COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 412

A GENERAL EQUILIBRIUM MODEL OF WORLD TRADE

PART I

FULL FORMAT COMPUTATION OF ECONOMIC EQUILIBRIA

Victor Ginsburgh and Jean Waelbroeck

November 18, 1975

A GENERAL EQUILIBRIUM MODEL OF WORLD TRADE

PART I

FULL FORMAT COMPUTATION OF ECONOMIC EQUILIBRIA

by

Victor Ginsburgh and Jean Waelbroeck*

This paper presents a method of computing general equilibria which is applicable to large models. It has been applied to a realistic model of the world economy described in [14].

The concept of equilibrium is, of course, important and many large models have been constructed, which faithfully represent the concept of partial equilibrium (e.g., Takayama and Judge [29], Duloy and Norton [6]). We do not know however of any large-scale realistic model which reflects all the complex interrelationships dealt with by general equilibrium theory. Yet such a model would provide a natural way of formulating numerous problems in international trade (see [9],[10]) and income distribution. It could be a practical tool for evaluating the impact of price distortions, such as tariffs and taxes (see [25], [26], [27], [31]) on income distribution and on production and consumption.

*University of Brussels and Center for Operations Research and Econometrics, Louvain. The paper was written while V. Ginsburgh was visiting the Cowles Foundation for Research in Economics, sponsored by the Ford Foundation, whose generous support is gratefully acknowledged, and while J. Waelbroeck was with the International Bank for Reconstruction and Development, Washington.

In Section 1 we recall fundamental concepts and theorems of general equilibrium theory to define the problem at hand and to classify possible computational approaches. Section 2 presents the theoretical foundations of our approach. Section 3 discusses the possibility of solving large general equilibrium problems by means of Scarf's algorithm or improved fixed point algorithms, while Section 4 examines whether the size of the problem can be reduced to save computational costs. Section 5 presents the three computational procedures which we have used. Section 6 indicates how general equilibrium can be formulated without loss of generality in a linear framework; finally Section 7 discusses our experience in applying the procedures to a large scale model.

1. Basic Concepts and Theorems of General Equilibrium Theory

Let x_i and w_i be r-vectors of consumption and initial endowments held by consumer i $(i = 1, 2, \ldots, m)$ and let y_j be an r-vector of production by producer j $(j = 1, 2, \ldots, n)$. Let X_i denote the consumption set of i , and Y_j the production set of j . Assume that the preferences of consumer i can be represented by a continuous real valued utility function $U_i(x_i)$. Finally let the r-vector p represent prices and θ_{ij} be the proportion of profit of firm j distributed to consumer i $(\theta_{ij} \geq 0 ,\ \sum_i \theta_{ij} = 1$ for $j = 1, 2, \ldots, n)$.

Then we define:

Definition 1.1. Competitive Equilibrium: The allocation $\{\bar{x}_i\}$, $\{\bar{y}_j\}$ supported by the prices $\bar{p}$, is a competitive equilibrium if the following conditions hold:

(a) Equality of demand and supply for non-free goods:

$$\sum_i \bar{x}_i - \sum_j \bar{y}_j - \sum_i w_i \leq 0 \ ; \quad \bar{p}\cdot(\sum_i \bar{x}_i - \sum_j \bar{y}_j - \sum_i w_i) = 0 \ ;$$

(b) utility maximization of consumers: $\bar{x}_i$ maximizes $U_i(x_i)$ subject to the budget constraint $\bar{p}\cdot x_i \leq \bar{p}\cdot w_i + \sum_j \theta_{ij}\bar{p}\cdot y_j$, and the feasibility constraints $x_i \in X_i$, for $i = 1, 2, \ldots, m$;

(c) profit maximization of firms: $\bar{y}_j$ maximizes $\bar{p}\cdot y_j$ subject to $y_j \in Y_j$ for $j = 1, 2, \ldots, n$.

Let us make the usual assumptions: (1) $U_i(x_i)$ is a continuous real valued increasing quasi-concave function; (2) the sets Y_j are convex, closed and $0 \in Y_j$; their vector sum $Y = \Sigma_j Y_j$ satisfies $Y \cap B = 0$

with B the closed positive orthant and $Y \cap (-Y) = 0$; (3) X_i is a convex, closed subset of R^r_+ ; (4) there exists an $x^0_i \in X_i$ such that $x^0_i < w_i$, all i . It can then be shown that:

Theorem 1.1. If the assumptions (1) to (4) hold, there exists a competitive equilibrium.

The proof proceeds by noting that utility and profit maximizing allocations satisfying (b) and (c) do not necessarily satisfy the balance condition (a). It then constructs an excess demand correspondence as $D(p) \ni d(p) = \Sigma w_i - \Sigma x_i + \Sigma y_j$, where x_i and y_j are consumer and production allocations which maximize utility and profits respectively at prices p . The mapping

$$h_k(p) = \frac{p_k + \max\{-d_k(p),\ 0\}}{1 + \sum_{\ell} \max\{-d_\ell(p),\ 0\}}$$

can be shown to have a fixed point $\bar{p}$ which obviously satisfies $d(\bar{p}) \geq 0$ so that conditions (a), (b), and (c) are satisfied by $\bar{p}$ and the corresponding allocations $\{\bar{x}_i\}$, $\{\bar{y}_j\}$.

Definition 1.2. Welfare Optimum: Let $W[U(x)] = W[U_1(x_1), \ldots, U_m(x_m)]$ be a concave increasing function of the $U_i(x_i)$ and consider the mathematical program $\max W[U(x)]$ subject to $\Sigma x_i - \Sigma y_j + \Sigma w_i \leq 0$, $x_i \in X_i$ $(i = 1, \ldots, m)$, $y_j \in Y_j$ $(j = 1, \ldots, n)$. An allocation $\{x^*_i\}$, $\{y^*_j\}$ which maximizes the objective function subject to the constraints is a welfare optimum.

Negishi [19] has considered the important special case where the welfare function is linear in the utilities of agents

$$W[U(x)] = \sum_{i=1}^{m} \alpha_i U_i(x_i) \ , \ (\alpha_i > 0 \ , \quad \Sigma \alpha_i = 1)$$

and where the utility functions $U_i(x_i)$ are concave rather than quasi-concave.[1]

Definition 1.3. Negishi Welfare Optimum: The allocation resulting from max $\Sigma \alpha_i U_i(x_i)$ subject to the same constraints as in Definition 1.2 will be called a Negishi Welfare Optimum (NWO). The α_i will be called welfare weights.

Theorem 1.2. Let the assumptions (1) to (4) be satisfied and let the welfare function have the properties assumed by Negishi. Then,

(1) For any set of admissible weights α_i , there exists a solution to the NWO satisfying conditions (a) and (c) of the competitive equilibrium.

(2) (Negishi [19]). There exists at least one set of strictly positive α_i for which the solution of the welfare optimum and its dual satisfy conditions (a), (b), and (c) of the competitive equilibrium.

