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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART 24

TRADE AND PRICES IN A CLOSED ECONOMY WITH EXOGENOUS UNCERTAINTY, DIFFERENT LEVELS OF INFORMATION, MONEY AND COMPOUND FUTURE MARKETS

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by

Pradeep Dubey and Martin Shubik**

1. THE COMPETITIVE EQUILIBRIUM

The basic general equilibrium closed static model of trade*** is as follows. Let there be n traders trading in $m+1$ goods. Assume initially that each trader i has a bundle of goods $(a^i_1, a^i_2, \ldots, a^i_{m+1})$.

Assume each trader i has a utility function u^i which is continuous, concave and nondecreasing

$$u^i = u^i(x^i_1, x^i_2, \ldots, x^i_{m+1}) \quad (1)$$

A competitive equilibrium price system exists† if there is a set of prices $p = (p^*_1, p^*_2, \ldots, p^*_m, p^*_{m+1})$ which has the property that for:

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*This work related to Department of the Navy Contract N00014-76-C-0085 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

**The authors are indebted to L. S. Shapley for many discussions, observations and specific suggestions as is indicated in several of the proofs which build on his work.

***In this paper, production is not discussed; there are several modelling problems in describing the sequence of production which we have not yet resolved.

†For further references, see Debreu [1].

$$\text{(2)} \qquad \max u_i(x_1^i, \ldots, x_{m+1}^i)$$

$$\text{subject to} \quad \sum_{j=1}^{m+1} p_j^*(x_j^i - a_j^i) = 0 \quad \text{for} \quad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} x_j^i = \sum_{i=1}^{n} a_j^i \qquad \text{for} \quad j = 1, \ldots, m+1 .$$

This formulation of the problem of trade is noninstitutional and nongame theoretic. No mechanism for the generation of prices is given which in any way restricts the possibility for trades.

In the 2 person 2 commodity model this means that all points in the Edgeworth box are feasible outcomes.

2. A RELATED TRADING GAME

Suppose that the traders in this economy were required to trade in a specially designated commodity in organized markets. There are many different market structures which can be considered such as the auction market, the double-auction market; markets in which individuals name price or quantity or both as their strategic variables or name contingent bids and so forth. Several of these have been discussed elsewhere [2]. Here we select the simplest which has already been considered by Shubik [2], Shapley [3] and Shapley and Shubik [4].

This game is based on an extension of the Cournot oligopoly model.

We call a commodity a "money" if it is accepted in trade for all and any other commodities. A "money" is called a "commodity money" if it would still maintain worth (beyond that of a collector's curiosum) if it were demonetized. It is a "fiat money" if its only utilitarian worth is in its use in trade and would lapse to having value only as a collector's item if demonetized.

Suppose that the $m+1^{st}$ commodity is selected as a "commodity money." We now consider a game in extensive form as follows.

There are m warehouses. At the start of the game all individuals are required to deposit all of their goods except money in the appropriate warehouse. They receive non-negotiable warehouse receipts which will be redeemed after trade for the amount of money obtained from the sale of each individual's goods.

The individuals know the size of the supply in each warehouse.

A strategy by each individual i is to select m bids b_j^i where $j = 1, \ldots, m$ and

$$(3) \qquad \sum_{j=1}^{m} b_j^i \leq a_{m+1}^i \; .$$

Thus each individual operates subject to a <u>cash constraint</u> rather than the classical <u>budget constraint</u> shown in (2). If we wish to relax the cash constraint this can be done by introducing the concept of <u>credit</u>. But in doing so we are immediately forced into considerations of banking, bankruptcy* and bank failure. Credit has been discussed elsewhere [4]. and we do not discuss it in detail here.

In these m markets there is an explicit simple way in which prices are formed. All individuals are required to bid simultaneously in all markets. After they have bid, all money taken in is added up then divided by the supply of the good

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*There are several problems involving bankruptcy which must eventually be solved. They can be avoided by forbidding credit and defining what is meant by "enough money." This device, although unrealistic, enables us to separate out the several difficulties in modelling trade and to concentrate on an easier problem.

(4) $$b_j = \sum_{i=1}^{n} b_j^i$$

(5) $$p_j = \frac{b_j}{a_j} .$$

The amount which each individual i obtains of commodity j $(j = 1, \ldots, m)$ is:*

(6) $$x_j^i = \frac{b_j^i}{p_j} = \frac{b_j^i}{b_j} a_j .$$

The payoff function to each trader i is given by:

(7) $$\pi^i(b^1, b^2, \ldots, b^n) = u^i(x_1^i, x_2^i, \ldots, x_m^i, x_{m+1}^i)$$

where $b^i = (b_1^i, b_2^i, \ldots, b_m^i)$ and

(8) $$x_{m+1}^i = a_{m+1}^i - \sum_{k=1}^{m} b_k^i + \sum_{k=1}^{m} a_k^i p_k .$$

The feasible set of trades and hence possibly the Pareto set of trades are restricted by the trading mechanism. This effect of a trading mechanism on the feasible set has been shown by Levitan and Shubik [5] and for this model by Shapley and Shubik elsewhere. An example however is given for the Edgeworth box where one of the two commodities is used as a money. The full discussion of this diagram is given in the paper by Shapley and Shubik noted above.

The completely shaded area in Figure 1 indicates the attainable

*There are some problems in evaluating this when $b_j = 0$. Here if $b_j = 0$ we define $x_j^i = 0$.

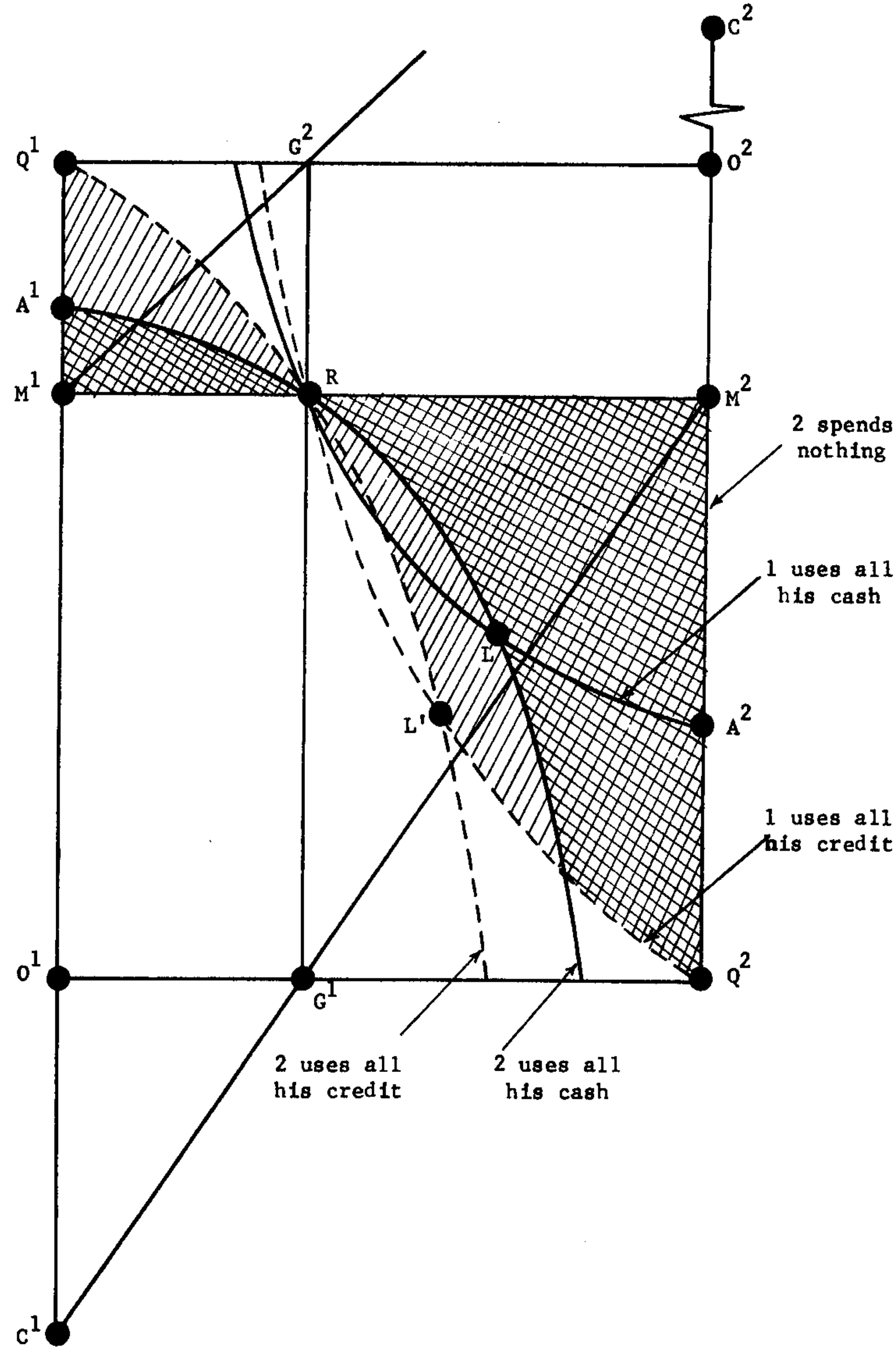

FIGURE 1. The Feasible Set with and without Credit*

*Reproduced from Figure 5 in Shapley and Shubik [4].

trades using one of the two commodities as a money. The half shading indicates the extra attainable trades when credit is permitted (i.e. individuals can bid more than they have cash on hand).

