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INVARIANT COMPETITIVE EQUILIBRIUM IN AN ∞-HORIZON ECONOMY WITH NEGOTIABLE SHARES

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Introduction

In the economic model under consideration, productive activities of the firms take place during a sequence of periods with equal duration and in such a manner that inputs at the beginning of a period result in outputs which become available at the end of that period. Inputs are financed by individuals which also play the role of consumers. The rewards of the outputs of a firm at the end of a period are distributed among the individuals in the same proportion as each of them contributes in financing the inputs at the beginning of that period. Under the assumption that only firms are able to transfer goods from preceeding periods to succeeding periods, the exchange of goods and services between individuals and firms take place at the moments of period changing--to be called time-points.

At each separate time-point, all agents (i.e. individuals and firms) are subject to budget constraints which are based on a price-system for each separate time-point. For an individual, the budget constraint requires that, at each time-point, the value of his consumption and his contribution

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in financing inputs of firms may not exceed his income which is constituted by the value of his (labor) supply and his part of the rewards generated by the outputs of the preceeding period. For a firm, the budget constraint requires that, at each time-point, the value of its inputs may not exceed the budget generated by the contributions of individuals at that time-point.

For each period, firms are supposed to maximize the value of the outputs, relative to given prices at the end of the period, by choosing a technologically feasible input-output combination which satisfies the budget constraint relative to given prices and a given budget at the beginning of that period. Apart from optimal input-output combinations, this process also determines, for each firm and for each period, a dividend-factor defined as the ratio between output value at the end of a period and the budget at the start of that period.

Individual's choice criterion consists of the discounted sum of a utility function representing his preference with respect to consumption-supply combinations for each separate period, and where the discount factor represents his time preference. Given a sequence of prices and dividend factors, individuals are supposed to maximize such a criterion function by choosing, within their consumption-supply possibilities and budget constraints, a sequence of consumption, supply, and contributions in financing the firms. It will be shown that, in the context an "invariant competitive equilibrium," to be defined later, these multi-period (or even ∞-horizon) decision processes can be replaced by a single-period decision process for each individual.

It is assumed that the economic system is invariant over time, i.e.: the number of firms and individuals, the technology of the firms, individual's consumption-supply possibilities, and individual's preferences are the inchangeable over time. Then, the concept of invariant competitive equilibrium can be defined as an invariant price-dividend system together with invariant action plans for each individual and for each firm such that: (1) the action plans are compatible with the optimization behavior of individuals and firms, (2) the dividend-factors represent the ratio between output-values and the budget of the firms, (3) for each period, the total supply of commodities is equal to the total demand.

Under assumptions which correspond with Debreu's suppositions concerning production and consumption sets, and under the assumptions of concave utility functions (for the sake of simplicity), the following results are deduced: (1) there exists an invariant competitive equilibrium, (2) the "physical" side of an invariant competitive equilibrium is compatible with any degree of inflation or deflation, (3) under certain circumstances (for instance linear technology) invariant competitive equilibria are Pareto efficient with respect to two different ∞-horizon optimality criteria.

The paper is organized as follows. Section 1 gives an axiomatic model of technology and of individual's consumption-supply possibilities, and analyzes the consequences of the overall balance of goods and services. In Section 2, the economic behavior of the firm is studied. In Section 3, we study the economic behavior of individuals. In addition, we show that, in invariant circumstances, the behavior can be characterized by a single-period decision processes. Finally, Section 4 gives the definition of the concept of invariant competitive equilibrium and deduces the existence and the properties concerning inflation and Pareto efficiency. The

Appendix contains some general properties concerning convex sets and ∞-horizon difference inequalities. A list of symbols is added at the end.

We emphasize that, for the sake of simplicity, a lot of important aspects are ignored. For instance: the case of multi-point input-output production technology, money and monetary institutions (which may be ruled out the property concerning inflation),* overlapping generations of the Samuelsonian type, minimum income and taxes (especially progressive income tax). Finally, we observe that the concept of invariant competitive equilibrium might be considered as a combination of the competitive equilibrium concept and the concept of invariant optimal solutions in convex ∞-horizon programs. (Viz. [3], [5], [8].)

1. Basic Elements of the Economic System

1.1. Periods, time-points, time-horizon

Economic activities take place at a sequence of "periods" with equal duration. The periods are numbered $t = 0, 1, \ldots$. The period, numbered 0, is considered as the last passed period. The moments of period changing are called "time-point." They will be indicated as "the start of period t ," "the end of period t ," or as "$(t-1)/t$." The total number of periods over which the economic activities take place is indicated by a positive integer h , and is called the "time-horizon." Without further specification, the time-horizon may be taken either as finite or as infinite.

*This aspect will be the topic of forthcoming studies by Shubik and Evers.

1.2. Commodities

Commodities are goods or services specified by a number of attributes which provide an adequate description with respect to the specific nature of the commodities in the context of the economic system. We assume that there is only a finite number g of distinguishable kinds of commodities; which specification is invariant over the time-horizon. In addition, it is assumed that the quantity of any kind of commodity at any time-point can be any real non-negative number; implying that, at each time-point, the commodity-space can be represented by R_+^g . Quantities of goods will be considered only at time-points; they will be denoted by non-negative g-dimensional vectors, sometimes endowed with a sub-index referring (dependent of the context) to the preceeding or succeeding period.

1.3. Prices

With each commodity--say the k^{th} one--real non-negative numbers $p_{k,t}$, $t = 1, 2, \ldots$ are associated, representing prices of commodity k at the starting points of the periods $t = 1, 2, \ldots$; i.e.: $p_{k,t}$ gives the value of one unit of commodity k at the start of a period t . Clearly, the price system at the start of a period t can be represented by a vector $p_t \in R_+^g$. The value of a bundle of commodities y relative to a price system p_t at time-point $(t-1)/t$ is the inner product $p_t'y$; i.e.: $\Sigma_{k=1}^g p_{k,t} y_k$. The effective meaning of the concept "value" will be clarified in the context of the budget constraints of the economic agents (viz. 2.2 and 3.2). In this study the concept of invariant prices (or invariant price system): $p_t = p_1$, $t = 1, 2, \ldots, h$, takes a central place.

1.4. Individuals: consumption and supply

All individuals choose, at the start of each period, a plan for consumption and (labor) supply, consisting of a specification of the quantities of their consumptions and supplies for (at least) the first forthcoming period. We assume that there are m individuals; they are all immortal (at least until the time-horizon). Each individual is indicated by an index $i = 1, 2, \ldots, m$. The consumption of the i^{th} individual at time-point $(t-1)/t$ will be represented by a vector $z_t^i \in R_+^g$, his supply at that moment by $w_t^i \in R_+^g$. A sequence of consumptions and supplies by the i^{th} individual over the whole time-horizon will be represented by $\{(z_t^i, w_t^i)\}_{t=1}^h$. Invariant consumptions and supply $(z_t^i, w_t^i) := (z^i, w^i)$, $t = 1, 2, \ldots, h$, will be denoted simply by the pair $(z^i, w^i) \in R_+^{2g}$. The period-index also will be omitted in cases where reference to time-points has no relevance.

Generally, individual's consumption consists of commodities related to food, clothing, housing, education, recreation, etc. That means, many components of the consumption vectors z_t^i always will be zero. The similar can be said of individual's supply vectors w_t^i, which generally will represent only labor. The use, and ownership, of durable consumption goods (i.e.: goods for consumptive purposes with a life-time of two or more periods) can be taken as a particular activity of a particular private firm (viz. §1.6).

In the light of these considerations, it should be clear that, at each time-point $(t-1)/t$, the set of consumption-supply combinations (z_t^i, w_t^i) available to the i^{th} individual will cover only a small part of $R_+^{2 \cdot g}$. We assume that, for each individual, these sets of consumption-supply combinations will be invariant over time; they will be represented by sets $C^i \subset R_+^{2g}$, $i = 1, 2, \ldots, m$.

1.5. <u>Assumptions on consumption-supply sets</u> C^i

1.5-A1: $C^i \subset R_+^{2g}$ (discussed above)

1.5-A2: $(0,0) \in C^i$ (possibility of inaction)

1.5-A3: $(z^i, w^i) \in C^i \Longrightarrow \forall \tilde{w}^i \geqq 0 | \tilde{w}^i \leqq w^i : (z^i, \tilde{w}^i) \in C^i$

(free disposal with respect to supply)

1.5-A4: $\exists \bar{w} \in R_+^g : \forall (z^i, w^i) \in C^i : w^i \leqq \bar{w}$

(the set of individual's supply is bounded)

Next two assumptions are imposed, rather on mathematical considerations than on economic relevance:

1.5-A5: C^i is closed

1.5-A6: C^i is convex

Later, the possibility of a particular consumption (non-saturation §3.3) and of a particular supply (productive supply §4.) will be added to these assumptions.

For some reason, to be clarified later, an arbitrary bound will be imposed on consumption, giving rise to the following sets:

1.5-D1: $\bar{C}^i := \{(z^i, w^i) \in C^i | z^i \leqq \bar{z}\}$

$\bar{z} \in R_+^g$ being the arbitrary bound. Evidently, we have the following property:

Proposition 1.5-P1: Each of the assumptions 1.5-A1 to 5, implies a similar property to the sets $\bar{C}^i$. In addition, definition 1.5-D1 together with assumption 1.5-A3 implies boundedness of each set $\bar{C}^i$.

1.6. Firms; inputs and outputs

Production is understood as an activity of taking inputs at the start of a period and transforming these into outputs which become available at the end of that period. The notion of inputs/outputs may be taken very general. For instance, inputs at the beginning of period t may include a machine at a certain age, which supplies productive capacity during that period, and leaves the process as a part of outputs at the end of period t, being another commodity, namely: the same machine but one period older.

In that context, firms are systems which choose, and carry out, producdive activities. We assume that there is a fixed number of n firms over the time-horizon h; each of them is indicated by an index $j = 1, 2, \ldots, n$.

The production of a firm j during period t will be denoted by a pair $(x_t^j, y_t^j) \in R_+^{2g}$, where x_t^j stands for the inputs at the start of period t, and where y_t^j represents the outputs coming out at the end of period t. The set of feasible productive activities of each firm j is assumed to be invariant over time and is denoted by $F^j \subset R_+^{2g}$, being a set of input-output combinations. Thus, a production plan $\{(x_t^j, y_t^j)\}_{t=1}^h$ of the j^{th} firm over a time-horizon h, always satisfies $(x_t^j, y_t^j) \in F^j$, $j = 1, 2, \ldots, h$. The period-index will be omitted for invariant productions, or in cases where the period-index is not relevant. The sets F^j will be called production sets.

1.7. <u>Assumptions on production sets</u> F^j

1.7-A1: $F^j \subset R_+^{2g}$ (discussed above)

1.7-A2: $(0,0) \in F^j$ (possibility of inaction)

1.7-A3: $(x^j, y^j) \in F^j \Longrightarrow \forall \tilde{y}^j \in R_+^g | \tilde{y}^j \leqq y^j : (x^j, \tilde{y}^j) \in F^j$

(free disposal)

1.7-A4: $(x^j, y^j) \in F^j \Longrightarrow \forall \delta > 1 : \exists \lambda > 1 : (\delta x^j, \lambda y^j) \in F^j$

(possibility of expansion)

1.7-A5: $(0, y^j) \in F^j \Longrightarrow y^j = 0$ (impossibility of free production)

1.7-A6: F^j is closed

1.7-A7: F^j is convex

<u>Proposition 1.7-P1</u>: Under 1.7-A1, 2, 6, 7 and boundedness of the set $(\{0\} \times R^g) \cap F^j$ (which is slightly weaker than 1.7-A5), positive numbers α, β exist such that, for every $(x^j, y^j) \in F^j$: $\|y^j\| \leqq \alpha + \beta \|x^j\|$. (Direct consequence of auxiliary proposition A-P3.)

With respect to any subset of F^j , defined by

1.7-D1: $\bar{F}^j := \{(x^j, y^j) \in F^j | \; x^j \leqq \bar{x}\}$, $j = 1, 2, \ldots, n$,

$\bar{x} \in R_+^g$ being an artificial bound we have:

<u>Proposition 1.7-P2</u>: Each of the assumptions 1.7-A1, 2, 3, 5, 6, 7 implies an equivalent property with respect to the sets $\bar{F}^j$. In addition, assumption 1.7-A5 implies (by 1.7-P1) boundedness of the sets $\bar{F}^j$.

1.8. Total balance of goods and the non-substitution condition of individual's supply

We assume that the economic system is closed; i.e.: all consumption and supplies $\{(z_t^i, w_t^i)\}_{t=1}^h$, $i = 1, 2, \ldots, m$ of the individuals, and all inputs-outputs $\{(x^j, y^j)\}_{t=1}^h$, $t = 1, 2, \ldots, n$, together have to satisfy:

$$(1.8.1) \qquad \sum_{i=1}^m z_t^i - \sum_{i=1}^m w_t^i + \sum_{j=1}^n x_t^j - \sum_{j=1}^n y_{t-1}^j \leqq 0 , \quad t = 1, 2, \ldots, h .$$

Briefly, demand excesses are not permitted. The time lag in the outputs of the firms is caused by the fact that production takes exactly one period. The initial outputs y_0^j , $j = 1, 2, \ldots, n$ are supposed to be the given result of the past. Note that stock-holding of goods may be considered as a particular productive activity which carries goods from preceeding time-points to succeeding time-points, implying that the inequalities (1.8.1) do not rule out the possibility of stockholding.

