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1. Introduction

Guilbaud [6] in his analysis of the problem of social choice takes as a primitive the notion of decisive set or winning coalition. That is, he starts out by defining those subsets of individuals who if they collectively prefer alternative x to alternative y , then the social preference is x over y . In the case of absolute simple majority rule, the winning coalitions are those containing more than half of the population.[1]

Assuming that individual's preferences are acyclic and people vote their true preferences, Guilbaud observes that a voting cycle can only occur when some finite family of winning coalitions has empty intersection. Hence if we consider voting rules, prescribed in terms of decisive sets, where the intersection of every finite family of winning coalitions is nonempty, then we will be certain these rules do not cycle.

One obvious way of achieving this finite intersection property (f.i.p.) is to require at least one individual to belong to every decisive

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[1]By absolute α majority rule, where $1/2 \leq \alpha < 1$, we mean that x is preferred to y iff the number of people who prefer x to y is at least $\alpha|I| + 1$. $|I|$ is the number of people in the population.

set, e.g. the chairman of a committee has a veto. In fact, if the set of alternatives is infinite, as is the case in most public good problems, and a few other mild restrictions are placed on the voting rule then Brown [2] has shown this is the only manner by which cycles can be prevented.

In many situations of interest, the set of alternatives is finite.[2] In this case, consideration of some typical voting rules show that the f.i.p. is really stronger than we need. We have in mind absolute simple majority over two alternatives or absolute two thirds majority where there are three alternatives. In both examples, the suggested voting rules do not cycle and do not have the f.i.p. They do satisfy a limited kind of f.i.p. in that all families of decisive sets having less than k members, for some integer k, have nonempty intersection.

This suggests a characterization of acyclic aggregation rules over finite sets of alternatives in terms of the number of alternatives, m, and the minimum number of decisive sets having empty intersection, n.

In Theorem 1, we show that an aggregation rule is acyclic only if $n \geq m+1$ or the family of decisive sets has the finite intersection property, i.e. there is a weak dictator.

The result quoted from [2] for infinite sets of alternatives may be viewed as the limiting case of Theorem 1.

In Theorem 2, we show that the necessary condition for acyclicity given in Theorem 1 is also sufficient for aggregation rules which are completely defined by their decisive sets.

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[2] Of course, the results in this paper apply to the case of an infinite set of alternatives, X, if social choices are always made from finite subsets of X whose cardinality is uniformly bounded by some integer m.

The first examples of acyclic aggregation rules, without weak dictators, were proposed by Craven [4]. If there are m alternatives, then Craven showed the following rule does not cycle, i.e. is acyclic: society prefers x to y iff the proportion of individuals who prefer x to y exceeds α, where α is some fixed number between 1 and m-1/m.[3] Ferejohn and Grether [5] have given necessary and sufficient "ethical" conditions for an acyclic aggregation rule to be a Craven rule. Of course, not every acyclic aggregation rule is of the form proposed by Craven.

If voting rules are defined in terms of their winning coalitions, as suggested by Guilbaud, then these rules are naturally ordered. Where for given rules σ_1 and σ_2, $\sigma_1 \geq \sigma_2$ iff every decisive set of σ_2 is a decisive set of σ_1. Under this ordering, if $\sigma_1 \geq \sigma_2$ then the society defined by σ_1 prefers x to y whenever the society defined by σ_2 prefers x to y, e.g. if σ_1 is absolute simple majority and σ_2 is unanimity then $\sigma_1 \geq \sigma_2$.

This ordering was introduced by Brown in [2] and [3] as a means of studying the relationship between social indecision and the degree of domination in a society.

A minimal winning coalition is a winning coalition where no proper subset of individuals is a winning coalition. Murakami [7] has argued that the size of minimal winning coalitions can be used as a proxy for the degree of domination in a society. The size of a minimal winning coalition being inversely proportional to that coalition's power in determing the social outcome. The obvious examples of his thesis are

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[3] Craven assumes that individual preferences are strong orders.

dictatorial rule and the rule of unanimity. These examples are not only extremes in terms of domination but they are also extremes with respect to social indecision. In the case of dictatorial rule, we have maximal domination and minimal social indecision. For the unanimity rule, we have just the converse. Hence there is a qualitative trade-off between social indecision and domination which can not be captured by considering only the size of minimal winning coalitions.

