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IMPLICATIONS OF MICROECONOMIC THEORY

FOR COMMUNITY EXCESS DEMAND FUNCTIONS

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ABSTRACT

A series of results, related to Sonnenschein's con-
jecture that Walras' Law is the only significant restric-
tion microeconomic theory imposes on community demand and
excess demand functions are surveyed and extended. These
results have important implications for general equili-
brium theory, both in questions of existence and stability,
and for empirical work.

0. Introduction

The usual assumptions made in general equilibrium theory on the
preference maximizing behavior of consumers subject to a budget restric-
tion imply, in strictly convex environments, that individual excess de-
mand functions satisfy Walras' Law, zero homogeneity in prices, and are
bounded from below. These properties are preserved by aggregation into
community or market excess demand functions. Hugo Sonnenschein (1972)
conjectured that these properties essentially characterize community

excess demand functions, in the sense that any function satisfying these three conditions can be obtained by aggregating individual excess demand functions of preference maximizing consumers in some conveniently constructed economy consisting of a finite number of agents.

The implications of such a result are evident. It has been stated often enough that meaningful aggregate economic models should be based on behavioral assumptions on microeconomic units. It follows from Sonnenschein's conjecture that one has to be extremely careful in stating the consequences of the hypotheses. Too often one finds empirical studies which use restrictive assumptions such as the symmetry of the Slutsky matrix, derived from individual maximizing behavior, to improve the efficiency of the estimators. Of course such procedures are not valid without a very carefully chosen sample. This has been known for a long time; what the results presented in the sequel show is that there are no a priori restrictions on the parameters to be obtained from economic theory if one stays within the framework of the general equilibrium model and no information on the structure of demand other than the usual textbook assumptions is given. An analysis of restrictions on individual behavior that are preserved by aggregation, such as gross substitutability, would be of great assistance in these cases.

To consider another example, the theory of international trade and welfare contains many conclusions drawn from models which treat countries as if they were single consumers. Exceptions such as Harry Johnson's (1959) floral patterns are infrequent, although Alfred Marshall (1879) already noted the possibility of multiple--an odd number of--equilibra. As will be seen at the end, one of the interesting questions that can

be solved with the results to be presented is what structure the sets
of equilibrium prices have, extending thus considerably the results known
to hold for the two commodity model.

Another field in which new answers are obtained is that of the
stability of multimarket equilibrium. It is not too long ago that the
optimistic view that the usual price adjustment process for competitive
economies is, as a rule, stable, could be read from works whose outstand-
ing representative is that of Kenneth Arrow, H. D. Block, and Leonid
Hurwicz (1959). Counterexamples with economies with a single unstable
equilibrium by Herbert Scarf (1960) and David Gale (1963) had a sobering
effect, without destroying the impression that the competitive pricing
processes show some kind of inherent stability. Here the question arises
whether such counterexamples are easily constructed, or whether they are
just unlikely exceptions.

Finally, in relation to the proof of existence of competitive equi-
librium, it has been hoped at one time that there might be some way to
dispense with fixed point theorems. Though Hirofumi Uzawa (1960) argued
that the existence of equilibrium is equivalent to Brouwer's fixed point
theorem, his argument was incomplete in that he did not show that for a
given, arbitrary market excess demand function it is always possible to
construct a competitive economy which generates it. The present author
(1968) is guilty of trying to circumvent fixed point theorems. That
effort, though permitting insights toward the computation of equilibria,
was from the logical point of view just a different, roundabout proof
of Brouwer's fixed point theorem.

1. Notation and Definitions

Commodity space is R^n ; frequently used subsets of commodity space are the non-negative orthant R^n_+ ; $P = \text{int } R^n_+$; $\bar{P} = R^n_+ \backslash \{0\}$; $\bar{S} = \{p \in R^n_+ : \|p\| = 1\}$; $S = \bar{S} \cap P$; and K will denote an arbitrary compact convex body in P . All these sets have the topology induced by the Euclidean norm $\|\cdot\|$. Let E^m denote the collection of m-consumer pure trade economies. A pure trade economy is a set of m consumers, indexed by i ; each consumer is described by a pair $(\underset{\sim}{\succ}_i, X^i)$.

