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A DYNAMICAL MODEL OF POLITICAL EQUILIBRIUM

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by

Gerald H. Kramer

In a democratic society many social decisions are made by a political mechanism based on majority voting by the citizenry. In many such choices the alternatives can be represented as points in an appropriately defined multi-dimensional commodity or policy space of some sort. Hence a fundamental task in analyzing the performance of a democratically structured public sector is to characterize the behavior of competitive voting processes over such multi-dimensional choice spaces, and in particular to see whether such processes lead to consistent social choices, or can be characterized by an equilibrium of some sort. These questions have been extensively studied, from several points of view. However the major results in this literature, which we review in more detail below, are either essentially negative, in the sense that they show that an equilibrium can exist only in very restrictive special cases, or rest on essentially ad hoc assumptions and formulations, or pose serious difficulties of interpretation (e.g. concerning the meaningfulness and existence of mixed strategies for certain agents in the process).

In this paper we take a different approach, more explicitly dynamical in character, in which we assume the political process to be driven

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by competition for votes extending across a series of elections. We show that the sequence of successively enacted policies generated by this process tends over time to converge on a relatively small subset of the feasible points. This subset, for which we give an explicit characterization, seems to provide a useful and natural equilibrium concept for this class of problems. It can also be given a more abstract social choice interpretation, as the set of maximal elements with respect to an essentially Arrowian social ordering. Before developing these results in detail, however, we first review the major results on voting over multi-dimensional choice spaces, to provide a setting and motivate the particular approach taken in this paper.

Majority Voting over Multi-Dimensional Choice Spaces

In the special case where there is only a single variable or public good to be decided upon, majority rule is generally well behaved. The convexity of voter preferences often implies satisfaction of Black's "single-peakedness" condition, which in turn implies that the majority preference relation is transitive, and that there will exist a majority equilibrium, i.e., an alternative which cannot be defeated by any other alternative in a pairwise majority vote (Black (1958)). Even when the single-peakedness condition fails because of non-convexities in individual preferences or technology, there will generally exist "local" equilibria, i.e., alternatives which cannot be defeated by neighboring alternatives (Kramer and Klevorick (1975)). Thus in one dimensional choice problems, the behavior of political mechanisms based on majority rule can normally be described in terms of an equilibrium.

The situation is quite different, unfortunately, when there are two or more goods or quantities to be voted on. The usual conditions for transitivity of majority rule (e.g. Black (1958), Sen (1966)) fail in higher dimensional spaces, in general (Kramer (1972)). A variety of rather different conditions for the existence of a "majority winner" in such situations have been proposed (Davis and Hinich (1966); Tullock (1967); Plott (1967); Davis, de Groot and Hinich (1972); Wendell and Thorson (1974)). Most of this "spatial modelling" literature has assumed voter preferences to be of particular form, in which each voter i has a unique satiation point s^i , and a point x is preferred to y by the voter if and only if x is closer to s^i (usually in the sense of ordinary Euclidean distance) than is y . (We shall refer to this type of preference ordering as a Type I ordering; most of our subsequent results will be for Type I

preferences.) The preference structure in the Plott conditions is more general, since preferences are assumed only to be representable by quasi-concave, continuously differentiable utility functions. All of these conditions, however, whether or not based on Type I preferences, are extremely restrictive. They in effect require the distribution of preferences to satisfy severe symmetry requirements, and are not robust against even small departures from the exact symmetry required. If a majority winner exists in a particular society, arbitrarily small perturbations in voters' preferences will destroy it (even if, in the case of the Type I conditions, the "perturbed" preferences are also Type I). In this sense, existence of a majority winner is not a stable property, and will virtually never occur in multi-dimensional choice problems. In game-theoretic terms, if the social choice process is viewed as an n-person majority game, its core will generally be empty.

Although in general majority rule is not transitive and does not yield a majority winner in multi-dimensional voting problems, it is still conceivable that it might be "well behaved" in some weaker sense. Buchanan (1968), for example, has argued that even when majority rule does cycle, the cycles will be confined to the set of Pareto Optimal alternatives, while Tullock (1967) has argued (at least for Type I preferences) that the cycling will tend to move toward a central area in the interior of the Pareto set, and remain there. If true, this would be an important result, for this central region would constitute a kind of equilibrium set, and we would have a useful if partial characterization of the behavior of majority voting on multi-dimensional problems. In the social choice literature there have been several proposals to define a social indifference class in terms of the majority cycles (so that any two alternatives

belonging to the same cycle are socially indifferent) (Kadane (1972), Good (1971)) presumably based on the hope that one of the cycles will be confined to a relatively small subset of the alternatives, akin to Tullock's "central region."

However, a result by Richard McKelvey (1975) shows that in multi-dimensional choice problems, in general there will be a single majority rule cycle extending over the entire feasible region. McKelvey's result is essentially the following: Suppose all voters have Type I preferences, and that no majority winner exists. Then for any two alternatives x , y , a sequence of points (x, x', x'', \dots, y) can be found, which begins with x and ends with y , such that each point is preferred by a majority to the preceding point. An illustration of a typical majority rule trajectory is given in Figure 1. There are seven

 Insert Figure 1 about here

voters, whose most-preferred points are labelled s^1 to s^7 , respectively. Each voter has a Type I preference ordering and hence, given the choice between any two points x and y , will prefer the point which is closest to his satiation point. The set of Pareto-optimal points is the shaded area. Beginning from an arbitrary initial point x^1 , each subsequent point is preferred by a majority to the previous one. The point x^2 is closer to s^1 , s^2 , s^3 and s^4 than is x^1 , so the majority composed of voters 1, 2, 3, 4 prefers x^2 to x^1 ; similarly $\{3,4,5,6\}$ prefers x^3 to x^2 , and so forth. The trajectory begins at a centrally located point x^1 , but soon moves outside the Pareto set itself, and clearly could be extended to move further and further