Returning to the framework proposed by Guilbaud, we see that the degree of overlap between decisive sets can be used as a measure of domination. This overlap, in turn, is related to the minimum number of decisive sets having empty intersection, the "n" in Theorem 1. If we are to prevent the most acute instance of social indecision, i.e. cycling, then Theorem 1 requires that $n \geq m+1$. If σ_1 and σ_2 are acyclic aggregation rules and $\sigma_1 \geq \sigma_2$, then σ_1 is at least as socially decisive as σ_2. Hence we are led to consider those acyclic aggregation rules, without weak dictators, which are maximal with respect to the ordering $\geq$.

These acyclic rules are "optimal" in the sense that we have first "minimized" the degree of domination, i.e., no weak dictators, and then "maximized" with respect to social decisiveness.

In Theorem 3, we consider an important class of acyclic aggregation rules, acyclic majority rules, and give necessary and sufficient conditions for a rule to be maximal in this class.

A classical condition in democratic theory used to prevent individual domination is anonymity. The anonymity condition forces the voting rule to depend only on the number of individuals who prefer x to y and not on who they are. This is an appealing condition and one that is consistent with Murakami's analysis of domination. But is the condition of anonymity consistent with our proposal, i.e., is every maximal acyclic majority rule anonymous? Using Theorem 3, we shall see that, in general, the answer is no.

In Theorem 4, we show that there exists an anonymous maximal acyclic majority rule Γ iff $|I| = m+1$. Moreover, Γ is unique and the winning coalitions for Γ are all sets of individuals having at least m members.

As a corollary, we see that the only Craven rules which are maximal acyclic majority rules are those where $|I| = m+1$ and $\alpha = m-1/m$.

One final remark, Craven suggests that the limiting acyclic rule, as the number of alternatives becomes large, is unanimity, i.e., if $\alpha \geq m-1/m$ then as m goes to infinity, α goes to 1 . It follows from Theorem 2 that Craven's rules are a small, albeit interesting, class of acyclic aggregation rules. The rules described in Theorem 2 have as their limit those rules which have a weak dictator. The "optimal" rules in this family, with respect to the trade-off between social indecision and domination, were characterized in [3]. The next example highlights the difference between Craven's approach and our own.

Consider a committee of three individuals with a large agenda. As opposed to the Craven rule of unanimity, we suggest the following weighted majority rule: the chairman has two votes, the other individuals each have one vote, and x defeats y iff x receives at least three votes.[4]

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[4] If we only assume that individual preferences are acyclic, then we must require the chairman to cast zero votes or two votes. But if we assume that individual preferences are weak orders, then the chairman can cast zero, one, or two votes. In the latter case, a motion can pass if the chairman is indifferent, i.e. he casts one vote, and the other two members of the committee vote in the affirmative.

2. Definitions and Assumptions

Let X be a nonempty finite set of alternatives, where $|X| = m \geq 2$ is the cardinality of X. A preference relation is a binary relation on X. $\mathcal{R}$ will denote the family of asymmetric preference relations. A preference relation P is acyclic if there does not exist alternatives $x_1, x_2, \ldots, x_n$ such that $x_1Px_2, x_2Px_3, \ldots, x_{n-1}Px_n$ and x_nPx_1. Let $\mathcal{A}$ be the family of acyclic preferences over X.

I is a nonempty finite set denoting the individuals in society.

$F_{\mathcal{A}}$ is the set of all functions from I to $\mathcal{A}$.

An aggregation rule is a mapping σ from $F_{\mathcal{A}}$ to $\mathcal{R}$.

An acyclic aggregation rule is a mapping σ from $F_{\mathcal{A}}$ to $\mathcal{A}$.

Given $f \in F_{\mathcal{A}}$, $\sigma(f)$ denotes the social preference.

Given $f \in F_{\mathcal{A}}$, $f(i)$ denotes the preference relation in $\mathcal{A}$ assigned by f to individual $i \in I$. If $a, b \in X$ and $J \subseteq I$, then $af(J)b \iff af(i)b$ for all $i \in J$. $f(i) = g(i)$ on $\{a,b\}$ means that $af(i)b \iff ag(i)b$ and that $bf(i)a \iff bg(i)a$. $f(J) = g(J)$ on $\{a,b\}$ means that $f(i) = g(i)$ on $\{a,b\}$ for all $i \in J$.