The game tree description of this game is shown in Figure 2 for two traders.*

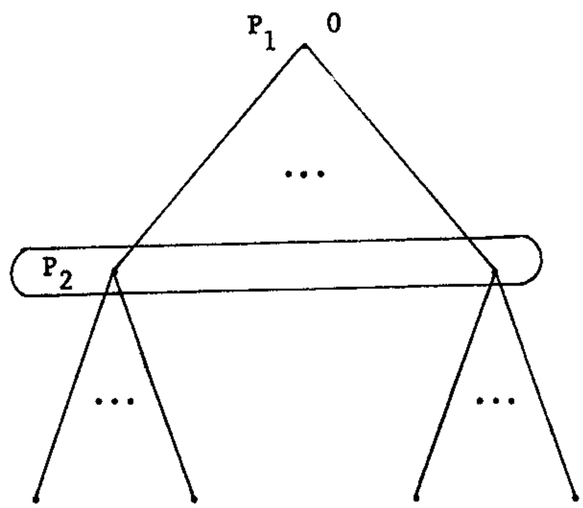

FIGURE 2

2.1. The Noncooperative Equilibrium

A noncooperative equilibrium will exist if there is a set of strategies $(\hat{b}^1, \hat{b}^2, \ldots, \hat{b}^n)$ such that:

$$\max_{b^i} \pi^i(\hat{b}^1, \hat{b}^{i-1}, b^i, \hat{b}^{i+1}, \ldots, \hat{b}^n) \dashrightarrow b^i = \hat{b}^i \quad \text{for } i = 1, \ldots, n .$$

It has been proved elsewhere [6] that the noncooperative equilibrium solution for the game whose payoffs are given in (7) always exists.

*Technically as this game has continuous strategy spaces we cannot use a finite tree, but for purposes of illustration we assume a finite grid on these strategic choices.

2.2. Convergence

Shapley and Shubik [6] have also shown that if the game is replicated by replacing each trader by k traders identical with each of the original traders, then as k increases the noncooperative equilibria will approach the competitive equilibria.

3. TRADE WITH EXOGENOUS UNCERTAINTY

3.1. Symmetric Information

Suppose that there is uncertainty in the economy concerning the endowments available to the traders. Suppose that there are s possible states for the initial endowments and that state u occurs with a probability of $\rho_u > 0$, where $\sum_{u=1}^{s} \rho_u = 1$. Suppose furthermore that all individuals are informed of these probabilities. Arrow [7] and Debreu [1] have shown that by enlarging the number of commodities to be traded, so that instead of $m+1$ commodities as in Section 2, there are $(m+1)s$ commodities, a competitive equilibrium price system can be shown to exist. The interpretation of this enlargement of commodities is that trade takes place in "contingent commodities" or futures contracts.

Consider an economy with two states of Nature: rain or shine. The commodity "oranges" can be regarded as two commodities "oranges if it rains" and "oranges if the sun shines."

Before we model this economy as a game we distinguish among <u>pure futures markets</u>, <u>compound futures markets</u> and <u>spot markets</u> (or certain futures markets). A simple example will help to make the distinctions clear. Suppose that there is one commodity "oranges" and two states of Nature: rain or shine. There is a probability ρ for state 1 (rain) and $1-\rho$ for state 2 (shine). If it rains there will be A oranges for sale and if it shines there will be B oranges for sale (assume that $B > A$).

We may consider three different market organizations where the first has two black boxes up for bids, the second has one black box and the third has one black box and one transparent box.

(1) Pure Futures

A with ρ 0 with $(1-\rho)$	B with $(1-\rho)$ 0 with ρ

The bidders are confronted with two black boxes whose contents they cannot see. They know that the first box contains A oranges with probability ρ; otherwise it is empty. The second box has B oranges with probability $(1-\rho)$; otherwise it is empty.

A pure future is a single black box which pays out for one contingency. If the contingency does not arise then all bids on that box are lost--they have turned out to be worthless. The way in which Arrow and Debreu have modelled uncertainty is this.

(2) Compound Futures

A with ρ B with $(1-\rho)$

The bidders are confronted with one black box which pays out varying amounts depending upon which contingency occurs. Individuals who have bid on the black box will obtain a percentage of the orange crop, but the amounts they obtain will vary with the contingency that occurs.

In this paper we carry out an analysis of this market structure where there are m compound futures markets, one for each commodity, where each box has the contents arising from s states. This contrasts with the Arrow and Debreu $(m+1)s$ markets.

(3) Spot Markets

A for certain	B-A with $(1-\rho)$ 0 with ρ

The market organizer could assign A oranges to a market which is independent of state and assign the remaining B-A oranges available in the second state to a pure futures market. Here we may assume that all individuals can see the contents of the first market.

A minor difficulty in terminology arises in talking about a spot market. Common usage refers to a market for a good to be obtained now with certainty. We might wish to consider a certain futures market where a good is obtained at a later date with certainty. As the models considered here are essentially static we do not make this distinction.

The Extensive Form

Figure 3 shows an economy with two contingent states. Nature moves first and selects the first state with probability ρ and the second with $1-\rho$. Then traders 1 and 2 move without further information. A more detailed description of the traders' moves would be required to distinguish the different markets noted above.

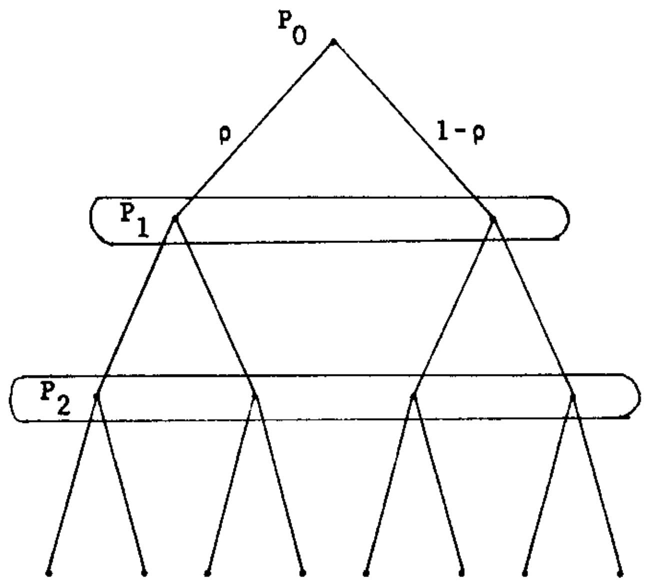

FIGURE 3

3.2. Nonsymmetric Information and a Trading Game

Suppose that one trader is informed about the move of Nature while the other is not informed. This possibility is illustrated in the extensive form shown in Figure 4. Here we note that Trader 1 moves

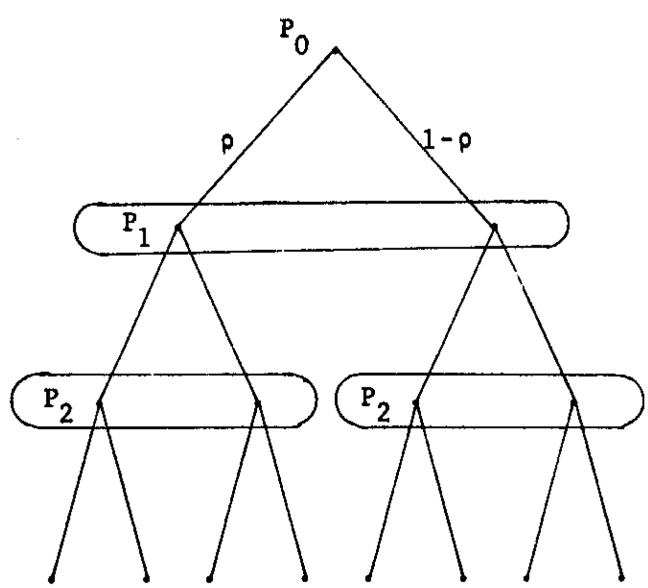

FIGURE 4

without knowing the outcome from Nature's move, then Trader 2 moves knowing Nature's move but not knowing the move of Trader 1.