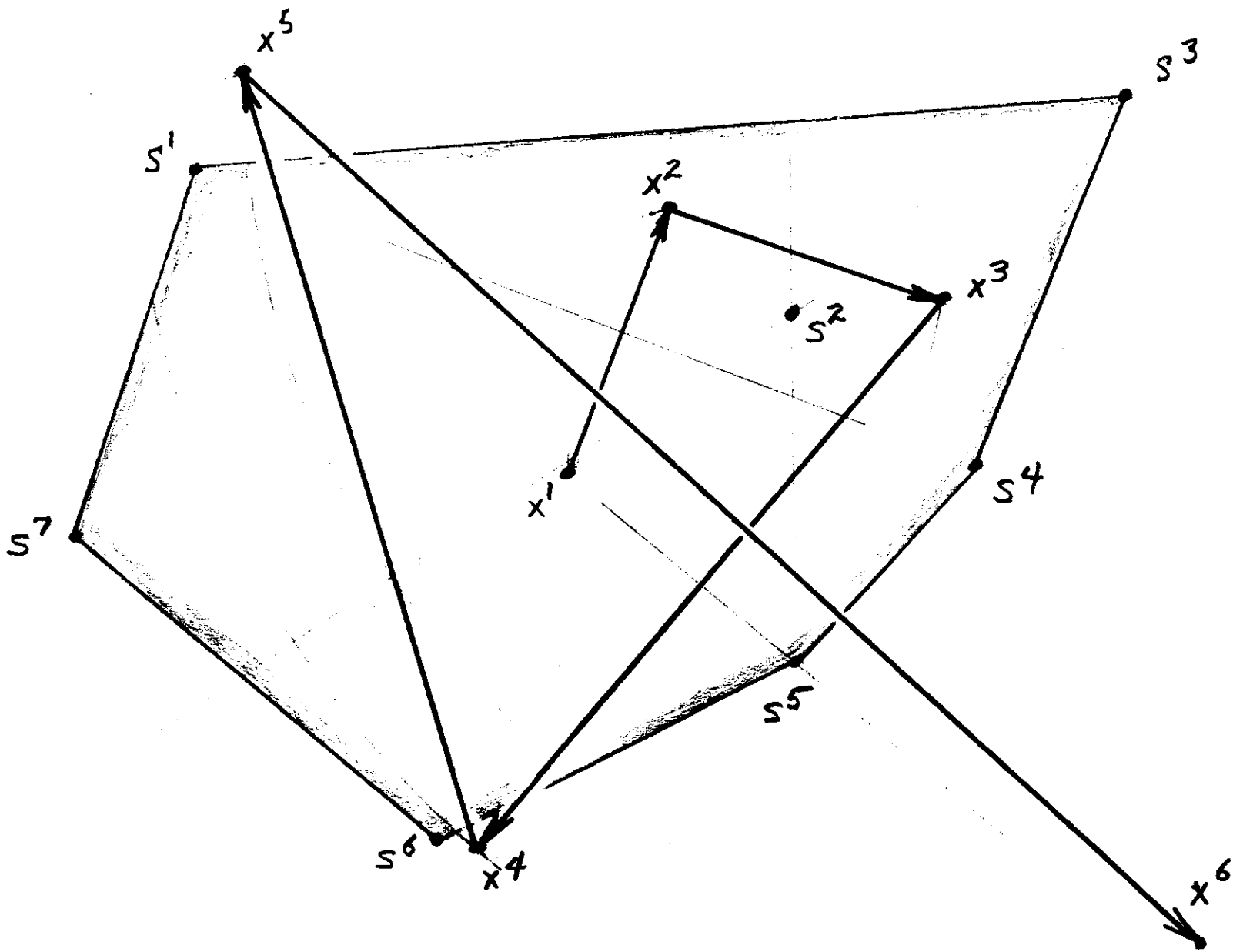


FIGURE 1. A majority rule trajectory

away from it, or to eventually reach an arbitrarily chosen point anywhere in the feasible region. Though the McKelvey result (and the example) invokes the assumption of Type I preferences, it seems clear that the conclusion is a general one, and that except under very special conditions or restrictions on voters' preferences, in multi-dimensional choice problems majority rule can quite literally wander from anywhere to anywhere in the space.

Competitive Political Processes

Most of the work described above has been primarily concerned with majority rule in the abstract, and not with the institutional mechanisms used to implement it. A related body of literature has been concerned with a two-party competitive political mechanism, in which two political parties are assumed to compete for votes by advocating particular policies, or alternatives. In every period an election is held, with each voter voting for the party whose platform he prefers. Whichever party receives a majority is elected, and enacts the alternative it advocated. The parties are interested only in winning elections, not in policies as such; thus each party is motivated to adopt policies which maximize its prospects in the next election. The original Downs-Hotelling analysis argued that when the policy space is one-dimensional, eventually both parties will tend to converge, in their platform choices, on the "majority winner" or "median" policy (Downs (1957)). From a more rigorous game-theoretic point of view, the competition for votes can be viewed as a two-player, zero-sum game, in which the feasible policies are the pure strategies available to each player, and the parties' vote shares (or perhaps their respective probabilities of winning) are the relevant payoffs. A pure strategy equilibrium for this game exists if, and only if, the electorate's preferences are such that some feasible policy is a majority winner. In the one-dimensional case this is ensured by the single-peakedness of voter preferences, implicitly assumed by Downs and Hotelling.

When the underlying policy space is multi-dimensional, however, the situation is quite different, since in general there will be no majority winner policy and hence no pure strategy equilibrium for the parties. Shubik (1970) and McKelvey and Ordeshook (1975) have explored the possibility

and nature of a mixed strategy equilibrium for the parties. It has been shown that the set of policies which are played with positive probability in such an equilibrium constitute a subset of the Pareto Optimal policies. No sharp characterization of this subset has yet been given, however, and indeed, it is still an open question whether such a mixed strategy equilibrium exists at all.* There are also serious interpretive difficulties with this approach. In particular, the mixed strategies cannot be interpreted as ambiguous or uncertain policy commitments by the parties** (Ordeshook (1971)), and it is not clear that the equilibrium mixed strategies, even if they do exist, are operationally or descriptively meaningful in the electoral competition context.***

A rather different approach proceeds from the premise that some citizens will (stochastically) abstain from voting under certain conditions, for example if they dislike both parties' policies. Particular versions of this type of assumption yield a pure strategy equilibrium for the parties (Hinich, Ledyard, and Ordeshook (1972), (1973); Riker

*The usual versions of the minmax theorem do not apply, since the pure strategy sets are infinite and (as is easily shown) the payoff function is neither continuous nor concave in each party's strategies.

**If they were so interpreted, voters would be in the position of voting on lotteries over policies. The relevant pure strategies for each party would then be the set of possible lotteries, and within these expanded strategy sets no pure strategy equilibrium will exist, in general.

***Mixed strategies require the parties to simultaneously and randomly choose policies in advance of the electoral campaign. However either party could benefit by postponing its choice until its opponent commits itself to a policy, and then choosing (non-stochastically) a policy which ensures its victory. Hence both parties have strong incentives to abandon their equilibrium mixed strategies.

and Ordeshook (1973)). However this type of equilibrium does not seem a very robust or compelling one. Its existence is sensitive to the specification used, and the particular formulations needed to ensure existence seem rather ad hoc, and not particularly consonant with the available empirical evidence on voter abstention (Slutsky (forthcoming)). In any event, such an equilibrium would fail to exist whenever compulsory voting laws or the salience of the election resulted in high voter turnout, and seems of little normative interest.

The approach taken in this paper is more dynamic in character. We assume the two parties to compete for votes repeatedly over an infinite series of elections. In each period one of the parties is elected, and enacts the policy it advocated. In the next election the incumbent party must defend this same policy, but the "out" party may adopt any policy it wishes, to maximize its prospects in the next election. Since in general there will be no "majority winner" policy, the incumbent's policy can always be defeated by some other alternative. Hence the "out" party will always win, and the two parties will alternate in office. Thus a sequence of successively enacted "winning" policies will be generated by this process, and in general will continue indefinitely to move about the space in some fashion. We shall refer to a sequence of points as a trajectory. The basic purpose of this paper is to investigate the behavior of the trajectories generated by a two-party competitive process.