If J is a subset of I, then $\bar{J}$ is the complement of J.

In the course of our discussion, we shall wish to impose one or more of the following conditions on a given aggregation rule, σ:

<u>Pareto</u> (P) For all $a, b \in X$ and $f \in F_{\mathcal{A}}$, $af(I)b \implies a\sigma(f)b$

<u>Positive Responsiveness</u> (PR) For all $a, b \in X$ and $f, g \in F_{\mathcal{A}}$, $\{i \in I | af(i)b\} \subseteq \{i \in I | ag(i)b\}$ and $a\sigma(f)b \implies a\sigma(g)b$.

<u>Independence</u> (I) For all $a, b \in X$ and $f, g \in F_{\mathcal{A}}$, $f = g$ on $\{a,b\} \implies \sigma(f) = \sigma(g)$ on $\{a,b\}$.

<u>Neutrality</u> (N) For all $a, b, c, d \in X$ and $f, g \in F_{\mathcal{A}}$, if $af(i)b \iff cg(i)d$ and $bf(i)a \iff dg(i)c$ then $a\sigma(f)b \iff c\sigma(g)d$

and $b\sigma(f)a \iff d\sigma(g)c$.

Weak Dictator (WD) i_0 is a weak dictator if for all $a, b \in X$ and $f \in F_{\mathcal{A}}$, $af(i_0)b \implies$ not $b\sigma(f)a$.

Anonymity (A) For all permutations η on I and $f \in F_{\mathcal{A}}$, if $g = f \circ \eta$ then $\sigma(f) = \sigma(g)$.

We note that (N) implies (I).

A set of individuals J is decisive for an aggregation rule σ if for all $a, b \in X$ and all $f \in F_{\mathcal{A}}$, $af(J)b \implies a\sigma(f)b$.

A set of individuals J is a minimal decisive set for an aggregation rule σ, if J is decisive for σ and no proper subset of J is decisive for σ .

If an aggregation rule σ has a nonempty family of decisive sets, then either some individual belongs to every decisive set, i.e. σ has property (WD) , or there exists a minimum number of decisive sets having empty intersection. In the latter event, $n(\sigma)$ is the minimum number of decisive sets having empty intersection. That is, every family of decisive sets of σ having less than $n(\sigma)$ members has nonempty intersection and there exists some family having $n(\sigma)$ members whose intersection is empty.

Every family of subsets of I , Γ, defines an aggregation rule σ_Γ where for all $a, b \in X$ and $f \in F_{\mathcal{A}}$, $a\sigma_\Gamma(f)b$ iff $\{i \in I \mid af(i)b\} \in \Gamma$. Note that the family of decisive sets for σ_Γ is Γ .

We shall call an aggregation rule, μ , an acyclic majority rule if $\mu = \sigma_\Gamma$ for some family of subsets Γ such that

(i) $I \in \Gamma$

(ii) $A \in \Gamma$, $A \subseteq B \implies B \in \Gamma$

(iii) $n(\sigma_\Gamma) \geq m+1$.

$\mathcal{A}$ will denote the family of acyclic majority rules over the set of individuals, I . If σ and μ are acyclic majority rules with Λ and Θ their respective families of decisive sets, then we say that $\sigma \geq \mu$ if $\Theta \subseteq \Lambda$. Under this ordering $\langle \mathcal{A}, \geq \rangle$ is a partially ordered set.

3. <u>The Partially Ordered Set of Acyclic Majority Rules</u>

<u>Theorem 1</u>. If σ is an acyclic aggregation rule having property (P) . Then either the family of decisive sets for σ has the f.i.p. or $n(\sigma) \geq m+1$.

<u>Proof</u>. Property (P) implies the existence of at least one decisive set, i.e. I is decisive. Now suppose the lemma is false, then the intersection of all the decisive sets is empty. Therefore $n(\sigma)$ exists. We need only consider the case where $n(\sigma) = m$. Hence there exists m decisive sets E_i , $i = 1, 2, \ldots, m$ where $\bigcap_{i=1}^{m} E_i = \emptyset$. Let $X = \{x_i\}_{i=1}^{m}$ and define the following preferences: x_i preferred to x_{i+1} for all individuals in E_i for $i = 1, 2, \ldots, m-1$ and x_m preferred to x_1 for all individuals in E_m . Hence each individual has an acyclic preference, but the aggregation preference for this society cycles.