If the information conditions faced by the traders are not symmetric (as is shown above) then the way in which we can think of them in terms of the "black boxes" described above is as though some of the traders are equiped with special glasses which enable them to see part or all of the contents of the various black boxes.

The Arrow-Debreu analysis runs into difficulties when the traders no longer perceive the same number of contingent commodities. Radner [8] has shown that with nonsymmetric information conditions it is possible that no competitive equilibrium price system exists.

The price system in the noncooperative game with markets and trade using a money, may nevertheless exist. In Section 4 we show that a noncooperative equilibrium always exists even with nonsymmetric information. In Section 5 a simple example is calculated and examined to show the behavior of the noncooperative equilibrium under different conditions.

3.3. Futures Markets

It is possible to construct a strategic game and solve for its noncooperative equilibria when any combinations of futures markets exist. In order to do so however we must completely specify the nature of the contracts and method of trade. Details such as who gets the money bid on contingent contracts when the contingency does not materialize must be noted. An explicit model of trade with simple futures markets is given in a subsequent paper [9].

4. THE EXISTENCE OF A NONCOOPERATIVE EQUILIBRIUM WITH NONSYMMETRIC INFORMATION

4.1. The Existence Proof

For a positive integer r we shall denote by I_r the set $\{1, \ldots, r\}$, by E^r the Euclidean space of dimension r, and by Ω^r the non-negative orthant of E^r.

Let

I_n = the set of traders

I_{m+1} = the set of commodities

I_s = the set of "states of nature," which occur with positive probabilities $\rho_1, \ldots, \rho_s$.

The initial allocation of trader i is a vector $a^i \in \Omega^{sm+1}$, where ${}^k a^i_j$ is the amount of commodity j available to i in state k (for $j \in I_{m+1}$, $k \in I_s$), and a^i_{m+1} represents the money held by i. The traders' utility functions are real-valued:

$$u^i : \Omega^{m+1} \longrightarrow \Omega^1 , \quad i \in I_n$$

and are assumed to be concave, continuous and non-decreasing. We shall say that trader i "desires" good j if $u^i(x^i)$ is an increasing function of the variable x^i_j, for any fixed choice of the other variables x^i_ℓ. A trader i for whom $a^i_{m+1} > 0$ will be called "moneyed." Finally, to complete the basic data of the market, we must specify the information of every trader i. This will consist of a partition $\mathcal{P}^i$ of I_s into non-empty sets, where the states in any member of $\mathcal{P}^i$ are precisely those that i cannot tell apart.

When we drop an index and use a bar it will indicate summation over the indexing set. Thus, for $x^i \in \Omega^{sm}$, ${}^{k}\bar{x}^i$ means $\sum_{j \in I_m} {}^{k}x^i_j$, etc. We assume that ${}^{k}\bar{a}_j > 0$ for all k and j.

To cast the rules of the market in the form of a game we define the strategy set of trader i to be

$$S^i = \{b^i \in \Omega^{sm} : {}^{k}\bar{b}^i \leq a^i_{m+1} \text{ for } k \in I_s \text{, and } {}^{k'}b^i = {}^{k}b^i \text{ whenever } k \text{ and } k' \text{ are both elements of the same member of } \mathcal{P}^i\}$$

Here ${}^{k}b^i_j$ is i's bid for commodity j in state k. But since i is uninformed about which state in $S \in P^i$ actually occurs, he is constrained to bid the same amount for each state in S (i.e. ${}^{k'}b^i_j = {}^{k}b^i_j$ if $k' \in S$ and $k \in S$). Note that S^i is a $|\mathcal{P}^i| \times m$ dimensional set if i is moneyed, and a single point if he is not. Thus the lack of information of a moneyed trader shows up in a reduction of the dimension of his strategy set.

The product $S^1 \times \ldots \times S^n$ will be denoted S; it is a compact convex set of dimension at most $(\sum_{i \in I_n} |\mathcal{P}^i|) \times m$.

The outcome of the game engendered by a particular $b \in S$ is determined in three simple steps. First, we calculate a "price vector for state k," ${}^{k}p \in \Omega^m$, by dividing the amount bid for each good by the total supply:

$${}^{k}p_j = {}^{k}\bar{b}_j / {}^{k}\bar{a}_j \text{, all } j \in I_m .$$

Next, we calculate the "final allocation" that results when the bids are executed:

$$ {}^k\xi^i_j(b) = {}^k x^i_j = \begin{cases} {}^k b^i_j / {}^k p_j & , \text{ if } j \in I_m \text{ and } {}^k p_j > 0 ; \\ 0 & , \text{ if } j \in I_m \text{ and } {}^k p_j = 0 ; \\ a^i_j - \sum_{\ell \in I_m} {}^k b^i_\ell + \sum_{\ell \in I_m} {}^k a^i_\ell \, {}^k p_\ell , & \text{ if } j = m+1 . \end{cases} $$

(Clearly this is nonnegative.) Finally, we calculate the final utilities of "payoffs" to the traders in state k :

$$ {}^k P^i(b) = u^i({}^k\xi^i(b)) , \quad \text{all } i \in I_n . $$

Thus the average payoff over all the states is clearly $P^i(b) = \sum_{k \in I_s} \rho_k \, {}^k P^i(b)$. In this way, we obtain an n-person game in the standard "strategic" (or "normal") form.

An "equilibrium point" (or "EP") of this game is defined to be a $b^* \in S$ with the property that, for each $i \in I_n$,

$$ P^i(b^*) = \max_{b^i \in S^i} P^i(b^* | b^i) , $$

where $b^*|b^i$ is b^* but with b^{*i} replaced by b^i .

Theorem 1. Assume that for any good in I_m there are at least two moneyed traders who desire it. Then an EP exists.

Proof. To prove this, it will be useful to define continuous games which approximate the game above. We assume, in effect, that some outside agency places a fixed bid of $\epsilon > 0$ in each of the sm markets. This does not change the players' strategy spaces, S^i , but it does of course change the outcomes and payoffs (which will be denoted P^i_ϵ). It is obvious that $P^i_\epsilon(b)$ is continuous. To see that it is concave in b^i (if we keep $\tilde{b}^h$ fixed for $h \neq i$) first note that

$$P^i_\epsilon(\tilde{b}|b^i) = \sum_{k \in I_s} \rho_k u^i({}^k\xi^i_{1,\epsilon}(\tilde{b}|b^i), \ldots, {}^k\xi^i_{m+1,\epsilon}(\tilde{b}|b^i)) \; *$$

where, for $\ell \in I_m$, ${}^k\xi^i_{\ell,\epsilon}$ (as a function of the variable b^i) has the form

$${}^k\xi^i_{\ell,\epsilon}(\tilde{b}|b^i) = \frac{{}^kC_\ell \, {}^kb^i_\ell}{{}^kD_\ell + {}^kb^i_\ell} \qquad ({}^kC_\ell \text{ and } {}^kD_\ell \text{ constants}).$$

This is concave in ${}^kb^i_\ell$ (hence in b^i) since ${}^kC_\ell > 0$ and ${}^kD_\ell > 0$. Moreover ${}^k\xi^i_{m+1,\epsilon}$ is a linear (hence also concave) function of b^i. Since u^i is nondecreasing this implies that $u^i({}^k\xi^i_\epsilon(\tilde{b}|b^i))$ is concave as a function of b^i. Then clearly so is $P^i_\epsilon(\tilde{b}|b^i)$. Now define

$$\phi^i(\hat{b}) = \{\tilde{b}^i \in S^i : P^i_\epsilon(\hat{b}|\tilde{b}^i) = \max_{b^i \in S^i} P^i_\epsilon(\hat{b}|b^i)\}$$

and consider the correspondence from S into S given by

$$\phi(\hat{b}) = \phi^1(\hat{b}) \times \ldots \times \phi^n(\hat{b}) .$$

It is easy to deduce that ϕ is convex valued and u.s.c. (using the fact that S^i is compact and convex, and P^i_ϵ is continuous in b and concave in b^i). Therefore by Kakutani's fixed point theorem there is a $b \in S$ such that

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* ${}^k\xi^i_{j,\epsilon}$ is the allocation computed on the assumption that there is an external bid of ϵ placed in each commodity for each state.

(9) $\qquad b \in \emptyset(b)$

which will clearly be an E.P.

Taking a convergent subsequence of these equilibrium points, as $\epsilon \to 0$, we would like to conclude that their limit is an equilibrium point of the original game. This will indeed be the case if the limit b^* is a point of continuity of P, i.e. ${}^k\bar{b}_j^* > 0$ for all $k \in I_s$ and $j \in I_m$. This is proved in Lemma 2 below.