A majority rule trajectory is one in which each point is preferred by a majority of voters to the preceding point. The McKelvey result referred to earlier shows that in general, the majority rule trajectories are not well behaved, since from any initial point there is a majority

rule trajectory which eventually reaches any other point. However many of the majority rule trajectories will not be generated by the political process described above, except under strong and implausible assumptions about the objective functions of the parties. In particular, to argue that every majority rule trajectory can be generated by such a process, it must be assumed that a party may choose any point y which defeats the incumbent's policy x , irrespective of the margin by which y defeats x . The party thus has zero marginal utility for votes, once a bare majority is achieved: it satisfies Riker's "size principal" with a vengeance (Riker (1958)). This assumption is not a very plausible one for electoral competition. Uncertainty about the election outcome itself, or in a parliamentary system, about future defections or deaths among the majority party, encourage parties to strive for a larger-than-minimal majorities, as risk insurance. A large winning margin is generally valued in itself, as a "mandate" for the victor, and in many systems brings tangible benefits such as increased patronage, and the election of legislators from marginal districts whose indebtedness to the party leadership ensures a more cooperative legislature. Recent U.S. Presidential elections, particularly the 1964 and 1972 contests, certainly provide little empirical evidence of any tendency for leading candidates to "satisfice" and settle for minimal margins.

The original Downs-Hotelling analysis of two-party competition was explicitly based on the contrary premise, that the parties are interested in maximizing the votes they receive. We shall also make this assumption. Under it, the trajectories generated by the political process will be such that each point is vote-maximizing against the preced-

ing point. Clearly these trajectories constitute a proper subset of the majority rule trajectories. We shall show that unlike the latter, the vote-maximizing trajectories are relatively well-behaved, and display strong convergence and stability properties.

Definitions and Assumptions

There are n voters, denoted by $N = \{1, 2, \dots, n\}$. The alternatives are points in the Euclidean k -space R^k , where $k < n/2$. Each voter i has a preference ordering over the points in R^k , representable by a Type I utility function, satiated at a unique point s^i . Thus $x \succ_i y$ iff $\|x - s^i\| \leq \|y - s^i\|$, where $\|z\| = (\sum_{i=1}^k z_i^2)^{1/2}$. The set of feasible points \mathcal{D} is a compact, convex body in R^k , and each voter is satiated in the interior of \mathcal{D} .

For any two points x, y , we define the vote for y against x , $n(x, y)$ as the number of voters for whom y is (strictly) preferred to x , i.e. $n(x, y) = |\{i : y \succ_i x\}|$. For any x , we denote by $v(x)$ the maximum vote against x , i.e. $v(x) = \max_y n(x, y)$. If y is a point for which $n(x, y) = v(x)$, then y is vote-maximizing against x . If $v(x) \leq n/2$ for any x , then (and only then) x is a majority winner, since no other point can defeat it in a pairwise majority vote. In the typical case, however, no majority winner will exist, so $v(x) > n/2$ at all x . The smallest value of $v(x)$ is the minmax number (Simpson (1969)), which we denote by n^* : thus $n^* = \min_x v(x)$. (It is straightforward to verify that the various maxima and minima referred to in these definitions all exist.) The difference between n^* and $n/2$ is a measure of "how close" a particular society comes to having a majority winner. Moreover, the set of points x for which $v(x) = n^*$, the minmax set, will turn out to be an important one.

We can now characterize the trajectories of interest. A trajectory from x to y is a finite sequence (x, z, z', \dots, y) . A trajectory

is an infinite sequence (x^1, x^2, \dots) . (Trajectories will be understood to be infinite unless stated otherwise.) A vote-maximizing trajectory is a sequence (x^i) in which x^{i+1} is vote-maximizing against x^i , i.e. $n(x^i, x^{i+1}) = v(x^i)$, all $i > 0$. (Other types of trajectories will be defined later.)

Some Preliminary Results

With these definitions in hand, we first note two straightforward but useful consequences of our assumptions, and then prove a fundamental result.

For any set of voters $C \subseteq N$, let $\mathcal{P}(C)$ be the set of points which are Pareto-Optimal with respect to C , i.e. $\mathcal{P}(C) = \{x : \text{for no } y \text{ is } y >_i x, \text{ all } i \in C\}$. Then we have:

Lemma 1. $\mathcal{P}(C)$ is the convex hull of the satiation points $\{s^i : i \in C\}$.

Proof. Omitted. (A straightforward consequence of the assumption of Type I preferences.)

Next, let $M(m)$ be the set of points which can be defeated by no more than m votes, i.e., $M(m) = \{x : v(x) \leq m\}$. Clearly these $M(\cdot)$ sets are nested, in the sense that $m' \geq m$ implies $M(m') \supseteq M(m)$.

Lemma 2. If $m < n$, then $M(m) = \bigcap_{\{C \subseteq N : |C| > m\}} \mathcal{P}(C)$.

Proof. Omitted. (Follows immediately from the definitions, and does not depend on the Type I assumption. In particular Lemma 2 would still be true if the Type I assumption were replaced by the weaker assumption that voter preferences are quasi-concave.)

To obtain $M(m)$ it thus suffices to examine the Pareto sets of each of the $m+1$ -membered coalitions. The set $M(m)$ will be given by the intersection of these Pareto sets. These $M(m)$ sets are important for characterizing the behavior of the vote-maximizing trajectories. A simple example (the same as that of Figure 1) with $k = 2$, $n = 7$

is given in Figure 2. $M(6)$ is simply the Pareto Optimal set $\mathcal{P}(N)$. $M(5)$ is given by the intersection of the $\mathcal{P}(C)$ of all the 6-membered coalitions, and $M(4)$ by that of the 5-membered coalitions. The Pareto

 Insert Figure 2 about here

sets of the 4-coalitions have no common point, so $M(3)$ is empty, and the minmax number n^* is 4. Note that the minmax set $M(4)$ lies well within the interior of the Pareto set $\mathcal{P}(N)$. The minmax set, as we shall see, is the "equilibrium set" on which the vote-maximizing trajectories converge. Though it is not a unique equilibrium, it is a small one relative to $\mathcal{P}(N)$.

We now use these two Lemmas to prove a fundamental result. Define the distance $d(x,A)$ from a point x to a (nonempty) set A in the obvious fashion: $d(x,A) = \inf_{y \in A} \|x-y\|$. Then we have:

Theorem 1. Let x, y be two feasible points for which $n(x,y) > n^*$, and m an integer such that $n(x,y) > m \geq n^*$. Then $d(y, M(m)) < d(x, M(m))$.