---

[1]This is more restrictive than the general assumption of quasi-concavity although, as Diewert points out in [2, p. 120], "from an empirical point of view, it is impossible to distinguish concave from quasi-concave preferences."

The proof proceeds by noting that any solution of an NWO satisfies all the equilibrium conditions, but not necessarily the budget constraints. It then constructs an excess budget correspondence $B_i(\alpha) \ni b_i(\alpha) = p\cdot(w_i + \sum_j \theta_{ij}y_j - x_i)$ where $\{x_i\}$, $\{y_j\}$ are optimal consumer and producer allocations, corresponding to the welfare weights α and p are Lagrangean multipliers associated with the constraints $\sum x_i - \sum y_j - \sum w_i \leq 0$. The mapping

$$g_i(\alpha) = \frac{\alpha_i + \max\{-b_i(\alpha),\ 0\}}{\sum_\ell [\alpha_\ell + \max\{-b_i(\alpha),\ 0\}]}$$

can be shown to have a fixed point $\bar{\alpha}$ which satisfies $b(\bar{\alpha}) \geq 0$, so that there exists an optimal solution $\{\bar{x}_i\}$, $\{\bar{y}_j\}$ and the associated multipliers $\bar{p}$ which satisfy all the equilibrium conditions.

These definitions and theorems suggest a possible classification of approaches to the computation of general equilibra.

(a) We will designate as direct computational methods procedures which search for a $\bar{p}$ such that $d(\bar{p}) \geq 0$.

(b) Indirect computation,[2] on the other hand, involves searching for the welfare weight vector such that $b(\bar{\alpha}) \geq 0$.

Each of these approaches may furthermore be undertaken in two ways:

(a) It is possible to grapple with the general equilibrium as a whole, in its direct or indirect form. This will be described as full format calculation.

---

[2] This terminology differs from Dixon's, who uses the term "joint maximization" instead of "indirect computation." It would be awkward to adopt his terminology, since we use a "joint maximizing" procedure to find both equilibrium p's and α's .

(b) It is possible to seek to reduce the format of the problem by obtaining analytical expressions representing the excess demand and excess budget correspondences $d(p)$ and $b(\alpha)$ and then to solve the systems $d(p) \geq 0$ or $b(\alpha) \geq 0$. This is the reduced format computational method.

2. The Master Program of the General Equilibrium Problem

The concept of the master program of the general equilibrium problem provides a useful way of clarifying the relation between different approaches to the computation of equilibria.

Definition 2.1. Master Program: The mathematical program $\max \sum_i \alpha_i U_i(x_i)$ $(\alpha_i > 0)$ subject to $\Sigma x_i - \Sigma w_i - \Sigma y_j \leq 0$, $\bar{p}\cdot(x_i - \sum_j \theta_{ij} y_j - w_i) \leq 0$, $x_i \in X_i$, $y_j \in Y_j$ for all i and j will be referred to as the master program of the general equilibrium problem. Remark that the master program is simply the Negishi welfare optimum further constrained by the conditions $\bar{p}\cdot(x_i - \Sigma \theta_{ij} y_j - w_i) \leq 0$.

Theorem 2.1. Consider the set of solutions $\{\bar{x}_i\}$, $\{\bar{y}_j\}$, $\bar{u}$, $\bar{\lambda}$ of the master program, where $\bar{u}$ and $\bar{\lambda}$ are the Lagrangean multipliers associated with the constraints $\Sigma x_i - \Sigma y_j - \Sigma w_i \leq 0$ and $\bar{p}\cdot(x_i - \sum_j \theta_{ij} y_j - w_i) \leq 0$, respectively. If $\bar{u} = k\bar{p}$, where k is a positive scalar, then for any choice of α the allocations $\{\bar{x}_i\}$, $\{\bar{y}_j\}$ and the prices $\bar{p}$ are a competitive equilibrium.

Proof. In view of the Kuhn-Tucker-Uzawa Theorem (see [20], p. 52), if $\{\bar{x}_i\}$, $\{\bar{y}_j\}$ is a solution, there exist $\bar{u} \geq 0$, $\bar{\lambda} \geq 0$ such that

$$(2.1)\qquad \Sigma\,\bar{x}_i - \Sigma\,\bar{y}_j - \Sigma\,w_i \le 0 \quad ; \quad \bar{u}\cdot(\Sigma\,\bar{x}_i - \Sigma\,\bar{y}_j - \Sigma\,w_i) = 0$$

$$(2.2a\text{-}b)\qquad \bar{p}\cdot(\bar{x}_i - \sum_j \theta_{ij}\bar{y}_j - w_i) \le 0 \;;\; \bar{\lambda}_i\bar{p}\cdot(\bar{x}_i - \sum_j \theta_{ij}\bar{y}_j - w_i) = 0\,, \quad i = 1, \ldots, m$$

and for $x_i \in X_i$ all i, $y_j \in Y_j$ all j,

$$(2.3)\qquad \Sigma\,\alpha_i U_i(x_i) + \bar{u}\cdot(\Sigma\,w_i + \Sigma\,y_j - \Sigma\,x_i) + \sum_i \bar{\lambda}_i\bar{p}\cdot(w_i + \Sigma\,\theta_{ij}y_j - x_i)$$
$$\le \Sigma\,\alpha_i U_i(\bar{x}_i) + \bar{u}\cdot(\Sigma\,w_i + \Sigma\,\bar{y}_j - \Sigma\,\bar{x}_i) + \sum_i \bar{\lambda}_i\bar{p}\cdot(w_i + \Sigma\,\theta_{ij}\bar{y}_j - \bar{x}_i)$$
$$\le \Sigma\,\alpha_i U_i(\bar{x}_i) + u\cdot(\Sigma\,w_i + \Sigma\,\bar{y}_j - \Sigma\,\bar{x}_i) + \sum_i \lambda_i\bar{p}\cdot(w_i + \Sigma\,\theta_{ij}\bar{y}_j - \bar{x}_i)\,.$$

Condition (a) of the equilibrium follows immediately from (2.1).

From (2.2.a) we see that the budget constraints are also satisfied. If we show that $U_i(x_i) \le U_i(\bar{x}_i)$ for every i, condition (b) of the equilibrium will also be satisfied. If $\bar{u} = k\bar{p}$, note first that $k\bar{p}\cdot\sum_i(w_i + \sum_j \theta_{ij}\bar{y}_j - x_i) = \bar{u}\cdot(\sum_i w_i + \sum_j \bar{y}_j - \sum_i \bar{x}_i) = 0$. Together with (2.2.a), this implies that for every i,

$$(2.4)\qquad \bar{p}\cdot(w_i + \sum_j \theta_{ij}\bar{y}_j - \bar{x}_i) = 0\,.$$

Now set $y_j = \bar{y}_j$ for all j, $x_i = \bar{x}_i$ for all i except s and $\bar{u} = k\bar{p}$; the first inequality in (2.3) implies then, for every $x_s \in X_s$

$$(2.5)\qquad \alpha_s U_s(x_s) + (k + \bar{\lambda}_s)\bar{p}\cdot(\bar{x}_s - x_s) \le \alpha_s U_s(\bar{x}_s)\,.$$

If x_s satisfies the budget constraint of equilibrium condition (b), then, using (2.4), it is easy to check that the second term on the left hand side of (2.5) is non-negative, and hence, since $\alpha_s > 0$, $U_s(x_s) \le U_s(\bar{x}_s)$ for all consumer allocations which can be attained by consumer s.