Q.E.D.

Remarks.

1. We could have made the model somewhat more general by stipulating that any good j will have different utilities in different states. Then

$$u^i = \sum_{k \in I_s} \rho_k {}^k u^i \quad \text{where} \quad {}^k u^i : \Omega^{m+1} \to \Omega^1$$

gives the utility of goods in state j. Theorem 1 remains true in this case as well (so does Theorem 2 stated later). Of course, we will now require that for any $(k,j) \in I_s \times I_m$ there exist two moneyed traders ℓ and r such that ${}^k u^\ell$ and ${}^k u^r$ are both increasing in the j^{th} variable.

2. Another variation,* for which Theorems 1 and 2 continue to hold, would be to let money be state dependent. Then we would say that trader i is moneyed in state h if $\min_{k \in S} {}^k a^i_{m+1} > 0$, where $h \in S \in \mathcal{P}^i$. He would be required to bid within this minimum in the states in S.

*This was suggested by L. S. Shapley.

3. Finally we may allow each trader to have his own subjective probability distribution $(\rho_1^i, \ldots, \rho_s^i)$ on I_s. His payoff would then involve his subjective probabilities. Clearly this does not affect Theorems 1 and 2.

It will be convenient to consider $\Omega^{s(m+1)}$ with an axis for each commodity in each state. By axis jk we will mean the axis for the j^{th} commodity in the k^{th} state. e^{jk} will denote the vector in $\Omega^{s(m+1)}$ whose jk^{th} component is 1 and all others are 0. Let us define for $i \in I_n$, $U^i : \Omega^{s(m+1)} \to \Omega^1$ by

$$U^i(x) = \sum_{k \in I_s} \rho_k u^i({}^k x)$$

Then $P^i(b) = U^i({}^1\xi^i(b), \ldots, {}^s\xi^i(b))$. Note that if i desires good j then U^i is increasing in the variables jk $(k \in I_s)$.

<u>Lemma 1</u>. Let $U : \Omega^{s(m+1)} \to \Omega^1$ be a continuous, non-decreasing function that is actually increasing in the variables jk (for a fixed $j \in I_m$, and all $k \in I_s$), and let H be a positive constant. Then a positive number $\hat{h} = \hat{h}(U, j, H)$ exists such that for all x and y in $\Omega^{s(m+1)}$ and $\eta_i \geq 0$, if

$$\|x\|^* \leq H \quad \text{and} \quad \|y-x\| \leq h\ ,$$

then

$$U(y + \sum_{i \neq jk} \eta_i e^i + e^{jk}) > U(x)$$

for any $k \in I_s$.

$^*\|\ \|$ stands for the maximum norm.

Proof. By Lemma C in [6], there exist positive numbers $h(U, jk, H)$ such that

$$U(y+e^j) > U(x)$$

if x and y satisfy the conditions of the lemma. Put $\hat{h} = \min_{k\in I_s}[h(U, jk, H)]$. Then, since U is non-decreasing, we get for any $t \in I_s$,

$$U(y + \sum_{i\neq jt} \eta_i e^i + e^{jt}) \geq U(y+e^{jt}) > U(x) .$$

Q.E.D.

Lemma 2.* Let u^i be continuous, concave and non-decreasing for each $i \in I_n$. Suppose that there are at least two moneyed traders that desire good $j^* \in I_m$. Then there is a positive constant C_{j^*} such that for every $\epsilon > 0$, every $k \in I_s$, and $b \in S$ satisfying (9) we have

$$^k\overline{b}_{j^*} + \epsilon > C_{j^*} .$$

Proof. Wlog $j^* = m$, and traders 1 and 2 desire it. Define

(10) $$H = \max\{^k\overline{a}_j : j \in I_{m+1}, k \in I_s\}$$

(11) $$h = \min\{\hat{h}(U^1, m, H), \hat{h}(U^2, m, H)\}$$

(12) $$A = \frac{1}{m+1}\min[a^1_{m+1}, a^2_{m+1}]$$

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*The proof of Lemma 2 closely follows the proof of Lemma 4 in [6].

where the function $\hat{h}$ in (11) is that defined in Lemma 1. Note that these three constants are independent of b and ϵ, and are positive. Let $\delta = \min_{k' \epsilon I_s} \{{}^{k'}\bar{b}_m\}$, and suppose that this minimum is attained for $t \epsilon I_s$. Let x be the allocation that results from the bid matrix b in the "ϵ-modified" game.

Suppose first that the joint condition

$$(13) \qquad {}^t b_m^i \leq \delta/2 \quad \text{and} \quad a_{m+1}^i - \sum_{j \epsilon I_m} {}^t b_j^i \geq A$$

holds for at least one of $i = 1, 2$; say for $i = 1$. Then an increase Δ in 1's bid for good m in states $k \epsilon S$ (where $t \epsilon S \epsilon \mathcal{P}^1$) would be feasible if $\Delta \leq A$ (for note that ${}^k b^1 = {}^t b^1$ for $k \epsilon S$, hence $a_{m+1}^1 - \sum_{j \epsilon I_m} {}^k b^1 \geq A$ for all $k \epsilon S$). Take $0 < \Delta \leq \min(\epsilon, A)$. Then the incremental effect on 1's final holding would be:

$$(15a) \qquad {}^k x_j^1(\Delta) - {}^k x_j^1 = 0 \quad \text{for } j \epsilon I_{m-1}, \; k \epsilon I_s$$

$$(15b) \qquad {}^k x_j^1(\Delta) - {}^k x_j^1 = 0 \quad \text{for } j = m \text{ or } m+1, \; k \notin S,$$

$$(15c) \qquad {}^k x_m^1(\Delta) - {}^k x_m^1 = \frac{{}^k\bar{a}_m({}^k b_m^1 + \Delta)}{{}^k\bar{b}_m + \Delta + \epsilon} - \frac{{}^k\bar{a}_m {}^k b_m^1}{{}^k\bar{b}_m + \epsilon}$$

$$= [{}^k\bar{a}_m \Delta] \left[\frac{{}^k\bar{b}_m - {}^k b_m^1 + \epsilon}{({}^k\bar{b}_m + \epsilon)({}^k\bar{b}_m + \epsilon + \Delta)}\right]$$

$$\geq [{}^k\bar{a}_m \Delta] \left[\frac{{}^k\bar{b}_m/2 + \epsilon/2 + \Delta/2}{({}^k\bar{b}_m + \epsilon)({}^k\bar{b}_m + \epsilon + \Delta)}\right]$$

$$= \frac{{}^k\bar{a}_m \Delta}{2({}^k\bar{b}_m + \epsilon)}, \quad \text{for } k \epsilon S.$$

$$(15d) \qquad {}^{k}x^{1}_{m+1}(\Delta) - {}^{k}x^{1}_{m+1} = \left(\frac{{}^{k}a^{1}_{m}}{{}^{k}\bar{a}_{m}} + 1 \right) \Delta \geq -\Delta \, , \quad \text{for} \quad k \in S \, .$$

Define

$$z = -\frac{2(\delta+\epsilon)}{{}^{t}\bar{a}_{m}} \sum_{k \in S} e^{(m+1)k} \, .$$

Then we have

$$(16) \qquad x^{1}(\Delta) \geq x^{1} + \frac{{}^{t}\bar{a}_{m}\Delta}{2(\delta+\epsilon)}(z + \sum_{k \in S} \eta_{k} e^{mk})$$

where $\eta_k \geq 0$, and $\eta_t = 1$. By Lemma 1, if both $x^1 + z \geq 0$ and $||z|| \leq h$, then

$$U^{1}(x^{1} + z + \sum_{k \in S} \eta_{k} e^{mk}) > U^{1}(x^{1}) \, .$$

Since U^1 is concave, this implies that

$$U^{1}\left(x^{1} + \frac{{}^{t}\bar{a}_{m}\Delta}{2(\delta+\epsilon)}(z + \sum_{k \in S} \eta_{k} e^{mk}) \right) > U^{1}(x^{1})$$

for sufficiently small Δ , and hence by (16), remembering that U^1 is non-decreasing, that

$$u^{1}(x^{1}(\Delta)) > u^{1}(x^{1})$$

for such Δ . But this means that trader 1 could have improved, contradicting (9). Hence either $x^1 + z \ngeq 0$ or $||z|| > h$. If the former, we have

$$ {}^{k}x^{1}_{m+1} - \frac{2(\delta+\varepsilon)}{{}^{t}\bar{a}_{m}} < 0 $$

for at least one $k \in S$, which implies (since ${}^{k}x^{1}_{m+1} \geq A$)

$$ \text{(17a)} \qquad \boxed{\delta+\varepsilon > \frac{{}^{t}\bar{a}_{m}\,{}^{k}x^{1}_{m+1}}{2} \geq \frac{{}^{t}\bar{a}_{m}A}{2}\ .} $$