In case of invariant consumption-supply (z^i, w^i) , $i = 1, 2, \ldots, m$, and of invariant inputs-outputs (x^j, y^j) , $j = 1, 2, \ldots, n$, the balance of goods represented by (1.8.1) reduced to:

$$(1.8.2) \qquad \sum_{i=1}^m z^i - \sum_{i=1}^m w^i + \sum_{j=1}^n x^j - \sum_{j=1}^n y^j \leqq 0 .$$

For that case, it is clear that the necessity of supply by individuals for invariant non-zero production can be expressed by the following assumption:

1.8-A1: There exists no invariant non-zero production $(x^j, y^j) \in F^j$, $j = 1, 2, \ldots, n$ satisfying:

$$\sum_{j=1}^{n} x^j - \sum_{j=1}^{n} y^j \leqq 0 \ . \qquad (1.8.3)$$

We observe that this assumption implies the condition formulated by 1.7-A5. Next propositions relate the non-substitution condition of individual's supply to the boundedness of the set of productive actions under limited supplies by individuals.

In order to simplify the notation, we introduce the concepts of "total-production set" F and "total consumption-supply set" C, defined by:

1.8-D1: $F := F^1 + F^2 + \ldots + F^n$

1.8-D2: $C := C^1 + C^2 + \ldots + C^m$.

Clearly, we have:

<u>Proposition 1.8-P1</u>: Each of the assumptions 1.7-A1 to 7 implies an equivalent property with respect to F . Assumption 1.8-A1 implies: $\{(x,y) \in F | x \leqq y\} = \{(0,0)\}$. Each of the assumptions 1.5-A1 to 6 implies an equivalent property with respect to C .

<u>Theorem 1.8-P2 and 3</u>: The assumptions 1.7-A1, 2, 3, 6, 7, and the assumptions 1.8-A1, 1.5-A4 imply the following properties:

P2: Vectors $\underline{x}, \underline{z} \in R^g_+$ exist such that, for every $(z,w) \in C$, $(x,y) \in F$, the inequality $z - w + x - y \leqq 0$ implies: $z \leqq \underline{z}$ and $x \leqq \underline{x}$.

P3: Sequences $\{(z_t, w_t)\}_{t=1}^{\infty} \subset C$, $\{(x_t, y_t)\}_{t=1}^{\infty} \subset F$, satisfying $z_t - w_t + x_t - y_{t-1} \leqq 0$, $t = 1, 2, \ldots$ for some $y_0 \in R_+^g$, are bounded.

<u>Proof of P2</u>: Assumption 1.5-A4 implies the existence of a vector $\bar{w}$ such that $C \subset (R_+^g \times [0,\bar{w}])$. Then:

(1) $\forall (z,w) \in C$, $(x,y) \in F \mid z - w + x - y \leqq 0 : z + x - y \leqq \bar{w}$

Defining $A := \{(a,v) := (z+x,y) \mid (z,w) \in C, (x,y) \in F\}$, we have:

(2) A is closed and convex. (To be deduced from closedness of C, F and $C, F \subset R_+^{2g}$, and from convexity of C, F.)

(3) $(0,0) \in A$. (By similar property of C and F.)

(4) $A \cap (\{0\} \times R^g)$ is bounded (By 1.8-A1, 1.8-P1, $C, F \subset R_+^{2g}$.)

(5) $\exists \alpha, \beta \in R_+^1 : \forall (a,v) \in A : \|v\|_1 \leqq \alpha + \beta \|a\|_1$. (By A-P3, and by the properties (2), (3), (4).)

Clearly, the definition of A and the properties (1), (5) imply P2.

<u>Proof of P3</u>: For the set A as defined above we have, in addition:

(6) $\{(a,v) \in A \mid a \leqq v\} = \{(0,0)\}$. (By 1.8-A1, 1.8-P1, and $C, F \subset R_+^{2g}$.)

For sequences $\{(z_t, w_t)\}_{t=1}^{\infty} \subset C$, $\{(x_t, y_t)\}_{t=1}^{\infty} \subset F$:

(7) $\{(a_t, v_t)\}_1^{\infty}$ defined by $(a_t, v_t) := (z_t + x_t, y_t)$, $t = 1, 2, \ldots$ is a sequence in A.

(8) In addition $z_t - w_t + x_t - y_{t-1} \leqq 0$, $t = 1, 2, \ldots$ implies $a_t - v_{t-1} \leqq \bar{w}$, $t = 1, 2, \ldots$, for $v_0 := y_0$. (By (1).)

(9) $\{(a_t, v_t)\}_{t=1}^{\infty}$ is bounded. (By A-P5, (2), (3), (6), (7), and (8).)

Clearly, (9), the definition of $\{(a_t, v_t)\}_{t=1}^{\infty}$ and non-negativity of the vectors z_t, x_t, imply P3.

Corollary of 1.8-P2: Invariant actions plans which have to satisfy the balance of goods are situated in bounded subsets of C^i and F^j defined by 1.5-D1, 1.7-D1, provided the artificial bounds are large enough.

Corollary of 1.8-P3: A similar statement can be made with respect to variable ∞-horizon action plans.

2. Economic Behavior of the Firms: Shares, Budget Constraints, Profit-Mazimization, Dividends

2.1. Introduction

Budget constraints include that, at each time-point, an agent is not allowed to expend more than he earns. All expenditures and earnings are expressed in units of values; i.e. as products of prices and quantities of goods; representing only a bookkeeping reality. All economic agents accept prices as given quantities, in addition, prices constitute the only information, concerning the system as a whole, agents use by choosing their action plans.

2.2. Budget constraints of the firm; shares

The proprietary rights over a firm during a period t, are distributed among the individuals in the same proportion as each of them contributes in financing the inputs at the beginning of that period. These contributions, from now on to be called shares, will be represented by a sequence of non-negative mxn-matrices $\{S_t\}_1^h$, where a matrix-element $s_t^{i,j}$ stands for the contribution of the i^{th} individual to the j^{th} firm at the beginning of period t. Thus, given a share-distribution $\{S_t\}_{t=1}^{\infty}$, the budget constraints of the j^{th} firm can be formulated as follows:

$$(2.2.1) \qquad p_t' x_t^j \leqq \sum_{i=1}^{m} s_t^{i,j} \ , \quad t = 1, 2, \ldots$$

Further, we introduce the possibility that shareholding of some of the firms is open to only one or to part of the individuals. Two extreme examples are: (1) a firm open to one single individual; all others are excluded from shareholding, (2) the complete "open-ownership" firm. Assuming that restrictions like this are invariant over time, we express the possibilities of share holding by a subset Ω in the space of real mxn-matrices M^{mxn}, defined by:

$$2.2\text{-D1:} \quad \Omega := \left\{ S \in M^{mxn} \;\middle|\; \begin{array}{ll} s^{i,j} = 0 \ , & \text{in case firm } j \text{ is closed for individual } i \\ s^{i,j} \geqq 0 \ , & \text{otherwise} \end{array} \right\}$$

The corresponding sets of shares $s^{i,j}$, owned by individual i with respect to firm j, will be denoted $\Omega^{i,j}$. Self-evident, we assume that, for each firm, share holding is open for at least one individual.

2.3. Choice criterion of the firm: profit maximization, dividends

Within the technological restrictions, represented by the production set, and within the budget constraints, the firms are supposed to maximize the value of the outputs by choosing feasible combinations of inputs and outputs. Thus, the economic behavior of a j^{th} firm can be characterized by the programs:

$$2.3\text{-D1:} \quad \left. \begin{array}{l} \sup \ p_{t+1}' y_t^j \quad \text{over} \quad (x_t^j, y_t^j) \in F^j \ , \\ \text{subject to:} \quad p_t' x_t^j \leqq \sum_{i=1}^{m} s_t^{i,j} \ . \end{array} \right\} \ t = 1, 2, \ldots, h \ .$$

Evidently, the yields of the outputs $p_{t+1}' y_t^j$ are distributed among share holders according to the ratio of shares they owned over period t. So, the i^{th} individual receives:

$$(2.3.1) \qquad (s_t^{i,j} / \sum_{i=1}^{m} s_t^{i,j}) p'_{t+1} \hat{y}_t^j \;, \quad t = 1, 2, \ldots, h \;,$$

where: $\hat{y}_t^j$, $t = 1, 2, \ldots, h$ are the optimal outputs of the programs 2.3-D1, provided optimal solutions exist and provided the amount of issued shares is positive. One can look at the quotients:

$$2.3\text{-D2:} \qquad d_{t+1}^j := p'_{t+1} \hat{y}_t / \sum_{i=1}^{m} s_t^{i,j} \;, \quad t = 1, 2, \ldots, h \;,$$

appearing in (2.3.1) as a sequence of <u>dividend-factors</u> of the j^{th} firm. With this concept, (2.3.1) can be written $d_{t+1}^j s_t^{i,j}$, $t = 1, 2, \ldots, h$.

2.4. <u>Economic behavior of the firm under invariant prices</u>

Under invariant prices p and invariant number of shares S, the economic behavior of the j^{th} firm can be described by:

$$2.4\text{-D1:} \qquad \Psi_j(p;S) := \sup p'y^j \;, \quad \text{over} \quad (x^j, y^j) \in F \;,$$
$$\text{subject to} \quad p'x^j \leqq \sum_{i=1}^{m} s^{i,j} \;.$$

Limiting ourselves to the bounded production sets $\bar{F}^j$ (viz. §1.8), and to an arbitrary maximum $\bar{\alpha}$ with respect to dividend-factors (to be clarified in §3.5), we replace the original description 2.4-D1 by:

$$2.4\text{-D2:} \qquad \bar{\Psi}_j(p,S) := \sup p'y^j \;, \quad \text{over} \quad (x^j, y^j) \in \bar{F}^j$$
$$\text{subject to} \quad p'x^j \leq \sum_{i=1}^{m} s^{i,j} \;, \quad p'y^j \leqq \bar{\alpha} \sum_{i=1}^{m} s^{i,j} \;.$$

The supremum $\overline{\Psi}_j$ will be taken as a function from $R_+^g \times M_+^{m\times n}$ to R^1.

Proposition 2.4-P1: Compactness of the sets $\overline{F}^j$ and the possibility of inaction, $(0,0) \in \overline{F}^j$ (viz. 1.7-A2), imply: for every $(p,S) \in R_+^g \times M_+^{m\times n}$, the existence of an optimal solution, implying that $\overline{\Psi}_j : R_+^g \times M_+^{m\times n} \to R^1$ is well-defined. Moreover, for every $(p,S) \in R_+^g \times M_+^{m\times n}$: $\overline{\Psi}_j(p,S) \in [0, \bar{\alpha}\Sigma_{i=1}^m s^{i,j}]$.

The existence of optimal solutions is a direct consequence of Weierstrass' theorem: a continuous function on a compact set possesses a maximum.

Proposition 2.4-P2 and 3: Under the expansion capability assumption 1.7-A4, the following properties hold: If the program 2.4-D1 possesses a feasible solution (x^j, y^j) such that $p'y^j > 0$, then:

P2: every optimal solution $(\hat{x}^j, \hat{y}^j)$ of 2.4-D1 satisfies:
$p'\hat{x}^j = \Sigma_{i=1}^m s^{i,j}$.

P3: every optimal solution $(\hat{x}^j, \hat{y}^j)$ of 2.4-D2 satisfies at least one of the equalities: $p'\hat{x}^j = \Sigma_{i=1}^m s^{i,j}$ or $p'\hat{y}^j = \bar{\alpha}\Sigma_{i=1}^m s^{i,j}$.

Proof: If (x^j, y^j) is a feasible solution of 2.4-D1, such that $p'y^j > 0$, $p'x^j < \Sigma_{i=1}^m s^{i,j}$, then a number $\delta > 1$ exists such that $p'(\delta x^j) \leqq \Sigma_{i=1}^m s^{i,j}$. Moreover, by 1.7-A4, a number $\lambda > 1$ exists such that $(\delta x^j, \lambda y^j) \in \overline{F}^j$. Clearly, $(\delta x^j, \lambda y^j)$ is a feasible solution of 2.4-D1, with $p'(\delta y^j) > p'y^j$. Obviously, (x^j, y^j) cannot be optimal, which proves P2. Taking into account the inequality $p'y^j \leqq \bar{\alpha}\Sigma_{i=1}^m s^{i,j}$ appearing in 2.4-D2, property P3 can be found in a similar manner.

For each firm, the set of optimal solutions of 2.4-D2 corresponding to any $(p,S) \in R_+^g \times M_+^{m\times n}$, will be represented by a multi-function

$\hat{F}^j : R_+^g \times M_+^{mxn} \twoheadrightarrow R_+^{2g}$ --further to be called input-output functions, formally defined by:

2.4-D3: $\hat{F}^j(p,S) := \{(x^j, y^j) \in \bar{F}_j | p'x^j \leqq \sum_{i=1}^{m} s^{i,j}, \; p'y^j = \bar{\Psi}_j(p,S)\}$.

Proposition 2.4-P4 to 6: If the sets $\bar{F}_j$ possess the following properties: (1) $\bar{F}^j \subset R_+^{2g}$, (2) $(0,0) \in \bar{F}^j$, (3) $\bar{F}_j$ is compact, (4) $\bar{F}^j$ is convex, then:

P4: The functions $\bar{\Psi}_j : R_+^g \times M_+^{mxn} \to R^1$ are concave and non-decreasing in the second argument.

P5: The functions $\bar{\Psi}_j$ are continuous in every $(p,S) \in R_+^g \times M^{mxn}$.

P6: The input-output functions $\hat{F}^j : R_+^g \times M^{mxn} \twoheadrightarrow R_+^{2g}$ are upper-semi-continuous* and convex in every $(p,S) \in R_+^g \times M^{mxn}$.