For each i , $\underset{\sim}{\succ}_i$ is a continuous, locally nonsatiated, strictly convex, complete preference preorder on his convex, closed, bounded below consumption set X^i . If the preferences are also monotone, it will be said that the corresponding consumer is monotone. A representable consumer has a continuous real valued utility function $u^i : X^i \to R$ such that for all $x, y \in X^i$ one has $u^i(x) \geq u^i(y)$ if, and only if, $x \underset{\sim}{\succ}_i y$.

An excess demand function $f : T \to R^n$, where $T \subset R^n$, by definition satisfies Walras' Law $p \cdot f(p) = 0$ for all $p \in T$. An individual excess demand function is an excess demand function with the property that there exists a consumer $(\underset{\sim}{\succ}, X)$ such that $f(p)$ maximizes his preferences over the budget set $\{x \in X : px \leq 0\}$. It is well known that individual excess demand functions are continuous, homogeneous of degree zero in prices, and satisfy Walras' Law.

If $f = \Sigma f^i$, where each f^i is an individual excess demand function, then f will be the market excess demand function generated by the economy whose consumers have excess demand functions f^i , and these individual excess demand functions decompose f . Similar terminology will be applied also to demand functions.

Remark: The present definition of a consumer does not explicitly incorporate the concept of an endowment. This is not needed if one is concerned with excess demand functions, since only the net trades are observed in the market. It is always possible to assume that the individual excess demand functions are well behaved in the sense that they correspond to positive demands by displacing the origin of the consumption sets, since these are bounded from below. When one is concerned with demand, then it is better to define consumers to be a triple, with some endowment vector w^i in X^i .

2. The Decomposition of Market Excess Demand

Most previous attempts at trying to characterize market excess demand functions were related to the solution of the simpler problem of counting the number of equilibria and determining which ones were stable. The few results known pertained mostly to the two commodity model.

Sonnenschein's (1972, 1973) analysis was a breakthrough. He showed that if all the coordinates but possibly one, of an excess demand function defined on a compact set K of positive prices, are polynomials, then there exists an economy in E^m consisting of a finite set of consumers with representable preferences generating it. This result permitted answering most of the questions raised before, at least approximately. Exact answers could not be given, because the number of consumers in his decomposition depended on the degree of the polynomials involved; the obvious resort to sequences of approximating economies was therefore not feasible.

Inspired by Sonnenschein's result, the present author (1974) obtained a decomposition of market excess demand functions, also defined

on some compact set K of positive prices, by an economy in E^{2n} with twice as many consumers as commodities, using the additional assumption --satisfied by Sonnenschein's polynomials--that the coordinates of the given excess demand function have continuous derivatives satisfying a one-sided Lipschitz condition. The proof is based on the differentiation of Walras' Law, which provides a natural decomposition of the given excess demand function into n gradients. These are then introduced as perturbations to the first n of 2n large consumers whose aggregate unperturbed market excess demand function is identically zero--all prices are equilibrium prices.

The next step forward was provided by Gerard Debreu (1974), who demonstrated that any continuous market excess demand function can be decomposed on a compact set of positive prices K , by an economy in E^n with representable, monotone consumers. His hindsight was to realize that the budget plane is spanned by a conveniently chosen set of n individual excess demand functions. Since a dilation of each of these produces an excess demand function of a monotone, representable consumer, the result follows. Formally, we have the following theorem.

Theorem. Let f be an excess demand function on K . Then f is generated by some economy in E^n with representable, monotone consumers.

Proof. The central argument in Debreu's proof runs as follows. Because of homogeneity, attention may be restricted to prices in S ; because of lower boundedness, there exists a continuous real valued positive function α on S such that for all valid arguments p one has

$$\beta(p) = f(p) + \alpha(p)p > 0 .$$

Set

$$f^i(p) = \beta_i(p)(e^i - p_i p) \; ,$$

where e^i stands for the ith unit vector. Because of Walras' Law these excess demand functions add up to f. It will be shown that they satisfy the strong axiom of revealed preference.

Let q and p be two price vectors for which f is defined. Then $q \cdot f^i(p) \leq 0$ if, and only if, $q_i/p_i \leq q \cdot p$, so that $q_i < p_i$ or $q = p$, and the revealed preference relation is acyclic.

