COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 393

COLLECTIVE RATIONALITY

Donald J. Brown

April 3, 1975

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by

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A central concern of the social sciences is the nature of choice or decision making. Economists, psychologists, sociologists, etc. spend much of their energies investigating how individuals or groups of individuals make or ought to make choices. A widely accepted point of view is that choices, however they are made, should be rational. This thesis has its origin in the utilitarian philosophy of Bentham and Edgeworth's application of that philosophy to problems of political economy.

The utilitarian approach to choice is simply that an action or decision should be judged by its consequences. Hence if one has well defined preferences over outcomes and each action is uniquely associated with an outcome, then the utilitarians prescribe that one ought to choose the action which produces the most preferred outcome.

Ignoring the questionable descriptive content of this prescription, there are still a number of questions regarding its normative meaning. First, how do we know that there is a most preferred outcome? Second, what does this prescription mean when the decision maker is a group of individuals?

The body of literature concerning the second question comprises the theory of social choice, where the seminal paper is Arrow's <u>Social</u>

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*This research was conducted under grants from the National Science Foundation and the Ford Foundation. The author would like to acknowledge the helpful suggestions and remarks of Steve Bloom.

Choice and Individual Values. One of the important distinctions made by Arrow is the difference between reconciling conflicting interests and tastes and summarizing differing estimates of the "general will" or "common good." As examples, he points out the difference between theories of committees or elections, i.e. preference aggregation, and the theory of juries. In the latter...'voters are judges of some truth rather than expressing their own preferences.' Although both the theories of committees and the theory of juries lie within the province of social choice theory, we shall follow Arrow and concern ourselves with the aggregation of preferences.

Arrow requires that the aggregation procedure, subject to a few reasonable ethical and institutional constraints, produce a weakly ordered social preference. This condition is called collective rationality. Since Arrow assumes individuals have weakly ordered preferences, i.e. individual rationality, Guilbaud [4] notes that the collective rationality condition is equivalent to requiring the preservation of individual rationality. This observation has been made by others, and has occasioned some confusion in the literature. Arrow has falsely been accused of subscribing to an organistic view of society. Even those social scientists who do support the "general will" theory are unhappy with the collective rationality condition.

Certainly, there is no a priori reason for the "general will," if it exists, to be rational in the way we prescribe individual rationality. But this observation does not apply to the essentially utilitarian position taken by Arrow. That is, the problem of social choice is the aggregation of preferences.

Hence in designing institutions, we are free to impose any condition which will facilitate the process of aggregation. As Arrow observes, collective rationality guarantees the existence of a socially preferred alternative,[1] and an unbiased procedure for realizing it.

Unfortunately, Arrow demonstrated in his Possibility Theorem that the only aggregation procedures satisfying his conditions are dictatorial.

Consequently, if we wish to construct non-dictatorial aggregation procedures, then we must modify some of Arrow's conditions. In [1], we argued that the collective rationality condition is too stringent. We shall not repeat that argument here, instead we will address the question raised by Guilbaud.

What properties of individual preferences are preserved by Arrowian aggregation procedures?

This question is necessarily prior to any collective rationality assumption.

Sen [7] produced the first example of an Arrowian procedure which preserved quasi-transitivity. A complete characterization of all the Arrowian procedures preserving quasi-transitivity was given by Hansson [5]. Brown [2] extended Hansson's results to acyclicity.

But what about semiorders or, in the case of economic applications, continuity, are these properties preserved by Arrowian procedures?

The purpose of this essay is to give a complete characterization of all first order properties of individual preferences, which are pre-

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[1] Here we are assuming that choices are always made from finite subsets of alternatives.

served by aggregation procedures satisfying the ethical and institutional conditions suggested by Arrow.

We will consider a preference relation as a particular type of relational structure. That is, if A is a nonempty set of alternatives, then a preference relation, P , on A is a binary relation over A , i.e. P is a subset of AxA . xPy is to be read " x is preferred to y ." More formally, $\mathfrak{M} = \langle A, P \rangle$ is a binary relational structure.

