COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 373

THE WALRASIAN AND VON NEUMANN EQUILIBRIA: A COMPARISON

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March 4, 1974

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by

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ABSTRACT

The two economic concepts mentioned in the title deal with different features of an economy, the first with the distribution of produced goods between consumers according to their preferences, the second with the growth of the economy through "production of commodities by means of commodities" (Sraffa), leaving the mechanism of consumption almost completely aside. It seems interesting that in spite of these differences many similarities between the models can be found and that the von Neumann model can be adjusted to account for the preferences of consumers. This seems to be the main result of the paper.

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This paper was prepared during the authors' stay in 1973 at the Cowles Foundation for Research in Economics at Yale University and at the Mathematics Research Center, University of Wisconsin. Their visit at the Cowles Foundation was sponsored by a grant from the Ford Foundation to the Cowles Foundation. Their work at the Mathematics Research Center was sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462.

1. Preliminaries

By a von Neumann model we shall understand a pair of matrices of the same dimensions: (A,B), $(\mathcal{A},\mathcal{B})$ and so on. The first one will be called the input matrix and the second the output matrix. In general, we do not make the (usual) assumption that the matrices of a von Neumann model are non-negative. In those cases in which this assumption is made, the model will be called non-negative.

Vectors acting on matrices from the left are row vectors $x = (x_1, \ldots, x_n)$ (intensities) and vectors acting on matrices from the right are column vectors $p = <p_1, \ldots, p_m>$ (prices).

We always denote by $(\ldots)$ rows and by $<\ldots>$ columns of objects between parentheses. This convention applies as well to numbers as to vectors and matrices: for instance, (x', x'') is a vector $(x'_1, \ldots, x'_n, x''_1, \ldots, x''_r)$, and $A = <A_1, \ldots, A_n>$, with A_i being $n_i \times m$ matrices, is a $\sum_i n_i \times m$ matrix.

All spaces are ordered by coordinates; thus $x = (x_1, \ldots, x_n) \leq$ $\leq x' = (x'_1, \ldots, x'_n)$ means $x_1 \leq x'_i$ for all $i = 1, \ldots, n$, and $x < x'$ means $x_i < x'_i$ for all $i = 1, \ldots, n$. This applies to all vectors (rows as well as columns) in all spaces.

By an equilibrium of a von Neumann model (A,B) of dimension $n \times m$ (non-negative or not) we understand a pair of vectors $x \geq 0$, $p \geq 0$ which together with a number $\lambda > 0$ (level of the equilibrium) satisfy the following inequalities:

(1.1) $\lambda xA \leq xB$ (conservation condition),

(1.2) $Bp \leq \lambda Ap$ (profitless economy condition),

(1.3) $xBp > 0$ (positive value of production).

If vectors $0 \neq x \geq 0$ and $0 \neq p \geq 0$ satisfy with a given $\lambda > 0$ only conditions (1.1) and (1.2) we shall say that they form a weak equilibrium. It is easy to see that even at a weak equilibrium

(1.4) $$\lambda xAp = xBp \ .$$

By a von Neumann model with labor constraints we understand a von Neumann model (A,B) of dimension say $n \times m$, together with a $n \times k$ matrix K (labor input-coefficient matrix) and a k-component row vector $\ell^* \geq 0$ (labor supply vector). In such model $xK \leq \ell^*$ is the condition for feasibility of a vector $x \geq 0$.

Let us consider now a Walrasian general equilibrium model involving s producers and r consumption units. The i^{th} producer is characterized by his set of production possibilities W_i, which is a subset of the space $R^m \times R^k$ of vectors (y,k), y being a commodity bundle and k being a vector of different kind of labor services. The j^{th} consumption unit is characterized by a similar set Z_j in $R^m \times R^k$. Since the producers need labor but do not produce it and consumption units consume commodities giving in return labor services, we shall assume that:

(1.5) If $(y,k) \in W_i$, then $k \leq 0$;

(1.6) If $(y,k) \in Z_j$, then $y \geq 0$ and $k \leq 0$.

We shall assume also

(1.7) $$(0,0) \in W_i \ , \quad i = 1, \ldots, s \ .$$

The vector of prices in $R^m \times R^k$ will be denoted by $\pi = \langle p,q \rangle$, $p = \langle p_1, \ldots, p_m \rangle$ being a vector of prices of commodities and $q = \langle q_1, \ldots, q_k \rangle$ a vector of wages paid per unit of different labor services.

With a given vector π, each producer aims to maximize his profit; thus he wants to have his activity vector in the set

$$T_i(\pi) = \{w \in W_i \mid w\pi \geq \bar{w}\pi \ , \quad \text{for all} \quad \bar{w} \in W_i\} \ .$$

(Note that $w = (y, -k)$ with $k \geq 0$; thus $w\pi = yp - kq$.)

The behavior of a consumption unit is given by its preference relation. Every consumption unit tries, when commodity prices and wages $\pi = \langle p,q \rangle$ are given, to choose the best $z \in Z_j$ following its preference, but is constrained to the set of those $z \in Z_j$ which are in the budget set:

$$S_j(\pi) = \{z \in Z_j \mid z\pi \leq 0\} \ .$$

(Note again that $z = (y, -k)$ with $y \geq 0$ and $k \geq 0$; thus $z\pi \leq 0$ means $yp \leq kq$.)

We shall not investigate the properties of preferences; instead, we shall assume that for every j a demand correspondence $D_j(\pi)$ is given, being the set of optimal (according to the preference of the consumption unit) vectors $z \in S_j(\pi)$. In this way, every consumption unit is described by a pair (Z_j, D_j) with

(1.8) $$D_j(\pi) \subset S_j(\pi) = \{z \in Z_j \mid z\pi \leq 0\} \quad \text{for every} \quad \pi \geq 0 .$$

The whole model is then an (s+r)-tuple

$$(W_1, \ldots, W_s, (Z_1, D_1), \ldots, (Z_r, D_r)) ,$$

with W_i satisfying (1.5) and (1.7), Z_j satisfying (1.6), and $D_j(\cdot)$ satisfying (1.8).