Finally, lower boundedness follows from the assumed continuity of f and the compactness of K; representability follows from the lemma to be stated. $\square$

Lemma. The individual excess demand functions in Debreu's decomposition stem from representable consumers.

Proof. For a proof, see Debreu (1974) or Andreu Mas-Collel (1975). $\square$

A careful analysis of the proof of the theorem reveals that continuity is used only to show that the individual excess demand functions are bounded from below. This was exploited in a joint paper by Daniel McFadden, Andreu Mas-Collel, Marcel Richter, and the present author (1974) to extend Debreu's decomposition to the whole positive orthant P, dropping the continuity requirement. They show how this can be achieved by using a different normalization rule, modifying slightly Debreu's argument. They also show that if one furthermore assumes that the given market excess demand function is bounded from above, then it can be decomposed into n individual excess demands satisfying the strong

axiom of revealed preference on $\bar{P}$, the closure of the nonnegative or-
thant with the origin deleted. Note that in these cases dropping the
continuity requirement implies that the resulting consumers may not be
representable; in fact, their preferences need not be continuous.

Recently, the present author (1974b) applied his perturbation tech-
nique to obtain a result which is not attainable using Debreu's technique.
It consists of decomposing excess demand functions restricting the class
of consumers to those with homothetic preferences. It is surprising that
this can be achieved by fixing the initial endowments except for a common
scaling factor and the condition that they be linearly independent vectors.
It is known--Eisenberg (1961), Chipman (1965, 1974)--that if all consumers
have endowments which are a scaled down replica of the aggregate endow-
ment vector, and if their preferences are homothetic, then their aggre-
gate excess demand is representable. As noted by Mas-Collel in a private
communication, the differentiability assumption in the following theorem
cannot be totally dispensed with, as can be seen in the two goods case,
where homotheticity implies that the ratio of quantities demanded is a
monotone function of relative price, hence is almost everywhere differ-
entiable.

Theorem. Let the excess demand function f have bounded second order
partial derivatives, and be defined on a compact set of positive prices
K . Then it is generated by some economy in E^n with representable,
monotone, homothetic consumers.

Proof. Let A be a regular matrix of order n with positive elements,
and columns a^i adding up to unity. Define the ith indirect utility
function v_i , a real valued function on K , by

$$v_i(p) = f_i(p)/k - a^i \cdot \log(Ap)$$

where the log of a vector means the vector of the logs of its coordinates. If the constant k is taken to be large enough, boundedness of the second order partial derivatives means that each v_i is strictly convex on K, so that the negative of its gradient satisfies the strong axiom of revealed preference. Endow consumer i with k units of commodity i. His excess demand function will thus be defined by

$$f^i(p) = -p_i Df_i(p) + kA' \widehat{Ap}^{-1} a^i p_i - ke^i$$

where Df_i is the gradient of f_i, the ith coordinate of f; the apostrophe indicates transposition, and the hat transforms a vector into the diagonal matrix with that vector along its main diagonal. Summing over i gives the desired result, since the last two terms cancel, and the first sums to f as can be verified by differentiating Walras' Law. The functions v_i are homogeneous, thus so are the direct utility functions defined by

$$u_i(x) = \inf[v_i(p): px \leq 1, p \text{ in } K] . \quad \square$$

This decomposition can be extended to any compact price set whose convex hull does not intersect the non-positive orthant by means of the following lemma, used implicitly by the author (1974b) and by Mas-Collel and Wilhelm Neuefeind (1975).

Lemma. Let T be a regular matrix of order n. The change in coordinates which transforms prices p into Tp and commodity bundles x

into $(T')^{-1}x$ does not perturb consumers, except possibly for the properties of lower boundedness and monotonicity.

Proof. Since the transformation is continuous and leaves inner products invariant, the result is quite obvious. Therefore the argument will only be given to prove that the weak axiom of revealed preference is not affected.

Assume f satisfies the weak axiom, and define $g(p) = T'f(Tp)$. Then, if $q \cdot g(p) \leq 0$, one has $(Tq)' \cdot f(Tp) \leq 0$, and because of the weak axiom, $(Tp)' \cdot f(Tq) > 0$ or $f(Tp) = f(Tq)$. Therefore, $p \cdot g(q) > 0$ or $g(p) = g(q)$. $\square$

Corollary. Let the hypothesis of the previous theorem hold, except that f is defined on an arbitrary subset of some closed, pointed, convex cone C containing P . Then the conclusion holds with "non-satiated" substituted for "monotone."