Proof: The proof of P4 is straightforward. In order to prove P5, let $\{(p^k, S^k)\}_1^\infty \subset R_+^g \times M_+^{mxn}$ be a sequence which converges to $(p^0, S^0) \in R_+^g \times M_+^{mxn}$. We distinguish the cases: (a) $\beta^0 := \Sigma_{i=1}^m s^{i,j,0} = 0$, (b) $\beta^0 > 0$. Since by 2.4-P1, for each (p^k, S^k) : $\bar{\Psi}_j(p^k, S^k) \in [0, \bar{\alpha}\Sigma_{i=1}^m s^{i,j,k}]$, convergency of $\{S^k\}_1^\infty$ to S^0 satisfying $\Sigma_{i=1}^m s^{i,j,0} = 0$, implies: $\bar{\Psi}_j(p^k, S^k) \to \bar{\Psi}_j(p^0, S^0)$ for $k \to \infty$. This proves P5 for case (a). In case $\beta^0 := \Sigma_{i=1}^m s^{i,j,0} > 0$, convergency of $\{(p^k, S^k)\}_1^\infty$ to (p^0, S^0) , boundedness of $\bar{F}^j$, $(0,0) \in \bar{F}^j$, and, finally, convexity of $\bar{F}^j$,

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*We follow the definition of Arrow and Hahn [1]: a multi-function $\emptyset : X \twoheadrightarrow Y$ (X and Y normed spaces) is called upper-semi-continuous, if for any $\{(x^k, y^k)\}_0^\infty \subset X \times Y$, the conditions $y^k \in \emptyset(x^k)$, $k = 1, 2, \ldots,$ and $(x^k, y^k) \to (x^0, y^0)$ for $k \to \infty$, imply: $y^0 \in \emptyset(x^0)$.

offers the possibility to construct a sequence of numbers $\{\delta^k\}_1^\infty \subset [0,1]$ with the following properties: (1) $\delta^k \to 1$ for $k \to \infty$, (2) $(x,y) \in \bar{F}^j$, $p^{k'}x \leqq \Sigma_{i=1}^m s^{i,j,k}$ implies $\delta^k(x,y) \in \bar{F}^j$, $\delta^k p^{k'} x \leqq \Sigma_{i=1}^m s^{i,j,0}$, (3) $(x,y) \in \bar{F}^j$, $p^{0'}x \leqq \Sigma_{i=1}^m s^{i,j,0}$ implies $\delta^k(x,y) \in \bar{F}^j$, $\delta^k p^{k'} x \leqq \Sigma_{i=1}^m s^{i,j,k}$. Clearly, (2) and (3) also hold for optimal solutions of 2.4-D2, implying--together with the fact that $(1, p^0)$ is the limit point of $\{(\delta^k, p^k)\}_1^\infty$ --that $\{\bar{\Psi}_j(p^k, S^k)\}_1^\infty$ converges to $\bar{\Psi}_j(p^0, S^0)$.

Proof of P6: Starting from the definition of the multi-function $\hat{F}^j$, using closedness of $\bar{F}_j$ and continuity of $\bar{\Psi}_j$, the proof that the condition for upper-semi-continuity are satisfied is a straightforward elaboration of the definition of upper-semi-continuity.

2.5. Firm's dividend-function under invariant prices

The dividend-functions of the j^{th} firm under invariant prices $p \in R_+^g$ and invariant shares $S \in M_+^{mxn}$, provided that the budget $\Sigma_{i=1}^m s^{i,j}$ is positive, will be represented by a function:

$$2.5\text{-D1:} \quad \gamma_j(p,S) := \bar{\Psi}_j(p,S) / \sum_{i=1}^{m} s^{i,j} .$$

Proposition 2.5-P1 to 3: If (1) $\bar{F}^j \subset R_+^{2g}$, (2) $(0,0) \in \bar{F}^j$, (3) $\bar{F}^j$ is compact, (4) $\bar{F}^j$ is convex, then:

P1: $\gamma_j(p,S)$ is non-increasing in the second argument.

P2: $\forall p \in R_+^g,\ S \in M_+^{mxn} | \Sigma_{i=1}^m s^{i,j} > 0 : \gamma_j(p,S) \in [0,\bar{\alpha}]$, $\bar{\alpha}$ being the artificial upper bound introduced in 2.4-D2.

P3: In every point $(p,S) \in R_+^g \times M_+^{mxn}$ satisfying $\Sigma_{i=1}^m s^{i,j} > 0$, the function γ_j is continuous.

Proof: P1 is implied by $\overline{\Psi}_j(p,S) \in [0, \overline{\alpha}\Sigma_{i=1}^m s^{i,j}]$ and concavely (viz. 2.4-P4) of $\overline{\Psi}_j$ in the second argument. P2 is implied by $\overline{\Psi}_j(p,S) \in [0, \overline{\alpha}\Sigma_{i=1}^m s^{i,j}]$. P3 is implied by continuity of $\overline{\Psi}_j$ (viz. 2.4-P5).

In order to include the zero-budget case in the dividend-factors, we define, for each firm, the following multi-valued function $D^j : R_+^g \times M_+^{m\times n} \multimap R^1$:

2.5-D2: $D^j(p,S) := \gamma_j(p,S)$, in case $\Sigma_{i=1}^m s^{i,j} > 0$,

otherwise:

$$D^j(p,S) := \left\{ \delta \in [0,\overline{\alpha}] \,\middle|\, \begin{array}{l} \forall S \in M_+^{m\times n} : \\ \overline{\Psi}_j(p,S) \leqq \delta\Sigma_{i=1}^m s^{i,j} \end{array} \right\}$$

where $\overline{\alpha}$ is the artificial upper bound for dividend-factors.

Proposition 2.5-P4 to 6: If (1) $\overline{F}^j \subset R_+^{2g}$, (2) $(0,0) \in \overline{F}^j$, (3) $\overline{F}^j$ is compact, (4) $\overline{F}^j$ is convex, then in every point $(p,S) \in R_+^g \times M_+^{m\times n}$, the multi-valued function $D^j : R_+^g \times M_+^{m\times n} \multimap R^1$, possesses the properties:

P4: $D^j(p,S)$ is upper-semi-continuous

P5: $D^j(p,S)$ is convex

P6: $D^j(p,S) \in [0,\overline{\alpha}]$.

Proof: P4: D^j is upper-semi-continuous in a point $(p^0, S^0) \in R_+^g \times M_+^{m\times n}$ if, for any sequence $\{(p^k, S^k, \delta^k)\}_1^\infty$ in $R_+^g \times M_+^{m\times n} \times R^1$ the conditions $\delta^k \in D^j(p^k, S^k)$, $k = 1, 2, \ldots$ and $(p^k, S^k, \delta^k) \to (p^0, S^0, \delta^0)$ for $k \to \infty$, imply $\delta^0 \in D^j(p^0, S^0)$. Let $\{(p^k, S^k, \delta^k)\}_1^\infty$ satisfy these conditions. If $\Sigma_{i=1}^m s^{i,j,0} > 0$, then $\delta^0 \in D^j(p^0, S^0)$ is implicated by continuity of the function γ_j (viz. 2.5-D1, 2 and 2.5-P3). Now, let $\Sigma_{i=1}^{m} s^{i,j,0} = 0$. Since $\gamma_j(p,S)$ is non-increasing in S (viz. 2.5-P1), the definitions of γ_j and D^j, and $\delta^k \in D^j(p^k, S^k)$, $k = 1, 2, \ldots$ imply: $\overline{\Psi}_j(p^k, S) \leqq \delta^k \Sigma_{i=1}^{m} s^{i,j}$, for every $S \geqq S^k$. Clearly, since the function $\overline{\Psi}^j$ is continuous in (p^0, S^0) (viz. 2.4-P5), and since $\{(p^k, S^k)\}_1^\infty$ converges to (p^0, S^0), we may conclude: $\overline{\Psi}_j(p^0, S) \leqq \delta^0 \Sigma_{i=1}^{m} s^{i,j}$ for every $S \geqq S^0$. In connection with the definition of D^j in points (p^0, S^0), satisfying $\Sigma_{i=1}^{m} s^{i,j,0} = 0$, this implies the validity of P4 in these particular points. The properties P5 and P6 can be deduced by straightforward elaboration of the definitions 2.4-D2, 2.5-D1 and 2.5-D2.

3. Economic Behavior of the Individuals: Budget-Constraints and Utility Maximization

3.1. Introduction

The budget constraints of the individuals differ in one remarkable aspect from that of the firm. Namely, individual's budget constraints on adjacent time-points are linked, by the delay in payments and proceeds of shares. In fact, the entire dynamic character of the economic system is concentrated in this feature. Another particular aspect comes out in the assumption of a utility function which covers, in the form of a discounted sum, the utility of all actions over the time-horizon.

3.2. Budget constraints of the individuals

Given a sequence of prices $\{p_t\}_1^\infty$ and a sequence of (liquidating) dividend-factors $\{d_t\}_1^h$ (where the j^{th} component d_t^j represents dividend-factor of the j^{th} firm over period $t-1$), the budget constraints of the individuals $i = 1, 2, \ldots, m$ are formulated:

$$(3.2.1) \qquad p_t' z_t^i - p_t w_t^i + \sum_{j=1}^{n} s_t^{i,j} - \sum_{j=1}^{n} d_t^j s_{t-1}^{i,j} \leqq 0 \ , \quad t = 1, 2, \ldots, h \ .$$

In this, the initial distribution of shares S_0 is supposed to be the given result of the past; the consumptions and supply $\{(z_t^i, w_t^i)\}_{t=1}^h$ together with the quantity of shares $\{(s^{i,1}, s^{i,2}, \ldots, s^{i,n})\}_{t=1}^h$ are his decision-variables.

Since the amount of shares an individual buys constitute a part of his decisions, the linked structure of the budget constraints caused by share holding, implies that all individual's budget constraints over the time-horizon has to be considered, simultaneously. However, in the case of invariant prices and dividends, some of the properties of the resulting budget constraints:

$$(3.2.2) \qquad p' z_t^i - p' w_t^i + \sum_{j=1}^{n} s_t^{i,j} - \sum_{j=1}^{n} d_t^j s_{t-1}^{i,j} \leqq 0 \ , \quad t = 1, 2, \ldots, h$$

can be transferred to the following single-period budget constraints:

$$(3.2.3) \qquad p' z^i - p' w^i + \sum_{j=1}^{n} (1 - \pi d^j) s^{i,j} \leqq (1-\pi) \sum_{j=1}^{n} d^j s_0^{i,j} \ ,$$

where π is any positive number smaller than one. Some relations between (3.2.2) and (3.2.3) will be deduced in §3.4.

3.3. Individual's choice criterion

Within the sets of admissible actions (i.e. consumption, supply and share holding), we assume that individual's choice criterions can be expressed by utility functions possessing the following structure*

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*For a fundamental study relating this structure to postulates about a preference ordering on a set of feasible actions, see Koopmans [6].

$$(3.3.1) \qquad \sum_{t=1}^{h} (\pi_i)^t \varphi_i(z_t^i, w_t^i) ,$$

where the scalars $\pi_i \in]0,1[$ are time-discount factors, representing individual's time preference. Assuming that prices are the only available information about the economic system as a whole, earning-capacity--being effectuated in the budget constraints--is the only attractive aspect of share holding. For that reason, shares are not adopted in the utility function. Concerning the single-period utility functions $\varphi_i : C^i \to R^1$, we make the following assumptions:

3.3-A1: φ_i is continuous

3.3-A2: φ_i is concave

3.3-A3: $\varphi_i(0,0) = 0$

3.3-A4: $\exists \underline{z}^i \in R_+^g : \forall (z^i, w^i) \in C^i , \; \lambda > 0 : (\lambda \underline{z}^i + z^i, w^i) \in C^i ,$ $\varphi_i(\lambda \underline{z}^i + z^i, w^i) > \varphi_i(z^i, w^i)$, (non-satuation condition; a vector $\underline{z}^i$ with these properties will be called a non-satuation direction)

3.3-A5: $(z^i, \bar{w}^i), (z^i, \tilde{w}^i) \in C^i, \bar{w}^i \leqq \tilde{w}^i \Longrightarrow \varphi_i(z^i, \bar{w}^i) \geqq \varphi_i(z^i, \tilde{w}^i)$ (non-increasing with respect to supply).

A next aspect which has to be specified is the time-horizon h , appearing in the choice criterion. Technically it makes sense to assume an infinite horizon, meeting the intuitive notion that individuals do not specify any terminal point, but, implying the dubious supposition that all individuals have, and actually use, full insight about future prices and dividends. Clearly, this objection can be relieved by assuming invariant prices and dividends. If, in addition, we restrict ourselves to invariant action plans under ∞-horizon optimization with invariant prices and dividends, then it turns out that the ∞-horizon can be reduced to single-period optimization problems.

3.4. Infinite horizon choice criterion versus single-period approximation under invariant prices and dividends

Summarizing: given the consumption-supply sets C^i, and possibilities of share holding Ω (viz. §2.2), the objective functions $\varphi_i : C^i \to R^1$, the time-discount factors $\pi_i \in]0,1[$, and given the invariant price-dividend system $(p, \{d^j\}_1^n)$, the economic behavior of each individual is characterized by the ∞-horizon programs:

3.4-D1: $\sup \sum_{t=1}^{\infty} (\pi_i)^t \varphi_i(z_t^i, w_t^i)$, over

$$[(z_t^i, w_t^i) \in \bar{C}^i, \; s_t^{i,1} \in \bar{\Omega}^{i,1}, \ldots, s_t^{i,n} \in \bar{\Omega}^{i,n}], \quad t = 1, 2, \ldots$$

$$\text{subject to:} \quad p'z_t^i - p'w_t^i + \sum_{j=1}^{n} (s_t^{i,j} - d^j s_{t-1}^{i,j}) \leqq 0, \quad t = 1, 2, \ldots,$$

where S_0 is the initial share distribution, and where, dependent of the context, $\bar{\Omega}^{i,j} := \Omega^{i,j}$ or $\bar{\Omega}^{i,j} := \Omega^{i,j} \cap [0,\bar{\omega}]$, $\bar{\omega}$ being an artificial upper bound. The single-period approximation is defined by:

3.4-D2: $\sup \varphi_i(z^i, w^i)$, over

$$(z^i, w^i) \in \bar{C}^i, \; s^{i,1} \in \bar{\Omega}^{i,1}, \ldots, s^{i,n} \in \bar{\Omega}^{i,n}, \quad \text{subject to}$$

$$p'z^i - p'w^i + \sum_{j=1}^{n} (1 - \pi_i d^j) s^{i,j} \leqq (1 - \pi_i) \sum_{j=1}^{n} d^j s_0^{i,j}.$$

Proposition 3.4-P1 to 4: For every feasible action $\{(z_t^i, w_t^i), \{s_t^{i,j}\}_{j=1}^n\}_{t=1}^{\infty}$ of 3.4-D1, the following properties hold:

P1: Under boundedness of the sets $\overline{C}^i$, $\overline{\Omega}^{i,j}$, the series $\{\Sigma_{t=1}^h (\pi_i)^t (z_t^i, w_t^i, s_t^{i,1}, \ldots, s_t^{i,n})\}_{h=1}^\infty$ converges. In the next proposition we denote the limit point: $(\tilde{z}^i, \tilde{w}^i, \{\tilde{s}^{i,j}\}_{j=1}^n)$.