Finally, set $y_j = \bar{y}_j$ for all j except s, $x_i = \bar{x}_i$ for all i and $\bar{u} = k\bar{p}$; the first inequality in (2.3) implies, for every $y_s \in Y_s$

$$(2.6) \qquad (k + \sum_i \bar{\lambda}_i \theta_{is})\bar{p}\cdot y_s \leq (k + \sum_i \bar{\lambda}_i \theta_{is})\bar{p}\cdot \bar{y}_s \ .$$

Since $(k + \sum_i \bar{\lambda}_i \theta_{is}) > 0$, clearly $\bar{p}\cdot y_s \leq \bar{p}\cdot \bar{y}_s$ which proves condition (c) of the equilibrium.

The following two corollaries clarify the theoretical implications of Theorem 2.1.

<u>Corollary 2.1.1</u> (tâtonnement in welfare weights space). The vectors $\{\bar{x}_i\}$, $\{\bar{y}_j\}$, $\bar{u} = k\bar{p}$ obtained by solving the master program are an equilibrium solution of the Negishi welfare optimum problem $\max \sum \alpha_i^* U_i(x_i)$, subject to $\sum_i x_i - \sum_j y_j - \sum_i w_i \leq 0$, $x_i \in X_i$ for all i, $y_j \in Y_j$ for all j, where the a_i^* are given the value $\alpha_i^* = \alpha_i : (1 + \bar{\lambda}_i/k)$.

<u>Proof</u>. We will verify that all vectors $x_i \in X_i$, $y_j \in Y_j$, $u \geq 0$ verify the saddle point inequalities

$$(2.7) \qquad \sum \alpha_i^* U_i(x_i) + \bar{u}\cdot(\sum w_i + \sum y_j - \sum x_i) \leq \sum \alpha_i^* U_i(\bar{x}_i) + \bar{u}\cdot(\sum w_i + \sum \bar{y}_j - \sum \bar{x}_i)$$
$$\leq \sum \alpha_i^* U_i(\bar{x}_i) + u\cdot(\sum w_i + \sum \bar{y}_j - \sum \bar{x}_i) \ .$$

That the second inequality of (2.7) is verified follows from the fact that $\{\bar{x}_i\}$, $\{\bar{y}_j\}$, $\bar{u}$ and $\bar{\lambda}$ verify the second inequality of (2.3). If the first inequality does not hold, then there exists one $y_s \in Y_s$ or one $x_s \in X_s$ such that $\bar{u}y_s > \bar{u}\bar{y}_s$ or $\alpha_s^* U_s(x_s) - \bar{u}\cdot x_s > \alpha_s^* U_s(\bar{x}_s) - \bar{u}\cdot\bar{x}_s$. The first possibility is excluded since $\bar{y}_s$ is a profit maximizing allocation at prices $\bar{p} = \frac{1}{k}\bar{u}$ for $s = 1, 2, \ldots, n$. Assume that the second possibility holds; then, given the definition of α_s^* , it must follow that $\alpha_s U_s(x_s) - (\bar{u} + \lambda_s \bar{p})\cdot x_s > \alpha_s U_s(\bar{x}_s) - (\bar{u} + \lambda_s \bar{p})\cdot \bar{x}_s$. Setting $y_j = \bar{y}_j$ for all j and $x_i = \bar{x}_i$ for all i except s , this contradicts the first saddle point condition of (2.3).

Corollary 2.1.2 (tâtonnement in prices space). The vectors $\{\bar{x}_i\}$, $\{\bar{y}_j\}$, λ^* where $\lambda_i^* = (\bar{\lambda}_i + k)/\alpha_i$ and where $\{\bar{x}_i\}$, $\{\bar{y}_j\}$, $\bar{u} = k\bar{p}$, $\bar{\lambda}$ are obtained by solving the master program, are a solution of the mathematical program $\max \sum_i U_i(x_i)$ subject to $\bar{p}\cdot(x_i - \sum_j \theta_{ij} y_j - w_i) \leq 0$, $x_i \in X_i$, $y_j \in Y_j$ for all i, j .

Proof. Subject to appropriate change of notation, the proof is similar to that of Corollary 2.1.1.[3]

From a computational point of view, it is convenient that the mathematical programs considered in the Theorem and its two Corollaries can be set up in the same basic format. The Negishi welfare optimum problem of Corollary 2.1.1 can be obtained by introducing slacks into the budget constraints of the master program. The values of the slacks then give the values of the excess budgets corresponding to each chosen value of α . Likewise the program of Corollary 2.1.2 can be obtained from the master program by introducing slacks into its resource balance constraints; the values of the slacks are the excess demands corresponding to each choice of p .

[3] It is useful to note that this mathematical program is decomposable.

From a theoretical point of view, the corollaries suggest a way of computing the excess budgets and excess demand correspondences involved in the proofs of existence of equilibrium by the Negishi indirect approach and by the more usual direct approach. Also, Corollary 2.1.1 indicates how the Negishi equilibrium weights can be computed from the solution of the master program. Corollary 2.1.2 shows how their inverses, the marginal utilities of income, can be computed from the solution of the same program.

3. Computation by the Scarf-Type Fixed Point Algorithms

The full format general equilibrium problem, as inspection shows, has an unusual structure which does not lend itself readily to computation. H. Scarf was the first to suggest a general algorithm which is capable of solving it. His beautifully elegant procedure not only is sure to find equilibrium solutions to any degree of approximation; but it has also provided a new approach to proving existence of such solutions and has yielded new information on the number of equilibrium points.