Proof. Apply the transformation of the lemma so that $C \backslash \{0\}$ is transformed into a subset of P . Use the theorem to obtain a decomposition with representable, monotone, homothetic consumers for the transformed excess demand function. Perform the inverse transformation to return to the original economy. Since the latter implies expanding C to its former size, this operation can only contract the individual consumption sets, so that lower boundedness is preserved, whereas monotonocity will possibly be lost, going over to the weaker property of local-nonsaciation. $\square$

Note that the same procedure can be used to transform the cone C into any closed convex cone with nonempty interior in P , giving

the result on initial endowments referred to before. Note also that the lemma can be applied to the other decompositions cited above, extending the results correspondingly.

It might be inquired whether it is possible to decompose excess demand functions with fewer consumers. That this is not so in general has been conjectured by the author (1972) and shown by Debreu (1974), who presented an example. The reader may try his hand at the excess demand function f defined by $f(p) = p - a/(a \cdot p)$, where $p \in S$, and $a \in S$ is fixed.

Nothing has been said up to now of the restrictions that might be imposed on the excess demand functions by nonnegativity restrictions on the resulting demand functions. Since a change in the origin of the consumption sets does not disturb excess demands, it is always possible to shift the origin so that the resulting demand functions, if continuous, are non-negative. This would of course not be allowed if one had information on the initial endowments of consumers. For example, the aggregate endowment vector of the economy might have been observed. The article by McFadden e.a. (1974) addresses itself to the question whether it is still possible to decompose excess demand functions if they have to be consistent with a given vector of aggregate endowments of commodities. The answer is in the affirmative in the case of two commodities, if aggregate total demand is strictly positive at all prices. If demand is allowed to drop to zero, or if there are more than two commodities, the nonnegativity restrictions on the individual demands impose effective limits to the possibility of decomposing such functions, as examples show. Nevertheless, a result stating that if total aggregate demand remains close to the observed aggregate endowment vector a decomposition

into n individual excess demand functions exists, is given there.

Finally, the article mentioned in the previous paragraph analyzes the situation when excess demand is not single-valued, as might be the case if preferences were not strictly convex. This is the situation usually assumed in general equilibrium theory. It is shown there that in such cases maximizing behavior indeed does impose restrictions on market excess demand correspondences. These are not just arbitrary convex valued correspondences. Almost all their values--i.e. all those values corresponding to prices which are not in a certain meager subset of P --are polyhedra.

Mas-Collel and Neuefeind (1975) sharpened this result. They analyzed the mean excess demand correspondences of atomless pure exchange economies whose consumers have complete, continuous, monotone, but not necessarily convex preferences. They demonstrated that the values of such correspondences belong to a very special class of convex set--i.e. the class of translates of ranges of atomless vector measures--except for a very small (i.e. meager and of Lebesgue measure zero) set of prices. Note that in the present case with a finite number of consumers, these sets are just images under affine mappings of the m-dimensional unit cube into n-dimensional commodity space, where m is the number of consumers, since each individual demand correspondence is a segment for most prices.

Stronger conditions can be imposed on consumers if it is desired to decompose an excess demand function at a single price vector p at which the value of the Jacobian matrix J of partial derivatives with respect to prices has been observed.

Assuming that p is an equilibrium price vector, Trout Rader (1975) shows that there exist   2n Leontief consumers--who desire commodities

in fixed proportions--whose aggregate excess demand functions have Jacobian J at p .

Rader's result can be considerably strengthened, extending it at the same time to non-equilibrium prices and reducing the number of consumers to n . The new result is surprising. It is possible to fix in advance not only the relative income distribution, but also the tastes of the n consumers, as long as the latter are independent.

Let thus p be in P , and J such that $Jp = 0$, because of homogeneity. Assume that the positive, regular matrix A of order n is given, column i giving the proportions in which the ith consumer desires the commodities, and normalized so that $A'p = e$, a column of unities. The relative income distribution is given by an n-vector $d > 0$ whose coordinates add up to unity. If the scale factor γ is chosen appropriately, it is easily checked that the matrix

$$W' = A^{-1}J + \gamma\hat{d}A'$$

is positive. Its columns represent the endowment of the corresponding consumer such that the aggregate excess demand function

$$f(p) = A(\widehat{A'p})^{-1}W'p - We$$

has Jacobian J at p .