If the latter, then

$$ \frac{2(\delta+\varepsilon)}{{}^{t}\bar{a}_{m}} > h $$

i.e.,

$$ \text{(17b)} \qquad \boxed{\delta+\varepsilon > \frac{h\,{}^{t}\bar{a}_{m}}{2}\ .} $$

Now consider the case where (13) fails for both $i = 1, 2$. Wlog we may assume that

$$ {}^{t}b^{1}_{m} \leq \delta/2\ . $$

From the failure of (13) for $i = 1$, we get

$$ \sum_{j\in I_m} {}^{t}b^{1}_{j} > a^{1}_{m+1} - A \geq mA\ . $$

Hence ${}^{t}b^{1}_{j} > A$ for at least one $j \in I_m$. If $j = m$, we have $\delta \geq 2\,{}^{t}b^{1}_{m} > 2A$ and so

(17c) $\boxed{\delta+\varepsilon > 2A\ .}$

If $j \neq m$, wlog let $j = 1$. Trader 1 could then decrease ${}^k b_1^1$ by a small $\Delta > 0$ and increase ${}^k b_m^1$ by the same amount (for $k \in S$), with the incremental effect:

$$\text{(18a)} \qquad {}^k x_1^1(\Delta) - {}^k x_1^1 = \frac{{}^k\bar{a}_1({}^k b_1^1 - \Delta)}{{}^k\bar{b}_1 - \Delta + \varepsilon} - \frac{{}^k\bar{a}_1\, {}^k b_1^1}{{}^k\bar{b}_1 + \varepsilon}$$

$$\geq -\left(\frac{{}^k\bar{a}_1 \Delta}{{}^k\bar{b}_1 + \varepsilon}\right) \qquad \text{for } k \in S$$

$$\text{(18b)} \qquad {}^k x_j^1(\Delta) - {}^k x_j^1 = 0 \quad \text{for } j = 2, \ldots, m-1 \text{ and } k \in I_s$$

$$\text{(18c)} \qquad {}^k x_j^1(\Delta) - {}^k x_j^1 = 0 \quad \text{for } j = 1, m, m+1 \text{ and } k \notin S$$

$$\text{(18d)} \qquad {}^k x_m^1(\Delta) - {}^k x_m^1 \geq \frac{{}^k\bar{a}_m \Delta}{2({}^k\bar{b}_m + \varepsilon)}$$

(by the same calculation as at (15c)); and

$$\text{(18e)} \qquad {}^k x_{m+1}^1(\Delta) - {}^k x_{m+1}^1 = \left(\frac{{}^k a_m^1}{{}^k\bar{a}_m} - \frac{{}^k a_1^1}{{}^k\bar{a}_1}\right)\Delta$$

$$\geq -\left(\frac{{}^k a_1^1}{{}^k\bar{a}_1}\Delta\right) .$$

Define

$$z = \frac{-2(\delta+\varepsilon)}{{}^t\bar{a}_m}\left[\sum_{k\in S} \frac{{}^k\bar{a}_1}{({}^k\bar{b}_1 + \varepsilon)} e^{1k} + \sum_{k\in S} \frac{{}^k a_1^1}{{}^k\bar{a}_1} e^{(m+1)k}\right] .$$

Then (16) is satisfied as before; and, arguing as before, we find that either $x^1 + z \ngeq 0$ or $\|z\| > h$. If the former, then either $|{}^k z_1| > {}^k x_1^1$ or $|{}^k z_{m+1}| > {}^k x_1^1$ for some $k \in S$. But

$$ {}^k x_1^1 = \frac{{}^k\bar{a}_1 \, {}^k b_1^1}{{}^k\bar{b}_1 + \epsilon} > \frac{{}^k\bar{a}_1 A}{{}^k\bar{b}_1 + \epsilon} $$

and

$$ {}^k x_{m+1}^1 \geq \frac{{}^k a_1^1 \, {}^k b_1^1}{{}^k\bar{a}_1} \geq \frac{{}^k a_1^1 A}{{}^k\bar{a}_1} \; . $$

In either case we get

$$ \text{(17d)} \qquad \boxed{\delta + \epsilon > \frac{{}^t\bar{a}_1 A}{2} \; .} $$

If the latter, then either $|{}^k z_1| > h$ or $|{}^k z_{m+1}| > h$ for some $k \in S$. In the first case the inequalities ${}^k\bar{b}_1 + \epsilon > A$ and ${}^k\bar{a}_1 < H$, yield

$$ \text{(17e)} \qquad \boxed{\delta + \epsilon > \frac{{}^t\bar{a}_m hA}{2H} \; ;} $$

in the second, ${}^k a_1^1 \leq {}^k\bar{a}_1$ yields

$$ \text{(17f)} \qquad \boxed{\delta + \epsilon \geq \frac{{}^t\bar{a}_m h}{2} \; .} $$

This exhausts all cases. Collecting (17a) through (17f) we see that it will suffice to take C_{j*} to be the minimum of $\{{}^k\bar{a}_m A/2,\ {}^k\bar{a}_m h/2,\ 2A,\ {}^k\bar{a}_m Ah/2H : k \in I_s\}$.

Q.E.D.

4.2. The Value of Information

A simple two person example serves to illustrate the possibilities of the worth of changes in information. In particular we show that the expected payoffs to the traders under symmetric incomplete information can be higher than under complete information, but that if one individual is informed but the other is not the former gains and the latter loses. Figures 3 and 4 show the game trees for symmetric incomplete information and nonsymmetric incomplete-complete information. Figure 5 shows the game tree for complete information. We now consider the three different 2-person models.

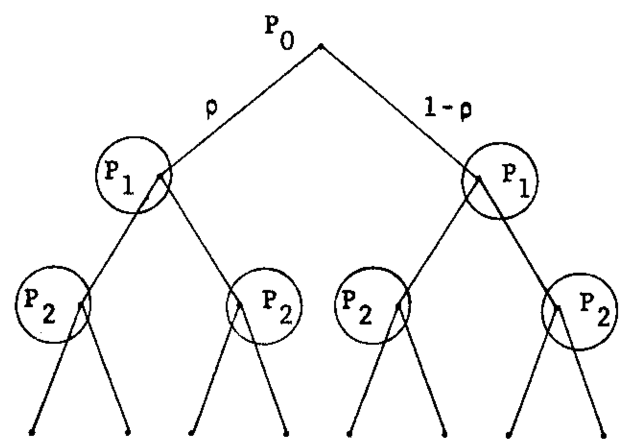

FIGURE 5

Suppose that there are two traders trading in one commodity using a commodity money. There are two states of the universe "rain" or "shine." Under the first state the first individual has 4 units of the good and the second has 12. Under the second state ownership is reversed. Each individual has 100 units of money under all circumstances. The utility function of individual i $(i = 1, 2)$ is:

$$U^i = \rho \log(q_1^i) + (1-\rho)\log(q_2^i) + M - m^i$$

where q_j^i is the amount of the consumer good in state j consumed by trader i. m^i is the net change in money of i after trade.

Let x_1 and x_2 and y_1 and y_2 be the amounts of money spent by traders 1 and 2 in states 1 and 2. If say, the first trader cannot distinguish between states 1 and 2 his decision variable will become x rather thsn x_1 and x_2. For simplicity we set $\rho = 1-\rho = 1/2$. There is a single compound futures market.

(a) Symmetric Incomplete Information

Here the payoff for trader 1 is:

$$\Pi_1 = \rho \log\left(\frac{16x}{x+y}\right) + (1-\rho)\log\left(\frac{16x}{x+y}\right) - x + 100 + \frac{1}{2}(x+y)$$

and similarly for trader 2. It is easy to check that $x = y = 1$ and that the final expected payoff for each $3 \log 2 + 100$, or $.90309 + 100$.

(b) Symmetric Complete Information

Here the payoff for trader 1 is:

$$\Pi_1 = \rho \log\left(\frac{16x_1}{x_1+y_1}\right) + (1-\rho)\log\left(\frac{16x_2}{x_2+y_2}\right) - \rho x_1 - (1-\rho)x_2 + 100$$
$$+ \frac{\rho}{4}(x_1+y_1) + \frac{3}{\rho}(1-\rho)(x_2+y_2)$$

and there is a similar expression for trader 2. Hence $x_1 = 4/(1+\sqrt{3}) = y_2$.

Here the final payoff for each is .8869 + 100 and $x_2 = 4/\sqrt{3}(1+\sqrt{3}) = y_1$. This is less than was obtained with incomplete information.