Scarf's algorithm is unfortunately costly to use. Even for small problems it involves examining excess demands (or excess budgets) at hundreds of points on a grid of prices (or welfare weights). Using the full format approach, it would be necessary to solve hundreds of times either the profit and utility maximizing problems of individual producers and consumers (in the direct computation), or the welfare optimum problem of the indirect approach. It would involve a large computational burden to solve problems other than classroom examples, and this explains why the procedure has not so far been applied in a planning context.[4]

4. Reduced Format Computation

It may happen that the form of the production and utility functions makes it possible to reduce the full format problem to a set of equations involving only prices or only welfare weights. For direct computation this would involve solving analytically the mathematical program of Corollary 2.1.2, to obtain expressions defining the optimal allocations as functions of prices; these expressions could be inserted into the excess demand constraints (a) of the general equilibrium problem, giving the set of conditions $d(p) \geq 0$ together possibly with side constraints, which are discussed below. The system can then be solved for equilibrium prices, which will determine the equilibrium allocations.[5]

For indirect calculation the reduced format of the system is obtained by solving analytically the mathematical program of Corollary 2.1.1. The optimal allocations and shadow prices are then functions of the welfare

[4] See H. Scarf [23], [24]. Mainly Eaves [7], [8] and Merrill [17] have proposed notable improvements of Scarf's original procedure, making it possible to start from any point of the price or welfare weight grid, and to refine the grid size as equilibrium is approached. These refinements have not so far improved the procedure to the point where it would become applicable to large realistic problems (see e.g. the computational experience reported in Chapter 3 of Wilmuth's dissertation [32]). In case of full format indirect or direct computation, every new mathematical program would differ very little from the previous one, if the grid with which the p^j or α^j vectors are chosen is small enough. This implies that very little computation would be necessary to go from one optimal solution to another. However, experience with relatively large linear programs shows that it takes a few seconds to check if a former solution is still optimal. For example for our problem, using the very efficient Apex I linear programming computer code developed by Control Data Corporation, it takes almost 6 seconds CP time on a CDC 6400 to check if a basis is still optimal, for a problem with some 400 constraints.

[5] As is well known, the excess demand functions are not independent. One of them can be dropped, while it is convenient to normalize prices.

weights α . Substituting these into the excess budget constraints gives a set of conditions $b(\alpha) \geq 0$ together possibly with side constraints which are discussed below. This system can also be solved for equilibrium weights, which determine the equilibrium allocations and prices.[6]

Even when format reduction is feasible the problem obtained may turn out to be quite complex. The resource balances are inequalities, and equilibrium prices of goods in excess supply must be zero. The non-negativity constraints on consumption and production are transformed by the reduction procedure into complicated non-linear inequalities involving prices.

That such side conditions can be neglected may be known on empirical or theoretical grounds. In empirical models the list of free and non-free goods is generally known in advance; the prices of free goods should be set equal to zero and attention confined to resource equations involving the non-free goods only. It may also be known in advance that certain agents do not consume certain goods, so that their demand can be disregarded. The knowledge that all agents require certain goods (e.g. food) makes it unnecessary to include non-negativity constraints for consumption of these goods.

The mathematical properties of the problem may also ensure that inequality and/or non-negativity constraints may be dropped. For example with input-output technologies it is known that under quite general conditions non-negative final demand implies non-negative production. Likewise some utility functions like the Cobb-Douglas have the property that the induced demand for a good rises indefinitely when its price tends to

[6]Likewise, the excess budget constraints are dependent and the α_i's can be normalized.

zero; such a good cannot be a free good, and the corresponding resource balance may be treated as an equation.

Altogether we feel that format reduction is a promising approach only when for empirical or theoretical grounds it is clear that resource constraints may be treated as equations, and non-negativity constraints on variables are superfluous. The range of cases for which this is true is so narrow that the approach is mainly applicable to illustrative cases.

One final remark is in order. In the favorable cases where the reduced format representation of the general equilibrium problem involves (under the direct approach) a system of r equations in r prices and (under indirect approaches) a system of m equations in m welfare weights, it may be worth comparing the number r of goods and the number m of consumers. For there is a presumption that the problem with fewer equations and unknowns is easier to solve than the other. There can of course be exceptions to this rule. And there is no reason to think that in the general case, when all conditions of the problem have to be taken into account, the format which involves fewer variables will be easier to tackle than the other.

5. Three Full Format Computational Procedures

Apart from Scarf-type fixed point algorithms, there exists no method which is guaranteed to find the solution of general equilibrium problems. It is well-known, however, that in solving non-linear problems engineers and economists often use algorithms which are not guaranteed to converge. Such algorithms usually include a number of options, as well as parameters which may be set arbitrarily by the user, and this makes them capable of solving successfully even ill-structured problems. Experience shows that the chances of convergence are best when the algorithms can be started close to the final solution of the problem.[7] The three algorithms described in this section do not have the mathematical elegance nor the convergence properties of Scarf-type procedures. But they have proved surprisingly effective in solving one rather large and realistic general equilibrium model, and seem flexible enough to provide a grip even on problems which prove more difficult to solve.

All three algorithms involve the same initial step, based on the straightforward idea that if we solve the program with prices $\bar{\bar{p}}$ and welfare weights $\bar{\bar{\alpha}}$ not too far from equilibrium the result will be a solution with dual prices $\bar{\bar{u}}$ close to $\bar{\bar{p}}$ and to equilibrium prices. Program P1 is defined as

<u>Program P1</u> (Near Equilibrium Program): max $\Sigma \bar{\bar{\alpha}}_i U_i(x_i)$, $(\bar{\bar{\alpha}}_i > 0)$ subject to $\sum_i x_i - \sum_i w_i - \sum_j y_j \leq 0$, $\bar{\bar{p}}\cdot(x_i - \sum_j \theta_{ij} y_j - w_i) \leq 0$, $x_i \in X_i$ for every i and $y_j \in Y_j$ for every j.

The first computational procedure follows immediately:

<u>Computational Procedure I</u> (Direct Computation in Prices Space)

- Step 1: Set the values of $\bar{\bar{\alpha}}, \bar{\bar{p}} > 0$.
- Step 2: Solve a mathematical program P1. Check whether $\bar{\bar{u}}$,

---

[7] An example of such a method is the Gauss-Seidel algorithm for solving systems of non-linear equations [15]. Other examples are given in a recent paper by McKenzie [18]. In real life economic problems one has, in general, a good guess at the final answer. In macro models, for example, the iterations are started by giving to the variables the value of the last observation. And, certainly, one would suspect an error in the computer program if the 1976 values of the variables were very different from the 1974 observations.

the optimal values of the Lagrange multipliers associated with the $\sum_i x_i - \sum_j w_i - \sum_i y_j \leq 0$ constraints of P1 are proportional to $\bar{\bar{p}}$.

- Step 3: Should this be true, we can consider $\bar{\bar{p}} = k\bar{\bar{u}}$ as a fixed point of the mapping $P : p \rightarrow p$. If this is not the case, put $\bar{\bar{p}} = \bar{\bar{u}}$ and go to Step 2.

In choosing initial values of prices it is reasonable to set $\bar{\bar{p}}$ equal to observed market prices. The initial $\bar{\bar{\alpha}}_i$ could be deduced by solving the utility maximization problems of equilibrium condition (b) on the basis of these prices and observed production levels, and setting $\bar{\bar{\alpha}}_i = 1/\bar{\bar{\lambda}}_i$, where $\bar{\bar{\lambda}}_i$ is the marginal utility of income so obtained. We have found that a rough guess of the proper value of $\bar{\bar{\alpha}}_i$ was good enough in practice.