P2: Under convexity and compactness of the sets $\overline{C}^i \subset R_+^{2g}$, $\overline{\Omega}^{i,j} \subset R_+^1$, $((1-\pi_i)/\pi_i)(\tilde{z}^i, \tilde{w}^i, \{\tilde{s}^{i,j}\}_{j=1}^n)$ is a feasible action with respect to 3.4-D2, provided the initial share distribution is the same.

P3: Under compactness of $\overline{C}^i$ and continuity of $\varphi_i : \overline{C}^i \to R^1$, the series $\{\Sigma_{t=1}^h (\pi_i)^t \varphi(z_t^i, w_t^i)\}_{h=1}^\infty$ converges.

P4: Under compactness of $\overline{C}^i$, continuity and convexity of φ_i: $\varphi_i(\tilde{z}^i, \tilde{w}^i) \geqq ((1-\pi_i)/\pi_i)\Sigma_{t=1}^\infty (\pi_i)^t \varphi_i(z_t^i, w_t^i)$.

<u>Proof of P1</u>: This property is a direct consequence of the boundedness of the sets $\overline{C}^i$, $\overline{\Omega}^{i,j}$ and of $\pi_i \in]0,1[$.

<u>Proof of P2</u>: Since the series $\{\Sigma_{h=1}^\infty (\pi_i)^t (z_t^i, w_t^i, s_t^{i,1}, \ldots, s_t^{i,n})\}_{h=1}^\infty$ converges, property P2 can be deduced from the auxiliary propositions A-P1 and A-P2.

<u>Proof of P3</u>: Compactness of $\overline{C}^i$ and continuity of $\varphi_i : \overline{C}^i \to R^1$ implies that, for every $\{(z_t^i, w_t^i)\}_{t=1}^\infty \subset \overline{C}^i$, the sequence $\{\varphi_i(z_t^i, w_t^i)\}_{t=1}^\infty$ is bounded, and henceforth (by $\pi_i \in]0,1[$) convergency of $\{\Sigma_{t=1}^h (\pi_i)^t \varphi_i(z_t^i, w_t^i)\}_{h=1}^\infty$.

<u>Proof of P4</u>: Let $\{(z_t^i, w_t^i)\}_{t=1}^\infty \subset \overline{C}^i$. Defining a sequence $\{(\overline{z}_h^i, \overline{w}_h^i)\}_{h=1}^\infty$ by the convex combinations: $(\overline{z}_h^i, \overline{w}_h^i) := ((1-\pi_i)/(\pi_i - \pi_i^{h+1}))\Sigma_{t=1}^h (\pi_i)^t (z_t^i, w_t^i)$, $h = 1, 2, \ldots$, concavity of φ_i implies:

(1) $\varphi_i(\overline{z}_h^i, \overline{w}_h^i) \geqq ((1-\pi_i)/(1-\pi_i^{h+1}))\Sigma_{t=1}^h (\pi_i)^t \varphi_i(z_t^i, w_t^i)$, $h = 1, 2, \ldots$

Further we have:

(2) $(\bar{z}^i_h, \bar{w}^i_h) \to (\tilde{z}^i, \tilde{w}^i)$ for $h \to \infty$. (By compactness of $\bar{C}^i$ and $\pi_i \in]0,1[$.)

(3) $\varphi_i(\bar{z}^i_h, \bar{w}^i_h) \to \varphi_i(\tilde{z}^i, \tilde{w}^i)$ for $h \to \infty$. (By (2) and continuity of φ_i .)

Combining P3, (3), and (1), one can find P4.

<u>Theorem 3.4-P5</u>: Under (1) compactness and convexity of the sets $\bar{C}^i$, $\bar{\Omega}^{i,1}, \bar{\Omega}^{i,2}, \ldots, \bar{\Omega}^{i,n}$, (2) continuity and concavity of the function $\varphi_i : \bar{C}^i \to R^1$, the following property holds:

If, for any initial share distribution $s_0^{i,j}$, $j = 1, 2, \ldots, n$, $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \hat{s}^{i,2}, \ldots, \hat{s}^{i,n})$ is an optimal action of the single-period program 3.4-D2 satisfying $\hat{s}^{i,j} = s_0^{i,j}$, $j = 1, 2, \ldots, n$, then, for the same $(s_0^{i,1}, s_0^{i,2}, \ldots, s_0^{i,n})$, $\{(\hat{z}^i_t, \hat{w}^i_t, \hat{s}^{i,1}_t, \ldots, \hat{s}^{i,n}_t)\}_{t=1}^{\infty}$ defined by $(\hat{z}^i_t, \hat{w}^i_t, \hat{s}^{i,1}_t, \ldots, \hat{s}^{i,n}_t) := (\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n})$, $t = 1, 2, \ldots,$ is an optimal action of the ∞-horizon program 3.4-D1.

<u>Proof</u>: It is clear that the invariant sequence $\{(\hat{z}^i_t, \hat{w}^i_t, \hat{s}^{i,1}_t, \ldots, \hat{s}^{i,n}_t)\}_{t=1}^{\infty}$ defined above is a feasible ∞-horizon action plan for the same initial share distribution, in addition: $\varphi_i(\hat{z}^i, \hat{w}^i) = ((1 - \pi_i)/\pi_i)\Sigma_{t=1}^{\infty}(\pi_i)^t\varphi_i(\hat{z}^i_t, \hat{w}^i_t)$. Now, suppose that the ∞-horizon program possesses a feasible solution $\{(\bar{z}^i_t, \bar{w}^i_t, \bar{s}^{i,1}_t, \ldots, \bar{s}^{i,n}_t)\}_{t=1}^{\infty}$ for which the corresponding value of the objective function is higher. Then, by 3.4-P1, 2, 3, and 4, the single-period program possesses a feasible solution--say $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^{i,1}, \ldots, \tilde{s}^{i,n})$ --such that $\varphi_i(\tilde{z}^i, \tilde{w}^i) > \varphi_i(\hat{z}^i, \hat{w}^i)$. However, this contradicts optimality of $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n})$.

Note: the opposite is <u>not</u> stated; i.e. an invariant optimal action plan of the ∞-horizon program will not necessarily generate an optimal action with respect to the single-period approximation. Anyway, theorem 3.4-P5 ensures that the "best" invariant ∞-horizon action plans will be selected by the single-period approximations with appropriate initial share distributions. For that reason we accept the single-period programs 3.4-D2 as adequate descriptions of the economic behavior of the individuals under invariant prices and dividends.*

3.5. Individual's invariant action-function under invariant prices and dividends

Starting from the single-period description 3.4-D2 and from the non-satuation assumption 3.3-A4, we call an invariant price system p consistent if, for all non-satuation directions $\{\underline{z}^i\}_1^m : p'\underline{z}^i > 0$, $i = 1, 2, \ldots, m$. An invariant dividend system $\{d^j\}_1^n$ will be called consistent if $\Omega^{i,j} \neq \{0\}$ implies: $\pi_i d^j \leqq 1$.

The meaning of this concept should be clear: if a price or a dividend system is non-consistent then, omitting the artificial bounds on consumption and share holding, at least one individual is able to increase his utility by increasing his consumption above every limitation. Formally, this would imply that the total balance of goods (viz. §1.8) cannot be satisfied.

Further, we observe that the consistency condition on invariant dividend-systems generates upper bounds with respect to the dividend-factors. These may be defined by:

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*For a fundamental study concerning convex ∞-horizon programming, see Evers [4].

3.5-D1: $\rho_j := \min(1/\pi_i)$, over $i = 1, 2, \ldots, m$,

subject to $\Omega^{i,j} \neq \{0\}$, $i = 1, 2, \ldots, m$.

Consequently, the artificial upper bound for dividend-factors $\bar{\alpha}$ appearing in the description of the behavior of the firm (2.4) can be any number larger than $\max(\rho_1, \rho_2, \ldots, \rho_n)$.

Proposition 3.5-P1 and 2: Under non-satuation (3.3-A4) and under a consistent price and consistent dividend system p , $\{d^j\}_{j=1}^n$, necessary conditions for any feasible solution $(z^i, w^i, s^{i,1}, \ldots, s^{i,n})$ of 3.4-D2 to be optimal, are:

P1: $p'z^i - p'w^i + \Sigma_{j=1}^n (1 - \pi_i d^j) s^{i,j} = (1 - \pi_i) \Sigma_{j=1}^n s_0^{i,j}$

P2: $s^{i,j} = 0$, in case $\pi_i d^j < 1$.

Proof: Both properties are straightforward consequences of the definitions of non-satuation and consistency.

Introducing the bounded sets $\bar{C}^i$ and $\bar{\Omega}^{i,j}$, the invariant economic behavior of the individuals give rise to the following definitions:

3.5-D1: $\mu_i(p, d^1, \ldots, d^n, S_0) := \sup \varphi_i(z^i, w^i)$, over

$(z^i, w^i) \in \bar{C}^i$, $s^{i,j} \in \bar{\Omega}^{i,j}$, $j = 1, 2, \ldots, n$, subject to:

$$p'z^i - p'w^i + \sum_{j=1}^{n} (1 - \pi_i d^j) s^{i,j} \leqq (1 - \pi_i) \sum_{j=1}^{n} d^j s_0^{i,j}$$

3.5-D4: $A^i(p, d^1, \ldots, d^n, S_0 := \{(z^i, w^i, s^{i,1}, \ldots, s^{i,n}) \in \bar{C}^i \times \bar{\Omega}^{i,1} \times \ldots \times \bar{\Omega}^{i,n} |$

$\varphi_i(z^i, w^i) = \mu_i(p, d, \hat{S}_0),\ p'z^i - p'w^i + \sum_{j=1}^{n} (1 - \pi_i d^j) s^{i,j} \leqq (1 - \pi_i) \sum_{j=1}^{n} d^j s_0^{i,j}\}$

Proposition 3.5-P3: Compactness of the sets $\bar{C}^i$, $\bar{\Omega}^{i,j}$, the properties

$(0,0) \in \bar{C}^i$, $0 \in \bar{\Omega}^{i,j}$, and continuity of $\varphi_i : \bar{C}^i \to R^1$ imply that, for every $p \geqq 0$, $d^1, \ldots, d^n \geqq 0$, $S_0 \in M_+^{mxn}$, the programs defined by 3.5-D1 possess optimal solutions. (Consequently, $A^i(p, d^1, \ldots, d^n, S_0) \neq \emptyset$.)

Proof: The properties $(0,0) \in \bar{C}^i$, $0 \in \bar{\Omega}^{i,j}$, imply that, for every $p \geqq 0$, $d^1, \ldots, d^n \geqq 0$, $S_0 \in M_+^{mxn}$, the problems defined by 3.5-D1 possess feasible solutions. Hence, compactness of $\bar{C}^i$, $\bar{\Omega}^{i,j}$, and continuity of φ_i imply (by Weierstrass' theorem) the existence of optimal solutions.

Proposition 3.5-P4: Compactness and convexity of the sets $\bar{C}^i \subset R_+^{2g}$ and $\bar{\Omega}^{i,j} \in R_+^1$, the properties $(0,0) \in \bar{C}^i$, $0 \in \bar{\Omega}^{i,j}$, continuity of the functions $\varphi_i : \bar{C}^i \to R^1$ implies: **the multi-functions** $A^i : R_+^g \times R_+^n \times M_+^{mxn} \twoheadrightarrow R_+^{2g} \times R_+^n$ defined by 3.5-D2 **are upper-semi-continuous** and convex in every point $(p, d^1, \ldots, d^n, S_0)$ of the domain, provided a $(z^i, w^i) \in \bar{C}^i$ exists such that $p'z^i - p'w^i < 0$.

Proof: This property is, in essence, a standard result in competitive equilibrium studies.

4. Invariant Competitive Equilibrium

4.1. Definition

Starting from the economic behavior of the agents as described in sections 2 and 3, an invariant competitive equilibrium of this economy is defined as a combination of: (1) prices $\hat{p} \in R_+^g$, (2) dividend-factors $\hat{d}^1, \hat{d}^2, \ldots, \hat{d}^n$, (3) shares $\hat{S} \in \Omega \subset M_+^{mxn}$, (4) consumption-supply: $(\hat{z}^1, \hat{w}^1) \in C^1, \ldots, (\hat{z}^m, \hat{w}^m) \in C^m$, (5) input-outputs $(\hat{x}^1, \hat{y}^1) \in F^1, \ldots, (\hat{x}^n, \hat{y}^n) \in F^n$, such that simultaneously:

(a) For each individual i : $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \hat{s}^{i,2}, \ldots, \hat{s}^{i,n})$ is an optimal action plan with respect to:

$\sup \varphi_i(z^i, w^i)$, over $(z^i, w^i) \in C^i$, $\hat{s}^{i,1} \in \Omega^{i,1}, \ldots, \hat{s}^{i,n} \in \Omega^{i,n}$,

subject to: $\hat{p}'z^i - \hat{p}'w^i + \Sigma_{j=1}^n (1 - \pi_i d^j)s^{i,j} \leqq (1 - \pi_i)\Sigma_{j=1}^n d^i \hat{s}^{i,j}$.