(c) <u>Nonsymmetric Incomplete-Complete Information</u>

Here the payoff for trader 1 is:

$$\Pi_1 = \rho \log\left(\frac{16x}{x+y_1}\right) + (1-\rho)\log\left(\frac{16x}{x+y_2}\right) - x + 100 + \frac{\rho}{4}(x+y_1) + \frac{3}{4}(1-\rho)(x+y_2)$$

and for trader 2 it is:

$$\Pi_2 = \rho \log\left(\frac{16y_1}{x+y_1}\right) + (1-\rho)\log\left(\frac{16y_2}{x+y_2}\right) - \rho y_1 - (1-\rho)y_2 + 100$$
$$+ \frac{3}{4}\rho(x+y_1) + \frac{1}{4}(1-\rho)(x+y_2) \ .$$

The maximization conditions give us:

$$\frac{y_1}{x(x+y_1)} + \frac{y_2}{x(x+y_2)} = 1 \ ,$$

$$\frac{x}{y_1(x+y_1)} = \frac{1}{4} \quad \text{and} \quad \frac{x}{y_2(x+y_2)} = \frac{3}{4}$$

which have a solution $x \simeq 1.032$, $y_1 = 1.580$ and $y_2 = .7655$ and the payoffs are $\Pi_1 = .8505 + 100$ and $\Pi_2 = 1.8506 + 100$.

A change from incomplete information to information for one benefits the recipient and hurts the other. A further increase to complete information may leave both worse off than with symmetric incomplete information.

These observations appear to be quite general and depend neither on the special example nor the two trader case. They raise several problems concerning the modeling of information which are discussed further in a subsequent paper.

It is, of course, obvious that increasing a trader's information while keeping that of others fixed will only improve his payoff at equilibrium, since this makes his strategy set larger, leaving those of the others unchanged.

5. AN EXAMPLE

5.1. The Replicated Market

There are two commodities and two states of the economy and a third commodity used as a money. Under State 1 endowments are $(A,0,M)$ and $(0,B,M)$ for traders 1 and 2 respectively. Under State 2 endowments are $(B,0,M)$ and $(0,A,M)$.

A trader i of type 1 cannot distinguish states hence his strategy consists of two numbers x_1^i and x_2^i which are the amounts he bids on goods 1 and 2 respectively. A trader j of type 2 can distinguish states hence his strategy consists of four numbers y_{11}^j, y_{21}^j and y_{12}^j, y_{22}^j.

The payoff to traders of types 1 and 2 are, respectively:

$$\Pi_1^i = \rho\frac{kAx_1^i}{x_1+y_{11}} + (1-\rho)\frac{kBx_1^i}{x_1+y_{12}} + \rho\frac{kBx_2^i}{x_2+y_{21}} + (1-\rho)\frac{kAx_2^i}{x_2+y_{22}} \tag{19}$$

$$- x_1^i - x_2^i + M + \frac{\rho}{k}(x_1+y_{11}) + \frac{(1-\rho)}{k}(x_1+y_{12})$$

where $x_s = \sum_{j=1}^{n} x_j^i$, $y_{sn} = \sum_{j=1}^{n} y_{sn}^j$

$$(20)\qquad \Pi_2^j = \rho\left[\log\left\{\frac{Ay_{11}^j}{x_1+y_{11}}\right\}\left\{\frac{By_{21}^j}{x_2+y_{21}}\right\}\right] + (1-\rho)\left[\log\left\{\frac{kBy_{12}^j}{x_1+y_{12}}\right\}\left\{\frac{kAy_{22}^j}{x_2+y_{22}}\right\}\right]$$

$$- \rho(y_{11}^j + y_{21}^j) - (1-\rho)(y_{12}^j + y_{22}^j) + M + \frac{\rho}{k}(x_2+y_{21}) + \frac{1-\rho}{k}(x_2+y_{22}) \ .$$

First order maximization conditions are as follows:

$$(21)\qquad \rho kA\left\{\frac{x_1+y_{11}-x_1^i}{(x_1+y_{11})^2}\right\} + (1-\rho)kB\left\{\frac{x_1+y_{12}-x_1^i}{(x_1+y_{12})^2}\right\} - 1 + \frac{1}{k} = 0$$

$$(22)\qquad \rho kB\left\{\frac{x_2+y_{21}-x_2^i}{(x_2+y_{21})^2}\right\} + (1-\rho)kA\left\{\frac{x_2+y_{22}-x_2^i}{(x_2+y_{22})^2}\right\} - 1 = 0$$

$$(23)\qquad \rho\frac{1}{y_{11}^j}\left\{\frac{x_1+y_{11}-y_{11}^j}{x_1+y_{11}}\right\} - \rho = 0$$

$$(24)\qquad \rho\frac{1}{y_{21}^j}\left\{\frac{x_2+y_{21}-y_{21}^j}{x_2+y_{21}}\right\} - \rho + \frac{\rho}{k} = 0$$

$$(25)\qquad \frac{(1-\rho)}{y_{21}^j}\left\{\frac{x_2+y_{21}-y_{21}^j}{x_1+y_{12}}\right\} - (1-\rho) = 0$$

$$(26)\qquad \frac{(1-\rho)}{y_{22}^j}\left\{\frac{x_2+y_{22}-y_{22}^j}{x_2+y_{22}}\right\} - (1-\rho) + \frac{1-\rho}{k} = 0 \ .$$

Assuming symmetric treatment of identical players we may set $x_j = kx_j^i$ and we may drop all superscripts. Equations (21) to (26) may be rewritten as:

$$\rho A\left\{\frac{1}{x_1+y_{11}} - \frac{x_1}{k(x_1+y_{11})^2}\right\} + (1-\rho)B\left\{\frac{1}{x_1+y_{12}} - \frac{x_1}{k(x_1+y_{12})^2}\right\} = \left(\frac{k-1}{k}\right) \tag{27}$$

$$\rho B\left\{\frac{1}{x_2+y_{21}} - \frac{x_2}{k(x_2+y_{21})^2}\right\} + (1-\rho)A\left\{\frac{1}{x_2+y_{22}} - \frac{x_2}{k(x_2+y_{22})^2}\right\} = 1 \tag{28}$$

$$\left\{1 - \frac{y_{11}}{k(x_1+y_{11})}\right\} = y_{11} \tag{29}$$

$$\left\{1 - \frac{y_{21}}{k(x_2+y_{21})}\right\} = y_{21}\left(\frac{k-1}{k}\right) \tag{30}$$

$$\left\{1 - \frac{y_{12}}{k(x_1+y_{12})}\right\} = y_{12} \tag{31}$$

$$\left\{1 - \frac{y_{22}}{k(x_2+y_{22})}\right\} = y_{22}\left(\frac{k-1}{k}\right). \tag{32}$$

For $k \to \infty$ and M sufficiently large (see below)

$$\begin{aligned} x_1 &= \rho A + (1-\rho)B - 1 \\ x_2 &= \rho B + (1-\rho)A - 1 \\ y_{11} &= y_{21} = y_{12} = y_{22} = 1\,. \end{aligned} \tag{33}$$

We note that for $\rho = 1$ or 0 this gives

$$x_1 = A-1 \text{ or } B-1 \quad \text{and} \quad y_{11} = y_{12} = y_{21} = y_{22} = 1$$

$$x_2 = B-1 \text{ or } A-1 \qquad\qquad y_{11} = y_{12} = y_{21} = y_{22} = 1$$

and $p_1 = p_2 = 1$ in either case, provided $M \geq \max[A+B-2, 2]$. If there is less of the monetary commodity a boundary solution must be considered.

This extremely special example was selected in order to illustrate the structure of the payoffs and strategies and the nature of the limiting equilibrium point. It does not depend upon the separability of money in the utility functions. This was selected for simplicity in computation.

5.2. A Sensitivity Analysis

In this simple example the effect of changing the "odds" on the uncertainty can be seen to be smooth in influencing the equilibrium, the x_1 and x_2 more continuously from their values at one competitive equilibrium state to that at another.

Owing to the special features of these simple utility functions the worth of extra information is not shown. However in 4.2 we have noted that all other things being equal more information available to a single individual will give him a payoff as great or higher than without that information. This is not necessarily true for two or more.