The second Computational Procedure is based on Theorem 2.1 and Corollary 2.1.1: it tries to make use of the steps of Computational Procedure I and to change at the same time the welfare weights α_i :

<u>Computational Procedure II</u> (Mixed Direct/Indirect Computation in the Space of Prices and Welfare Weights)

- Steps 1 and 2: are the same as in Computational Procedure I.
- Step 3: If $\bar{\bar{u}} \neq k\bar{\bar{p}}$, set $\bar{\bar{p}} = \bar{\bar{u}}$ as in Procedure I. Also compute a new value for $\bar{\bar{\alpha}}_i$ as suggested by Corollary 2.1.1; i.e. choose $\bar{\bar{\alpha}}_i = \bar{\bar{\alpha}}_i : (1 + \bar{\bar{\lambda}}_i/\bar{\bar{k}})$ where, by convention, the left and right hand side $\bar{\bar{\alpha}}_i$ are the old and new values of the variable; $\bar{\bar{\lambda}}_i$ is the value of the earlier optimal value of $\bar{\bar{\lambda}}_i$. It is reasonable

to set $\bar{\bar{k}} = \sum_{\ell} \bar{\bar{u}}_{\ell} / \sum_{\ell} \bar{\bar{p}}_{\ell}$, but in practice since the values of $\bar{\bar{u}}$ and $\bar{\bar{p}}$ are close together we found that $k = 1$ gave good results. Go to Step 2.

We now come to the third procedure, based on an equivalent formulation to the welfare optimum; define Program P2:

<u>Program P2</u> (Pareto Optimum): max $U_1(x_1)$ subject to $U_i(x_i) \geq U_i(x_i^0)$ $(i = 2, \ldots, m)$, $\sum_i x_i - \sum_i w_i - \sum_j y_j \leq 0$, $x_i \in X_i$ for every i , $y_j \in Y_j$ for every j ; $U_i(x_i^0)$ is the value of consumer's i utility when he consumes a given bundle x_i^0 .

Clearly, to every optimal solution of P2, there corresponds a solution of the Negishi welfare optimum in which $\alpha_1 = 1$ and the weights $\alpha_2, \ldots, \alpha_m$ of which are the dual values picked up by the $U_i(x_i) \geq U_i(x_i^0)$ constraints of P2. Thus finding the fixed point in the welfare weights space is the same as finding the $U_i(x_i^0)$ associated to the equilibrium solution of the welfare optimum.

Since there are some reasons to believe that the utility frontier in the welfare space has a flat shape, a very small change in the welfare weights may induce large welfare changes; and, in a sense, it may seem easier to control the mapping from the excess budgets into the welfares, rather than the usual mapping from the excess budgets into the welfare weights. We now define:

Computational Procedure III (Indirect Computation in Utility Space). This procedure is inspired by Corollary 2.1.1.

- Steps 1 and 2: are the same as in Computational Procedure I.
- Step 3: Set $\bar{\bar{p}} = \bar{\bar{u}}$ where $\bar{\bar{u}}$ are the dual prices given by the solution of the last mathematical program solved, and compute excess budgets $b_i(\bar{\bar{p}}) = \bar{\bar{p}} \cdot (w_i + \Sigma\, \theta_{ij} \bar{\bar{y}}_j - \bar{\bar{x}}_i)$ at these prices. If the excess budgets are negligibly small, go to Step 6 (or end). If not go to
- Step 4: Compute the welfares $\bar{\bar{U}}_i = U_i(\bar{\bar{x}}_i)$ of the last mathematical program solved, and set $\bar{\bar{U}}_i = \bar{\bar{U}}_i + \mu_i b_i(\bar{\bar{p}})$, where, by convention, the left and right hand side $\bar{\bar{U}}_i$ refer to the old and new values of $\bar{\bar{U}}_i$.
- Step 5: Set up a new program P2, solve it, and go to Step 3.
- Step 6: (Optional) as a way of refining the approximation, set up a program P1 where the $\bar{\bar{\alpha}}$ and $\bar{\bar{p}}$ vectors are the dual solutions of the last Step 3 program.

A reasonable choice of μ_i is suggested by the fact that if $\bar{\bar{\lambda}}_i$ is the marginal utility of income, a small unit increase in utility requires, if prices are constant, an increase $1/\bar{\bar{\lambda}}_i = \bar{\bar{\alpha}}_i$ in spending. The procedure is quite robust, however, and in our case we found that setting $\mu_i = 1$ gave good results.

6. Analytical Representation of Production Sets, Consumption Sets, and Utility Functions[8]

We have used up to now an abstract set theoretic representation of production and consumption sets, which has the advantage of focusing attention on the resource and budget balances which lie at the heart of the general equilibrium problem. In actual computation it is necessary to be more specific. It is well-known that production and consumption sets can be represented by functional inequalities.[9] Computer applications require functional rather than set theoretic representations of production and consumption sets.

It is obviously possible to implement procedures I, II, and III using non-linear programming codes. For practical reasons the choice of functions must be limited to analytic functions which computers can understand. A certain approximation error is therefore necessarily involved in the representation of the general equilibrium problem; this error is of course small compared to our ignorance of the structure of the problem.

In fact this ignorance is so great that using linear rather than non-linear programming involves no meaningful loss of accuracy. The former takes advantage of the existence of remarkably efficient matrix generation and solution codes which can be used to set up and solve problems; furthermore, in spite of a widespread opinion, using linear programming involves no loss in generality.

It is well-known that any convex production set can be approximated

---

[8] See also [13] and [14] for a more comprehensive discussion.

[9] See e.g. T. Negishi [19].

by the polyhedron generated by a set of linear inequalities. Under constant returns the production sets are cones and the system of inequalities has the form specified by activity analysis.

It is less well-known--but by no means novel[10]--that any concave utility function can be represented by the r-dimensional efficient frontier of a convex set generated in an r+1 space of which the coordinates are $(U, x_1, \ldots, x_r)$. Such a function is represented by max U subject to

$$Gx - \ell U \geq g \, , \quad x \geq 0$$

where ℓ is an s-column vector of ones, $[G\ell]$ is an $s \cdot (r+1)$ matrix of which the rows are the coefficients of the s hyper-planes bounding the convex set and g is an s-vector of intercepts of the hyperplanes with the U-axis. For the case of separable utility functions it is more convenient to use a stepwise approximation.

$$\max \sum_{j=1}^{r} \sum_{k=1}^{s_r} g_{jk} x_{jk} \quad \text{subject to} \quad 0 \leq x_{jk} \leq \bar{x}_{jk} \quad \text{all} \quad j, k$$

where g_{jk} is the marginal utility of commodity j over the step x_{ij} , bounded above by $\bar{x}_{jk}$; another special case is the homothetic function which may be represented by the activity analysis representation $\max \Sigma z_j$ subject to $z_j \geq 0$ and $\sum_j g_{jk} z_j \leq x_k$ where the g_{jk} are baskets of commodities on the indifference surface $U(x) = 1$.

From a theoretical point of view it is clear that linear constraints can represent production and consumption sets to any degree of approximation.