Let us note that with given sets W_i and Z_j the supply correspondences $T_i(\cdot)$ and the budget correspondences $S_j(\cdot)$ are well defined, but the demand correspondences $D_j(\cdot)$ are certainly not. Because of the special character of the problems considered here, we need assume almost nothing about them. However, for use in Section 3, it will be necessary to assume the following property:

(1.9) If $(y, -k) \in D_j(\pi)$, $(y', -k') \in S_j(\pi)$ and $y' \geq y$, $k' \leq k$,

$$\text{then} \quad (y', -k') \in D_j(\pi) .$$

By an equilibrium for a Walrasian model (Walrasian equilibrium) we mean an (s+r+1)-tuple of vectors $(w^1, \ldots, w^s, z^1, \ldots, z^r, \pi)$ such that $\pi \geq 0$ and

(1.10) $$w^i \in T_i(\pi) , \quad i = 1, \ldots, s ,$$

(1.11) $$z^j \in D_j(\pi) , \quad j = 1, \ldots, r ,$$

(1.12) $$\sum_{j=1}^{r} z^j - \sum_{i=1}^{s} w^i \leq 0 \quad \text{(allocation property)},$$

(1.13) $$\sum_{j=1}^{r} z^j\pi - \sum_{i=1}^{s} w^i\pi = 0 \quad \text{(Walras law)}.$$

Let us note the well-known fact that:

(1.14) If $w^1, \ldots, w^s, z^1, \ldots, z^r, \pi$ are in equilibrium, then

$$w^i\pi = 0 , \quad i = 1, \ldots, s \quad \text{and} \quad z^j\pi = 0 , \quad j = 1, \ldots, r .$$

It follows, since $z^j\pi \leq 0$ by (1.11) and (1.8) and $w^i\pi \geq 0$ by (1.7) and (1.10), that (1.13) is a sum of non-positive numbers and equals 0 ; hence all summands must be equal to 0 .

2. Walrasian Model with Linear Production

Let us suppose that the i^{th} producer is in control of n_i linear technologies forming a non-negative von Neumann model with labor constraints: A_i , B_i , K_i , ℓ^* $(i = 1, \ldots, s)$, where A_i , B_i are of dimension $n_i \times m$, K_i is of dimension $n_i \times k$ and $\ell^* = (\ell^*_1, \ldots, \ell^*_k) \geq 0$ is the labor supply vector common for all producers. Suppose further that ℓ^* splits into labor endowment vectors ℓ^{*j}, $(j = 1, \ldots, r)$, each attributed to one of the consumption units, so $\ell^* = \sum_{j=1}^{r} \ell^{*j}$ and $\ell^{*j} \geq 0$. Finally, let $\alpha > 0$ be an exogenously given factor of growth.

The Walrasian model with sets

$$(2.1)\quad W_i = \{(y, -k) | y = x^i(B_i - \alpha A_i), k = x^i K_i \leq \ell^*, x^i \geq 0\}, \quad (i = 1, \ldots, s),$$

$$(2.2)\quad Z_j = \{(c, -\ell) | c \geq 0, 0 \leq \ell \leq \ell^{*j}\}, \quad (j = 1, \ldots, r),$$

and suitably chosen demand correspondences $D_j(\cdot)$ will be called a Walrasian model with linear production.

Let us note the role of α in the definition of sets W_i. With $\alpha = 1$ and $x^i \geq 0$ as intensities, the whole vector $w^i = (x^i(B_i - A_i), -x^i K_i)$ $= (y_1, \ldots, y_m, -k_1, \ldots, -k_k)$ enters into exchange with other producers and consumption units. However, if $\alpha > 1$, then $x^i(B_i - A_i) = x^i(B_i - \alpha A_i)$ $+ (\alpha-1)x^i A_i$ and only $x^i(B_i - \alpha A_i)$ enters into the vector $w^i = (x^i(B_i - A_i), -x^i K_i)$ while $(\alpha-1)x^i A_i$ is left over for increase of production in future periods.

We shall assume that:

$$(2.3)\qquad \ell^* > 0 .$$

$$(2.4)\qquad \text{If } 0 \neq x^i \geq 0, \text{ then } 0 \neq x^i K_i \geq 0 .$$

Under these conditions, the set of x's which can be applied in (2.1) to define the set W_i will be non-empty, compact and convex; thus the W_i are also non-empty, compact and convex.

The sets Z_j defined by (2.2) are convex and bounded from below.

The von Neumann models of all producers together form a von Neumann model with labor constraints:

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix} = \langle A_1, \ldots, A_s \rangle , \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_s \end{pmatrix} = \langle B_1, \ldots, B_s \rangle ,$$

$$K = \begin{pmatrix} K_1 \\ \vdots \\ K_s \end{pmatrix} = \langle K_1, \ldots, K_s \rangle, \ \ell^* ,$$

the dimensions of the matrices being $n \times m$ for A and B and $n \times k$ for K, with $n = \sum_i n_i$. The intensity vectors for this model are then n-vectors $x = (x^1, \ldots, x^s)$, with the x^i being n_i-vectors.

For further convenience, we shall identify the $n_i \times m$ matrix A_i with the $n \times m$ matrix $\langle 0, \ldots, A_i, \ldots, 0\rangle$, and similarly B_i with $\langle 0, \ldots, B_i, \ldots, 0\rangle$, K_i with $\langle 0, \ldots, K_i, \ldots, 0\rangle$. The n_i-vector will be identified with the n-vector $x = (0, \ldots, x^i, \ldots, 0)$. With this convention, we have $A = \sum_{i=1}^{s} A_i$, $B = \sum_{i=1}^{s} B_i$, $K = \sum_{i=1}^{s} K_i$ and for every x an unique representation $x = \sum_{i=1}^{s} x^i$.