<u>Theorem 2</u>. If σ_Λ is an aggregation rule. Then σ_Λ is acyclic iff σ_Λ has property (WD) or $n(\sigma_\Lambda) \geq m+1$.

<u>Proof</u>. Necessity follows from Theorem 1. Let Λ by the family of decisive sets for σ_Λ . If σ_Λ is cyclic then for some $f \in F_{\mathcal{a}}$ and $x_1, x_2, \ldots, x_r$ we have $x_1 \sigma_\Lambda(f) x_2$, $x_2 \sigma_\Lambda(f) x_3$, $\ldots$, $x_{r-1} \sigma_\Lambda(f) x_r$, and $x_r \sigma_\Lambda(f) x_1$. Let $E_j = \{i \in I | x_j f(i) x_{j+1}\}$ for $j = 1, 2, \ldots, r-1$ and $E_r = \{i \in I | x_r f(i) x_1\}$, then the $E_\ell \in \Lambda$ for $\ell = 1, 2, \ldots, r$.

Since individuals are assumed to have acyclic preferences, $\bigcap_{\ell=1}^{r} E_\ell = \emptyset$.

Therefore $n(\sigma_\Lambda) \leq r \leq m < m+1$ and $\bigcap_{D \in \Lambda} D = \emptyset$.

Lemma 1. If an aggregation rule σ has properties (A) and (PR) . Then all the minimal decisive sets for σ have the same cardinality, ℓ , and a set $J \subseteq I$ is decisive iff $|J| \geq \ell$.

Proof. Omitted.

Corollary 2.1. If σ is an acyclic majority rule and Λ is the family of decisive sets for σ . Then σ has property (A) , iff there exists an integer r less than $|I|$ such that $\Lambda = \{D \subseteq I \mid |D| > r\}$.

Proof. Omitted.

Craven [4] gives a sharp lower bound for r which is $\left[\frac{m-1}{m} \cdot |I|\right]$, where $[x]$ is the greatest integer y such that $y \leq x$.

Let $\Gamma(r) = \{D \subseteq I \mid |D| > r\}$ where r is an integer.

Ferejohn and Grether [5] show that for all $r \geq \left[\frac{m-1}{m} \cdot |I|\right]$, if individual preferences are complete and acyclic then $\sigma_{\Gamma(r)}$ is an acyclic majority rule.

Lemma 2. If σ is an acyclic majority rule, then σ has properties (P) , (PR) , and (N) .

Proof. Omitted.

Lemma 3. If $\mathcal{G}$ is a nonempty upward directed subfamily of $\mathcal{D}$, i.e. for all $\Lambda, \Theta \in \mathcal{G}$ there exists $\Gamma \in \mathcal{G}$ such that $\Lambda \subseteq \Gamma$ and $\Theta \subseteq \Gamma$. Then $\bigcup \mathcal{G}$ is in $\mathcal{D}$.

Proof. Omitted.

Proposition 1 [Birkhoff [1]]. If $\mathcal{L}$ is a family of families of subsets ordered by set inclusion and closed under upward directed unions. Then $\mathcal{L}$ has a maximal element, i.e. an element not property contained in any other element in $\mathcal{L}$.

Lemma 4. Every acyclic majority rule can be extended to a maximal acyclic majority rule.

Proof. Omitted.

Lemma 5. If an acyclic majority rule Γ is maximal, then $n(\sigma_\Gamma) = m+1$.

Proof. Suppose $n(\sigma_\Gamma) = k > m+1$ and let $\{E_i\}_{i=1}^k$ be a subfamily of decisive sets for σ_Γ such that $\bigcap_{i=1}^k E_i = \emptyset$, $m \geq 2$ implies that $k > 3$. If $B = E_1 \cap E_2$, then $B \neq \emptyset$; $B \not\in \Gamma$; any subfamily of Γ having empty intersection with B must have at least $k-2$ members. If not we contradict the assumption that $n(\sigma_\Gamma) = k$. If $\Lambda = \{J \subseteq I | E \subseteq J$ where $E \in \Gamma$ or $E = B\}$, then Λ is an acyclic majority rule properly containing Γ , i.e., Γ is not maximal.