As pointed out in §3.4, such an optimal action plan may be considered as a "best" invariant ∞-horizon action plan.

(b) For each firm j : $(\hat{x}^j, \hat{y}^j)$ is an optimal action plan with respect to:

$\sup \hat{p}'y^j$, over $(x^j, y^j) \in F^j$, subject to: $\hat{p}'x^j \leqq \Sigma_{i=1}^m \hat{s}^{i,j}$.

(c) In addition, the dividend-factors $\hat{d}^1, \hat{d}^2, \ldots, \hat{d}^n$ satisfy

$\hat{p}'\hat{y}^j = \hat{d}^j \Sigma_{i=1}^m \hat{s}^{i,j}$, $j = 1, 2, \ldots, n$.

(d) The total demand and total supply of commodities are equal; i.e.:

$\Sigma_{i=1}^m (\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n (\hat{x}^j - \hat{y}^j) = 0$.

It will appear that, under certain assumptions (viz. 4.2-P7), condition (d), can be replaced by:

(d') Total demand is smaller or equal than total supply of commodities;

i.e.: $\Sigma_{i=1}^m (\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n (\hat{x}^j - \hat{y}^j) \leqq 0$.

A direct consequence of the equilibrium conditions (a) to (d) (or (d')) is the following homogeneity property:

<u>Proposition 4.1-P1</u>: If $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ is an invariant competitive equilibrium, then, for every $\lambda > 0$, $[\lambda\hat{p}, \{\hat{d}^j\}_1^n, \lambda\hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ is an invariant competitive equilibrium, as well.

The existence proof of the equilibrium is organized in two parts. In the first part a number of necessary conditions are deduced for combinations to be an invariant competitive equilibrium. Most of these conditions contain some economic relevance. The second part gives the final proof. Finally, some results are deduced concerning inflation and Pareto efficiency.

4.2. Total effects of non-satuation, expansion capacity, and of individual's productive supply

Starting from the non-satuation assumption (3.3-A4) we introduced already (§3.5), in the context of individuals' economic behavior, the concepts of consistent price-systems and consistent dividend-systems. Now, extending these concepts to firm's economic behavior, we call a price-dividend-system $(p, d^1, d^2, \ldots, d^n)$ consistent if, simultaneously: (1) p is a consistent price-system, (2) $(d^1, d^2, \ldots, d^n)$ is a consistent dividend-system, (3) there is a number $\omega > 0$ such that for each firm--say the j^{th} one--the relations $(x^j, y^j) \in F^j$, $p'x^j \leqq \omega$ imply $p'y^j \leqq d^j\omega$. Violating (3) means that, under price-system p, there is at least one firm j with a profit ratio higher than d^j, implying that d^j cannot be a well-defined dividend-factor. It should be clear that under the non-satuation assumption, consistency is a necessary condition for a price-dividend system to enter into an invariant competitive equilibrium. We memorize the following implications of the consistency concept (see also §3.5).

<u>Proposition 4.2-P1 to 4</u>: Under non-satuation (3.3-A4) the following properties hold:

P1: Consistency of p implies $p \neq 0$.

P2: Consistency of a dividend-system $(d^1, d^2, \ldots, d^n)$ implies $d^j \leqq \rho^j$, $j = 1, 2, \ldots, n$, where the numbers $\rho_1, \rho_2, \ldots, \rho_n$ are defined by 3.5-D1.

P3: If there is a production $(\underset{\sim}{x}^j, \underset{\sim}{y}^j) \in F^j$ such that, for all $\lambda \geqq 0$: $\lambda\ (\underset{\sim}{x}^j, \underset{\sim}{y}^j) \in F^j$, then $p'\underset{\sim}{y}^j \leqq d^j p'\underset{\sim}{x}^j$ is a necessary condition for a price-dividend system $(p, d^1, \ldots, d^n)$ to be consistent.

P4: If the combination $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ satisfies the equilibrium conditions (a), (b), (c), and (d'), then the price-dividend system is consistent.

Proof: The properties P1 and P2 are discussed in §3.5. The properties P3 and P4 are straightforward implications of the concepts of consistency.

Proposition 4.2-P5: If the assumptions 1.7-A1, 1.7-A4, and 3.3-A4 are satisfied, then every combination $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ satisfying the equilibrium conditions (a), (b), (c), and (d'), also satisfies $\hat{p}'\hat{x}^j = \Sigma_{i=1}^m \hat{s}^{i,j}$, $j = 1, 2, \ldots, n$.

Proof: For every $j = 1, 2, \ldots, n$, we distinguish three cases: (1) $\Sigma_{i=1}^m \hat{s}^{i,j} = 0$, (2) $\Sigma_{i=1}^m \hat{s}^{i,j} > 0$, $\hat{p}'\hat{y}^j = 0$, (3) $\Sigma_{i=1}^m \hat{s}^{i,j} > 0$, $\hat{p}'\hat{y}^j > 0$. In case (1), non-negativity of $\hat{p}$ and $\hat{x}^j$ implies $\hat{p}'\hat{x}^j = \Sigma_{i=1}^m \hat{s}^{i,j}$. In case (2), we have $\hat{d}^j = 0$, implying (by 3.5-P2): $\hat{s}^{i,j} = 0$, $i = 1, 2, \ldots, m$. Clearly, this contradicts the assumption $\Sigma_{i=1}^m \hat{s}^{i,j} > 0$. In case (3), proposition 2.4-P2 implies $\hat{p}'\hat{x}^j = \Sigma_{i=1}^m \hat{s}^{i,j}$.

Proposition 4.2-P6: Under non-satuation (3.3-A4) and the possibility of expansion (1.7-A4), a necessary condition for a combination

$[\hat{p},\ \{\hat{d}^j\}_1^n,\ \hat{S},\ \{(\hat{z}^i,\ \hat{w}^i)\}_1^m,\ \{(\hat{x}^j,\ \hat{y}^j)\}_1^n]$ --the price-dividend system consistent--to satisfy the equilibrium conditions (a), (b), (c), is constituted by the equality: $\hat{p}'(\Sigma_{i=1}^m \hat{z}^i + \Sigma_{j=1}^n \hat{x}^j - \Sigma_{i=1}^m \hat{w}^i - \Sigma_{j=1}^n \hat{y}^j) = 0$. (This property may be considered as an analogy of Walras' law.)

<u>Proof</u>: Non-satuation and a consistent price-dividend system $[\hat{p},\ \{\hat{d}^j\}_1^n]$ implies (viz. 3.5-P1) that $\hat{p}'z^i - \hat{p}'w^i + \Sigma_{j=1}^n (1 - \pi_i d^j)\hat{s}^{i,j} = (1 - \pi_i)\Sigma_{j=1}^n \hat{s}^{i,j}$, $i = 1, 2, \ldots, m$, is a necessary condition for $[\hat{S},\ \{(\hat{z}^i,\ \hat{w}^i)\}_1^m]$ to satisfy equilibrium condition (a).

Trivially, a necessary condition to satisfy equilibrium condition (c) is: $-\hat{p}\ \hat{y}^j + \hat{d}^j \Sigma_{i=1}^m \hat{s}^{i,j} = 0$, $j = 1, 2, \ldots, n$. Next, by 1.7-P4 and 3.3-A4 we may conclude (viz. 4.2-P5): $\hat{p}'\hat{x}^j - \Sigma_{i=1}^m \hat{s}^{i,j} = 0$, $j = 1, 2, \ldots, n$. Adding these three necessary conditions, one will find: $\hat{p}'[\Sigma_{i=1}^m (\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n (\hat{x}^j - \hat{y}^j)] = 0$.

<u>Proposition 4.2-P7</u>: If the assumptions 1.5-A1, 1.5-A3, 1.7-A1, 1.7-A3, 1.7-A4, 3.3-A4, and 3.3-A5 are satisfied, then for every combination $[\hat{p},\ \{\hat{d}^j\}_1^n,\ \hat{S},\ \{(\hat{z}^i,\ \hat{w}^i)\}_1^m,\ \{(\hat{x}^j,\ \hat{y}^j)\}_1^n]$ which satisfies the equilibrium conditions (a), (b), (c), and (d'), supply and output vectors $\{\bar{w}^i\}_1^m$, $\{\bar{y}^j\}_1^n$, $(\bar{w}^i \leqq \hat{w}^i$, $i = 1, 2, \ldots, m)$, $(\bar{y}^j \leqq \hat{y}^j$, $j = 1, 2, \ldots, n)$ exists such that: $[\hat{p},\ \{\hat{d}^j\}_1^n,\ \bar{S},\ \{(\hat{z}^i,\ \bar{w}^i)\}_1^m,\ \{(\hat{x}^j,\ \bar{y}^j)\}_1^n]$ satisfies the equilibrium conditions (a), (b), (c), (d).

<u>Proof</u>: Suppose condition d' is satisfied with at least one inequality. Then, 1.5-A3 and 1.7-A3 implies the existence of vectors $\{\bar{w}^i\}_1^m$, $\{\bar{y}^i\}_1^n$ satisfying: (1) $\Sigma_{i=1}^m (\hat{z}^i - \bar{w}^i) + \Sigma_{j=1}^n (\hat{x}^j - \bar{y}^j) = 0$, (2) $(\hat{z}^i,\ \bar{w}^i) \in C^i$, $i = 1, 2, \ldots, m$, (3) $(\hat{x}^j,\ \bar{y}^j) \in F^j$, $j = 1, 2, \ldots, n$, (4) $\bar{w}^i \leqq \hat{w}^i$, $i = 1, 2, \ldots, m$, (5) $\bar{y}^j \leqq \hat{y}^j$, $j = 1, 2, \ldots, n$. Moreover, by

4.2-P6, and by the properties (1), (4), (5), we may conclude:

(6) $\hat{p}'\bar{w}^i = \hat{p}'\hat{w}^i$, $i = 1, 2, \ldots, m$, (7) $\hat{p}'\bar{y}^j = \hat{p}'\hat{y}^j$, $j = 1, 2, \ldots, n$.

Further, 3.3-A5 and property (4) imply: (8) $\varphi_i(\hat{z}^i, \bar{w}^i) \geqq \varphi_i(\hat{z}^i, \hat{w}^i)$, $i = 1, 2, \ldots, m$. Clearly, if $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ satisfies the equilibrium conditions (a), (b), (c), and (d'), then by the relations (1), (2), (3), (6), (7), (8), the combination $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \bar{w}^i)\}_1^m, \{(\hat{x}^j, \bar{y}^j)\}_1^n]$ satisfies the equilibrium conditions (a) to (d).

Although 4.2-P1 ensures that, under non-satuation, the price system is not zero, the possibility of an invariant competitive equilibrium where none of the consumers earns any income is still present. Next assumption--to be called "productive supply capacity"--rules out the possibility of such an equilibrium and, at the same time, indicates how an equilibrium with complete inactions can be excluded.

4.2-A1: Each individual--say the i^{th} --has the capability of a supply $\underset{\sim}{w}^i \neq 0$, $(0, \underset{\sim}{w}^i) \in C^i$, such that a production $\{(\underset{\sim}{x}^j, \underset{\sim}{y}^j)\}_1^n$ exists satisfying the conditions:

(a) $\forall\lambda \geqq 0 : \lambda(\underset{\sim}{x}^j, \underset{\sim}{y}^j) \in F^j$, $j = 1, 2, \ldots, n$

(b) $\Sigma_{j=1}^n(\underset{\sim}{y}^j - \rho_j\underset{\sim}{x}^j) + \underset{\sim}{w}^j > 0$, where $\rho_1, \rho_2, \ldots, \rho_n$ are the numbers defined by 3.5-D1.

Clearly, condition (a) states that $\{(\underset{\sim}{x}^j, \underset{\sim}{y}^j)\}_1^n$ can be produced in one multiple; (b) requires a net output of all kinds of commodities with a rate of productivity which is high enough to attract share holders (viz. 3.5-P2).

<u>Proposition 4.2-P8</u>: Under non-satuation (3.3-A4) and productive supply capability (4.2-A1), a positive number ν exists such that, for every consistent price-dividend system $(p, d^1, \ldots, d^n)$, there is a sequence of vectors $\{\underset{\sim}{w}^i\}_1^m \subset R^g$ satisfying: $(0, \underset{\sim}{w}^i) \in C^i$, $p'\underset{\sim}{w}^i \geqq \nu \|p\|_1$, $i = 1, 2, \ldots, m$.

<u>Proof</u>: Let $\underset{\sim}{w}^i$ be a supply vector of the i^{th} individual as described in 4.2-A1, and let $\{(\underset{\sim}{x}^j, \underset{\sim}{y}^j)\}_{j=1}^n$ be the corresponding productive actions. Then, defining $u^i := \Sigma_{j=1}^n (\underset{\sim}{y}^i - \rho_j \underset{\sim}{x}^j) + \underset{\sim}{w}^i$, and $\nu_i := \min(u^i_1, u^i_2, \ldots, u^i_n)$ which is positive by 4.2-A1, we have, for every $p \in R_+^g$:

$$(4.2.1) \qquad \sum_{j=1}^{n} (p'\underset{\sim}{y}^i - \rho_j' \underset{\sim}{x}^j) + p'\underset{\sim}{w}^i \geqq \nu^i \|p\|_1 .$$

Since by consistency of $[p, \{d^j\}_1^n]$: $p'\underset{\sim}{y}^j \leqq d^j p'\underset{\sim}{x}^j \leqq \rho_j p'\underset{\sim}{x}^j$, $j = 1, 2, \ldots, n$ (viz. 4.2-P2), the inequalities (4.2.1) imply $p'\underset{\sim}{w}^i \geqq \nu^i \|p\|_1$.