Having an arbitrary von Neumann model, A, B, K, ℓ^* with labor constraints, we can proceed backwards and form from it a Walrasian model with linear production simply by:

a. Splitting the rows of matrices A, B, K between products and forming in this way sub-models A_i, B_i, K_i such that $\sum_i A_i = A$, $\sum_i B_i = B$, $\sum_i K_i = K$;

b. Splitting the vector ℓ^* into a sum $\ell^* = \sum_j \ell^{*j}$, $\ell^{*j} \geq 0$ and assigning each ℓ^{*j} as a labor endowment to the j^{th} consumption unit;

c. Accepting a factor of growth $\alpha > 0$;

d. Accepting suitable demand correspondences $D_j(\cdot)$.

3. Walrasian Equilibria with Profitless Prices

The prices $\pi = \langle p,q \rangle \geq 0$ are profitless for the i^{th} producer if and only if

$$(3.1) \qquad (B_i - \alpha A_i)p = K_i q \ .$$

They are profitless for the whole model (or simply profitless), if and only if

$$(3.2) \qquad (B - \alpha A)p = Kq \ .$$

Obviously, (3.2) is equivalent to (3.1) for all $i = 1, \ldots, s$.

We now proceed to study equilibria of Walrasian models with profitless prices. The main theorem is the following:

Theorem 1. In a Walrasian model with linear production and with the demand correspondences satisfying (1.9), a given profitless price $\hat{\pi} = \langle \hat{p},\hat{q} \rangle \geq 0$ is in equilibrium with some $\hat{w}^i \in W_i$ $(i = 1, \ldots, s)$ and $\hat{z}^j \in Z_j$ $(j = 1, \ldots, r)$, if and only if there is at least one choice of consumption bundles $z^j = (c^j, -\ell^j) \in D_j(\hat{\pi})$, $j = 1, \ldots, r$

such that the total consumption can be produced within the supplied labor, i.e. for some $x \geq 0$: $x(B - \alpha A) \geq \sum_{j=1}^{r} c^j$ and $xK \leq \sum_{j=1}^{r} \ell^j$.

The proof of this theorem will be preceded by two lemmas, the first of which is of a rather general character.

Lemma 1. Let $\bar{y}, \bar{y}^1, \ldots, \bar{y}^r$ be non-negative vectors in R^m , $\bar{v}, \bar{v}^1, \ldots, \bar{v}^r$ non-negative vectors in R^k , p a price vector for R^m and q a price vector for R^k . If

(3.3) $$\sum_{j=1}^{r} \bar{y}^j \leq \bar{y} ,$$

(3.4) $$\bar{v} \leq \sum_{j=1}^{r} \bar{v}^j ,$$

and

(3.5) $$\bar{y}p = \bar{v}q ,$$

(3.6) $$\bar{y}^j p \leq \bar{v}^j q , \quad j = 1, \ldots, r ,$$

then there exist vectors $y^1, \ldots, y^r$ in R^m and $v^1, \ldots, v^r$ in R^k such that

(3.7) $$\bar{y}^j \leq y^j , \quad j = 1, \ldots, r ,$$

(3.8) $$0 \leq v^j \leq \bar{v}^j , \quad j = 1, \ldots, r ,$$

(3.9) $$\sum_{j=1}^{r} y^j \leq \bar{y} ,$$

(3.10) $$\sum_{j=1}^{r} v^j \geq \bar{v} ,$$

and such that

(3.11) $$y^j p = v^j q , \quad j = 1, \ldots, r .$$

Proof. Let us consider the following linear programming problem: find vectors $y^1, \ldots, y^r \geq 0$, $v^1, \ldots, v^r \geq 0$ which minimize

$$f(y^1, \ldots, y^r, v^1, \ldots, v^r) = \sum_{j=1}^{r} (v^j q - y^j p) ,$$

under constraints (3.7)-(3.10) and

(3.11*) $$y^j p \leq v^j q , \quad j = 1, \ldots, r .$$

The vectors $\bar{y}^j$, $\bar{v}^j$, $j = 1, \ldots, r$ are a feasible solution of the problem. Any feasible solution $\sigma = (y^1, \ldots, y^r, -v^1, \ldots, -v^r)$ is bounded from above by (3.9) and (3.10). It follows that there exists a feasible Pareto-optimal solution σ , i.e. such that for any feasible solution $\tilde{\sigma}$ if $\sigma \leq \tilde{\sigma}$, then $\sigma = \tilde{\sigma}$. Suppose that for a feasible Pareto-optimal solution σ , the value of the program $f > 0$. Then one of the summands in the objective function must be positive and by (3.5) one of the inequalities (3.9) and (3.10) is satisfied as the strict inequality for at least one coordinate. Suppose that $y^{j_0}p < v^{j_0}q$ and $\sum_j y^j_{i_0} < \bar{y}_{i_0}$. Then we can find an $\epsilon > 0$ such that the vector $\tilde{y}^{j_0} = (y_1^{j_0}, \ldots, y_{i_0}^{j_0} + \epsilon, \ldots, y_m^{j_0})$ satisfies constraints: $\tilde{y}^{j_0}p \leq v^{j_0}q$ and $\sum_j y^j_{i_0} + \epsilon \leq \bar{y}_{i_0}$. Hence the solution $\tilde{\sigma} = (y^1, \ldots, \tilde{y}^{j_0}, \ldots, y^r, v^1, \ldots, v^r)$ is feasible and $\sigma \leq \tilde{\sigma}$ but $\sigma \neq \tilde{\sigma}$, which contradicts the assumed Pareto-optimality of σ . In the case when (3.10) is satisfied as a strict inequality for one coordinate, the reasoning is analogous. It shows that for feasible Pareto-optimal solutions the value of the program is $f = 0$; hence (3.11) is satisfied.

Lemma 2. A price vector $\pi = \langle p,q \rangle \geq 0$ is profitless for the i^{th} producer if and only if $T_i(\pi) = W_i$.