Proof. The general idea of the proof is illustrated in Figure 3. The set $\{z : \|z-y\| < \|z-x\|\}$ of points closer to y than to x constitutes an open halfspace H , bounded by the set of perpendicular bisectors

 Insert Figure 3 about here

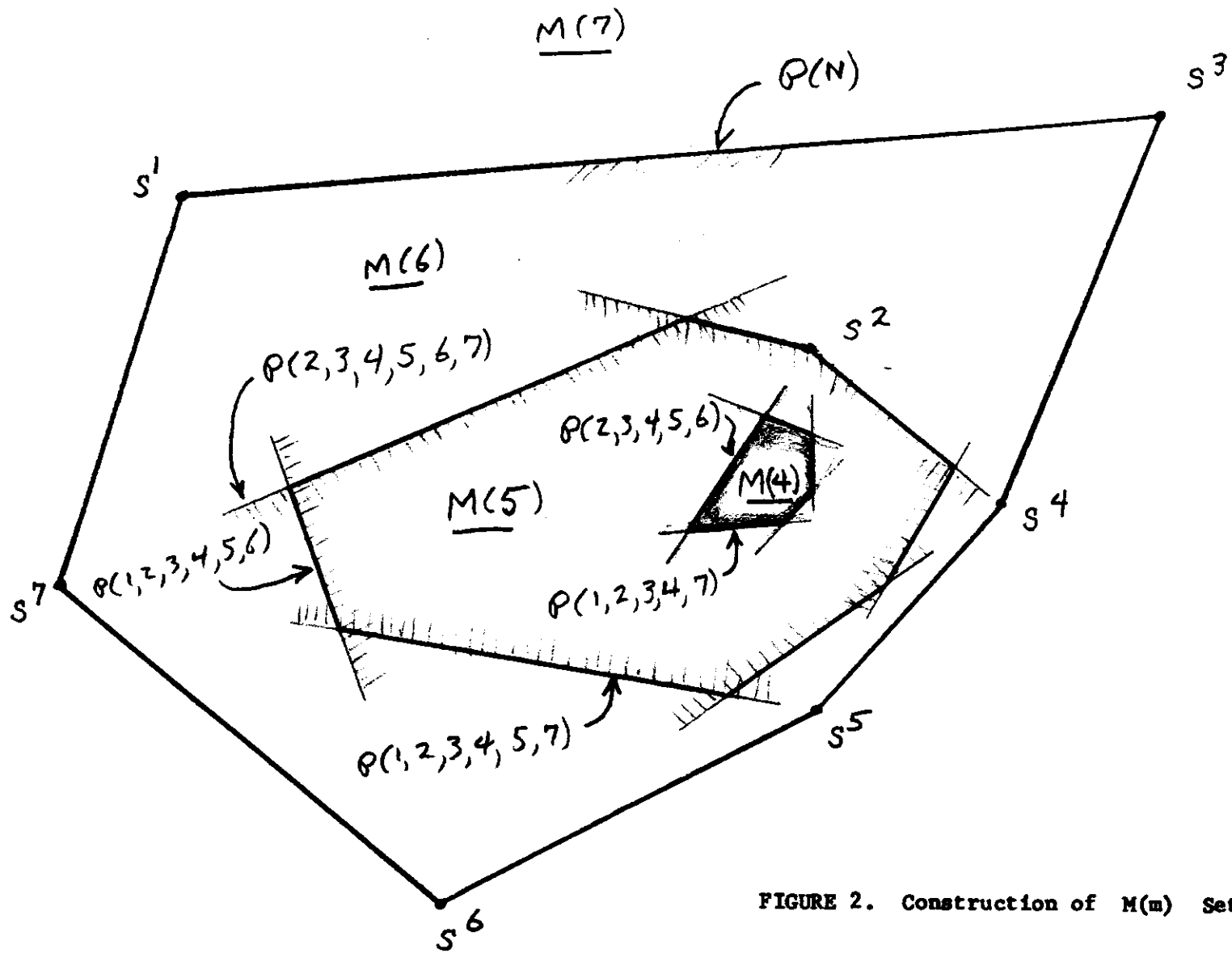


FIGURE 2. Construction of $M(m)$ Sets

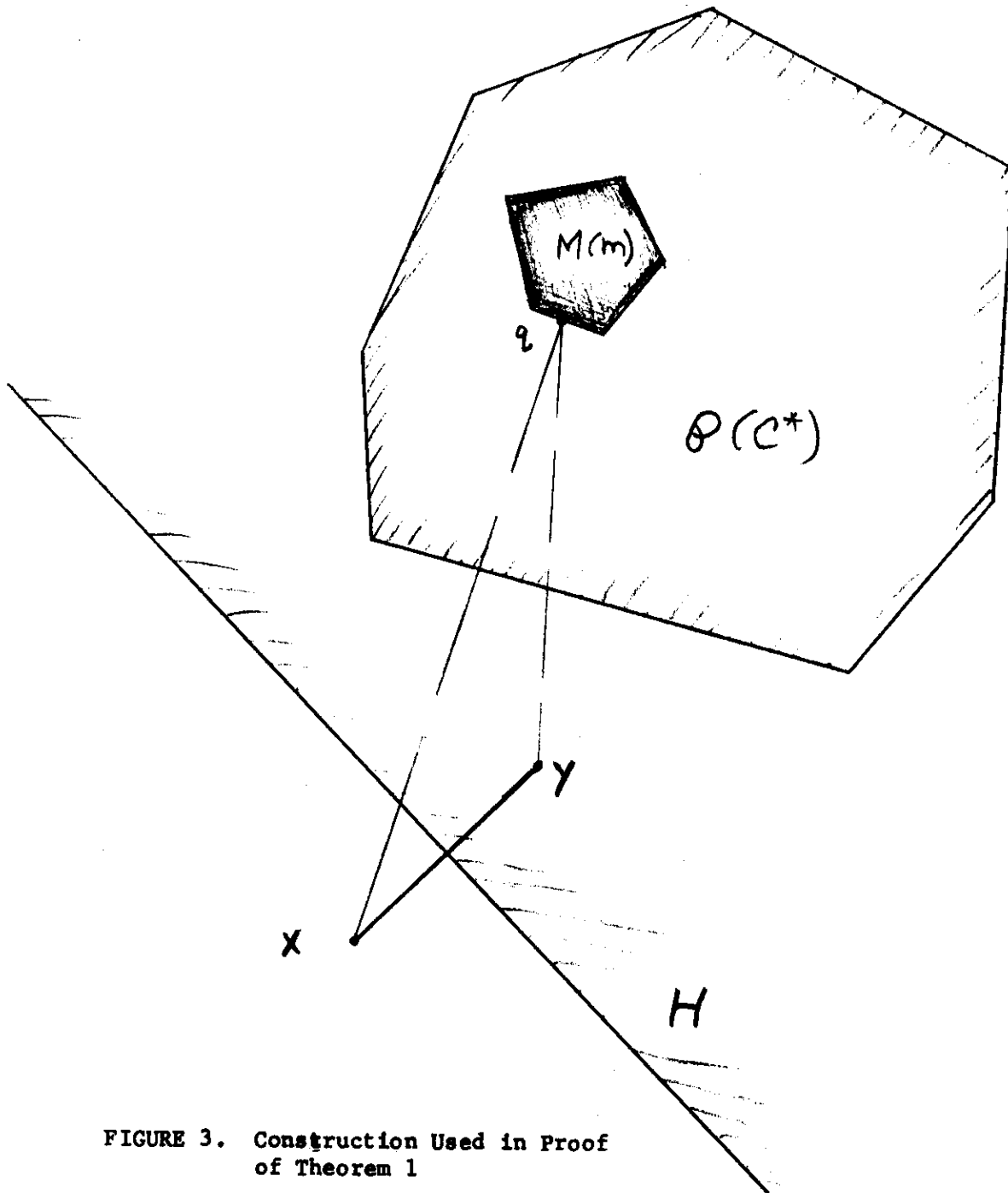


FIGURE 3. Construction Used in Proof of Theorem 1

of the line segment $[x, y]$. A voter j prefers y to x if and only if his satiation point s^j is closer to y than to x , so the set C^* of voters who vote for y is the set whose satiation points lie in H . From Lemma 1, their Pareto set $\mathcal{P}(C^*)$ is a convex hull of these satiation points, which is therefore contained in H also, i.e.