First order properties of $\mathfrak{M}$ are those properties which are expressible as sentences in a formal language $\mathcal{L}$, i.e. the (lower or first order) predicate calculus. This language will contain a symbol, called a binary predicate, corresponding to the preference relation P . $\mathcal{L}$ also contains variables ranging over the elements of A . Finally the language contains a number of logical symbols such as connectives: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\Longrightarrow$ (implies) and quantifiers: $\exists x$ (there exists x) and $\forall x$ (for all x). $\mathcal{L}$ is called first order since variables range only over the elements of A (and not over subsets of A , or relations over A , etc.). Hence quantification is only with respect to elements of A .

If x and y are variables of $\mathcal{L}$ and P denotes the binary predicate, then xPy is said to be an atomic formula.

A formula α of $\mathcal{L}$ is said to be a basic Horn formula iff α is a disjunction of formulas θ_i , i.e. $\alpha = \theta_1 \vee \ldots \vee \theta_m$, where at most one of the formulas θ_i is an atomic formula, the rest being negations of atomic formulas.[2] If m = 1 , then α is either an atomic formula or the negation of an atomic formula. If $m > 1$ and θ_m is an

[2] θ is the negation of an atomic formula iff $\theta = \neg\beta$ where β is an atomic formula.

atomic formula, say, then α is written as $(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_{m-1} \Longrightarrow \theta_m)$ where $\theta_i = \neg\beta_i$ for $i = 1, 2, \ldots, m-1$ and each β_i is an atomic formula. If the basic Horn formula α consists of only the negations of atomic formulas, then α can be written as $(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_{m-1} \Longrightarrow \neg\beta_m)$ where $\theta_i = \neg\beta_i$ for $i = 1, 2, \ldots, m$.

A Horn formula γ is the conjunction of basic Horn formula, i.e. $\gamma = \alpha_1 \wedge \ldots \wedge \alpha_n$ where the α_i are basic Horn formula. (Note n may be 1.)

ζ is said to be a Horn sentence if ζ is of the form $(Q_1x_1) \ldots (Q_nx_n)\gamma$ where γ is a Horn formula, each Q_i is a quantifier symbol and every variable is quantified.

A universal Horn sentence is a Horn sentence where all the quantifiers are of the form $\forall x$ (for all x).

Transitivity is a first order property of a preference relation which can be expressed as a universal Horn sentence. Recall that P is said to be transitive iff for all x,y,z in A , if x is preferred to y and y is preferred to z , then x is preferred to z . More formally, $(\forall x)(\forall y)(\forall z)(xPy \wedge yPz \Longrightarrow xPz)$.

There are first order properties of preference relations which cannot be expressed as Horn sentences, and therefore we must consider a larger class of sentences. A disjunction of atomic and negations of atomic formulas $\theta_1 \vee \ldots \vee \theta_m$ is said to be weakly Horn iff at most two of the terms θ_i , θ_j are atomic formulas, the rest being negations of atomic formulas. Weak Horn sentences and universal weak Horn sentences are defined in the obvious manner.

The following set of axioms for weakly ordered preferences are universal weak Horn sentences:

(i) $(\forall x)(\forall y)(xPy \Longrightarrow \neg yPx)$

(ii) $(\forall x)(\forall y)(\forall z)(\neg xPy \wedge \neg yPz \Longrightarrow \neg xPz)$

In contrast, the following set of axioms for quasi-transitive preferences are universal Horn sentences:

(iii) $(\forall x)(\forall y)(xPy \Longrightarrow \neg yPx)$

(iv) $(\forall x)(\forall y)(\forall z)(xPy \wedge yPz \Longrightarrow xPz)$

To characterize acyclicity, we will define a special class of Horn sentences. A negative Horn formula is a disjunction of negations of atomic formulas. Again, negative Horn sentences and universal negative Horn sentences are defined in the obvious manner.