6. CONVERGENCE OF THE EQUILIBRIUM WITH REPLICATION

Let us consider a "replication sequence" of economies $\Gamma_1, \ldots, \Gamma_k, \ldots$. There are a fixed number n of types of traders, characterized by their utilities u^i , endowments a^i , and information $\mathcal{P}^i$. The economy Γ_k has $t = nk$ traders, k of each type. An E.P. in which traders of the same type choose the same strategies is called type-symmetric (denoted T.S.E.P.)* and such an E.P. b can be represented as a vector $\hat{b}$ in $S = \overset{n}{\underset{i=1}{X}} S^i$ where S^i is the strategy set of trader-

*That a T.S.E.P. always exists is shown in [6].

type i in Γ_k. Thus, for each k, a T.S.E.P. ${}_k b$ in Γ_k gives us a price and an allocation in Γ_1 via ${}_k\hat{b} \in S$. We will say that ${}_k\hat{b}$ is an interior E.P. if it is in the relative interior of S.* Given a price vector $p \in \Omega^{sm}$, we will construct a $\tilde{p} \in \Omega^{s(m+1)}$ by putting ${}^{\ell}p_{m+1} = 1$ for all $\ell \in I_s$. (This is tantamount to having the "price" of money equal to 1.)

Define the budget set of trader i at prices $\tilde{p}$ to be

$$B^i(\tilde{p}) = \{x \in \Omega^{s(m+1)} : {}^k\tilde{p}\cdot{}^k x \le {}^k\tilde{p}\cdot{}^k a^i \text{ , and } {}^k\tilde{p}_j{}^k x_j = {}^{k'}\tilde{p}_j{}^{k'}x_j \text{ for } j \in I_m \text{ , whenever } k \text{ and } k' \text{ are in the same member of } P^i\} \text{ .}$$

The reason for putting ${}^k\tilde{p}_j\cdot{}^k x_j = {}^{k'}\tilde{p}_j\cdot{}^{k'}x_j$ is that trader i, since he cannot distinguish between k and k', must act (i.e. "bid" in our framework) identically in both states, and therefore the value (under $\tilde{p}$) of ${}^k x_j$ and ${}^{k'}x_j$ must be equal. Then we define competitive prices in the usual way.

When the inequality in the definition of $B^i(\tilde{p})$ is replaced by an equality, the resulting set will be denoted $\tilde{B}^i(\tilde{p})$.

Given a symmetric allocation $(x^1, \ldots, x^k, \ldots, x^{(n-1)k+1}, \ldots, x^{nk})$ for the traders in Γ_k, we shall denote by $\hat{x}^i$ the common value of $x^{(i-1)k+1}, \ldots, x^{ik}$.

<u>Theorem 2</u>. Suppose we have a sequence ${}_k b$ of interior T.S.E.P.'s such that ${}_k\hat{b} \to \hat{b} \in S$. Let p be the prices resulting from $\hat{b}$. Then $\tilde{p}$ is competitive for Γ_1 and indeed for each Γ_k.

---

*Conditions for having interior E.P.'s are given in [6].

Before the proof we will require a series of lemmas.

<u>Lemma 3</u>. Let ${}_kp \in \Omega^{sm}$ be the prices given by ${}_kb$, using the notation of Theorem 2, and let ${}_k\hat{b} \to \hat{b}$. Then ${}_kp \to p > 0$.

<u>Proof</u>. We first show that there is a positive constant D such that $\sum_{i \in I_n} {}_k^{\ell}\hat{b}_j^i > D$ for all k , $\ell \in I_s$ and $j \in I_m$. By Lemma 2, ${}_k^{\ell}\bar{b}_j$ must satisfy one of (17a)-(17f).

$$\text{(17a)} \qquad \Longrightarrow {}_k^{\ell}\bar{b}_j > \frac{{}^{\ell}\bar{a}_j A}{2}$$

$$\Longrightarrow k \sum_{i \in I_n} {}_k^{\ell}\hat{b}_j^{i*} > (k \sum_{i \in I_n} {}^{\ell}a_j^i)A/2 \ , \quad \text{since } {}_kb \text{ is symmetric}$$

$$\Longrightarrow \sum_{i \in I_n} {}_k^{\ell}\hat{b}_j^i > (\sum_{i \in I_n} {}^{\ell}a_j^i)A/2 \ .$$

(Here $A = a_{m+1}^t$, where t is a moneyed trader-type who desires j -- there is no need to require two distinct types of traders desiring j). (17b), (17d), (17e), (17f) lead in a similar way to lower bounds for $\sum_{i \in I_n} {}_k^{\ell}\hat{b}_j^i$. (17c) does not, but--recalling its proof--we see that it can be sharpened to ${}_k^{\ell}\hat{b}_j^t > A$; hence afortiori $\sum_{i \in I_n} {}_k^{\ell}\hat{b}_j^i > A$. We can then pick D to be the minimum of all these lower bounds.

Now

$$ {}_k^{\ell}p_j = \frac{{}_k^{\ell}\bar{b}_j}{{}^{\ell}\bar{a}_j} \geq \frac{kD}{k \sum_{i \in I_n} {}^{\ell}a_j^i} = \frac{D}{\sum_{i \in I_n} {}^{\ell}a_j^i} > 0 \ , $$

---

*Here I_n is the set of trader-types.

for all k . The theorem easily follows.

Q.E.D.

For Lemma 4, we need to introduce some further notation. For any $a \in \Omega^{s(m+1)}$ and any $\delta > 0$, let $(a)^\delta = \{x \in \Omega^{s(m+1)} : \|x-a\| < \delta\}$; for $A \subset \Omega^{s(m+1)}$ let $(A)^\delta = \bigcup_{a \in A} (a)^\delta$.

Now note that since $p > 0$ * there is a $T > 0$ such that $B^i(\tilde{p}) \subset (0)^T$, and $B^i({}_k p) \subset (0)^T$ for large enough k.

We will denote $B^i({}_k\tilde{p})$ by ${}_kB^i$ and $\tilde{B}^i(\tilde{p})$ by B henceforth.

<u>Lemma 4</u>. For any $\epsilon > 0$, there is a k_ϵ such that for all $k \geq k_\epsilon$, $\tilde{B}^i \subset ({}_kB^i)^\epsilon$.

<u>Proof</u>. There exist a k_1 and positive numbers L and M such that, for $k \geq k_1$, $Le < {}_k\tilde{p} < Me$ (where e is the unit vector in $\Omega^{s(m+1)}$) and ${}_kB^i \subset (0)^T$.

Let $G = \min\{{}^\ell\tilde{p} \cdot {}^\ell a^i : \ell \in I_s\}$. For simplicity we assume that $G > 0$. (The proof can be easily modified otherwise.) We denote by S_ℓ the set in $\mathcal{P}^i$ which contains $\ell \in I_s$.

Consider any $y \in \tilde{B}^i$. For any $\ell \in I_s$, let ${}^\ell y_{r(\ell)} = \max\{{}^\ell y_j : j \in I_{m+1}\}$. Then ${}^\ell y_{r(\ell)} \geq G/(m+1)M$ for any $\ell \in I_s$. Define ${}_kz(y)$ as follows

$$
{}^\ell_k z_j(y) = \begin{cases} \dfrac{({}^\ell\tilde{p}_j\, {}^\ell y_j - \epsilon)}{{}^\ell_k\tilde{p}_j} & \begin{array}{l} \text{if } j \in I_m,\ j = r(\tilde{\ell}) \text{ for some } \tilde{\ell} \in S_\ell \\ \text{OR} \\ \text{if } j = m+1, \text{ and } j = r(\ell) \end{array} \\[2ex] \dfrac{{}^\ell\tilde{p}_j\, {}^\ell y_j}{{}^\ell_k\tilde{p}_j} & \text{otherwise.} \end{cases}
$$

---

* 0 denotes the origin of $\Omega^{s(m+1)}$.

Let $\epsilon < GL/(m+1)D$. Then there is a $k_\epsilon \geq k_1$ such that for $k \geq k_\epsilon$

$$1-\epsilon < \frac{{}^{\ell}\tilde{p}_j}{{}^{\ell}_{k}\tilde{p}_j} < 1+\epsilon\ , \quad |{}^{\ell}_{k}\tilde{p}\cdot{}^{\ell}a^i - {}^{\ell}\tilde{p}\cdot{}^{\ell}a^i| < \epsilon$$

for all $\ell \in I_s$, $j \in I_m$.

It is easy to verify that for such k , ${}_kz(y) \in {}_kB^i$, and $\|{}_kz(y) - y\| < (T+2/L)\epsilon$ for any $y \in \tilde{\hat{B}}^i$. Since ϵ was arbitrary this proves the theorem.

Q.E.D.

<u>Lemma 5</u>. Let $p({}_kb)$ denote the price formed by the bid vector ${}_kb$. Let F be any positive constant. Then

$$\frac{{}^{\ell}p_j({}_kb)}{{}^{\ell}p_j({}_kb|b'^i)} \to 1$$

<u>uniformly</u> for all b'^i such that $\|b'^i\| \leq F$.