Proof. If π satisfies (3.1), then for every vector $w = (x(B_i - \alpha A_i), -xK_i) \in W_i$ we have $w\pi = x(B_i - \alpha A_i)p - xK_iq = 0$, which implies $T_i(\pi) = W_i$. Suppose now that $T_i(\pi) = W_i$. Then for every $w \in W_i$ we have $w\pi = 0\pi = 0$, since by (1.7) $0 \in W_i$. By assumption (2.3), it is possible to find $x^* > 0$ such that $x^*K_i \leq \ell^*$. Thus for all x's in the open set $0 < x < x^*$ we have $(x(B_i - \alpha A_i), -xK_i) \in W_i$ and $x(B_i - \alpha A_i)p = xK_iq$, which implies (3.1).

Now, we proceed to the

Proof of Theorem 1. Let us assume that the conditions of the theorem are satisfied. In order to apply Lemma 1, let us take $\bar{y} = x(B - \alpha A)$, $\bar{y}^j = c^j$, $\bar{v}^j = \ell^j$, $j = 1, \ldots, r$, $\bar{v} = xK$ and $\langle p,q \rangle = \langle \hat{p},\hat{q} \rangle$.

It is a routine matter to check that all requirements of Lemma 1 are assumed in the theorem. Therefore, there exist vectors y^j , v^j , $j = 1, \ldots, r$ satisfying the assertion of Lemma 1. Denoting y^j by $\hat{c}^j$ and v^j by $\hat{\ell}^j$, we obtain consumption bundles $\hat{z}^j = (\hat{c}^j, -\hat{\ell}^j)$ which belong to Z_j , since by (3.8) $\hat{\ell}^j \leq \ell^j$ and $\ell^j \leq \ell^{*j}$, since $z^j \in Z_j$. By (3.11) $\hat{z}^j\hat{\pi} = 0$ (i.e. $\hat{z}^j \in S_j(\hat{\pi})$) and following (3.7) and (3.8) $\hat{z}^j \geq z^j$; thus it follows from property (1.9) of the demand correspondences that $\hat{z}^j \in D_j(\hat{\pi})$, $j = 1, \ldots, r$, and hence (1.11) holds true. From (3.10) and the fact that $z^j \in Z_j$ it follows that the vectors $\hat{w}^i = (x^i(B_i - \alpha A_i), -x^i K_i)$, $i = 1, \ldots, s$, belong to W_i ; by Lemma 2 they satisfy (1.10). Finally, from Lemma 2 and (3.11) we obtain the Walras law (1.13), and from (3.9) and (3.10) the allocation property (1.12).

Suppose that $\hat{w}^i$, $\hat{z}^j$ and $\hat{\pi}$ are in equilibrium; then by (1.14) the prices $\hat{\pi}$ are profitless and the allocation property (1.13) implies the conditions of the theorem.

In the case of so-called Leontief-type models, we may obtain using Theorem 1, a slight generalization of the result by Gale (see Gale [1955], and Arrow and Hahn [1971], pp. 40-45).

Let us suppose that the von Neumann model describing the production side in the Walrasian model is of Leontief type; i.e., A is a square matrix and $B = I$ is the identity matrix. If the matrix αA is producible (i.e. $x\alpha A < x$ for some $x \geq 0$), then it is known from the Perron-Frobenius theory that the inverse matrix $(I - \alpha A)^{-1}$ exists and is non-negative.

(3.12) If the production in a Walrasian model is described by a Leontief-type model (A, I) and if αA is producible, then in order to have equilibrium with given profitless prices $\hat{\pi} = \langle \hat{p}, \hat{q} \rangle$, it is necessary and sufficient that there exist a choice of $z^j = (c^j, -\ell^j) \in D_j(\hat{\pi})$, $j = 1, \ldots, r$, such that

$$\sum_{j=1}^{r} c^j (I - \alpha A)^{-1} K \leq \sum_{j=1}^{r} \ell^j .$$

<u>Proof</u>. Suppose that $w^i = (x^i(I - \alpha A_i), -x^i K_i)$, $z^j = (c^j, -\ell^j)$ and $\hat{\pi}$ are in equilibrium. Then from the allocation property (1.12) we have for $x = \Sigma x^i$, $c = \Sigma c^j$, $\ell = \Sigma \ell^j$: $xK \leq \ell$ and $c \leq x(I - \alpha A)$. Since $(I - \alpha A)^{-1}$ and K are non-negative matrices, it follows that $c(I - \alpha A)^{-1} \leq x$ and $c(I - \alpha A)^{-1} K \leq xK \leq \ell = \Sigma \ell^j$.

Let us assume now that the condition is satisfied for some $z^j \in D_j(\hat{\pi})$, $j = 1, \ldots, r$. We define $x = \sum_{j=1}^{r} c^j (I - \alpha A)^{-1}$, so $x(I - \alpha A) = \sum_{j=1}^{r} c^j$. From the condition, $xK = \sum_{j=1}^{r} c^j (I - \alpha A)^{-1} K \leq \sum_{j=1}^{r} \ell^j$. The existence of an equilibrium with $\hat{\pi}$ as prices now follows from Theorem 1.

For a Leontief-type model with producible αA, for any $q \geq 0$ a profitless $\pi = \langle p, q \rangle$ can be found simply by setting $p = (I - \alpha A)^{-1} Kq$. Moreover, if the model involves only one kind of labor, then q is a number which can be assumed to equal 1 and then $p = (I - \alpha A)^{-1} K$ is the corresponding price of commodities, making $\pi = \langle p, 1 \rangle$ profitless. Now, if $z^j = (c^j, -\ell^j) \in D_j(\pi)$, then for $c = \Sigma c^j$, $\ell = \Sigma \ell^j$, we have $cp \leq \ell q = \ell$, so that $c(I - \alpha A)^{-1}K \leq \ell$ and the condition of (3.12) is satisfied. It follows then that:

(3.13) If in the Walrasian model there is only one kind of labor, then under the assumptions of Theorem (3.12) there exists an equilibrium with profitless prices and with the wage equal to 1.