P is acyclic iff for all integers n :

(v) $(\forall x_1)(\forall x_2) \ldots (\forall x_n)(x_1Px_2 \wedge x_2Px_3 \wedge \ldots \wedge x_{n-1}Px_n \Longrightarrow \neg x_nPx_1)$

It is important to note that the notion of acyclicity requires an infinite set of axioms. Also observe that the axioms are universal negative Horn sentences.

We now recall the ethical and institutional conditions for the aggregation of preferences suggested by Arrow.

I is a finite or infinite set of individuals.

A is an infinite set of alternatives.

$\mathcal{W}$ denotes the set of weak orders on A .

$\mathcal{P}$ denotes the set of quasi-transitive orders on A .

$\mathcal{A}$ denotes the set of acyclic orders on A .

$F_{\mathcal{R}}$ is the set of all functions from I to $\mathcal{R}$, where $\mathcal{R} = \mathcal{A}, \mathcal{P}$, or $\mathcal{W}$.[3]

An aggregation rule is a mapping σ from $F_{\mathcal{R}}$ to $\mathcal{R}$.

Given $f \in R_{\mathcal{R}}$, $f(i)$ denotes the preference relation in $\mathcal{R}$ assigned

[3] Arrow only considered the case $\mathcal{R} = \mathcal{W}$.

by f to individual $i \in I$. If $a,b \in A$ and $J \subseteq I$, then $af(J)b \iff af(i)b$ for all $i \in J$. Given $f,g \in F_{\mathcal{R}}$, $f(i) = g(i)$ on $\{a,b\}$ means that $af(i)b \iff ag(i)b$ and that $bf(i)a \iff bg(i)a$. $f(J) = g(J)$ on $\{a,b\}$ means that $f(i) = g(i)$ on $\{a,b\}$ for all $i \in J$.

σ is said to be an Arrowian aggregation rule if it has the following properties:

<u>Pareto</u> (P) For all $a,b \in A$ and $f \in F_{\mathcal{R}}$, $af(I)b \Longrightarrow a\sigma(f)b$.

<u>Positive Responsiveness</u> (PR) For all $a,b \in A$ and $f,g \in F_{\mathcal{R}}$, $\{i \in I | af(i)b\} \subseteq \{i \in I | ag(i)b\}$ and $a\sigma(f)b \Longrightarrow a\sigma(g)b$.

<u>Independence</u> (I) For all $a,b \in A$ and $f,g \in F_{\mathcal{R}}$, $f = g$ on $\{a,b\} \Longrightarrow \sigma(f) = \sigma(g)$ on $\{a,b\}$.

The notion of decisive set allows us to construct a large and important class of Arrowian rules.

A set of individuals J is decisive for an aggregation rule σ if for all $a,b \in A$ and all $f \in F_{\mathcal{R}}$, $af(J)b \Longrightarrow a\sigma(f)b$.

Every family of subsets of I, Γ, defines an aggregation rule σ_Γ where for all $a,b \in A$ and $f \in F_{\mathcal{R}}$, $a\sigma(f)b$ iff $\{i \in I | af(i)b\} \in \Gamma$. Note that the family of decisive sets for σ_Γ is Γ.

We say that an aggregation rule μ is completely defined by its decisive sets if $\mu = \sigma_\Gamma$ for some family of subsets Γ. For the remainder of this essay, we shall restrict our attention to Arrowian aggregation rules completely defined by their decisive sets.

If σ_Γ is an Arrowian rule, then

(1) $I \in \Gamma$

(2) $A \in \Gamma$, $A \subseteq B \Longrightarrow B \in \Gamma$.

A less obvious, but more important fact is that Γ will possess some kind of finite intersection property as a consequence of being an $\mathcal{R}$-aggregation rule for $\mathcal{R} = \mathcal{Q}, \mathcal{P}$, or $\mathcal{W}$.