---

[10] See e.g. Diewert [3].

The price of greater precision is of course a greater number of constraints, so that an accurate model might be very costly. This is especially true for production and consumption sets which cannot be described by separable or homothetic functions. In practice what matters of course is behavior in the neighborhood of equilibrium: it is not necessary to describe the economy accurately elsewhere. Identifying a region within which equilibrium is expected to lie and setting up constraints which describe that region is a crucial part of the construction of the model. This inevitably involves trial and error, in which additional constraints are added as a result of preliminary experiments with the model.[11]

7. Computational Results

Computations have been made using an international equilibrium model described in [14]. This model includes some 360 constraints and 400 upper or lower bounded variables out of the 880 variables in all.

The model was carefully constructed to provide a believable picture of the world economy. It computes international equilibrium prices of close to 50 traded goods among four groups of countries representing the non-socialist world; the total number of equilibrium prices computed is however much larger. The results reported were obtained for model variants which corresponded to interesting alternative states of the world economy. Among others, various hypotheses of free trade have been examined, which we hope to describe in a later paper.

---

[11] This is of course a feature of all linear programming models and not of the general equilibrium procedure *per se*. In practice we found that in the trial and error phase it was sufficient to solve the "near equilibrium" model P1, using general equilibrium procedures only to refine the solution obtained.

The computations were run on a CDC 6400 computer on which a canned version of the linear programming code Apex I was available. It is useful, at this stage, to go into some detail concerning the code described at full length in [1]. This will shed some light on the interpretation of the performances.

7.1. Some Basic Features on the Computer Program

Apex I can be used as such, by simply submitting to the code any linear program in a suitable form (MPS format) and by specifying how the problem should be solved: direction of optimization, name of the objective function, name of the right hand side and the bounds vector, type of desired printed output; a precise description of the files on which are located the various components of the problem--input data, eventual revision, advanced basis, etc...--should, of course, also be supplied to the computer. The optimal solution can then be transferred onto a magnetic disc which is readable by a user supplied FORTRAN program (USP). This program can compute whatever is needed, change some of the data of the problem and submit it for a new solution. The general scheme is thus a series of as many (Apex → USP → Apex) iterations as needed.[12] Of course, such a procedure requires a fairly large number of files stored on disc to go from one major iteration to the next; this means also that at each of these major iterations the linear programming problem matrix must be reconstituted and the optimal solution, including the optimal basis, has to be stored on an external (not in central memory) file.

[12] This notion of iteration should not be confused with the usual simplex iteration. To avoid confusion, we use in the sequel the term "major iteration" as opposed to "simplex iteration."

Apex I can also be used as a callable subroutine. This means that the user has his own FORTRAN program and wants to employ special subroutines of the Apex I system, for instance, the optimization subroutine. With this option, the problem can reside in central memory during the major iterations; this simplifies the management of the various files, since they stay in central core, and also reduces the computing time: there is neither reconstruction of the constraints matrix, nor construction of an output file at each major iteration. Of course, there is no free lunch and this way of using Apex requires much more programming work and skill; in fact, it amounts to updating and recompiling the partly FORTRAN-written Apex I program.

We chose to use the first option; it is clear that if several dozens of runs have to be made, the second option would be preferred. We shall try to give an idea of the possible gains in computing time, using the second option.

The computer programs have been written in such a way that every one of the computational procedures required a maximum of eight major iterations, the first of them being in each case the following P1 type program: max $\sum_i \alpha_i U_i(x_i)$ subject to $\sum_i x_i - \sum_j y_j - \sum_i w_i \leq 0$, $\bar{\bar{p}} \cdot (x_i - \sum_j \theta_{ij} y_j - w_i) \leq 0$, $x_i \in X_i$, $y_j \in Y_j$, for every i and j. Since in the empirical model constructed prices were normalized to one, the initial guess $\bar{\bar{p}}$ is a vector of ones; likewise, utility is almost equal to income, so that the initial value of α_i was also set equal to one.[13] The following seven major iterations for Procedures I and II

[13] The data concerning this first iteration will not be given; they are not very meaningful since anyway the computations start with an advanced basis and not from a completely artificial one.

are described in enough detail in Section 5 of this paper. For Procedure III there are six iterations of the program P2-type; the eighth and last iteration is a P1 program constructed according to Step 6 of the procedure.

7.2. Comparative Performance of the Procedures

The three procedures discussed in Section 5 have been tested on five different variants of the same model.[14] The main results are summarized in Table 1 which gives for every variant a measure of the speed of convergence and the total number of simplex iterations needed. The measure of the speed of convergence is taken to be the sum of absolute values of excess budgets (i.e. in this case the balance of payments) in millions of U.S. dollars. It may help the reader to evaluate the convergence to know that in variant 1 the equilibrium value of world imports is 178.1 billion.

In all cases, convergence was achieved more or less quickly, except for run 2 with Procedure I, which might have diverged if iterations had been added. On the average the procedure in the utility space (Procedure III) performs best with an average of 130 simplex iterations to reach equilibrium, against 165 for Procedure I and 172 for Procedure II. The number of simplex iterations to achieve equilibrium is small anyway

[14] The variants were chosen for their economic meaning and for their value in testing the robustness of the procedure to sharp changes in starting values. The problems are (ordered from A to L in Tables 1 and 2): reference solution; free trade in Latin America; free trade in Asia; free trade in Africa; free trade in the developing countries; world-wide free trade; free trade in the developed world; free trade in developed and 50% tariff cut in developing countries; effects of the oil crisis; effects of the oil crisis I and recession in developed world; effects of the oil crisis II; recession in developed world, without oil crisis.

compared to the solution of the P1-type linear program (between 800 and 1000 simplex iterations to achieve near equilibrium), and is only loosely related to the initial "distance" from equilibrium: compare for instance run 1 with run 5.

From a practical point of view it is important to note that the "near equilibrium" prices and allocations obtained by solving problem P1 were indeed close enough to equilibrium to make it possible to use only the near equilibrium model in the initial exploration of the system. General equilibrium did, however, diverge from the preliminary near equilibrium solutions, especially for the values of excess budgets. As noted above, obtaining true general equilibrium solutions was little more costly than remaining satisfied with calculations which stopped at the near equilibrium stage.

The results shed light on the idea stressed by P. Dixon [4], [5] (and also by Ginsburgh and Waelbroeck in an earlier paper [11] written before they knew of Dixon's work) that if the number of agents is less than the number of goods, indirect computation should be cheaper than direct computation, because the fixed point research is restricted to a smaller dimensional space. Procedure III is indeed cheaper than the others, but the gain is small. As there did seem to be a gain and our funds were limited, the other methods were momentarily not further explored; we preferred to gain more insight into Procedure III.