4. The von Neumann Model with Incorporated Labor and Consumption

Let us consider a von Neumann model with labor constraints A , B , K , ℓ^* . Suppose that r consumption units, labeled $j = 1, \ldots, r$, are willing to offer labor services for consumption bundles, in particular the j^{th} unit offers the labor vector ℓ^j in return for the consumption bundle c^j . Then with non-negative matrices

$$C = \begin{pmatrix} c^1 \\ \vdots \\ c^r \end{pmatrix} = \langle c^1, \ldots, c^r \rangle , \quad L = \begin{pmatrix} \ell^1 \\ \vdots \\ \ell^r \end{pmatrix} = \langle \ell^1, \ldots, \ell^r \rangle$$

we can form a von Neumann model

$$\mathcal{A} = \begin{pmatrix} A , & 0 \\ 0 , & 0 \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} B , & -K \\ -C , & L \end{pmatrix}$$

of dimension $(n+r) \times (m+k)$, where $n \times m$ is the dimension of A and B , $n \times k$ is the dimension of K , $r \times m$ is the dimension of C and $r \times k$ is the dimension of L .

This is a model without labor constraints; instead, it has negative entries in the matrix $\mathcal{B}$, which therefore can hardly be called an output matrix.

The intensity vectors for this model are of the form $x = (x', x'')$, with the n-vector x' applicable to A, B and K and the r-vector x'' applicable to C and L. The price vectors are, as in the Walrasian model, $\pi = \langle p,q \rangle$.

Let us recall that, following the definition in Section 1 (see (1.1) and (1.2)), a weak equilibrium for the model $(\mathcal{A},\mathcal{B})$ consists of an intensity vector $0 \neq x = (x', x'') \geq 0$ and a price vector $0 \neq \pi = \langle p,q \rangle \geq 0$, which together with a number $\lambda > 0$ satisfy:

$$\lambda x'A \leq x'B - x''C , \tag{4.1}$$

$$x'K \leq x''L , \tag{4.2}$$

$$Bp - Kq \leq \lambda Ap , \tag{4.3}$$

$$Lq \leq Cp . \tag{4.4}$$

In order to have an equilibrium (see (1.3)), we have to require additionally

$$x'Bp - x'Kq - x''Cp + x''Lq > 0 , \tag{4.5}$$

or equivalently

$$x'Ap > 0 . \tag{4.6}$$

The equality (1.4), which holds even at a weak equilibrium is now of the form

$$(4.7) \qquad \lambda x'Ap = x'Bp - x'Kq - x''Cp + x''Lq \ .$$

It can be written in two different forms:

$$x'[(B - \lambda A)p - Kq] = x''[Cp - Lq] \ ,$$

and

$$[x''C - x'(B - \lambda A)]p = [x''L - x'K]q \ .$$

By (4.3) and (4.1) the left sides of both equations are non-positive and by (4.4) and (4.2) the right sides of both equations are non-negative. Therefore, they are equal to 0 and we have a weak equilibrium

$$(4.8) \qquad x''C = x'(B - \lambda A)p = x''Lq = x'Kq \ .$$

The concept of incorporating consumption units into a von Neumann model as additional technological processes is not new. It can be found, e.g. in Gale [1955] and Weil [1967]. Let us remark that at equilibrium we adjust not only the proportions in which the production processes are to be used (the x' vector), but also the proportions of sizes of consumption units (the x'' vector). Therefore, it will be more appropriate to interpret consumption units as professional groups for which the number of representatives has to be chosen at equilibrium, than to interpret them as fixed groups of people (households) with some capacities to work and some requirements for consumption. However, we can accept the latter interpretation, but then we have to ask for an intensity vector $x = (x', x'')$ at equilibrium with a previously-given vector x'', for instance $x'' = e = (1, 1, \ldots, 1)$, which fixes the size of consumption units.

We shall now state the theorem on existence of equilibria for the model $(\mathcal{A},\mathcal{B})$, giving only the outlines of the proof. Let us assume that the model $(\mathcal{A},\mathcal{B})$ has the following properties:

(4.9) All matrices A , B , K , C , L are non-negative;

(4.10) All rows of A as well as all rows of C are non-zero;

(4.11) There exists an $x = (x'x'') \geq 0$ such that $x\mathcal{B} > 0$ (i.e. $x'B - x''C > 0$ and $x''L - x'K > 0$).

Under these assumptions the following theorem can be proved:

<u>Theorem 2</u>. There exists a positive maximal factor of growth of the model $(\mathcal{A},\mathcal{B})$ (i.e. such a number $\check{\lambda} > 0$ such that $\check{\lambda}x\mathcal{A} \leq x\mathcal{B}$, for some $0 \neq x \geq 0$, but for every $\lambda > \check{\lambda}$ and $x \geq 0$ if $\lambda x\mathcal{A} \leq x\mathcal{B}$, then $x = 0$) and there exists a minimal factor of interest $\hat{\lambda}$ (i.e. $\hat{\lambda} = \sup\{\xi > 0 | \text{there exists an } x \geq 0 \text{ such that } \xi x\mathcal{A} < x\mathcal{B}\}$), which are levels of weak equilibria, i.e. there exist vectors $x = (x', x'')$ and $\pi \geq 0$ which together with $\check{\lambda}$ (respectively with $\hat{\lambda}$) satisfy (4.1)-(4.4). Every other possible weak equilibrium level λ (i.e. such number λ that (4.1)-(4.4) hold with some $0 \neq x \geq 0$ and $0 \neq \pi \geq 0$) is between $\hat{\lambda}$ and $\check{\lambda}$ $(\hat{\lambda} \leq \lambda \leq \check{\lambda})$, and conversely every number between $\hat{\lambda}$ and $\check{\lambda}$ is a weak equilibrium level.

It follows from (4.11) that the maximal factor of growth $\check{\lambda}$ is positive, but it could be $+\infty$. That possibility is excluded by (4.9) and (4.10); hence $\check{\lambda}$ is a positive number. To show that $\hat{\lambda}$ is an equilibrium level, we can apply methods very similar to those used in Łoś [1971], for the case of non-negative models. The fact that A is a non-negative matrix is essential here. Similar arguments apply in showing that $\hat{\lambda}$ is a weak equilibrium level. The rest of the theorem is more or less trivial.