<u>Proposition 4.2-P9</u>: If the sets C^i satisfy the assumption: 1.5-A2, and 6, and if the functions φ_i satisfy the assumptions 3.3-A1, 2, 3, and 4, and if, for some individual i , there is a supply $\underset{\sim}{w}^i$ as described in 4.2-A1 satisfying

$$(4.2.2) \qquad \lim_{\lambda \to 0, \lambda > 0} \varphi_i(0, \lambda \underset{\sim}{w}^i)/\lambda \geqq 0 ,$$

then, under a consistent price dividend system $[\hat{p}, \{\hat{d}^j\}_1^n]$, a necessary condition for $[\hat{z}^i, \hat{w}^i, \{\hat{s}^{i,j}\}_{j=1}^n]$ to satisfy equilibrium condition (a) is: $(\hat{z}^i, \hat{w}^i) \neq 0$.

Proof: Let $\underset{\sim}{w}^i$ be a supply vector as mentioned in 4.2-A1. Then a positive number ν^i exists (viz. the proof of 4.2-P8) such that, for every consistent price-dividend system $[\hat{p}, \{\hat{d}^j\}_1^n]$,

$$(4.2.3) \qquad \hat{p}'\underset{\sim}{w}^i \geqq \nu^i \|\hat{p}\|_1 \ .$$

Let $\underset{\sim}{z}^i$ be a non-satuation direction of individual i (viz. 3.3-A4). Then, by convexity of C^i (1.5-A6), concavity of φ_i (3.3-A2), $(0,0) \in C^i$ (1.5-A2), $\varphi_i(0,0) = 0$ (3.3-A3), and by positivity of ν^i in (4.2.2), for every $\lambda \in]0,1[$, we have:

(1) $\lambda(\delta\underset{\sim}{z}^i, (1-\delta)\underset{\sim}{w}^i) \in C^i$

(2) $\varphi_i(\lambda\delta\underset{\sim}{z}^i, \lambda(1-\delta)\underset{\sim}{w}^i) \geqq \lambda[\delta\varphi_i(\underset{\sim}{z}^i, 0) + (1-\delta)\varphi_i(0, \lambda w^i)/\lambda]$

(3) $\lambda\delta\hat{p}'\underset{\sim}{z}^i - \lambda(1-\delta)\hat{p}'\underset{\sim}{w}^i \leqq 0$.

By continuity of φ_i (3.3-A1), assumption (4.2.2), $\varphi_i(\underset{\sim}{z}^i, 0) > 0$ (3.3-A4), and by property (2), it is possible to choose a λ (close enough to zero) such that:

(4) $\varphi_i(\lambda\delta\underset{\sim}{z}^i, \lambda(1-\delta)\underset{\sim}{w}^i) > 0$.

By virtue of (1), (3), (4), and $\varphi_i(0,0) = 0$ (viz. 3.3-A3) we may conclude that a $(z^i, w^i) \in C^i$ exists such that $\hat{p}'z^i - \hat{p}'w^i \leqq 0$, $\varphi_i(z^i, w^i) > 0$, implying that a necessary condition for $[(\hat{z}^i, \hat{w}^i), \{\hat{s}^{i,j}\}_{j=1}^n]$ to satisfy equilibrium condition (a) is: $(\hat{z}^i, \hat{w}^i) \neq 0$.

4.3. The existence of an invariant competitive equilibrium

Theorem 4.3-P1: The assumptions 1.5-A1 to 6, 1.7-A1 to 7, 1.8-A1, 3.3-A1 to 5, and assumption 4.2-A1, imply the existence of an invariant competitive equilibrium.

Before giving the final proof, we construct some auxiliary definitions and propositions. First of all, we observe that we may restrict ourselves to consistent price-dividend systems $[\hat{p}, \{\hat{d}^j\}_1^n]$ (viz. 4.2-P4), implying that $\hat{p} \neq 0$, and next, by the homogeneity property 4.1-P1, that it is sufficient to consider invariant prices p of the set:

4.3-D1: $P := \{p \in R_+^g \mid \|p\|_1 = 1\}$,

only. Starting from this set of prices we have the following property:

Proposition 4.3-P2: Under the assumptions 1.5-A4, 1.7-A1 to 7, 1.8-A1 and 3.3-A4, the artificial bounds appearing in the programs 2.4-D2 and 3.5-D3 can be taken such large that: **a combination** $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ -- $\hat{p}$ being a vector in P -- satisfies the equilibrium conditions (a), (b), (c), (d') if and only if simultaneously:

(1) $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n}) \in A^i(\hat{p}, \hat{d}^1, \ldots, \hat{d}^2, \hat{S})$, $i = 1, 2, \ldots, m$

(2) $(\hat{x}^j, \hat{y}^j) \in \hat{F}^j(\hat{p}, \hat{S})$, $j = 1, 2, \ldots, n$

(3) $\hat{d}^j \in D^j(\hat{p}, \hat{S})$, $j = 1, 2, \ldots, n$

(4) $\Sigma_{i=1}^m(\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n(\hat{x}^j - \hat{y}^j) \leqq 0$

Proof: The property is a straightforward consequence of the propositions 1.8-P2, 4.2-P2, and 4.2-P5.

In order to combine the multi-functions A^i , $\hat{F}^j$, and D^j , we define a multi-function $G : P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \twoheadrightarrow [0,\bar{\alpha}]^n \times \bar{\Omega} \times U$ by the relation:

4.3-D2: $G(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S}) := \{(d^1, \ldots, d^n, S, u) \in [1,\bar{\alpha}]^n \times \bar{\Omega} \times R_g \mid$
$\exists \{(z^i, w^i)\}_1^m, \{(x^j, y^j)\}_1^n \subset R^{2g}$:
$(z^i, w^i, s^{i,1}, \ldots, s^{i,n}) \in A^i(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S})$, $i = 1, 2, \ldots, m$,
$(x^j, y^j) \in \hat{F}^j(\underline{p}, \underline{S})$, $d^j \in D^j(\underline{p}, \underline{S})$, $j = 1, 2, \ldots, n$,
$u = \Sigma_{i=1}^m(z^i - w^i) + \Sigma_{j=1}^n(x^j - y^j)\}$.

With respect to the domain, the set U is defined by:

4.3-D3: $U := \{u \in R^g \mid \|u\|_1 \leq \|\bar{x}\|_1 + \|\bar{z}\|_1\}$,

$\bar{x}$ and $\bar{z}$ being vectors as described in 4.3-P2. Further, the numbers $\bar{\omega}$ (appearing in the definition of $\bar{\Omega}$, 3.4-D1) and $\bar{\alpha}$ are chosen in such a manner as mentioned in 4.3-P2. The definition of the multi-function G implies the following property:

Proposition 4.3-P3: If the multi-functions A^i (3.5-D4), $\hat{F}^j$ (2.4-D3) and D^j (2.5-D2) are upper-semi-continuous and convex in a point $(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S}) \in P \times [0,\bar{\alpha}]^n \times \bar{\Omega}$, then the multi-function $G : P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \to [0,\bar{\alpha}]^n \times \bar{\Omega} \times u$, is upper-semi-continuous and convex in $(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S})$.

Finally, we introduce multi-functions $\hat{P} : U \twoheadrightarrow P$ and $GP : P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \times U \twoheadrightarrow P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \times U$, by the relations:

4.3-D4: $\hat{P}(\underline{u}) := \{\hat{p} \in P \mid \hat{p}'\underline{u} = (\max p'\underline{u}, \text{ over } p \in P)\}$.

4.3-D5: $GP(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S}, \underline{u}) := \{(p, d^1, \ldots, d^n, S, u) \in \hat{P}(\underline{u}) \times G(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S})\}$

Proposition 4.3-P4 and 5: If the assumptions 1.5-A1 to 6, 1.7-A1 to 7, 1.8-A1, 3.3-A1 to 5, and 4.2-A1 are satisfied, then

P4: A combination $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ is an invariant competitive equilibrium (i.e. satisfies the conditions (a), (b), (c), (d') mentioned in §4.1) if, and only if $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \hat{u}]$ --where $\hat{u} := \Sigma_{i=1}^m(\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n(\hat{x}^j - \hat{y}^j)$ --is a fixed point of the multi-function GP defined by 4.3-D5 (i.e. $(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u}) \in GP(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u})$).

P5: Multi-function GP possesses a fixed point.

<u>Proof</u>: P4: If $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ satisfies the equilibrium conditions (a), (b), (c), (d') then, by 4.3-P2:

$(\hat{x}^j, \hat{y}^j) \in \hat{F}^j(\hat{p},\hat{S})$, $\hat{d}^j \in D^j(\hat{p},\hat{S})$, $j = 1, 2, \ldots, n$

$(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n}) \in A^i(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S})$. Defining

$\hat{u} := \Sigma_{i=1}^m(\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n(\hat{x}^j - \hat{y}^j)$, we may conclude:

(1) $(\hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u}) \in G(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S})$ (viz. 4.3-D2)

(2) $\hat{u} \leqq 0$

(3) $\hat{p}'\hat{u} = 0$, (by (2) and 4.2-P6)

(4) $\hat{p} \in \hat{P}(\hat{u})$ by (2), (3) and 4.3-D4

(5) $(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u}) \in GP(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u})$, by (1), (3) and by 4.3-D5.

The other way round, if $(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u}) \in GP(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S}, \hat{u})$, then (viz. 4.3-D5 and 4.3-D2) there is a sequence $\{(\hat{z}^i, \hat{w}^i)\}_1^m$ and a sequence $\{(\hat{x}^j, \hat{y}^j)\}_1^n$ such that:

(6) $(\hat{x}^j, \hat{y}^j) \in \hat{F}^j(\hat{p},\hat{S})$, $j = 1, 2, \ldots, n$

(7) $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n}) \in A^i(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S})$, $i = 1, 2, \ldots, m$

(8) $\Sigma_{i=1}^m(\hat{z}^i - \hat{w}^i) + \Sigma_{j=1}^n(\hat{x}^j - \hat{y}^j) = \hat{u}$.

In addition we have

(9) $\hat{d}^j \in D^j(\hat{p},\hat{S})$, $j = 1, 2, \ldots, n$ (by 4.3-D5 and 4.3-D2)

(10) $\hat{p}'\hat{u} = 0$ (by 4.2-P6, (8), and by (6), (7), (8), and 4.3-P2)

(11) $\hat{u} \leqq 0$ (by (10) and definition 4.3-D4).

Clearly, $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)_1^n\}]$ satisfies the equilibrium conditions (a), (b), (c), (d').

P6: The domain of the multi-function GP is compact and convex. Further, the propositions 2.4-P6, 2.5-P4, 2.5-P6, 3.5-P4, 4.2-Pi, and the definition of GP (4.3-D5) imply that the multi-function GP is upper-semi-continuous and convex in every point of its domain. Hence, by Kakutani's fixed point theorem we may conclude: GP possesses a fixed point.

Corollary of 4.3-P4, 4.3-P5, and 4.2-P7: The assumptions 1.5-A1 to 6, 1.7-A1 to 7, 1.8-A1, 3.3-A1 to 5, and 4.2-A1 imply the existence of an invariant competitive equilibrium.

4.4. Invariant competitive equilibrium and inflation*

As pointed out in 2.4 and 3.4, an invariant competitive equilibrium $[\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n]$ contains optimal ∞-horizon action plans $(z_t^i, w_t^i, s_t^{i,1}, \ldots, s_t^{i,n}) := (\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n})$, $t = 1, 2, \ldots$ and $(x_t^j, y_t^j) := (\hat{x}^j, \hat{y}^j)$, $t = 1, 2, \ldots$, for individuals and firms under an invariant price-dividend system $(p_t, d_t^1, \ldots, d_t^n) := (\hat{p}, \hat{d}^1, \ldots, \hat{d}^n)$, $t = 1, 2, \ldots$. Now, it will be shown that, with an appropriate modification of the sequence of share-distribution, these action plans remain optimal under every degree of inflation (or deflation) with respect to the price-system. The latter is described by:

$$(4.4.1) \qquad p_t := \delta_t \hat{p}\ , \quad t = 1, 2, \ldots ,$$

where $\{\delta_t\}_1^\infty$ is any sequence of positive scalars. The basis of this property is constituted by the fact that, the set of actions $\{(z_t^i, w_t^i)\}_{t=1}^\infty$, $i = 1, 2, \ldots, m$, $\{(x_t^j, y_t^j)\}_{t=1}^\infty$, $j = 1, 2, \ldots, n$, satisfying

*This section is based on a suggestion of Martin Shubik.

$$(4.4.2)\qquad \left.\begin{aligned} &\delta_t\hat{p}'(z_t^i - w_t^i) + \sum_{j=1}^{n}[\delta_t\hat{s}_t^{i,j} - (\delta_t/\delta_{t-1})\hat{d}^j(\delta_{t-1}\hat{s}_t^{i,j})] \leqq 0 \qquad i = 1, 2, \ldots, m \\ &\delta_t\hat{p}'x_t^j \leqq \sum_{i=1}^{m}\delta_t\hat{s}_t^{i,j}, \qquad j = 1, 2, \ldots, n \\ &\delta_t'\hat{p}'\hat{y}^j = (\delta_t/\delta_{t-1})\hat{d}^j\sum_{i=1}^{m}\delta_{t-1}\hat{s}_{t-1}^{i,j}, \qquad j = 1, 2, \ldots, n \end{aligned}\right\} t = 1, 2, \ldots$$

with $\delta_0 := 1$, is independent of the sequence of positive scalars, $\{\delta_t\}_1^\infty$. From (4.4.2) one can deduce the following property:

Theorem 4.4-P1: If $(z_t^i, w_t^i, s_t^{i,1}, \ldots, s_t^{i,n}) := (\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n})$, $t = 1, 2, \ldots$ and $(x_t^j, y_t^j) := (\hat{x}^j, \hat{y}^j)$, $t = 1, 2, \ldots$, are optimal action plans relative to the invariant price-dividend system $(p_t, d_t^1, \ldots, d_t^n) := (\hat{p}, \hat{d}^1, \ldots, \hat{d}^n)$, $t = 1, 2, \ldots$, then, for every sequence of positive scalars $\{\delta_t\}_1^\infty$ the action plans $(z_t^i, w_t^i, s_t^{i,1}, \ldots, s_t^{i,n}) := (\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n})$, $t = 1, 2, \ldots$ and $(x_t^j, y_t^j) := (\hat{x}^j, \hat{y}^j)$, $t = 1, 2, \ldots$ are optimal under the price-dividend system $(p_t, d_t^1, \ldots, d_t^n) := (\delta_t\hat{p}, (\delta_t/\delta_{t-1})\hat{d}^1, \ldots, (\delta_t/\delta_{t-1})\hat{d}^n)$, $t = 1, 2, \ldots$.