Proposition 1[4]

(α) If $\sigma_\Gamma : F_{\mathcal{Q}} \rightarrow \mathcal{Q}$, then Γ is a prefilter

(β) If $\sigma_\Gamma : F_{\mathcal{P}} \rightarrow \mathcal{P}$, then Γ is a filter

(γ) If $\sigma_\Gamma : F_{\mathcal{W}} \rightarrow \mathcal{W}$, then Γ is an ultrafilter

Recall that Γ is a prefilter if it satisfies (1) and (2) above and the intersection of any finite subfamily of Γ is nonempty.

Γ is a filter, if it is a prefilter and the intersection of any finite subfamily of Γ belongs to Γ.

Γ is an ultrafiler, if it is a filter and for every subset $J \subseteq I$, either $J \in \Gamma$ or $\bar{J}$, the complement of $J \in \Gamma$.

In all cases, we assume that $\emptyset$, the empty set, $\notin \Gamma$.

Originally, we introduced the formal language $\mathcal{L}$ as a means of expressing properties of a given preference relation P over a set A of alternatives. If α is a weak Horn sentence in $\mathcal{L}$ then we shall now consider all the binary relational structures $\mathcal{M} = \langle X,R \rangle$ such that α is true in $\mathcal{M}$. In this event, $\mathcal{M}$ is said to be a model of α.[5] It is important to note that if $\mathcal{K}(\alpha)$ is the class of models of α, then $\mathcal{M} = \langle A,P \rangle$, the original binary structure, may or may not belong to $\mathcal{K}(\alpha)$. Similarly, if Θ is a set of weak Horn sentences in $\mathcal{L}$, then $\mathcal{K}(\Theta)$ will denote the class of models for Θ, that is $\mathcal{M} \in \mathcal{K}(\Theta)$ iff

[4] See [1] for a discussion of the normative content of the assumption that Γ is a prefilter, filter, or ultrafilter. See [2] for a proof of Proposition 1.

[5] The definitions and concepts of model theory as they are used in this paper may be found in [3].

every sentence in Θ is true in $\mathcal{M}$.

Consider the sentence $(\forall x)(\forall y)(\forall x)(xPy \wedge yPz \Longrightarrow xPz)$. A model for this sentence is the set of commodity bundles, where P is the relation of "preferred to" induced by a utility function. Another model is the set of integers where P is the relation of "greater than." A binary relational structure which is not a model of this sentence is the set of living males and we define the relation as "is a father of."

If $\{P_i\}_{i \in I}$ is a family of preferences over A , then we shall consider them as binary relational structures $\mathcal{M}_i = \langle A, P_i \rangle$, $i \in I$. If Γ is a prefilter, filter, or ultrafilter over I . Then $\langle A, P_\Gamma \rangle$ is called a prereduced subproduct, reduced subproduct, or ultrasubproduct respectively. Where for all $x, y \in A$, $xP_\Gamma y$ iff $\{i \in I | xP_i y\} \in \Gamma$.

Let Θ be a set of weak Horn sentences in $\mathcal{L}$ and $\mathcal{K}(\Theta)$ the class of models for Θ. An arbitrary class of models $\mathcal{K}$ is said to be closed under prereduced subproducts, if every prereduced subproduct of every family of binary relationsl structures $\{\mathcal{M}_i\}_{i \in I}$, where $\mathcal{M}_i = \langle A, P_i \rangle$, belongs to $\mathcal{K}$. If $\mathcal{K} = \mathcal{K}(\Theta)$ for some family of weak Horn sentences and $\mathcal{K}$ is closed under prereduced subproducts, then we will say that the sentences Θ are preserved by prereduced subproducts. Similarly for reduced subproducts or ultrasubproducts.

Clearly, if $\mathcal{K}(\Theta)$ is closed under reduced subproducts then any Arrowian aggregation rule σ_Γ defined by a filter Γ preserves Θ . That is, for all assignments f , $\sigma_\Gamma(f)$ is a model of Θ . Moreover, if $\mathcal{K}(\Theta)$ is not closed under reduced subproducts, then there exists a set of alternatives A , a set of individuals I , an assignment g , and a filter Γ over I such that σ_Γ is not a model of Θ .