5. The Walrasian Equilibria and the von Neumann Equilibria

From a given Walrasian model with linear production we shall construct now a von Neumann model with incorporated labor and consumption. The Walrasian model is already defined with the help of the matrices $A = \sum_{i=1}^{s} A_i$, $B = \sum_{i=1}^{s} B_i$, $K = \sum_{i=1}^{s} K_i$. They can enter the matrices $\mathcal{A}$ and $\mathcal{B}$. We need only matrices C and L, and they can be constructed from any choice of elements $z^j = (c^j, -\ell^j)$ in Z_j, $j = 1, \ldots, r$, as $C = \langle c^1, \ldots, c^r\rangle$ and $L = \langle \ell^1, \ldots, \ell^r\rangle$. Thus the constructed model depends on the z^j's through $\mathcal{B} = \mathcal{B}(z^1, \ldots, z^r)$.

<u>Theorem 3</u>. If the vectors

$$\hat{w}^i = (\hat{x}^i(B_i - \alpha A_i), -\hat{x}^i K_i), \quad i = 1, \ldots, s,$$

$$\hat{z}^j = (c^j, -\ell^j) \qquad\qquad, \quad j = 1, \ldots, r,$$

$$\hat{\pi} = \langle \hat{p}, \hat{q}\rangle \geq 0$$

are an equilibrium of the Walrasian model, then $x = (x', x'')$, with

$x' = \sum_i \hat{x}^i$ and $x'' = e = (1, 1, \ldots, 1)$ together with $\hat{\pi}$ form an equilibrium of the von Neumann model $(\mathcal{A}, \mathcal{B} = \mathcal{B}(\hat{z}^1, \ldots, \hat{z}^r))$ at the level $\lambda = \alpha$.

Proof. The matrices C and L are defined so that $eC = \Sigma\, c^j$ and $eL = \Sigma\, \ell^j$. From the allocation property (1.12) it follows that $0 \geq \Sigma\, \hat{z}^j - \Sigma\, \hat{w}^i = \Sigma(\hat{c}^j, -\hat{\ell}^j) - \Sigma(\hat{x}^i(B_i - \alpha A_i), -\hat{x}^i K_i) = (eC, -eL) - (\Sigma\, \hat{x}^i(B_i - \alpha A_i), -\Sigma\, \hat{x}^i K_i) = (x''C - x'(B - \alpha A), -x''L + x'K)$. Thus we have $x''C - x'B + \alpha x'A \leq 0$, which is (4.1) with $\lambda = \alpha$, and $x'K - x''L \leq 0$, which is (4.2).

From (1.10) it follows that $\hat{w}^i\hat{\pi} \geq w^i\hat{\pi}$ for all $w^i \in W_i$. By (1.14) $\hat{w}^i\hat{\pi} = 0$, so using the definition (2.1) of W_i we obtain

$$w^i\hat{\pi} = x^i(B_i - \alpha A_i, -K_i)\hat{\pi} = x^i[(B_i - \alpha A_i)\hat{p} - K_i\hat{q}] \leq 0$$

for all $x^i \geq 0$ such that $x^i K_i \leq \ell^*$. This holds for all $i = 1, \ldots, s$; hence we obtain

$$x[(B - \alpha A)\hat{p} - K\hat{q}] \leq 0, \tag{5.1}$$

for at least such $x \geq 0$ as satisfy $xK \leq \ell^*$. But we have assumed $\ell^* > 0$, so there exists an $\bar{x} > 0$ satisfying $\bar{x}K \leq \ell^*$, and since K is a non-negative matrix, every vector from the open interval $0 < x < \bar{x}$ will satisfy (5.1). This shows that $B\hat{p} - \alpha A\hat{p} - K\hat{q} \leq 0$, which is (4.3) with $p = \hat{p}$, $q = \hat{q}$ and $\lambda = \alpha$.

Finally, to prove (4.4) we use (1.14), from which it follows that $0 = z^j\hat{\pi} = \hat{c}^j\hat{p} - \hat{\ell}^j\hat{q}$, $j = 1, \ldots, r$. Thus $C\hat{p} - L\hat{q} = \langle\hat{c}^1, \ldots, \hat{c}^r\rangle\hat{p} - \langle\hat{\ell}^1, \ldots, \hat{\ell}^r\rangle\hat{q} = \langle\hat{c}^1\hat{p} - \hat{\ell}^1\hat{q}, \ldots, \hat{c}^r\hat{p} - \hat{\ell}^r\hat{q}\rangle = \langle 0, \ldots, 0\rangle = 0$, which shows that (4.4) is satisfied as an equality with $p = \hat{p}$ and $q = \hat{q}$.

Before we proceed to the theorem stating an inverse relation between Walrasian and von Neumann equilibria, some remarks ought to be made.

First, let us note that (1.11) was used in the proof in a weaker form: $z^j \in S_j(\pi)$, which is enough to prove (4.4). This is the only way in which (1.11) is used in the proof of Theorem 3. It shows the obvious fact that the model $(\mathcal{A}, \mathcal{B}(z^1, \ldots, z^r))$ can choose the composition neither of consumption nor of labor: they must be given in advance by $z^1, \ldots, z^r$. Therefore, to a given Walrasian model with linear production there correspond a whole family of von Neumann models with different matrices C and L depending on the choice of z^j in Z_j , $j = 1, \ldots, r$.

The second remark concerns the role of α . This number was an exogenously accepted factor of growth in the Walrasian model and it is reconfirmed now as the factor of growth of the corresponding von Neumann model at a weak equilibrium. But the acceptance of linear production in a Walrasian model specifies not only sets W_i (see (2.1)), but also technologies which are allowed to work at different levels of the factor of growth, α . Thus a von Neumann model with incorporated labor and consumption expresses more economic reality than the production side of the Walrasian model does. Indeed, we deal with a family of Walrasian models depending on the parameter α , through the production sets

$W_i = W_i(\alpha)$. For a particular α , they depend only on the matrix $(B_i - \alpha A_i)$; thus for any matrix M_i the same set $W_i(\alpha)$ will be defined by matrices A_i , B_i and by matrices $A_i + \alpha^{-1}M_i$ and $B_i + M_i$. Such a change, however, affects the production sets at other $\alpha' \neq \alpha$ and affects models $(\mathcal{A},\mathcal{B})$, which themselves do not depend on α . In particular, such a change affects both the maximal factor of growth $\check{\lambda}$ and the minimal factor of interest $\hat{\lambda}$ of the model $(\mathcal{A},\mathcal{B})$ (see Theorem 2); therefore it changes the equilibria.