Theorem 3. The following condition is necessary and sufficient for an acyclic majority rule Γ to be maximal: for all $D \subseteq I$, $D \notin \Gamma$ iff there exists $E_i \in \Gamma$ where $i = 1, 2, \ldots, m-1$ such that $\bar{D} \supseteq \bigcap_{i=1}^{m-1} E_i$.

Proof. Suppose Γ is not maximal, then there exists $D \notin \Gamma$ such that $\Lambda = \{J \subseteq I | E \subseteq J$ where $E \in \Gamma$ or $E = D\}$ is an acyclic majority rule. Hence $n(\sigma_\Lambda) \geq m+1$ and for all $\{E_i\}_{i=1}^{m-1}$ in Λ , $D \cap (\bigcap_{i=1}^{m-1} E_i) \neq \emptyset$. That is, $\bar{D} \not\supseteq \bigcap_{i=1}^{m-1} E_i$. Since $\Gamma \subseteq \Lambda$, this completes the sufficiency part of the proof.

Suppose the stated condition holds and Γ is not maximal. Then applying Lemma 4, Γ can be extended to a maximal acyclic majority rule Λ which properly contains Γ. Let $D \in \Lambda/\Gamma$. But then there exists $\{E_i\}_{i=1}^{m-1}$ in Γ such that $D \cap (\bigcap_{i=1}^{m-1} E_i) = \emptyset$. Therefore $n(\sigma_\Lambda) \leq m$ which contradicts the assumption that Λ is an acyclic majority rule.

A natural question to raise at this juncture is the relationship between anonymous acyclic majority rules, e.g. Craven rules, and maximal acyclic majority rules. The following example shows that maximal acyclic majority rules need not have property (A).

Let $m = 2$ and $I = \{1,2,3,4\}$, then it suffices to give the minimal winning coalitions. Let $\Gamma = \{\{1,2\}, \{1,3\}, \{2,3\}\}$, then it is easy to check, using Theorem 3, that Γ is maximal. Clearly Γ is not anonymous, since individual "4" does not belong to any minimal winning coalition. Moreover, no maximal acyclic majority rule in this example has property (A), i.e. they all have the same structure as Γ.

<u>Theorem 4</u>. The following conditions are equivalent:

(a) $|I| = m+1$

(b) If $\Gamma = \{J \subseteq I \mid |J| \geq m\}$, then Γ is the unique anonymous maximal acyclic majority rule.

(c) There exists an anonymous maximal acyclic majority rule.

<u>Proof</u>. The proof that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) is omitted. Suppose σ is an anonymous maximal acyclic majority rule and Λ is the family of decisive sets for σ. By Lemma 1, there exists an integer ℓ where a set $J \subseteq I \in \Lambda$ iff $|J| > \ell$. By Lemma 2, σ satisfies the "ethical" conditions of Ferejohn and Grether[5] and is therefore a Craven rule.

[5] That is, σ has properties (P), (PR) and (N) in addition to property (A).

Since σ is maximal in the class of acyclic majority rules, it must a fortiori be maximal in the class of Craven rules. That is, $\ell = \left\lceil \frac{m-1}{m} \cdot |I| \right\rceil$. If $|I| < m+1$, then we need only consider the case $|I| = m$. In this event, σ has only one decisive set, I , which contradicts the assumption that σ has no weak dictators.

If $|I| > m+1$, then consider the case $|I| = m+k$ where $2 \leq k \leq m$. Hence $\frac{m-1}{m} \cdot (m+k) = m + (k-1) - \frac{k}{m}$. Therefore $\ell = m+k-2$ and $n(\sigma_\Gamma) = m+k$ which contradicts Lemma 5, i.e., we assumed that σ_Γ is a maximal acyclic majority rule. If $k > m$, then there exists a coalition E of size ℓ such that the condition in Theorem 3 is not met. Hence σ_Γ cannot be maximal. Therefore (c) $\Longrightarrow$ (a).

Note that, in fact, Γ is the only anonymous acyclic majority rule if $|I| = m+1$. Hence anonymity and maximality, although not inconsistent, are in a sense incompatible. Recalling the discussion in the Introduction concerning domination and social indecision, where we noted their inverse relationship; it is reasonable that maximality which "minimizes" social indecision is at odds with anonymity which "minimizes" individual domination.

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