$\mathcal{P}(C^*) \subset H$. Since $|C^*| = n(x, y) > m$, it follows from Lemma 2 that $M(m) = \bigcap_{\{C \subset N: |C| > m\}} \mathcal{P}(C) \subseteq \mathcal{P}(C^*)$, so $M(m)$ is contained in H also.

Let q be the point in $M(m)$ closest to x ($M(m)$ is non-empty since $m \leq n^*$ and clearly compact from Lemma 1, so such a point exists), i.e. such that $d(x, M(m)) = \|x - q\|$. Then since $q \in M(m) \subset H$, q must be closer to y than to x , i.e. $d(x, M(m)) = \|x - q\| > \|y - q\| \geq d(y, M(m))$, which proves the Theorem. Q.E.D.

Vote-Maximizing Trajectories

We now turn to the task of explicitly characterizing the behavior of the vote-maximizing trajectories. The number of such trajectories is enormous, since at any point x^i there are many points which are vote-maximizing against x^i , any one of which could be chosen as the next point x^{i+1} in a vote-maximizing trajectory. The point a in Figure 4 lies outside of the Pareto optimal set $P(N)$, so any point preferred to a by all seven voters is vote-maximizing against a . The set of such points lies in the interior of the shaded region, A , the intersection

 Insert Figure 4 about here

of the sets of points preferred to a by each voter, i.e. $\bigcap_i \{x : x \succ a\}$. At the point b , $v(b) = 6$, and there are two six-membered coalitions $C_1 = \{1,2,3,4,5,6\}$ and $C_2 = \{2,3,4,5,6,7\}$ whose Pareto sets do not contain b . Any point in B_1 is preferred by all members of C_1 to b , and B_2 is the set preferred to b by all $i \in C_2$, so the set of vote-maximizing points is $B_1 \cup B_2$. At point c , $v(c) = 4$, and the set of vote-maximizing points is even more complex, as shown on the figure. Though at any point x^i the number of votes $v(x^i)$ by which x^{i+1} must defeat x^i is unique, it is clear that the set of individuals who cast these votes, and the direction and distance from x^i to x^{i+1} , are quite indeterminate. Despite the multiplicity and indeterminacy of the vote-maximizing trajectories, however, we can nevertheless obtain a useful characterization of their behavior. In particular, every vote-maximizing trajectory must tend to move inside the nested $M(m)$ sets, in the following sense:

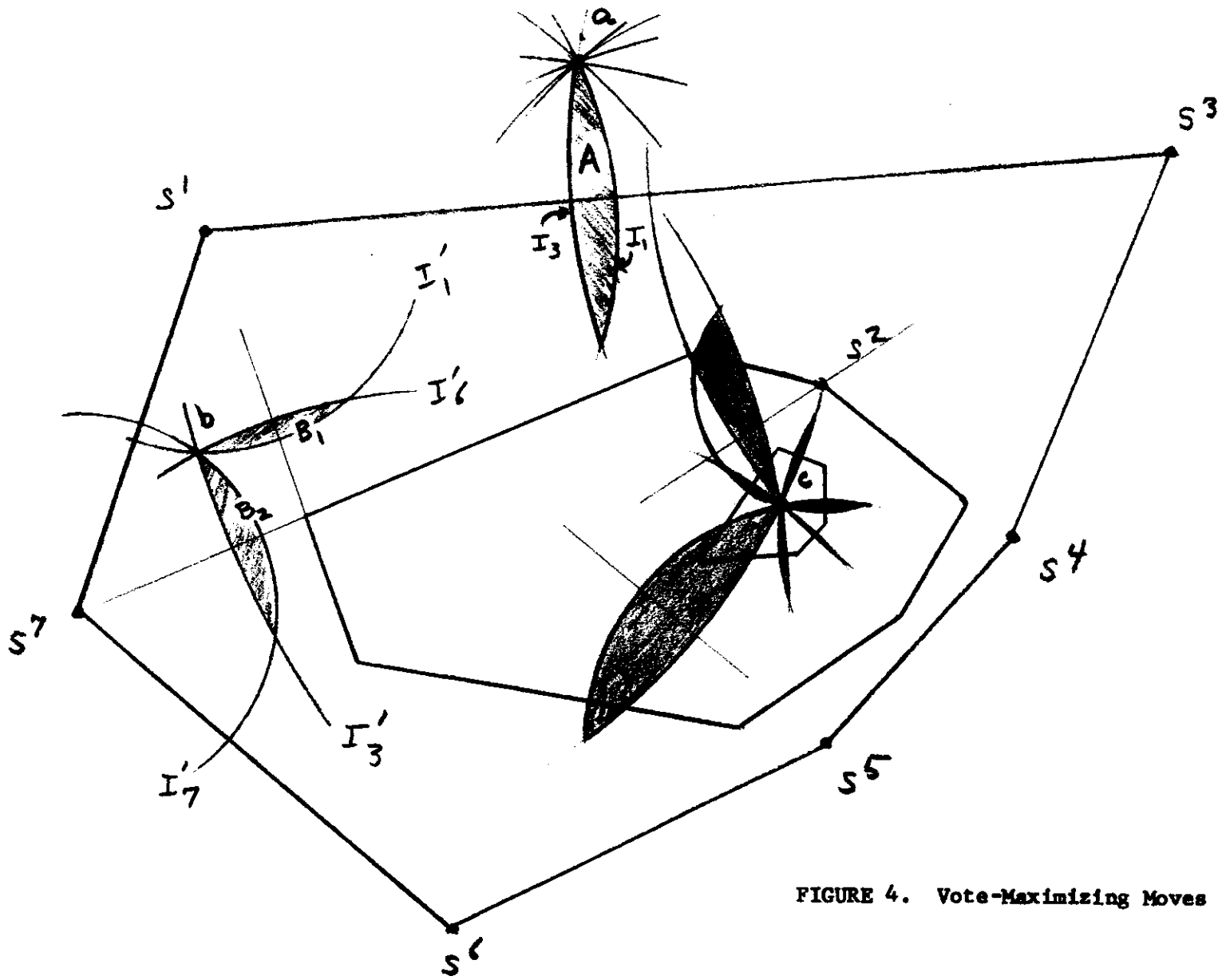


FIGURE 4. Vote-Maximizing Moves

Theorem 2. Let (x^i) be a vote-maximizing trajectory and $M(m) \neq \emptyset$. Then for any i , $x^i \notin M(m)$ implies $d(x^{i+1}, M(m)) < d(x^i, M(m))$.

Proof. Since x^{i+1} is vote-maximizing against x^i , $n(x^i, x^{i+1}) = v(x^i)$. $x^i \notin M(m)$ implies $v(x^i) > m$, and $M(m) \neq \emptyset$ implies $m \leq n^*$, so the conditions of Theorem 1 hold and the result follows immediately. Q.E.D.