Similar results had already been given by the present author (1972) postulating quadratic consumers; for the latter, the following result, stated without proof, holds.

Lemma. Let $J = S - kx'$, with $x \in P$ and $k \in R^n$; S a symmetric matrix of order n , negative semi-definite of rank n-1 , such that

for some p in P satisfying $p \cdot x = 1$ one verifies that $Sp = 0$.
Then there exists an unsatiated consumer with preferences representable
by a quadratic utility function, whose demand function has Jacobian J
at p . ☐

Note that even though demand functions depend not only on prices
as excess demand functions but also on income, the term Jacobian as used
here, in both cases, refers to the square matrix of order n of price
derivatives.

Assuming that at a given set of prices p the Jacobian J of
the aggregate excess demand function has been observed; that n symmetric
negative semi-definite matrices S^i are given with p as characteristic
vector for its unique zero root; that each consumer's demand $x^i > 0$
is known except for a common scaling factor γ ; and that the vector
k^i of derivatives of his demand function with respect to income is given
(with elements adding up to unity), then one can verify the following.

Let K , X be the matrices with columns k^i , x^i , and set

$$ W' = \gamma X' + K^{-1}(J - \Sigma S^i) $$

choosing γ so that this matrix is positive. Then

$$ J = \Sigma[S^i - k^i(\gamma x^i - w^i)'] = \Sigma[J^i + k^i(w^i)'] \ , $$

and each $J^i = S^i - k^i(\gamma x^i)'$ satisfies the lemma. The construction is
feasible whenever the matrix K is nonsingular.

It is noteworthy that even Cobb-Douglas consumers are sufficiently
varied so as to explain arbitrary excess demand functions if commodities
are gross substitutes--recall that gross substitution is preserved by

aggregation, and that individual demand functions from Cobb-Douglas utility functions satisfy that property, so that more cannot be hoped for. Let

$$u^i(x) = a^i \cdot \log x$$

be the utility function of the ith consumer, where the column a^i of matrix A is assumed to be positive with coordinates summing to unity. The aggregate excess demand function is given by the relation

$$f(p) = \hat{p}^{-1} AW'p - We .$$

Assuming, without loss of generality, that $p = e$, the Jacobian at p is

$$J = AW' - \widehat{(Ad\gamma)}$$

which has a solution with A and W positive for given J and d satisfying the usual restrictions, if γ is chosen sufficiently large.

3. The Decomposition of Market Demand

Up to now, reference has been made to demand as a function of prices and income, assuming that the latter is determined by the value of the initial endowment of the corresponding consumer at the current prices, as is customary in general equilibrium theory. Empirical studies, on the other hand, often refer to demand as an explicit function of prices and aggregate income, assuming some rule for the distribution of income among consumers has been given. It is thus of some importance to analyze the possibility of decomposing demand functions given in this sense.

In this field, the state of the art is considerably less satisfactory.

It has been shown by Sonnenschein (1973) that in a two commodity world

it is possible to decompose arbitrary demand functions, if it is assumed

that income is shared equally by two consumers. But his argument does

not take into account non-negativity restrictions on the quantities de-

manded by each individual, nor is it generalizable to more commodities.

Non-negativity requirements impose effective restrictions if the

decomposition is to hold for all prices. For example, consider the fol-

lowing situation.

Two price vectors in P are observed, p and q. Commodities

are classified into two nonempty groups, such that the prices of the first

group satisfy the inequality $p_1 > q_1$, those of the second group

$p_2 < q_2$. Subindices refer to the two groups.

Consider any consumer whose income is the same in both situations;

for simplicity set it equal to unity. Let the commodity bundles x, y

be chosen at prices p, q respectively. Then the weak axiom of re-

vealed preference implies, even if weak convexity is allowed, that at

least one of the inequalities $p \cdot y \geq 1$ or $q \cdot x \geq 1$ holds.

If the first inequality holds, we have $(p_1 - q_1) \cdot y_1 \geq (q_2 - p_2) \cdot y_2$.