7.3. Computational Procedure III

Twelve variants of the model have been computed (this includes the five we have spoken of earlier). The results are reproduced in Table 2 which, basically, gives the same data as Table 1: column (1) contains the sum of the absolute values of the excess budgets during the iterative process; column (2) gives the number of LP simplex iterations; for each problem we indicate also the total time taken in central processing seconds (input-output time is negligible since Apex works in core) and the time taken for solving the various linear programs involved; the difference between the two numbers is due to:[15]

- the time taken for the construction of the LP matrices, which amounts to some 63 seconds per problem; this time could be saved if Apex was used as a subroutine, so that the problem would always remain in the memory;
- the time taken to prepare the final complete solution of the linear program: 7 seconds CP time;
- the time for reading each LP solution, computing the excess budgets and revising the data of the linear program before going to the next major iteration: 32 seconds CP time.

It can be seen that, even when departing from very far from equilibrium (cases I and J), convergence is achieved (case J would have needed 2 or 3 more iterations) and that computing time is far from being prohibitive, once a solution of the linear program including budget constraints is available. As we have already written, this amounts to a thousand simplex iterations for our model; however, when alternative solutions are computed, we can start from a fairly good basis, for instance the one for the reference solution.

---

[15] These numbers are averages: the variance is, however, very small (about 2 or 3 percent around the mean).

7.4. Dealing with Convergence Difficulties

We do not know of a convergence proof of the procedures described, nor do we think that such a proof is possible.[16] That such procedures are likely to behave well is suggested by heuristic reasoning; for example, common sense suggests that increasing the welfare weight of one consumer in a Negishi welfare optimum will increase the excess spending of that consumer and reduce that of the others. The theoretical literature, however, devises counterexamples which show that unexpected behavior is possible. Close examination of these examples shows that they imply perverse income effects. If relative prices do not change, shifting welfare weights in favor of a consumer leads to a shift in allocation in favor of that consumer in a way which increases his excess spending but reduces that of the others. But if prices change, this tendency may be counteracted by income effects reflecting differences between the spending patterns and initial resources of the various consumers.

We have not encountered convergence problems in solving our model, and have therefore not had a chance to grapple with such difficulties. The algorithms are, however, designed in such a way that it is possible to govern their behavior to a certain extent, by control parameters. Also, there exist three alternative procedures[17] and this also offers

---

[16] It is not difficult to think of conditions which guarantee convergence. For example, as proved by Mantel [16] and Dixon [4] if in the indirect procedure increasing one consumer's welfare weight reduces the excess spendings of other consumers the indirect procedure converges (the argument is the same as that used to prove convergence of price tâtonnement in the gross substitutes case). As Mantel points out, this is not a useful condition for it is not possible in general to check in advance whether the condition holds.

[17] And it is not difficult to imagine several others.

scope for a trial and error process of experimenting with alternative algorithms. As all the procedures use the same master problem format this would not be difficult in practice.

The most powerful approach is to use a true gradient procedure instead of the heuristic adjustment of prices, welfare weights, and required utility levels of the three procedures. M. Osterrieth [21] has tried this approach to compute the solution of the well-known unstable equilibrium problem described by Scarf [22]. Mantel's paper [16] and numerical experiments show that the simple procedures described in Section 5 do not converge for this problem. Osterrieth has found that a gradient method which adjusts welfare weights to reduce the sum of the absolute excess spendings yields a solution quite quickly.

For linear programming problems gradients cannot be obtained analytically. But quasi-gradients could be obtained by changing welfare weights one by one and observing the effect on all excess budgets. This would be an expensive procedure. Extracting the solution of models is not a game of blind man's bluff however, and the builder of a realistic model usually has quite a bit of insight into its properties. If difficulties are encountered, concentrated thinking on what the model is doing and why, will probably be the cheapest and most effective way of understanding how the algorithm (or the model) should be modified to obtain the desired solution.

7.5. Representation of Price Distortions, External Economies

The solution procedures alternate between "Revision phases" in which prices, welfare weights, or minimum utility levels are revised, and "Solution phases" in which a (linear) programming code is used to compute new shadow prices and allocations. It is important to notice that the Revision phases give an opportunity to change other parameters of the problem. In the model solved this possibility was used to revise ad valorem tariffs to keep them proportional to prices. Income taxes and other indirect taxes could be treated in the same way, and some types of public spending could be connected by appropriate functions to the values of variables of the model. External economies and diseconomies could also be dealt with: for example it would be possible to represent increases in efficiency which are associated with increases in the size of new industries, provided that these increases accrued to producers via external effects. Finally, some types of monopoly behavior could be represented by using the Revision phase to change the prices of the monopolist in response to sales and to shadow prices of other goods, while in each Solution phase purchases from the monopolist would be treated as purchases from outside the system.

TABLE 1: Comparative Performances of the Procedures

	A			B			E			F			K		
	I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
Major Iteration 1	3184	3184	3184	10367	10367	10367	17900	17900	17900	13771	13771	13771	8260	8260	8260
2	3342	3342	1154	2322	2322	1742	3751	3751	2794	4405	4405	3237	2917	2917	480
3	1680	1811	930	714	539	799	629	414	1088	210	334	384	919	969	482
4	270	501	584	168	389	152	137	30	180	24	70	49	931	132	134
5	70	214	359	38	294	142	52	3	18	5	5	7	18	40	25
6	18	15	72	185	70	40	4	3	4	2	1	3	5	14	5
7	3	2	26	184	12	5	2	3	2	1	1	2	1	10	1
8	3	1	1	188	2	2	1	3	2	1	1	1	1	2	2
Number of Simplex Iterations	154	175	113	231	239	160	166	147	153	110	120	121	164	182	94

Each column gives the sum of the absolute values of the excess budgets during the iterative procedure.

TABLE 2: Performance of Procedure III

	A		B		C		D		E		F	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
Iteration 1	3184		10367		6370		5750		17900		13771	
2	1154	51	1742	73	2899	67	1246	50	2794	101	3237	94
3	930	33	799	44	959	49	232	18	1088	37	384	19
4	584	10	152	25	124	7	212	8	180	8	49	0
5	359	10	142	6	54	0	50	3	18	0	7	0
6	72	0	40	0	1	0	25	0	4	0	3	0
7	26	0	5	0	1	0	12	0	2	0	2	0
8	1	9	2	12	1	6	1	7	2	7	1	8
Total Simplex Iterations		113		160		129		86		153		121
<u>Computing Time</u>												
Total	307		332		319		309		327		310	
of which LP	205		230		217		207		225		208	

Column (1) gives the sum of the absolute values of the excess budgets.
Column (2) gives the number of simplex iterations in every major iteration.