This result implies that on any vote-maximizing trajectory the distance to any non-empty $M(m)$ set (which the trajectory has not already entered) must be strictly decreasing, and in this sense the trajectory must approach the set. Theorem 2 does not guarantee that the trajectory will actually reach such a set, since the trajectory could conceivably get "stuck" outside, by taking ever-shorter steps or getting caught in a limit cycle of some sort. Only a slight strengthening of the conditions of Theorem 2 is required to preclude this possibility, however. We can think of $\|x^{i+1} - x^i\|$ as the distance a trajectory moves in period i , and of $\sum_{i=1}^t \|x^{i+1} - x^i\|$ as the total distance it travels in the first t periods. It is then natural to think of a non-degenerate trajectory as one which keeps moving forever, in the sense that this distance is unbounded. More precisely, a trajectory (x^i) is non-degenerate if for all $K > 0$ there exists some t^* for which $\sum_{i=1}^{t^*} \|x^{i+1} - x^i\| > K$. We shall also say a trajectory (x^i) enters a set A if $x^{i^*} \in A$ for some i^* , and that a set $A \subseteq R^k$ is a body if it contains an interior (relative to R^k). With these definitions, we have:

Theorem 3. If $M(m)$ is a body, every non-degenerate vote-maximizing trajectory must enter it.

 Insert Figure 5 about here

Proof. Let $M(m)$ be a body and x^i, x^{i+1} be any two points such that $x^i \notin M(m)$ and x^{i+1} is vote-maximizing with respect to x^i . On Figure 5, L is the line through x^i and x^{i+1} , $\delta_i = \|x^{i+1} - x^i\|$, and $H = \{z : \|z - x^{i+1}\| < \|z - x^i\|\}$ is the set of points closer to x^{i+1} than to x^i , an open half space bounded by the plane P formed by the perpendicular bisectors of the line segment $[x^i, x^{i+1}]$. From Theorem 1, $M(m) \subset H$, as shown. By assumption $M(m)$ is a body, and from Lemmas 1 and 2 is compact, so there exists a largest ball B contained in $M(m)$. Let ρ be the radius of such a ball, and the point c be its center. Define $h = d(c, L)$, the distance from c to L , and a , the point in L such that $\|a - c\| = h$, and similarly let k be the distance from c to P , and $b \in P$, such that $\|b - c\| = k$. Finally let $d_i = \|x^i - c\|$ and $d_{i+1} = \|x^{i+1} - c\|$. Clearly the line segment $[a, c]$ is parallel to P , so $d(a, P) = k$, and is perpendicular to L , which implies $d_i^2 = h^2 + \|a - x^i\|^2 = h^2 + (k + \frac{1}{2}\delta_i)^2$, and $d_{i+1}^2 = h^2 + \|a - x^{i+1}\|^2 = h^2 + (k - \frac{1}{2}\delta_i)^2$. Hence $d_i^2 - d_{i+1}^2 = 2\delta_i k > 2\delta_i \rho$, since $M(m) \subset H$ implies $k > \rho$.

Suppose now there existed a non-degenerate trajectory (x^i) which never entered $M(m)$. Then the above inequality would hold for all $i > 0$, which after rewriting and summing for $i = 1, 2, \dots, t$, would imply

$$2\rho \sum_{i=1}^t \delta_i < \sum_{i=1}^t (d_i^2 - d_{i+1}^2) = d_1^2 - d_{t+1}^2,$$

and hence that

$$d_{t+1}^2 < d_1^2 - 2\rho \sum_{i=1}^t \|x^i - x^{i+1}\|$$

(since $\delta_i = \|x^i - x^{i+1}\|$ by definition). But since the trajectory is non-degenerate the sum on the right is unbounded, so for sufficiently large t it would be true that $d_{t+1}^2 < \rho^2$, contradicting the hypothesis that the trajectory never enters $M(m)$. Q.E.D.

The condition that $M(m)$ be a body is an extremely weak one, which will nearly always hold. In special cases--for example if there happens to exist a majority winner, or if all voters' satiation points lie in some affine subspace of R^k --there may exist an $M(m)$ which does not contain interior points. It is clear that such instances are exceptional, however, and are not preserved under arbitrarily small perturbations of voters' utility functions.* For all practical purposes, Theorem 3 ensures that every non-degenerate vote-maximizing trajectory moves inside the nested $M(m)$ sets, and hence eventually enters the innermost one, the minmax set $M(n^*)$.

These remarks can be made precise, for example as follows: Let \mathcal{S} be the set of all n -membered societies, and induce a topology on \mathcal{S} by defining the distance between two societies S, S^ as

$$\sum_{i,j} \min \|s^i - s^{*j}\|,$$

where s^i and s^{*j} are the satiation points of voter i in S and j in S^* , respectively. It can then be shown that the set of societies for which $M(n^*)$ is a body is open dense in \mathcal{S} . It follows from this that the set of societies for which some non-empty $M(m)$ is not a body is of measure zero in \mathcal{S} , and hence that this property is essentially unobservable. The converse property, that every $M(m)$ is a body, is generic, and will hold for "almost all" societies. See Hirsch and Smale (1974).

An illustration of a typical vote-maximizing trajectory is given in Figure 6. The trajectory enters the $P(N) = M(6)$ and $M(5)$ sets

 Insert Figure 6 about here

in straightforward fashion, and then (at x^5) temporarily jumps outside of the $M(5)$ set. This does not contradict Theorem 2, since $x^4 \in M(5)$ and Theorem 2 pertains to $M(m)$ sets the trajectory has not yet entered. After returning to the $M(5)$ set, the trajectory then alternates between the five-membered coalitions $\{1,2,3,4,7\}$ and $\{2,3,4,5,6\}$, approaching and eventually entering the minmax set $M(4)$. The continuation of the trajectory after x^{12} is not shown, but it is clear (for example by examination of the point c in Figure 4) that having entered the minmax set $M(4)$, a vote-maximizing trajectory may then jump outside of it. Such departures may occur repeatedly, and as Figure 4 shows, they may be substantial in magnitude. In this sense the minmax set is not a stable equilibrium. It is also true, however, that Theorems 2 and 3 ensure that after each such departure the trajectory must immediately begin to move back toward the minmax set, and eventually reenter it. Hence there is another sense in which the equilibrium is stable, since it tends to restore itself whenever it is disturbed.

