. This model will be presented in a subsequent paper.*

Although our results appear to show a computational equivalence between a market economy and one with centralized prices, this does not take into account differences which might exist at the level of communication, bureaucracy and the incentive system for agents or fiduciaries. Thus it should not be construed as showing a full equivalence between the systems.

*There are extra difficulties encountered in defining short term profits, bad loans and roll over conditions on loans, as well as defining the strategies, payoffs and bank failure rules for the inside bank.

APPENDIX A

To compute u^i and v^j , there are four cases to consider

- (A) $u^i = Mv^j$; $v^j = Mu^i$
- (B) $u^i = Mv^j$; $v^j > Mu^i$
- (C) $u^i > Mv^j$; $v^j = Mu^i$
- (D) $u^i > Mv^j$; $v^j > Mu^i$.

Case A: $u = Mv = MMu \implies M = 1 \implies u^i + v^j = K$. Therefore:

(1)
$$u^i = v^j = K/2$$

Therefore: $\frac{1-\alpha}{\lambda_1} = \frac{\beta}{\lambda_2} = \frac{K}{2}$. But $\lambda_1 \leq \mu_1$ and $\lambda_2 \leq \mu_2$ require:

(2)
$$\frac{1-\alpha}{\mu_1} \leq \frac{K}{2} ; \frac{\beta}{\mu_2} \leq \frac{K}{2}$$

So (1) holds only if (2) holds.

Case B: $Mu^i > v^j \implies \lambda_2 = \mu_2$ and from (41) in 2.4 $u^i = Mv^j$ implies:

$$K \left(\frac{\beta}{\mu_2} \right) = \left(\frac{1-\alpha}{\lambda_1} \right) \left(\frac{1-\alpha}{\lambda_1} + \frac{\beta}{\mu_2} \right)$$

or

$$\lambda_1 = \frac{1-\alpha}{2K} \left(1 + \sqrt{1 + 4K\mu_2/\beta} \right) .$$

Therefore:

$$(3) \quad \begin{cases} u^i = \frac{2K}{1 + \sqrt{1 + 4K\mu_2/\beta}} \\ v^j = \beta/\mu_2 \end{cases}$$

But $\lambda_1 \leq \mu_1$. Therefore, if (3) holds:

$$(4) \quad \left\{ \begin{array}{l} \mu_1 \geq \frac{1-\alpha}{2K}(1 + \sqrt{1 + 4K\mu_2/\beta}) \\ \text{and } M(u^i + v^j) < (u^i + v^j) \implies K < \frac{1-\alpha}{\lambda_1} + \frac{\beta}{\mu_2} \\ \text{So: } \mu_2 < \frac{2\beta}{K} \end{array} \right.$$

Case C: By symmetry $\lambda_1 = \mu_1$; $v^j = \mu u^i$

$$(5) \quad v^i = \frac{2K}{1 + \sqrt{1 + 4K(\mu_1/1-\alpha)}}; \quad u^i = \frac{1-\alpha}{\mu_1}$$

only if

$$(6) \quad \mu_2 \geq \frac{\beta}{2K}(1 + \sqrt{1 + 4K(\mu_1/1-\alpha)}) \quad ; \quad \mu_1 < \frac{2(1-\alpha)}{K}$$

Case D:

$$(7) \quad \begin{cases} u^i = \frac{1-\alpha}{\mu_1}; \quad v^j = \frac{\beta}{\mu_2} \\ \mu_1 = \lambda_1; \quad \mu_2 = \lambda_2 \end{cases}$$

But for (7) to hold; (41) and (D) implies:

$$\frac{\mu_1 \beta K}{\mu_2 (1-\alpha) + \mu_1 \beta} < \frac{1-\alpha}{\mu_1}$$

$$\frac{\mu_2 (1-\alpha) K}{\mu_2 (1-\alpha) + \mu_1 \beta} < \frac{\beta}{\mu_2}$$

or

$$(8) \quad \boxed{\begin{aligned} \mu_1 &< \frac{1-\alpha}{2K} (1 + \sqrt{1+4(\mu_2/\beta)K}) \\ \mu_2 &< \frac{\beta}{2K} (1 + \sqrt{1+4(\mu_1/(1-\alpha))K}) \end{aligned}}$$

It is easy to check that (8) implies:

$$(9) \quad \left\{ \begin{aligned} \mu_1 &< \frac{1-\alpha}{2K} \implies u^i > \frac{K}{2} \\ \mu_2 &< \frac{\beta}{2K} \implies v^j > \frac{K}{2} \end{aligned} \right.$$

Using the expression for x^i in (36) of 4.2 and letting $n \rightarrow \infty$ we can show:

$$x^i(\mu_1, \mu_2) = A(1-\alpha) \quad \text{if } (\mu_1, \mu_2) \in \text{area A or B (page 15)}$$

$$\frac{\partial x^i}{\partial \mu_2} < 0 ; \quad \frac{\partial x^i}{\partial \mu_1} > 0 \quad \text{if } (\mu_1, \mu_2) \in \text{area D}$$

$$x^i(\mu_1, \mu_2) = A \frac{\mu_1^2 K (1 + \sqrt{1+4K(\mu_1/(1-\alpha))})}{(\alpha(1-\alpha) + \mu_1^2 K) (1 + \sqrt{1+4K(\mu_1/(1-\alpha))}) + \alpha \mu_1} < A(1-\alpha)$$

if $(\mu_1, \mu_2) \in \text{area C} .$

Similar results hold for y^j .

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