Therefore, $1 = q \cdot y \leq q_1 \cdot y_1 + \alpha(q_2 - p_2) \cdot y_2 \leq a \cdot y_1$ for some $\alpha > 1$,

where $a = q_1 + \alpha(p_1 - q_1) > 0$. Similarly, if the second inequality holds

one deduces that there exists some $\beta > 1$, so that $1 \leq b \cdot x_2$, where

$b = p_2 + \beta(q_2 - p_2) > 0$. Thus in any case one has that $a \cdot y_1 + b \cdot x_2$

should not be less than the consumer's income. Note that abundant use

has been made of the assumption that the commodity bundles are non-

negative.

It is easily seen that the relation obtained is linear, with coef-

ficients given by a and b which are independent of the preferences

of the consumer. Thus this relation can be aggregated over any number

of consumers. It must therefore also hold for the aggregate demand func-

tion. It is very easy to present two positive vectors x , y such that

$p \cdot x = q \cdot y > a \cdot y_1 + b \cdot x_2$. The demand functions having these values at

p and q cannot be decomposed if aggregate income is assumed to be dis-

tributed independently of prices without violating some non-negativity

constraint.

The foregoing example shows that the most one might hope for is

to be able to decompose demand functions in the neighborhood of some given

price income pair. But even then, excepting Sonnenschein's result for

the two commodity case, no such decomposition is known to hold. Only in

the limiting situation in which this neighborhood shrinks to a single

point, and information is given only on the derivatives of the demand

functions is it possible to "decompose" them, in the sense that one can

decompose a demand function which has the same value and derivatives at

the chosen point. Furthermore, it has been remarked earlier that a con-

stant relative income distribution and homothetic preferences imply that

the market demand function satisfies the strong axiom of revealed pre-

ference. Thus the decomposition of excess demand functions into indivi-

dual excess demand functions of homothetic consumers has no counterpart

for demand functions when the relative income distribution is constant.

The situation is qualitatively different from the case of excess

demand functions. Intuitively, Walras' Law for demand functions requires

that the total value of the quantities demanded be always equal to income,

a positive number, whereas for excess demand functions the net budget

of the consumer is zero. Thus only in the second situation is it possible

to multiply an individual excess demand function by a positive real valued

function without disturbing the budget restriction.

In spite of these negative remarks, there exist theorems for the

case of continuously differentiable demand functions, which give some

grounds for the conjecture that stronger results will eventually be found.

In fact, it may be conjectured that demand functions can be decomposed

if non-negativity restrictions are not taken into account, thus providing

a local decomposition holding for a neighborhood of some fixed price in-

come pair.

Sonnenschein (1974) showed that given an arbitrary matrix J of

order n , and a positive price vector p , there exists an economy

E^m with representable consumers which generates an aggregate demand func-

tion whose Jacobian, evaluated at the given price vector p , coincides

with J when income is shared equally among consumers.

This statement can be sharpened; the number of consumers m need

not exceed the number of commodities n , and also the relative income

distribution can be given arbitrarily--aggregate income, though shared

in fixed proportions, need not be shared equally. It is furthermore suf-

ficient to consider only quadratic consumers.

In fact, Sonnenschein's result is a particular case of a more general

theorem. W. E. Diewert (1973) has demonstrated that if the number of

consumers m does not exceed the number of commodities n , and if con-

sumers have differentiable demand functions, then the Jacobian J of

the aggregate demand function, when the income distribution is fixed,

has the property that there exists some $n \times (n-m)$ regular--i.e. with

full column rank--matrix U which is orthogonal to each individual's demand vector, and such that the matrix $L = U'JU$ is symmetric and negative semi-definite. The usual properties of the Slutsky matrix are then obtained by setting $m = 1$; for $m = n$ the statement is empty.

The following theorem shows that this generalized Slutsky condition in fact characterizes demand functions generated by m consumer economies locally, in the sense that it characterizes their Jacobians at some given price vector p.

Theorem. Let J be a matrix of order n and let p be in P. Let U be a regular $n \times (n-m)$ matrix such that $L = U'JU$ is symmetric and negative semi-definite, and with the property that $U'x = 0$, where $x = -J'p > 0$ represents, due to the differentiation of Walras' Law, total aggregate demand. Let d in P represent the given relative income distribution; thus its coordinates add up to unity. Then there exists an economy in E^n with quadratic consumers generating an aggregate demand function with Jacobian J at p.