4.5. Pareto efficiency

In order to study the concept of Pareto efficiency in the context of ∞-horizon action plans, we introduce two different concepts of efficiency: Given the initial outputs $\{y_0^j\}_{j=1}^n$, an action plan $\{\{(z_t^i, w_t^i)\}_{i=1}^m, \{(x_t^j, y_t^j)\}_{j=1}^n\}_{t=1}^\infty$ will be called strictly efficient if:

(a) $\{(z_t^i, w_t^i)\}_{t=1}^\infty \subset \bar{C}^i$, $i = 1, 2, \ldots, m$

(b) $\{(x_t^j, y_t^i)\}_{t=1}^\infty \subset \bar{F}^j$, $j = 1, 2, \ldots, n$

(c) $\Sigma_{i=1}^m (z_t^i - w_t^i) + \Sigma_{j=1}^n (x_t^j - y_{t-1}^j) = 0$, $t = 1, 2, \ldots$

(d) There is no action plan $\{\{(\bar{z}_t^i, \bar{w}_t^i)\}_{i=1}^m, \{(\bar{x}_t^j, \bar{y}_t^j)\}_{j=1}^n\}_{t=1}^\infty$, which satisfies the conditions (a), (b), (c), and, in addition: $\Sigma_{t=1}^\infty (\pi_i)^t \varphi_i(\bar{z}_t^i, \bar{w}_t^i) \geqq \Sigma_{t=1}^\infty (\pi_i)^t \varphi_i(z_t^i, w_t^i)$, $i = 1, 2, \ldots, m$, with strict inequality for at least one i .

In the concept of <u>weak efficiency</u>, condition (d) is replaced by:

(d') There is no action plan $\{\{(\bar{z}_t^i, \bar{w}_t^i)\}_{i=1}^m, \{(\bar{x}_t^j, \bar{y}_t^j)\}_{j=1}^n\}_{t=1}^\infty$, which satisfies the conditions (a), (b), (c), and, in addition: $\varphi_i(\bar{z}_t^i, \bar{w}_t^i) \geqq \varphi_i(z_t^i, w_t^i)$, $i = 1, 2, \ldots, m$, $t = 1, 2, \ldots$, with strict inequality for at least one (i,t) .

Note that, by virtue of 1.8-P4, we may restrict ourselves to bounded consumption-supply sets and to bounded input-outputs sets. Evidently, we have the following property:

<u>Proposition 4.5-P1</u>: Strict efficiency implies weak efficiency.

In the next part it will be shown that, under some extra assumptions, every invariant ∞-horizon action plans generated by an invariant competitive equilibrium is strictly or weakly efficient. These assumptions are:

4.5-A1: For each production set F^j : $(x^j, y^j) \in F^j$ implies, for every $\lambda \geqq 0$: $\lambda(x^j, y^j) \in F^j$ (such a technology will be called linear). Note that this assumption implies the assumption 1.7-A4.

4.5-A2: The numbers $\{\rho_j\}_1^n$ defined by 3.5-D1 are equal; i.e. $\rho_1 = \rho_2 = \ldots = \rho_n$

4.5-A3: The time-discount factors $\{\pi_i\}_1^m$ are equal; i.e. $\pi_1 = \pi_2 = \ldots = \pi_m$. (Note: 4.5-A3 implies 4.5-A2 in such a manner that: $1/\pi_1 = \rho_1$.)

Proposition 4.5-P2: Assumption 4.5-A1 implies the following property: If $(\hat{x}^j, \hat{y}^j)$ is an optimal solution of

$$(4.5.1) \qquad \max \hat{p}'y^j , \quad \text{over} \quad (x^j, y^j) \in F^j , \quad \text{subject to} \quad \hat{p}'x^j \leqq \omega ,$$

ω being some non-negative number, then, for every δ satisfying $\hat{p}'\hat{y}^j = \delta\omega$, the vector $(\hat{x}^j, \hat{y}^j)$ is an optimal solution of:

$$(4.5.2) \qquad \min(-\hat{p}'y^j + \delta\hat{p}'x^j) , \quad \text{over} \quad (x^j, y^j) \in F^j .$$

Proof: Let $(\hat{x}^j, \hat{y}^j)$ be optimal with respect to (4.5.1) and let δ satisfy $\hat{p}'\hat{y}^j = \delta\omega$. Suppose $(\hat{x}^j, \hat{y}^j)$ is not optimal with respect to (4.5.2). Then a $(\bar{x}^j, \bar{y}^j) \in F^j$ exists such that $(-\hat{p}'\bar{y}^j + \delta\hat{p}'\bar{x}^j) < 0$, implying the existence of a number $\lambda \geqq 0$, satisfying: $\hat{p}'(\lambda\bar{x}^j) \leqq \omega$, $\hat{p}'(\lambda\bar{y}^j) > \delta\omega$. Since $\lambda(\bar{x}^j, \bar{y}^j) \in F^j$ (by 4.5-A1), these relations contradict optimality of $(\hat{x}^j, \hat{y}^j)$ with respect to (4.5.1).

Proposition 4.5-P3 and 4: Given a price-system $\hat{p} \in R_+^g$, consider, for each $i = 1, 2, \ldots, m$, the following optimization problems:

$$(4.5.3) \qquad \mu_i(\gamma_i) := \max \varphi_i(z^i, w^i) , \quad \text{over} \quad (z^i, w^i) \in C^i , \quad \text{subject to} \quad \hat{p}'z^i - \hat{p}'w^i \leqq \gamma_i .$$

$$(4.5.4) \qquad \nu_i(\delta_i) := \min \hat{p}'z^i - \hat{p}'w^i , \quad \text{over} \quad (z^i, w^i) \in C^i , \quad \text{subject to} \quad \varphi_i(z^i, w^i) \geqq \delta_i .$$

Under non-satuation (3.3-A4) the following properties hold:

P3: If, for some γ_i, $(\hat{z}^i, \hat{w}^i)$ is optimal for (4.5.3) then $(\hat{z}^i, \hat{w}^i)$ is optimal for (4.5.4), with $\delta_i := \mu_i(\gamma_i)$

P4: For every γ_i such that (4.5.3) possesses an optimal solution: every optimal solution of (4.5.4), with $\delta_i := \mu_i(\gamma_i)$, is optimal for (4.5.3).

Proof: Let, for some γ_i, $(\hat{z}^i, \hat{w}^i)$ be optimal for (4.5.3). Then, from 3.3-A4, one can deduce:

(1) $\hat{p}'\hat{z}^i - \hat{p}'\hat{w}^i = \gamma_i$, implying

(2) $\nu_i(\mu_i(\gamma_i)) = \gamma_i$, and hence:

(3) $(\hat{z}^i, \hat{w}^i)$ is optimal for (4.5.4) with $\delta_i := \mu_i(\gamma_i)$, which proves P3.

In order to prove P4, let, in addition, $(\bar{z}^i, \bar{w}^i)$ be optimal for (4.5.4) with $\delta_i := \mu_i(\gamma_i)$. Then:

(4) $\hat{p}'\bar{z}^i - \hat{p}'\bar{w}^i \leqq \hat{p}'\hat{z}^i - \hat{p}'\hat{w}^i$, implying (by $\delta_i := \mu_i(\gamma_i)$):

(5) $(\bar{z}^i, \bar{w}^i)$ is a feasible solution of (4.5.3) such that

$\varphi_i(\bar{z}^i, \bar{w}^i) \geqq \mu_i(\gamma_i)$.

Clearly $\varphi_i(\bar{z}^i, \bar{w}^i) = \mu_i(\gamma_i)$, implying optimality of $(\bar{z}^i, \bar{w}^i)$ with respect to (4.5.4).

Proposition 4.5-P5: Under the assumptions 3.3-A4 (non-satuation) and 4.5-A2, every invariant competitive equilibrium $[\hat{p}, \{\hat{d}^j\}_{j=1}^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_{i=1}^m, \{(\hat{x}^j, \hat{y}^j)\}_{j=1}^n]$ satisfies: $\hat{p}'\hat{y}^j = \bar{\rho}\Sigma_{i=1}^m \hat{s}^{i,j}$, $j = 1, 2, \ldots, n$, where $\bar{\rho}$ is the number such that $\bar{\rho} = \rho_j$, $j = 1, 2, \ldots, n$ (viz. 4.5-A2).

Proof: Equilibrium condition 4.1-c states:

(1) $\hat{p}'\hat{y}^j = \hat{d}^j\Sigma_{i=1}^m \hat{s}^{i,j}$, $j = 1, 2, \ldots, n$.

Under 3.3-A4, equilibrium condition 4.1-a and the propositions **4.2-p2 and** 4 imply:

(2) $\hat{d}^j \leqq \rho^j$, $j = 1, 2, \ldots, n$.

In addition, from 3.5-P2 and from definitions 3.5-D1 one can deduce:

(3) $\hat{d}^j < \rho^j$ implies $\Sigma_{i=1}^m \hat{s}^{i,j} = 0$.

Clearly, with $\rho_j = \bar{\rho}$, $j = 1, 2, \ldots, n$, the relations (1), (2), and (3) imply: $\hat{p}'\hat{y}^j = \bar{\rho}\Sigma_{i=1}^m \hat{s}^{i,j}$, $j = 1, 2, \ldots, n$.

<u>Theorem 4.5-P6</u>: Under the assumptions 3.3-A4 (non-satuation), 4.5-A1 (linear technology), and 4.5-A2: every invariant ∞-horizon action plan $\{\{(\hat{z}_t^i, \hat{w}_t^i)\}_{i=1}^m, \{(\hat{x}_t^j, \hat{y}_t^j)\}_{j=1}^n\}_{t=1}^\infty$ generated by an invariant competitive equilibrium is weakly efficient. If, in addition, 4.5-A3 is satisfied, then such an ∞-horizon action plan is strictly efficient.

<u>Proof</u>: Let $[\hat{p}, \{\hat{d}^j\}_{j=1}^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_{i=1}^m, \{(\hat{x}^j, \hat{y}^j)\}_{j=1}^n]$ be an invariant competitive equilibrium. Let $\{\{(\hat{z}_t^i, \hat{w}_t^i)\}_{i=1}^m, \{(\hat{x}_t^j, \hat{y}_t^j)\}_{j=1}^n\}_{t=1}^\infty$ be the ∞-horizon action plan generated by this equilibrium. Then:

(1) Each $(\hat{z}_t^i, \hat{w}_t^i)$ is optimal with respect to max. problem (4.5.3) with $\gamma_i := \Sigma_{j=1}^n \hat{s}^{i,j}$. (By equilibrium condition 4.1-a.)

(2) Each $(\hat{z}_t^i, \hat{w}_t^i)$ is optimal with respect to min. problem (4.5.4) with $\delta_i := \varphi_i(\hat{z}^i, \hat{w}^i)$. (By 3.3-A4, (1), and 4.5-P3.)

(3) Each $(\hat{x}_t^j, \hat{y}_t^j)$ is optimal with respect to min. problem (4.5.2) with $\delta = \bar{\rho}$, $\bar{\rho}$ such that $\rho_j = \bar{\rho}$, $j = 1, 2, \ldots, n$. (By 3.3-A4, 4.5-A1, 4.5-A2, 4.5-P5, and 4.5-P2.)

(4) $\Sigma_{i=1}^m(\hat{z}_t^i - \hat{w}_t^i) + \Sigma_{j=1}^n(\hat{x}_t^j - \hat{y}_{t-1}^j) = 0$, $t = 1, 2, \ldots$ with $\hat{y}_0^j = \hat{y}^j$, $j = 1, 2, \ldots, n$. (By equilibrium condition 4.1-d.)

Defining a set $H \subset R^g$ by:

$$(4.5.5) \qquad H := \{v \in R^g \mid \exists \{(z_t^i, w_t^i)\}_{t=1}^{\infty} \in \bar{C}^i,\ i = 1, 2, \ldots, m,$$

$$\{(x_t^j, y_t^j)\}_{t=1}^{\infty} \in \bar{F}^j,\ j = 1, 2, \ldots, n :$$

$$\varphi_i(z_t^i, w_t^i) \geqq \varphi_i(\hat{z}^i, \hat{w}^i),\ i = 1, 2, \ldots, m,$$

$$v = \sum_{t=1}^{\infty} (1/\bar{\rho})^t [\sum_{i=1}^{m} (z_t^i - w_t^i) + \sum_{j=1}^{n} (x_t^j - (1/\bar{\rho}) y_t^j]\} ,$$

one can deduce from (2) and (3) that

$\hat{v} := \Sigma_{t=1}^{\infty}(1/\bar{\rho})^t[(\hat{x}_t^i - \hat{w}_t^i) + \Sigma_{j=1}^{n}(\hat{x}_t^j - (1/\bar{\rho})\hat{y}_t^j]$ is an optimal solution of the min. problem:

$$(4.5.6) \qquad \min \hat{p}'v , \quad \text{over} \quad v \in H .$$

Further, from (4), one can deduce that $\hat{p}'\hat{v} = (1/\bar{\rho})\hat{p}'\Sigma_{j=1}^{n}\hat{y}_0^j$, implying (by optimality of $\hat{v}$):

(5) $\forall v \in H : \hat{p}'v \geqq (1/\bar{\rho})\hat{p}'\Sigma_{j=1}^{n}\hat{y}_0^j$.