<u>Proof</u>.

$$\frac{{}^{\ell}p_j({}_kb)}{{}^{\ell}p_j({}_kb|b^i)} = \overline{{}^{\ell}_{k}b_j}/\overline{{}^{\ell}({}_kb|b'^i)_j}$$

$$= \frac{k \sum_{t \in I_n} {}^{\ell}_{k}\hat{b}^t_j}{k \sum_{t \in I_n \setminus \{i\}} {}^{\ell}_{k}\hat{b}^t_j + (k-1)\,{}^{\ell}_{k}\hat{b}^i_j + {}^{\ell}_{k}b'^i_j}$$

$$= \frac{1}{1 + [({}^{\ell}_{k}b'^i_j - {}^{\ell}_{k}\hat{b}^i_j)/k \sum_{t \in I_n} {}^{\ell}_{k}\hat{b}^t_j]} = \frac{1}{1+\gamma_k}\ .$$

Since $\sum_{i\in I_n} {}_k^{\ell}\hat{b}_j^i \geq D$ by the proof of Lemma 3, $|{}_k^{\ell}b'^i_j| \leq F$ and $|{}_k^{\ell}\hat{b}_j^i| \leq {}^{\ell}a^i_{m+1}$, we get

$$|\gamma_k| \leq \frac{F + {}^{\ell}a^i_{m+1}}{kD} \to 0 .$$

Q.E.D.

Let

$$X^i = \{b \in \Omega^{sm} : {}^{\ell}b = {}^{\ell'}b \quad \forall\ \ell, \ell' \in S \in \mathcal{P}^i\} .$$

This is clearly a convex set.

Consider the E.P. ${}_kb$. Now ${}_k\hat{b}^i$, for $i \in I_n$, is a solution to the following maximization problem:

$$\max_{b^i \in X^i} P^i({}_kb|b^i)$$

$$\text{subject to} \quad a^i_{m+1} - \sum_{j\in I_m} {}^{\ell}b^i_j \geq 0 \quad \text{for all} \quad \ell \in I_s .$$

By the Kuhn-Tucker Theorem* there exist non-negative numbers $\lambda_1, \ldots, \lambda_s$ such that the above problem is equivalent to

$$\max_{b^i \in X^i} P^i({}_kb|b^i) + \sum_{\ell\in I_s} \lambda_\ell (a^i_{m+1} - \sum_{j\in I_m} {}^{\ell}b^i_j) .$$

Moreover, since ${}_k\hat{b}^i$ is the solution,

$$\lambda_\ell \cdot (a^i_{m+1} - \sum_{j\in I_m} {}^{\ell}\hat{b}^i_j) = 0 \quad \text{for all} \quad \ell \in I_s .$$

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*We are grateful to L. S. Shapley for suggesting the use of this theorem.

But ${}_k\hat{b}$ is an interior point by assumption, hence $a^i_{m+1} - \sum_{j \in I_m} {}^{\ell}\hat{b}^i_j > 0$ for all $\ell \in I_s$, which implies $\lambda_\ell = 0$ for all $\ell \in I_s$.

Thus, in fact, ${}_k\hat{b}^i$ maximizes $P^i({}_kb|b^i)$ on X^i.

Let ${}_kA^i = \{\xi^i({}_kb|b^i) \in \Omega^{s(m+1)} : b^i \in X^i\}$.

Lemma 6. For any $\epsilon > 0$, there is a k_ϵ such that for $k \geq k_\epsilon$

$${}_k\tilde{B}^i \subset ({}_kA^i)^\epsilon .$$

Take any $y \in {}_k\tilde{B}^i$.

Define $c^i \in \Omega^{sm}$ by

$${}^{\ell}c^i_j = {}^{\ell}_kp_j\, {}^{\ell}y_j .$$

Then clearly $c^i \in X^i$.

For $1 \leq j \leq m$

$$|{}^{\ell}\xi^i_j({}_kb|c^i) - {}^{\ell}y_j| = \left|\frac{{}^{\ell}_kp_j\, {}^{\ell}y_j}{{}^{\ell}p_j({}_kb|c^i)} - {}^{\ell}y_j\right| .$$

Since $\|c^i\| \leq MT$ for any $y \in {}_kB^i$, if k is large enough, by Lemma 5 we get

$$\frac{{}^{\ell}_kp_j}{{}^{\ell}p_j({}_kb|c^i)} \to 1$$

uniformly for $\ell \in I_s$, $j \in I_m$, and the choice of y in ${}_k\tilde{B}^i$. Hence $|{}^{\ell}\xi^i_j({}_kb|c^i) - {}^{\ell}y_j|$ can be made uniformly small. But since $y \in {}_k\tilde{B}^i$

$$ {}^{\ell}y_{m+1} = a^i_{m+1} - \sum_{j\in I_m} {}^{\ell}_{k}p_j\, {}^{\ell}y_j + \sum_{j\in I_m} {}^{\ell}_{k}p_j\, {}^{\ell}a^i_j \,. $$

Comparing this with

$$ {}^{\ell}\xi^i_{m+1}({}_kb|c^i) = a^i_{m+1} - \sum_{j\in I_m} {}^{\ell}p_j({}_kb|c^i)\, {}^{\ell}\xi_j({}_kb|c^i) $$

$$ + \sum_{j\in I_m} {}^{\ell}_{k}p_j({}_kb|c^i)\, {}^{\ell}a^i_{m+1} $$

we see that

$$ |{}^{\ell}\xi^i_{m+1}({}_kb|c^i) - {}^{\ell}y_{m+1}| $$

can be made uniformly small.

Since $\xi^i({}_kb|c^i) \in {}_kA^i$, this establishes that ${}_k\tilde{B}^i \subset ({}_kA^i)^{\epsilon}$ for large enough k .

Q.E.D.

<u>Proof of Theorem 2</u>

It is straightforward to verify that, for all $\ell \in S$, and for all k

$$ \sum_{i\in I_n} {}^{\ell}_{k}\hat{x}^i = \sum_{i\in I_n} {}^{\ell}a^i \,, ^{*} \quad \text{and} $$

$$ {}^{\ell}_{k}p \cdot {}^{\ell}_{k}x^i = {}^{\ell}_{k}p \cdot {}^{\ell}a^i \quad \text{for all} \quad i \in I_n \,. $$

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* ${}_kx$ denotes the allocation resulting from the T.S.E.P ${}_kb$. If ${}_k\hat{b} \to \hat{b}$, clearly ${}_k\hat{x}^i$ converges with k , and we denote the limit $\hat{x}^i$.

By taking limits these equalities hold if we drop the k .

To prove that $(\hat{x}^1, \ldots, \hat{x}^n, \tilde{p})$ is a C.E. for Γ_1 , we need to show that $\hat{x}^i$ maximizes U^i on B^i .

Let

$$M = \max \{U^i(x) : x \in B^i\}$$

$$M^k = \max \{U^i(x) : x \in {}_kB^i\}$$

$$\tilde{M}^k = \max \{U^i(x) : x \in {}_kA^i\} .$$

By Lemmas 4 and 6, and noting that

$$M^k = \max \{U^i(x) : x \in {}_k\tilde{B}^i\}$$

and

$$M = \max \{U^i(x) : x \in \tilde{B}^i\}$$

since U^i is non-decreasing, we get

$$\lim \tilde{M}^k \geq \lim M^k \geq M .$$

But it is easily verified that $U^i(\hat{x}^i) = \lim \tilde{M}^k$. This proves that $\hat{x}^i$ maximizes U^i on B^i . It is straightforward to show that if $\tilde{p}$ is competitive for Γ_1 then it is competitive for each Γ_k .

Q.E.D.

Remark. If we take $s = 1$, we get the analogues of Theorems 1 and 2 in [6].

7. FUTURES MARKETS AND THE DISTRIBUTION OF INFORMATION

Formal mathematical models are frequently best used to present extreme cases. The general equilibrium models clearly tell us what an economy with perfect trust and costless information looks like. In contrast the noncooperative trading games described here switch from a budget constraint to a cash constraint, and as has been done here they can reflect clear differences in the availability of information and structure of markets.

Most of the actual markets lie between these two extremes. The cash constraint is too binding, the budget constraint is too loose. The loosening of the cash constraint usually requires the introduction of a credit mechanism into the economy. In the models here this is avoided implicitly by assuming each trader had enough of a commodity money that credit was not needed. If this assumption is not made, details concerning banking and bankruptcy must be specified.

There are other market structures than the one suggested here which can reflect trade with uncertainty. Two others specifically illustrating trading in futures are discussed in the next paper. They serve to stress the modeling of information and communication conditions. In the next paper we also examine the relationship between these noncooperative market models and Radner's treatment of nonsymmetric information.

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