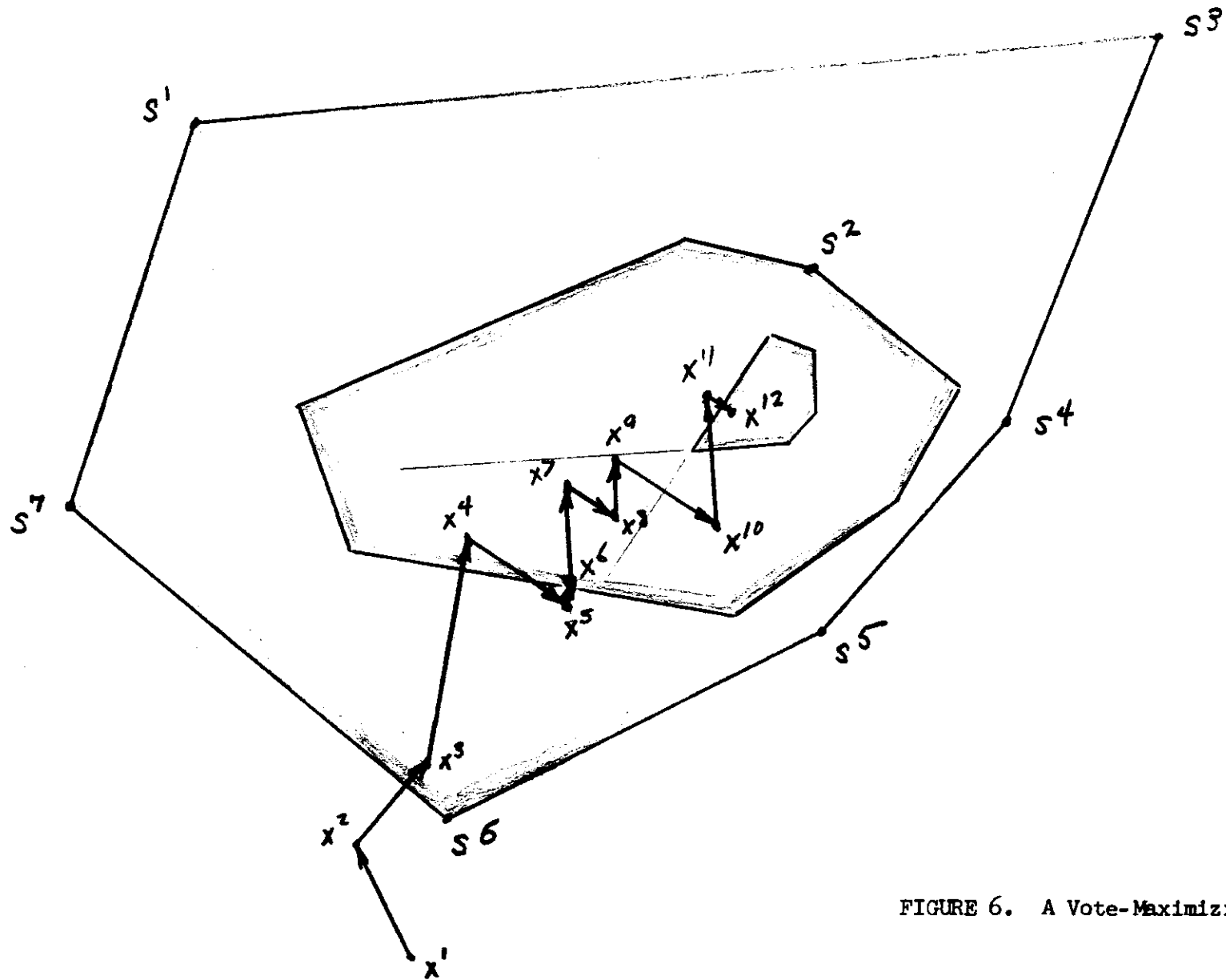


FIGURE 6. A Vote-Maximizing Trajectory

Discussion, Interpretation and Some Further Results

The results reported above provide a rather general characterization of the behavior of a Downsian two-party political system acting over a multi-dimensional issue or policy space. Competitive vote maximization in such a situation will lead the two parties to converge on the minmax set. Most previous work on party competition in multi-dimensional choice spaces has been based on the much more restrictive equilibrium concept of a "majority winner" (or Plott equilibrium, or "multivariate median"). The minmax set is identical to the majority winner if the latter exists. As we pointed out earlier, however, in multi-dimensional issue spaces the existence of a majority winner is an extremely rare, and unstable, phenomenon. Thus any attempt to characterize the behavior of party competition in terms of such an equilibrium will be of little practical interest. The equilibrium suggested here, the minmax set, always exists, and hence our convergence results will apply quite generally. Hence these results generalize the Downsian model of party competition to multiple-issue elections, and show that the minmax set is the natural generalization of the "median" (of the distribution of voters' most-preferred points), the point on which the parties' platform choices tend to converge in the unidimensional case.

The size of the minmax set is in some rough sense a measure of the degree of concensus within the society. In a purely distributional question, in which a limited amount of a single private good must be distributed among the citizenry, each citizen will want to maximize his own share, at the expense of everyone else. In this case, the minmax set will coincide with the entire Pareto optimal set, $\mathcal{P}(N)$, and the voting process will be relatively indeterminate. Such redistributive questions

are the most divisive a society must face, and it is not surprising that voting is unable to resolve them. In a more general public good-type problem, in which the interests of the citizens are not so diametrically opposed, we will have a situation more like that of Figure 2, where the minmax set lies well within the interior of $P(N)$, and the voting process becomes more determinate. Moreover if the number of voters increases in such a way that their satiation points become spread more smoothly across the space, the minmax set will shrink, and there is thus a sense in which the process will tend to become more determinate in large societies.* These qualitative characteristics of minmax sets seem eminently reasonable to us, and reinforce our feeling that the minmax equilibrium concept is a compelling one which deserves to be taken seriously.

The basic motivational premise of our model is that parties are primarily interested in maximizing votes over the current election period. The vote maximization assumption is a plausible abstraction, which we think is at least as defensible (for a competitive two-party system) as the assumption of profit maximization by firms. Nevertheless it is clear that political parties (like firms) may have other objectives as well, and in particular may have preferences for policies as such, independently of their electoral consequences. In a more general treatment, we might suppose the parties to be maximizing an objective function involving both policies and votes, and to be willing to sacrifice some extra votes in favor of these policy objectives in elections in which a comfortable margin of victory is already assured. Some generalization of our results

*These assertions are precisely formulated, and proved, in a forthcoming paper.

in this direction is possible. In particular, if the vote-policy trade-offs are such that no "out" party would ever settle for fewer than n^* votes, then the sequence of successively enacted policies would constitute a subset of the n^* -trajectories (i.e. trajectories (x^i) for which $n(x^i, x^{i+1}) \geq n^*$). Theorems 2 and 3 imply that every n^* -trajectory converges on the minmax set, so the model generalizes immediately this extent.

A somewhat different issue concerns the myopia of our model. In each period the "out" party is assumed to maximize its prospects in the coming election, without regard to the possible consequences in subsequent periods. While this is clearly a strong assumption, we do not think it is an unrealistic one, in an electoral context. Many observers have noted the relatively short horizons of elected officials, and the fact that their preoccupations rarely extend beyond the next election (in this respect they differ from private firms and non-elected officials). Indeed, this shortsightedness is often cited as a major shortcoming of the decision-making process in a democracy. We think it interesting, in this regard, that our completely myopic democratic process nevertheless does succeed, over time, in attaining what is in many respects an eminently reasonable social optimum, the minmax set.

The myopia assumption could be relaxed in various ways. One might be to suppose that parties are interested in maximizing their votes in the current election, and also in guarding themselves against unnecessary losses in the following election. It would then adopt a policy x^{i+1} which is vote-maximizing-loss-minimizing against the incumbent's policy x^i , i.e. such that

$$n(x^i, x^{i+1}) = v(x^i)$$

and

$$v(x^{i+1}) = \min_{\{y : n(x^i, y) = v(x^i)\}} v(y) .$$

The sequence of policies enacted would then constitute a maxmin trajectory (x^i) , i.e. a trajectory satisfying the above relation for all $i \geq 0$.