TABLE 2 (continued)

	G		H		I		J		K		L	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
Iteration 1	4069		6824		34740		27202		8260		6414	
2	2071	54	1672	52	35600	152	33246	168	480	61	531	56
3	854	28	296	10	9350	118	5758	138	482	10	76	1
4	199	6	15	0	1350	50	1910	26	134	11	2	0
5	570	9	2	0	758	11	1008	29	25	1	1	0
6	189	0	1	0	266	1	210	13	5	0	1	0
7	60	0	1	0	90	0	204	3	1	0	1	0
8	2	11	1	10	2	14	315	15	2	11	2	9
Total Simplex Iterations		108		72		345		392		94		66
Computing Time												
Total	313		265		404		451		312		230	
of which LP	211		163		302		349		210		128	

ACKNOWLEDGMENTS

The original inspiration of this paper should be attributed to H. Scarf, who in a brilliant seminar many years ago impressed one of the authors with the importance of implementing in a correct way the concepts of general equilibrium theory. The ideas discussed in this paper were developed over several years. An early version of Theorem 2.1 in particular was presented in September 1973 in a paper presented at the Oslo Meeting of the Econometric Society and at a conference in Namur [13] while the first equilibrium solution of the model was computed in May 1974 [11]. While our ideas were developed independently, and often at great pain, we have found that each of them had already been anticipated somewhere in the literature, often in a different context. In particular indirect computation had been suggested by Dixon [4], [5], Mantel [16], Scarf [24], Takayama [28], Trzeciakowski [30], Woodland [33]. The use of tâtonnement in utility space had been suggested by Dixon [4], [5] and Mantel [16], while the idea of reaching equilibrium by using tâtonnement in goods space is of course very old. The linear programming representation of utility functions was used, e.g. by Diewert [3], but did not originate with him. All that can be claimed, it seems, is that these ideas have been fitted together in a new way, that the paper innovates in the Schumpeterian sense of representing a "new combination of factors of production." Also, this seems to be the first successful computation of a general equilibrium for a large realistic model.

We wish to acknowledge material and intellectual support which has made our work possible. Construction of the model was supported by the IBRD as part of the research program of its Development Research

Center, and by the Belgian Fonds de la Recherche Fondamentale Collective. Of equal importance to this material support have been very stimulating discussions with B. Balassa, H. Chenery, E. Diewert, J. Drèze, L. Goreux, and H. Scarf, each of whom has made a decisive imprint on some aspects of our thinking. We, of course, are responsible for the usual remaining errors.

REFERENCES

[1] Apex-I Reference Manual, Control Data Corporation, 1972.

[2] Diewert, W. E., "On a Theorem of Negishi," Metroeconomica, 25 (1973), 119-135.

[3] ________, "Afriat and Revealed Preference Theory," The Review of Economic Studies, 40 (1973), 419-425.

[4] Dixon, P. E., The Theory of Joint Maximization, North-Holland, 1975.

[5] ________, "The Computation of Economic Equilibria: A Joint Maximization Approach," paper to be delivered at the Third World Congress of the Econometric Society, Toronto, May 1975.

[6] Duloy, J., and R. D. Norton, "Chac, a Programming Model of Mexican Agriculture," in Goreux, L. and A. Manne (eds.), Multi-Level Planning, Case Studies for Mexico, North-Holland, 1973.

[7] Eaves, B. C., "Computing Kakutani Fixed Points," SIAM Journal of Appl. Math., 21 (1971), 236-244.

[8] ________, "Homotopies for the Computation of Fixed Points," Mathematical Programming, 3 (1972), 1-23.

[9] Hansen, T., and J. E. Spencer, "On the Calculation of Competitive Equilibrium in Models of International Trade," Norwegian School of Economics and Business Administration Discussion Paper, December 1969.

[10] Houthakker, H., "The Calculation of Bilateral Trade Patterns in a Ricardian Model with Intermediate Products and Barriers to Trade," Harvard University-Institute of Economic Research Discussion Paper 399, February 1975.

[11] Ginsburgh, V. A., and J. Waelbroeck, "Computational Experience with a Large General Equilibrium Model," Core Discussion Paper 7420 (1974), forthcoming in J. Los (ed.), Computing Equilibria, How and Why, North-Holland.

[12] ________, "Linear Programming Planning Models and General Equilibrium Theory," Core Discussion Paper 7421 (1974).

[13] ________, "Linear Programming Formulation of International Trade Mechanisms and Trade Theory," in H. Glejser (ed.), Quantitative Studies of International Economic Relations, North-Holland, forthcoming.

[14] ________, "A General Equilibrium Model of World Trade. Part II: The Empirical Specification,"

[15] Klein, L. R., M. K. Evans, and M. Hartley, Econometric Gaming: A Computer Kit for Macroeconomic Analysis, Macmillan, 1969.

[16] Mantel, R. R., "The Welfare Adjustment Process: Its Stability Properties," International Economic Review, 12 (1971), 415-430.

[17] Merrill, O. H., "Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Non Empty, Convex, Upper Semi-Continuous Point to Set Mappings," Technical Report 71-7, Department of Industrial Engineering, University of Michigan, 1971.

[18] McKenzie, L. W., "Why Compute Equilibria," paper delivered to the Conference "Computing Equilibria: How and Why?" Torun, July 1974, University of Rochester, January 1975.

[19] Negishi, T., "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," Metroeconomica, 12 (1960), 92-97.

[20] Nikaido, H., Convex Structures and Economic Theory, Academic Press, 1968.

[21] Osterrieth, M., "Indirect Computation of Equilibrium for Scarf's Tâtonnement Problem, A Preliminary Note," World Bank, September 1975.

[22] Scarf, H., "Some Examples of Global Instability of the Competitive Equilibrium," International Economic Review, 1 (1960), 157-172.

[23] ________, "The Approximation of Fixed Points of a Continuous Mapping," SIAM Journal of Appl. Math., 15 (1967), 1328-1342.

[24] ________, with the collaboration of T. Hansen, The Computation of Economic Equilibria, Yale University Press, 1973.

[25] Shoven, J. B., "A Proof of the Existence of General Equilibrium with ad valorem Commodity Taxes," Journal of Economic Theory, 8 (1974), 1-25.

[26] ________, and J. Whalley, "General Equilibrium with Taxes: A Computational Procedure and an Existence Proof," The Review of Economic Studies, 40 (1973), 475-489.

[27] ________, "On the Computation of Competitive Equilibrium on International Markets with Tariffs," Journal of International Economics, 4 (1974), 341-354.

[28] Takayama, T., "A Note on Existence Proofs of a General Equilibrium of an Abstract Economy," April 1967.

[29] ________, and G. G. Judge, Studies in Economic Planning over Space and Time, North-Holland, 1973.

[30] Trzeciakowski, W., "Systems of Indirect Management in a Planned Economy," Cowles Foundation Discussion Paper 312 (1971).

[31] Whalley, J., "Fiscal Harmonization in the EEC: Some Preliminary Findings of Fixed Point Calculations," in Proceedings of the Clemson Conference on "Computation of Fixed Points with Applications," June 1974.

[32] Wilmuth, R. J., "The Computation of Fixed Points," Ph.D Dissertation, Stanford University, 1973.

[33] Woodland, A. D., "Competitive Equilibrium in International Trade," Ph.D Dissertation, University of New England, 1969.