Proof. Only an outline will be given. Assume for simplicity that aggregate income $p \cdot x = 1$. The proof can be divided into three main steps.

The first step consists in showing that total demand can be allocated among the consumers in such a way that the matrix X of order n --each column represents the demand of the corresponding consumer--is positive and satisfies the following conditions:

$Xe = x$, aggregate demand is as given;

$X'p = d$, relative income distribution is as given;

$U'X = 0$, Diewert's orthogonality condition is satisfied.

The second step in the proof consists in defining the Slutsky matrix

$$S = JUL^{-1}U'J' - XVX'$$

and the matrix of derivatives of individual demand functions with respect to income

$$K = -[J(I - UL^{-1}U'J')(X^*)' + XV] \; ,$$

where V is any positive semi-definite matrix of order m and rank $m-1$ satisfying the equation $Vd = 0$, and X^* stands for the pseudo-inverse of X .

It can then be checked that S is symmetric and negative semi-definite of rank $n-1$, that $Sp = 0$, and that K satisfies $K'p = e$, and $Kd = -Jp$. The latter corresponds, due to the homogeneity of demand with respect to income and prices, to the derivative of aggregate demand with respect to aggregate income, given the income distribution.

The third step in the proof consists in noting that

$$J^i = (1/m)S - k^i(x^i)'$$

is the Jacobian of a quadratic consumer evaluated at prices p , and that the sum of these Jacobians is J . ☐

Note finally, that for $m = 1$ the problem of the characterization of demand functions has been analyzed intensely in the literature on the problem of the integrability of demand functions.

4. Application to the Description of Equilibrium Price Sets

   One of the oldest preoccupations related to multimarket equilibria
is the description of equilibrium price sets, especially in regard to
cardinality and stability. Some developments related to this problem
have run parallel to the main lines of research mentioned above. Under
the usual assumptions of general competitive analysis, it is known that
such sets are compact, if prices are normalized. Strong assumptions such
as gross substitutability, or that the weak axiom of revealed preference
be satisfied at equilibrium, guarantee uniqueness and stability. Is it
possible to deduce similar results from more general assumptions?

   By using his polynomial decomposition, Sonnenschein (1971) out-
lined the first such result in the setting of the n commodity model.
He showed that any finite set of positive price vectors is the equilibrium
price set of some economy with a finite number of consumers. He also
showed (1973) that Samuelson's example of an excess demand function that
is Hicksian stable but not dynamically stable can in fact arise from some
finite economy.

   The most complete analysis of the problem is to be found in recent
work by Mas-Collel (1975), who proved that given any compact set K of
positive prices, there exists a market excess demand function generated
by n monotone consumers whose equilibrium price set coincides with K .
Our lemma on coordinate changes can then be applied to adapt his result
to showing that any set of prices which is the subset of some pointed
convex closed cone with the origin deleted containing P can be the
equilibrium price set of some economy in E^n . A simplified version of
Mas-Collel's theorem follows.

Theorem. Let K be a compact subset of $\bar{S}$. There exists an economy in E^n generating an excess demand function on $\bar{S}$ for which the equilibrium price set coincides with K .

Proof. Due to the results previously stated, it is sufficient to show that there exists a continuous excess demand function f on $\bar{S}$ with the value zero for those, and only those, prices which are in K .

Take a fixed point q in K and let $g(p) = q - (p \cdot q)p$ for $p \in \bar{S}$. Obviously, g is a continuous excess demand function which is not zero on the complement of K . To obtain f(p) , multiply g(p) by the distance between the point p and the set K . $\square$

Mas-Collel also showed that the usual assumptions of economic theory do not imply much even in the case in which excess demand functions are restricted to be continuously differentiable, with regular Jacobians at equilibrium prices--the case of regular economies. If the excess demand function f is given on a compact set K of positive prices, and if $S \backslash K$ is connected, then the only additional restriction that can be deduced from the requirement that there exist an economy in E^n with strictly monotone consumers generating an excess demand function on S whose restriction to K agrees with f is that it satisfy Egbert Dierker's (1972) sign condition. If that condition is met, the economy can be chosen so that it has no equilibria outside the set of definition of the proposed excess demand function f .

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