To summarize briefly: the choice of consumption and offered labor services is endogenous and the factor of growth is exogenous for a Walrasian equilibrium, while on the other hand consumption and labor supply are exogenous for a von Neumann equilibrium and the factor of growth is endogenous for it. In this respect nothing can be changed as long as we deal with a von Neumann model with constant matrices.

Now, we shall prove a theorem in some sense converse to Theorem 3.

Theorem 4. If the model $(\mathcal{A},\mathcal{B}(z^1, \ldots, z^r))$ with $z^j \in Z_j$, $j = 1, \ldots, r$, has a weak equilibrium $x = (x', x'') \geq 0$, $\pi = \langle p,q \rangle \geq 0$ at the level $\lambda > 0$ and $x'' \leq e = (1, 1, \ldots, 1)$, then the vectors

$$\hat{w}^i = (x'^i(B_i - \lambda A_i), -x'^i K_i) , \quad i = 1, \ldots, s$$

$$\hat{z}^j = x''^j z^j = x''^j(c^j, -\ell^j) \quad , \quad j = 1, \ldots, r$$

together with the prices $\hat{\pi} = \pi$ satisfy conditions (1.10), (1.12) and (1.13) for the equilibrium in the Walrasian model with $\alpha = \lambda$, and moreover, $\hat{z}^j \in S_j(\hat{\pi})$, $j = 1, \ldots, r$ (which is obviously weaker than the condition (1.11)).

Proof. It follows from (4.1) and (4.2) that

$$0 \geq (-x'(B - \lambda A) + x''C, x'K - x''L) = -(x'(B - \lambda A), -x'K) + (x''C - x''L)$$

$$= \sum_{j=1}^{r} x''^j (c^j, -\ell^j) - \sum_{i=1}^{s} (x'^i (B_i - \lambda A_i), -x'^i K_i)$$

$$= \sum_{j=1}^{r} \hat{z}^j - \sum_{i=1}^{s} \hat{w}^i ;$$

Thus the allocation property (1.12) is satisfied. In the same way, we can show that the Walras law (1.13) is equivalent to (4.8) which holds at a weak equilibrium. Since $z^j \in Z_j$ and $x'' \leq e$, we have $x''^j \ell^j \leq \ell^{*j}$, which shows that $\hat{z}^j \in Z_j$. By (4.4), for every j , $\hat{z}^j \hat{\pi} = x''^j (c^j p - \ell^j q) \geq 0$, but according to (4.8) $\sum_{j=1}^{r} \hat{z}^j \hat{\pi} = x''Cp - x''Lq = 0$, which implies $\hat{z}^j \hat{\pi} = 0$ and shows that $\hat{z}^j \in S_j(\hat{\pi})$.

Similarly, it follows from (4.2) and the fact that $z^j \in Z_j$ that $x'K \leq x''L \leq \Sigma\, \ell^j \leq \Sigma\, \ell^{*j} = \ell^*$, hence $\hat{w}^i \in W_i$, $i = 1, \ldots, s$. For every $w^i = (x(B_i - \lambda A_i), -xK_i) \in W_i$ we have $w^i \hat{\pi} = x(B_i - \lambda A_i)\hat{p} - xK\hat{q} \leq 0$ by (4.3). But according to (4.8), for weak equilibrium vectors $\Sigma\, \hat{w}^i \hat{\pi} = x'(B - \lambda A)\hat{p} - x'K\hat{q} = 0$, which shows that $\hat{w}^j \hat{\pi} = 0$ and $\hat{w}^i \in T_i(\hat{\pi})$, $i = 1, \ldots, s$.

6. The Walrasian Equilibria and the von Neumann Equilibria (Continued)

Since von Neumann models with incorporated labor and consumption, as presented in Section 5, do not relate Walrasian equilibria with von Neumann equilibria in a satisfactory way, we shall generalize them again by introducing variable coefficients in the consumption-labor part of the models.

The demand correspondences $D_j(\pi)$ in the Walrasian model, which give to every consumption unit the optimal consumption-labor vectors at given prices, are supposed to be invariant under positive scalar multiplication; i.e.,

(6.1) If $\pi \geq 0$ and $\xi > 0$, then $D_j(\pi) = D_j(\xi\pi)$ for $j = 1, \ldots, r$.

This allows us to investigate them on a normal set of price vectors containing all non-negative directions, for instance on the set

$$S = \{\pi = \langle p,q\rangle \geq 0 \mid \sum_i p_i + \sum_j q_j = 1\} .$$

This set is a simplex, thus a convex and compact set. Let $d_j(\pi)$ be a selection from $D_j(\pi)$ on S, i.e. a function which to every $\pi \in S$ assigns a consumption-labor vector $d_j(\pi) \in D_j(\pi)$. From the assumption (1.8) on demand correspondences, it follows that the demand function $d_j(\pi)$ satisfies the budget constraint $d_j(\pi)\pi \leq 0$, for every $\pi \in S$. We shall assume here something more, namely:

(6.2) $d_j(\pi)\pi = 0$ for every $\pi \in S$, $j = 1, \ldots, r$.

From $d_j(\pi) = (c^j(\pi), -\ell^j(\pi)) \in D_j(\pi)$, $j = 1, \ldots, r$, we form the matrices

$$C(\pi) = \langle c^1(\pi), \ldots, c^r(\pi)\rangle ,$$

$$L(\pi) = \langle \ell^1(\pi), \ldots, \ell^r(\pi)\rangle ,$$

and insert them into the $\mathcal{B}$-matrix of the von Neumann model $(\mathcal{A},\mathcal{B})$. We obtain the model

$$\mathcal{A} = \begin{pmatrix} A , & 0 \\ 0 , & 0 \end{pmatrix} \quad \mathcal{B}(\pi) = \begin{pmatrix} B & , & -K \\ -C(\pi) & , & L(\pi) \end{pmatrix}$$

with entries in the r lowest rows of $\mathcal{B}$ depending on $\pi \in S$.