If the parties were willing to sacrifice some current votes to ensure against next-period losses, a larger class of trajectories would be generated; one such class is the set of n^* -min-trajectories, in which x^{i+1} minimizes $v(x)^{i+1}$ subject to the constraint of winning n^* or more votes against x^i . Since the maxmin and n^* -min trajectories are subsets of the vote-maximizing and n^* -trajectories, respectively, Theorems 1 through 3 imply immediately that such trajectories converge on the minmax set. In a fully general treatment we might model the competition as a dynamic game in which each party strives to maximize a discounted stream of future election returns. While we do not think the gain in descriptive realism in such a treatment is necessarily great, it is nevertheless an interesting, and open, question as to whether the minmax set would still emerge as an equilibrium in this more general framework.

Though our primary concern in this paper has been to characterize the equilibrium behavior of a two-party competitive political system this equilibrium can also be characterized in a more abstract voting theory framework. In particular, for any $\lambda \in (0,1)$, we can define the special majority preference relation P_λ in the obvious fashion: for any $x, y \in \mathcal{D}$, $x P_\lambda y$ iff $n(x,y) > \lambda n$. Typically the simple majority relation $P_{1/2}$ is intransitive, and indeed cyclic, because of the "voters' paradox" phenomenon. The unanimity or Pareto ordering

(given by $\lambda^u = 1 - \frac{1}{n}$) in contrast, does not cycle, and the set of maximal elements of P_{λ^u} is the Pareto Optimal set $P(N)$, which is always non-empty, though typically large. A natural question is to ask whether there are intermediate values of λ for which P_{λ} is also acyclic, and to characterize the smallest such value. The following result answers to these questions.

Theorem 4. A necessary and sufficient condition for P_{λ} to be acyclic is that $\lambda \geq n^*/n$.

Proof. Necessity: For any $x \in \mathcal{D}$, clearly

$$\{z \in \mathcal{D} : z P_{\lambda} x\} = \bigcup_{\{C_j \subseteq N : |C_j| > \lambda n\}} \left(\bigcap_{i \in C_j} \{z \in \mathcal{D} : z > x\} \right)$$

it is composed of finite unions and intersections of open sets. \mathcal{D} is compact, so if P_{λ} is acyclic, there exists a maximal element in \mathcal{D} (Brown (1973), Theorem 7, p. 8), i.e. an element $x^* \in \mathcal{D}$ such that for no $y \in \mathcal{D}$ is $y P_{\lambda} x^*$. This implies $v(x^*) \leq \lambda n$, which by definition of the minmax number is possible only if $\lambda n \geq n^*$.

Sufficiency: If P_{λ} were not acyclic there would exist $r \geq 2$ points $x^1, x^2, \dots, x^r \in \mathcal{D}$ such that $x^{i+1} P_{\lambda} x^i$, all $i = 1, 2, \dots, r-1$, and moreover $x^1 P_{\lambda} x^r$. Since $\lambda \geq n^*/n$, Theorem 1 implies that $d(x^{i+1}, M(n^*)) < d(x^i, M(n^*))$, all $i = 1, 2, \dots, r-1$, and hence that $d(x^r, M(n^*)) < d(x^1, M(n^*))$. But if $x^1 P_{\lambda} x^r$, Theorem 1 would also imply $d(x^1, M(n^*)) < d(x^r, M(n^*))$. This is a contradiction, which shows that P_{λ} must be acyclic. Q.E.D.

The relation P_{λ^*} given by $\lambda^* = n^*/n$ has some attractive features as a social preference relation. The family of P_{λ} relations is clearly

monotonic, in the sense that if $\lambda' \geq \lambda''$, then $P_{\lambda'} \subseteq P_{\lambda''}$ (where $P_{\lambda} \subseteq \mathcal{D} \times \mathcal{D}$), and hence $M_{\lambda''} \supseteq M_{\lambda'}$, where $M_{\lambda} = \{x \in \mathcal{D} : \text{for no } y \in \mathcal{D} \text{ is } y P_{\lambda} x\}$ is the set of maximal or undominated elements of the P_{λ} relation. Theorem 4 shows that λ^* is the smallest λ for which P_{λ} is acyclic, so P_{λ^*} is the largest or most decisive of the acyclic P_{λ} relations, and yields the smallest or most sharply characterized (non-empty) set of social optima.*

In an axiomatic social choice framework (Arrow (1963)), the P_{λ^*} ordering satisfies the Pareto axiom (since $P_{\lambda^u} \subseteq P_{\lambda^*}$) and is clearly Non-Dictatorial (and indeed satisfies the much stronger Anonymity axiom, since the P_{λ} relations do not depend on the labelling of voters). It also satisfies natural weakenings of the Domain axiom (to the class of Type I societies) and Arrow's Rationality condition (acyclicity). The controversial Independence of Irrelevant Alternatives axiom is not satisfied, since P_{λ^*} is defined in terms of the global parameter n^* , which depends on alternatives which may be "irrelevant" to a particular choice. The social ordering can be constructed in a fairly decentralized fashion,

*Craven (1971) and Ferejohn and Grether (1974) have explored the structure of the P_{λ} relations over finite sets of discrete alternatives, and have shown (essentially) that if r is the number alternatives, a necessary and sufficient condition for P_{λ} to be acyclic for all societies is that $\lambda \geq \frac{r-1}{r}$. This is a conservative bound, since in many societies lower values will still yield acyclic P_{λ} , and in any event it does not apply to multidimensional choice spaces in which the feasible set is infinite. However we believe the topological structure of the multidimensional case can be exploited to obtain a comparable bound, and in particular conjecture that such a bound will be given by $k/(k+1)$, where k is the dimensionality of the space.

however, without requiring complete knowledge of all voter's preferences for all alternatives, and to that extent fulfills the basic idea behind the Independence axiom, of informational decentralization (Arrow (1963), pp. 110 ff.), and it should not be difficult to reformulate the axiom into a more flexible form which still captures this idea. It follows from the definitions that the M_{λ^*} set is precisely the minmax set $M(n^*)$, so Theorems 2 and 3 in effect show that the "hidden hand" of party competition results in optimization of the essentially Arrowian social preference relation P_{λ^*} .

All of the results proven here (except Lemma 2) make essential use of the assumption that all voters have Type I preferences. Most previous work in this area has been based on this or essentially equivalent assumptions. The Type I societies are an important class, within which all of the intrinsic difficulties of voting intransitivities emerge. Nevertheless the Type I assumption is a restrictive one, and the extent to which it can be relaxed is clearly an important issue. Some obvious immediate generalizations can be obtained by replacing the Euclidean metric by other metrics. Clearly the minmax and $M(m)$ sets exist and are characterized by Lemma 2 under much weaker assumptions on preferences. On the basis of an extensive and unsuccessful search for counter examples, we conjecture that the essential qualitative properties expressed in Theorems 1 through 4, concerning the acyclicity of the P_{λ} relations and the convergence of the vote-maximizing trajectories on the minmax set, also hold under much weaker assumptions, such as, for example, the assumption that voter preferences are representable by smooth and strictly quasi-concave utility functions.

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