Similar statements hold for ultrasubproducts and prereduced subproducts.

Hence our preservation theorems are immediate corollaries of the following two theorems of S. R. Kogalovskij [6].

<u>Proposition 2</u>. $\mathcal{K}$ is closed under ultrasubproducts iff there exists a set of universal weak Horn sentences such that $\mathcal{K} = \mathcal{K}(\Theta)$.

If $\{\mathfrak{M}_i\}_{i \in \mathcal{I}}$ is a family of binary relational structures, $\mathfrak{M} = \langle A_i, P_i \rangle$, then the direct product $\mathfrak{M} = \langle A, P \rangle$ is the set $A = \prod_{i \in I} A_i$ and for all $f, g \in A$, fPg iff $f(i)P_i g(i)$ for all $i \in I$.

<u>Proposition 3</u>. $\mathcal{K}$ is closed under reduced subproducts iff there exists a set of universal weak Horn sentences such that $\mathcal{K} = \mathcal{K}(\Theta)$ and $\mathcal{K}$ is closed under direct products.

The following lemma due to Chang and Keisler [3] allows us to give a simpler version of Proposition 2.

<u>Lemma 1</u>. Let α be a universal weak Horn sentence. Then α is preserved under finite direct products iff α is a universal Horn sentence.

<u>Proposition 4</u>. $\mathcal{K}$ is closed under reduced subproducts iff there exists a set of universal Horn sentences, Θ , such that $\mathcal{K} = \mathcal{K}(\Theta)$.

Our final proposition, concerning prereduced subproducts, is a straightforward consequence of Kogalovskij's two theorems.

<u>Proposition 5</u>. $\mathcal{K}$ is closed under prereduced subproducts iff there exists a set of universal negative Horn sentences, Θ , such that $\mathcal{K} = \mathcal{K}(\Theta)$.

In general, Propositions 2, 4, and 5 are most useful as sufficient conditions for preserving certain types of sentences under Arrowian aggregation rules.

If one has a given property, like semiordered, which can be expressed in terms of the following universal weak Horn sentences, Θ ,

(1) $(\forall x)\neg xPx$

(2) $(\forall x)(\forall y)(\forall z)(\forall w)(xPy \wedge zPw \Longrightarrow xPw \vee zPy)$

(3) $(\forall x)(\forall y)(\forall z)(\forall w)(xPy \wedge yPz \Longrightarrow xPw \vee wPz)$

Then, by Proposition 2, we know that they are preserved under ultrasubproducts. Hence σ_Γ preserves Θ for all ultrafilters Γ . If σ_Γ is the aggregation procedure, then we can extend Hansson's proof that the preservation of quasi-transitivity forces Γ to be a filter to the case of semiorders. The following example shows that if σ_Γ is defined by a filter Γ , then σ_Γ need not preserve semiorders:

$A = \{x,y,w,z\}$

$I = \{1,2\}$

$P_1 = \{(x,y),\ (z,w),\ (x,w)\}$

$P_2 = \{(x,y),\ (z,w),\ (z,y)\}$

$\Gamma = \{I\}$

P_1 and P_2 are semiorders over A and the social preference defined by σ_Γ is $P_1 \cap P_2 = \{(x,y),\ (z,w)\}$, which is not a semiorder.

Hence, we have shown that, in general, the only Arrowian procedure σ_Γ , over a finite set of individuals I , which aggregates semiorders into semiorders, is dictatorial.[6] See [8] for a similar result.

In the introduction, we asked is continuity of preference relations preserved under aggregation. This question cannot be resolved by the propositions in this paper as continuity is a second order notion. That is, it requires quantification over sequences (sets) of ordered pairs of alternatives.

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[6] Every ultrafilter Γ over a finite set I is of the form $\Gamma = \{A \subseteq I | i_0 \in A\}$, where i_0 is the "dictator."

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