By an equilibrium for such model we shall understand a pair of vectors $x = (x', x'') \geq 0$, $\pi = \langle p,q\rangle \in S$ which together with an equilibrium level $\lambda > 0$ satisfy conditions (1.1), (1.2) and (1.3), in which $A = \mathcal{A}$ and $B = \mathcal{B}(\pi)$: specifically,

$$\lambda x\mathcal{A} \leq x\mathcal{B}(\pi) \ ; \ \mathcal{B}(\pi)\pi \leq \lambda\mathcal{A}\pi \ ; \ x\mathcal{B}(\pi)\pi > 0 \ .$$

Since assumption (6.2) implies $-C(\pi)p + L(\pi)q = 0$ for every $\pi = \langle p,q\rangle \in S$, the condition (4.4) in the conditions for equilibrium (4.1)-(4.5) is always satisfied and therefore it can be omitted. The remaining conditions for a weak equilibrium are:

$$\lambda x'A \leq x'B - x''C(\pi) , \tag{6.3}$$

(6.4) $$x'K \leq x''L(\pi) ,$$

(6.5) $$Bp - Kq \leq \lambda Ap ,$$

and additionally, for an equilibrium,

(6.6) $$x'Ap > 0 \quad \text{(see (4.6))}.$$

From Theorem 4, assumption (6.2) and the construction of the model $(\mathcal{Q},\mathcal{B}(\pi))$, it follows immediately that:

Theorem 5. If the model $(\mathcal{Q},\mathcal{B}(\pi))$ has a weak equilibrium $x = (x',e) \geq 0$, $\pi = \langle p,q \rangle \in S$ at the level $\lambda > 0$, then the vectors

$$\hat{w}^i = (x'^i(B_i - \lambda A_i), -x'^i K_i) , \quad i = 1, \ldots, s ,$$

$$\hat{z}^j = (c^j(\pi), -\ell^j(\pi)) \qquad , \quad j = 1, \ldots, r ,$$

together with $\pi = \hat{\pi}$ form an equilibrium of the Walrasian model with $\alpha = \lambda$.

The inconvenience of Theorem 5 is that in order to have a Walrasian equilibrium we must find not only a weak von Neumann equilibrium, but moreover one such that the vector $x = (x', x'')$ at equilibrium has $x'' = e = (1, 1, \ldots, 1)$. This is connected with the fact (see remarks in Section 4) that the Walrasian model does not allow for adjustment of the sizes of consumption units but keeps these fixed, and thus requires consumption of definite amounts for them.

It is extremely difficult to see under what conditions a weak equilibrium, as required by Theorem 5, would exist. The theorems below throw some light on this problem:

(6.7) For a given price vector $\pi = \langle p, q \rangle \in S$ and $\lambda > 0$,

$$\mathcal{B}(\pi)\pi \leq \lambda \mathcal{A}\pi \quad \text{iff} \quad Bp - Kq \leq \lambda Ap \ .$$

Proof. This is an immediate consequence of the assumption (6.2).

(6.8) If for a given $\lambda > 0$ there exists an $\bar{\pi} \in S$ such that for every $\epsilon > 0$,

$$(\lambda+\epsilon)x\mathcal{A} \leq x\mathcal{B}(\bar{\pi}) \ , \quad x = (x', e) \geq 0$$

has no solution, then there exists $\tilde{\pi} \in S$ such that

$$\mathcal{B}(\tilde{\pi})\tilde{\pi} \leq \lambda \mathcal{A}\tilde{\pi} \ .$$

Proof. By assumption the inequalities

$$x'[B - (\lambda+\epsilon)A] \geq eC(\bar{\pi}) \ ,$$

$$x'(-K) \geq -eL(\bar{\pi}) \ ,$$

have no solutions $x' \geq 0$, when $\epsilon > 0$. It follows from a variant of the Farkas Lemma (see Gale [1960], p. 47, th. 2.8) that

(6.9) $$[B - (\lambda+\epsilon)A]p - Kq \leq 0 \ ,$$

(6.10) $$eC(\bar{\pi})p - eL(\bar{\pi})q > 0$$

have, for every $\epsilon > 0$, a solution $\pi = \pi(\epsilon) = \langle p,q\rangle \geq 0$. Since π satisfies (6.10) it cannot be zero; thus we may assume $\pi(\epsilon) \in S$ But S is a compact set, so a sequence $0 < \epsilon_n \to 0$ can be found such that $\pi(\epsilon_n) \to \tilde{\pi} = \langle\tilde{p},\tilde{q}\rangle$. Since all $\pi(\epsilon_n)$ satisfy (6.9), each with its particular ϵ_n , we obtain in the limit

$$(B - \lambda A)\tilde{p} - K\tilde{q} \leq 0 ,$$

which together with (6.7) show that the assertion of Theorem (6.8) holds true.

Theorems (6.7) and (6.8) can be applied in different ways to find a weak equilibrium. For instance, let us suppose that for every $\pi \in S$, the number

$$\lambda(\pi) = \sup\{\lambda \mid \text{there exists an } x = (x',e) \geq 0 \text{ such that } \lambda x\mathcal{A} \leq x\mathcal{B}(\pi)\}$$

is positive, and moreover that

$$\bar{\lambda} = \min\{\lambda(\pi) \mid \pi \in S\}$$

is positive also. If for every $\pi \in S$ with $\lambda(\pi) = \bar{\lambda}$ we can solve $\bar{\lambda}x\mathcal{A} \leq x\mathcal{B}(\pi)$, then a weak equilibrium $x = (x',e) \geq 0$, $\pi \in S$, exists at the level $\bar{\lambda}$.

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