Now, let $\{\{(\bar{z}_t^i, \bar{w}_t^i)\}_{i=1}^{m}, \{(\bar{x}_t^j, \bar{y}_t^j)\}_{j=1}^{n}\}_{t=1}^{\infty}$ be an action plan satisfying the conditions 4.5-a, b, c (with $\bar{y}_0^j := \hat{y}_0^j$, $j = 1, 2, \ldots, n$) and in addition: $\varphi_i(\bar{z}_t^i, \bar{w}_t^i) \geqq \varphi_i(\hat{z}^i, \hat{w}^i)$, $i = 1, 2, \ldots, m$, $t = 1, 2, \ldots$. Then, defining $\bar{v} := \Sigma_{t=1}^{\infty}(1/\bar{\rho})^t[\Sigma_{i=1}^{m}(\bar{z}_t^i - \bar{w}_t^i) + \Sigma_{j=1}^{n}(\bar{x}_t^j - (1/\bar{\rho})\bar{y}_t^j]$, we have:

(6) $\bar{v} \in H$.

(7) $\bar{v} \leqq (1/\bar{\rho})\Sigma_{j=1}^{n}\hat{y}_0^j$. (By 4.5-c and auxiliary proposition)

(8) $\hat{p}'\bar{v} = (1/\bar{\rho})\hat{p}'\Sigma_{j=1}^{n}\hat{y}_0^j$; i.e. $\bar{v}$ is an optimal solution of (4.5.5). (By (5), (6), (7).)

(9) Each $(\bar{z}_t^i, \bar{w}_t^i)$ is an optimal solution of a corresponding problem (4.5.4) with $\delta_i := \varphi_i(\hat{z}^i, \hat{w}^i)$. (By (8) and the definition of min. problem (4.5.6) and (4.5.5).)

(10) Each $(\bar{z}_t^i, \bar{w}_t^i)$ is optimal with respect to (4.5.3) with $\gamma_i := \Sigma_j^n \hat{s}^{i,j}$. (By (8), proposition 4.5-P4 and (1).)

Hence, the suppositions imply $\varphi_i(\bar{z}^i_t, \bar{w}^i_t) = \varphi_i(\hat{z}^i_t, \hat{y}^i_t)$, proving the first part of the proposition (viz. the definition of weak efficiency).

In order to prove the second part concerning strict efficiency, we observe that assumption 4.5-A3 implies $\pi_i = 1/\bar{\rho}$, $i = 1, 2, \ldots, m$. Putting $\pi := 1/\bar{\rho}$, we replace set H (definition 4.5.5) by:

$$H^* := \{v \in R^g \mid \exists \{(z^i_t, w^i_t)\}^{\infty}_{t=1} \in \bar{C}^i,\ i = 1, 2, \ldots, m,$$

$$\{(x^j_t, y^j_t)\}^{\infty}_{t=1} \in \bar{F}^j,\ j = 1, 2, \ldots, n$$

$$\sum_{t=1}^{\infty} \pi^t \varphi_i(z^i_t, w^i_t) \geqq \sum_{t=1}^{\infty} \pi^t \varphi_i(\hat{z}^i, \hat{w}^i),\ i = 1, 2, \ldots, m$$

$$v := \sum_{t=1}^{\infty} \pi^t [\sum_{i=1}^{m} (z^i_t - w^i_t) + \sum_{j=1}^{n} (x^j_t - \pi y^j_t)]\} .$$

Further, strict efficiency can be deduced in a similar manner as weak efficiency.

APPENDIX

Auxiliary Propositions A-P1 to 5:

P1: Let $Q \subset R^k$ be a closed convex set, and let $\{q_t\}_{t=1}^{\infty} \subset Q$. Then, for every $\rho \in]0,1[$ such that $\{\Sigma_{t=1}^{r} \rho^t q_t\}_{r=1}^{\infty}$ converges: $((1-\rho)/\rho)\Sigma_{t=1}^{\infty} \rho^t q_t \in Q$.

P2: Let $\{(u_t, v_t)\}_{t=1}^{\infty} \subset R^k \times R_+^k$ be a sequence which satisfies $u_t - v_{t-1} \leqq w$, $t = 1, 2, \ldots,$ for some $v_0 \in R_+^k$ and some $w \in R^k$. Then, for every $\rho \in]0,1[$ such that $\{\Sigma_{t=1}^{r} \rho^t (u_t - \rho v_t)\}_{r=1}^{\infty}$ converges: $\Sigma_{t=1}^{\infty} \rho^t (u_t - \rho v_t) \leqq \rho v_0 + (\rho/1-\rho)w$.

P3: Let $W \subset R^{2k}$ be a closed convex set containing the origin $(0,0)$, and such that $W \cap (\{0\} \times R^k)$ is bounded. Then numbers $\alpha, \beta \geqq 0$ exist such that, for every $(w,v) \in W$: $\|v\|_1 \leqq \alpha + \beta\|w\|_1$.

P4: Let $U \subset R^{2k}$ be a closed convex set containing the origin, and such that $U \cap \{(u,v) \in R^{2k} | u \leqq v\} = \{(0,0)\}$. Then numbers $\alpha \geqq 0$, $\beta \geqq 0$, $\gamma > 1$ exist such that the relations: $(u,v) \in U$, $u - \gamma v \leqq w$, imply: $\|v\|_1 \leqq \alpha + \beta\|w\|_1$.

P5: For a set U as mentioned above, every non-negative sequence $\{(u_t, v_t)\}_{t=1}^{\infty} \subset U$ satisfying $u_t - v_{t-1} \leqq w$, $t = 1, 2, \ldots,$ for some $v_0 \in R_+^k$ and some $w \in R^k$, is bounded.

Proof of P1: Suppose $\{q_t\}_{t=1}^{\infty}$ is a sequence in Q, and suppose $\rho \in]0,1[$ is a number such that $\{\Sigma_{t=1}^{r} \rho^t q_t\}_{r=1}^{\infty}$ converges. Defining a sequence $\{\bar{q}_r\}_{r=1}^{\infty}$ by the convex combinations:

(A.1.1) $\quad \bar{q}_r := ((1-\rho)/(\rho-\rho^{r+1}))\sum_{t=1}^{r}\rho^t q_t$, $r = 1, 2, \ldots,$

we may conclude: $\bar{q}_r \in Q$, $r = 1, 2, \ldots$ (by convexity of Q). In addition, convergency of $\{\Sigma_{t=1}^{r}\rho^r q_t\}_{r=1}^{\infty}$ and $\rho \in]0,1[$ implies convergency of $\{\bar{q}_r\}_{r=1}^{\infty}$. Clearly, by closedness of Q , this implies $((1-\rho)/\rho)\Sigma_{r=1}^{\infty}\rho^t q_t \in Q$.

<u>Proof of P2</u>: $u_t - v_{t-1} \leqq w$, $v_t \geqq 0$, $t = 1, 2, \ldots$ implies, for every $\rho > 0$: $\Sigma_{t=1}^{r}\rho^t(u_t - \rho v_t) \leqq \rho v_0 + ((\rho-\rho^{r+1})/(1-\rho))w$, $r = 1, 2, \ldots$ Clearly, $\rho \in]0,1[$ and convergency of $\{\Sigma_{t=1}^{r}\rho^t(u_t - \rho v_t)\}_{r=1}^{\infty}$ implies: $\Sigma_{t=1}^{\infty}\rho^t(u_t - \rho v_t) \leqq \rho v_0 + (\rho/(1-\rho))w$.

<u>Proof of P3</u>: Following Rockafellar [7], we define the recession cone rec(A) of a set $A \subset R^k$ by: $\mathrm{rec}(A) := \{a \in R^k | \forall b \in A, \lambda \geqq 0 : b + \lambda a \in A\}$. Now, defining a set $V := \{(w,w) \in R^{2k} | \|w\|_1 \leqq 1\}$, we have, with respect to a set W as assumed in P3, the following properties:

(1) $\mathrm{rec}(W \cap V) = \mathrm{rec}(W) \cap \mathrm{rec}(V)$. (By closedness and convexity of W and V . See Rockafellar 8.3.3.)

(2) $\mathrm{rec}(V) \subset \{0\} \times R^k$. (By the definition of V .)

(3) $\mathrm{rec}(W) \cap (\{0\} \times R^k) = \{(0,0)\}$. (By boundedness of the set $W \cap (\{0\} \times R^k)$.)

(4) $\mathrm{rec}(W \cap V) = \{(0,0)\}$. (By (1), (2), and (3).)

(5) $(W \cap V)$ is bounded. (By (4), and by closedness of the set $W \cap V$. See Rockafellar 8.4.)

(6) $\exists \delta > 0 : \forall (w,v) \in W \cap V : \|v\|_1 \leqq \delta$. (By (5).)

(7) $\forall (w,v) \in W : (1 + \|w\|_1)^{-1}(w,v) \in W \cap V$. (By the definition fo set V , convexity of W , and by $(0,0) \in W$.)

(8) $\exists \delta > 0 : \forall (w,v) \in W : \|v\|_1 \leqq \delta(1 + \|w\|_1)$. (By (6), (7), and by the definition of set V .)

<u>Proof of P4</u>: Defining $T := \{(u,v) \in R^{2k} | u \leqq v\}$, we have, for a set U as assumed in P4, the following properties:

(1) $\text{rec}(U \cap T) = \{(0,0)\}$. (By closedness and convexity of U , T , and by boundedness of $U \cap T$. See Rockafellar 8.4.)

(2) $\text{rec}(U) \cap T = \{(0,0)\}$. (By closedness and convexity of U , T , by (1) and $\text{rec}(T) = T$. See Rockafellar 8.3.3.)

(3) $\exists \gamma > 1 : \{(u,v) \in R^{2k} | \ u \leqq \gamma v\} \cap \text{rec}(U) = \{(0,0)\}$. (By (2) and closedness of $\text{rec}(U)$. See Rockafellar 8.2.)

(4) $\exists \gamma > 1 : \text{rec}(\{(u,v) \in R^{2k} | u \leqq \gamma v\} \cap U) = \{(0,0)\}$. (By (4), and by closedness and convexity of the sets U and $\{(u,v) \in R^{2k} | u \leqq \gamma v\}$. See Rockafellar 8.3.3.)

Now, defining for a $\gamma > 1$ the set W by:

$W := \{(w,v) \in R^{2k} | \exists u \in R^k : (u,v) \in U, \ u - \gamma v \leqq w\}$, we have:

(5) W is closed, convex, and contains the origin. (By the assumptions concerning U .)

(6) $W \cap (\{0\} \times R^k)$ is bounded. (By (4), (5), and by the definition of W . See Rockafellar 8.4.)

(7) $\exists \gamma > 1 , \ \delta \geqq 0 : \forall u, v, w \in R^k | (u,v) \in U , \ u - \gamma v \leqq w :$
$\|v\|_1 \leqq \delta \|w\|_1 + \delta$. (By A-P3, (6) and the definition of W .)

<u>Proof of P5</u>: Let $\alpha \geqq 0$, $\beta \geqq 0$, $\gamma > 1$ be the numbers as mentioned in A-P4. Let $\{(u_t, v_t)\}_{t=1}^{\infty} \subset U$ be a sequence satisfying $u_t - v_{t-1} \leqq w$, $t = 1, 2, \ldots$, for some $v_0 \in R^k_+$ and some $w \in R^k$. Defining a sequence $\{(\bar{u}_r, \bar{v}_r)\}_{r=1}^{\infty}$ by the convex combinations

$(\bar{u}_r, \bar{v}_r) := ((\gamma-1)/(\gamma-\gamma^{1-r}))\Sigma_{t=1}^{r}\gamma^{t-r}(u_t, v_t)$, $r = 1, 2, \ldots$, we have:

(1) $(\bar{u}_r, \bar{v}_r) \in U$, $r = 1, 2, \ldots$. (By convexity of U .)

(2) $\bar{u}_r - \gamma\bar{v}_r \leqq w + (\gamma^{-r}(\gamma-1)/(\gamma-\gamma^{1-r}))v_0$, $r = 1, 2, \ldots$. (By $u_t - v_{t-1} \leqq w$, $t = 1, 2, \ldots$ and $v_t \geqq 0$, $t = 1, 2, \ldots$.)

Since the coefficient $\gamma^{-r}(\gamma-1)/(\gamma-\gamma^{1-r})$ is non-increasing with respect to r (implied by $\gamma > 1$), property (2) implies the existence of vector $\bar{w}$ such that:

(3) $\bar{u}_r - \gamma\bar{v}_r \leqq \bar{w}$, $= 1, 2, \ldots$, and henceforth.

By virtue of A-P4, the relations (1) and (3) imply: $\|\bar{v}_r\|_1 \leqq \alpha+\beta\|\bar{w}\|_1$, $r = 1, 2, \ldots$, and henceforth (by the definition of $\{(\bar{u}_t, \bar{v}_t)\}_{r=1}^{\infty}$, and by $u_t, v_t \geqq 0$, $u_t - v_{t-1} \leqq w$, $t = 1, 2, \ldots$) boundedness of $\{(u_t, v_t)\}_{t=1}^{\infty}$.

List of Symbols

R^n , n-dimensional vectorspace.

$R_+^n := \{x \in R^n | x_i \geqq 0, i = 1, 2, \ldots, n\}$, the non-negative orthant.

$\|\cdot\|_1$, the ℓ_1-norm, for $x \in R^n$ defined by $\|x\|_1 := |x_1| + |x_2| + \ldots + |x_n|$.

$x'y$, the inner product of a pair of finite dimensional vectors.

$[x,y] := \{z \in R^n | z \geqq x, z \leqq y\}$,

$]x,y] := \{z \in R^1 | z > x, z \leqq y\}$.

$[x,y[:= \{z \in R^1 | z \geqq x, z < y\}$.

$]x,y[=]x,y] \cap [x,y[$.

$A^1 + A^2 + \ldots + A^n := \{x := (x^1 + x^2 + \ldots + x^k) | x^r \in A^r, r = 1, 2, \ldots, n\}$.

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