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MONEY AND THE DECENTRALIZATION OF EXCHANGE

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ABSTRACT

A pairwise trading process is formulated subject to conditions of nonnegativity of traders' holdings and quid pro quo. It is shown that:

- (i) There is a centralized procedure that achieves the equilibrium allocation for an arbitrary economy.
- (ii) It is not in general possible to find a decentralized procedure that achieves the equilibrium allocation for an arbitrary economy.
- (iii) In a monetary economy there is a decentralized procedure that achieves the equilibrium allocation.

The usefulness of money is that it allows decentralization of the trading process.

MONEY AND THE DECENTRALIZATION OF EXCHANGE

by

Joseph M. Ostroy and Ross M. Starr*

I. The Process of Trade

When we first inquired into the microeconomic usefulness of money an eminent scholar related an anecdote elaborating the "inconveniences of barter." The gist of the tale was this:

Consider the eminent scholar travelling far from home. He stops at a hotel and asks for lodging for the night. The clerk replies "That will be fifteen dollars (unit of account)." E. S. agrees and extracts from the trunk of his car a copy of his latest textbook.

"Here's a copy of my latest textbook. It sells for fifteen dollars (unit of account)."

"Good, here's your room key. Have a pleasant stay."

The hotel keeper trades the book for fifteen dollars worth of soap. The soap distributor sends it to his supplier for fifteen dollars worth of supplies. The supplier sends the book, as allowance, to his son, studying at a major university where E. S.'s text is used in a large lecture course. The boy trades the book to a student in the course in exchange for fifteen dollars worth of contraband, which he consumes.

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The authors are assistant professors of economics at UCLA and Yale, respectively. Collaboration initiated while Professor Starr was consultant to the Western Management Science Institute, UCLA, whose support and that of its director, Jacob Marschak, is gratefully acknowledged. We are deeply indebted to Gordon Bradley for the formulation of trading rule α . We have also benefitted from discussions with Robert Clower, Peter Howitt, Herbert Scarf and James Tobin. Errors are the authors' responsibility.

That is how trade would take place in a smoothly functioning barter economy. The inconveniences of barter consist in the information and coordination implicit in the story at each stage of trade. Only if the hotel keeper knows that his distributor's supplier will accept textbooks in trade is he likely to accept E. S.'s book in exchange for lodging. To make a substantial number of transactions depend on trading partner's demands, trading partner's trading partner's demands, trading partner's trading partners'...trading partners' demands, would make even the simplest trade depend on the communication of massive amounts of data about who trades with whom, when, and what they want. As long as there is a generally acceptable, universally held medium of exchange no such communication is necessary. Each trade merely consists in the exchange of a desired commodity for the medium of exchange. All one need know about one's trading partner's trading partners is that, like everyone else, they accept the medium of exchange. The informational requirements of barter imply the need for a central coordination of trade; the function of a common medium of exchange is to allow decentralization of the trading process.*

II. A Model of the Trading Process

Consider a standard general equilibrium model of a pure exchange economy. Let there be N commodities indexed by $n = 1, \dots, N$ and a set of J traders, I , indexed by $i = 1, \dots, J$. Let each trader have an

*See [8] for some strategic aspects of this statement.

endowment bundle $b_i \in R_+^N$.* The vector $p \in R_+^N$ is said to be an equilibrium price vector if there is a $d_i \in R_+^N$ so that

(1) $p \cdot d_i = p \cdot b_i$ and d_i is maximal according to i 's preferences among all d_i' such that $p \cdot d_i' = p \cdot b_i$, for each $i \in I$, and,

(2)
$$\sum_i d_i = \sum_i b_i .$$

Conditions (1) and (2) constitute the definition of an equilibrium when all prices are strictly positive, a restriction which will be retained throughout.

Trader i 's desired net trade vector at price vector p is $z_i = d_i - b_i$. In terms of net trades, the equilibrium conditions are

(1')
$$p \cdot z_i = 0 , \text{ each } i \in I , \text{ and}$$

(2')
$$\sum_i z_i = 0 .$$

The complex of initial endowments will be represented by the $J \times N$ matrix $B = \|b_{in}\|$. Similarly, let $D = \|d_{in}\|$ and $Z = \|z_{in}\|$, where $Z = D - B$.

For purposes of this paper, a competitive equilibrium for an exchange economy may be characterized by its price vector, its excess demands, and its initial endowments, (p, Z, B) which are connected by the following, and only the following, restrictions:

* R_+^N is the non-negative orthant of N -dimensional Euclidean space.

$$(U.1) \quad Zp = 0 ,$$

$$(U.2) \quad \sum_i z_{in} = 0 , \quad n = 1, \dots, N , \quad \text{and}$$

$$(U.3) \quad Z \geq -B \quad (\text{the inequality holds entry-wise}).$$

The qualification that z_i is maximal among all allowable excess demands need not concern us since any (p, Z, B) satisfying (U) is a competitive equilibrium for some pattern of preferences.

Condition (U.3) makes the obvious point that a trader cannot plan to supply what he does not have. For any $J \times N$ matrix Z , call the matrix $B_Z = \|b_{in}^Z\|$ minimally sufficient for Z if*

$$[z_i]^- = -b_i , \quad \text{all } i = 1, \dots, J ,$$

Suppose $B = B_Z$; then each trader will want to sell all of his initial endowments in exchange for commodities of which he has none to start. A matrix of initial endowments is minimally sufficient if before trade begins, no fraction of any commodity is in the hands of the trader with whom it will end up when all desired trades have been completed.

Trade is supposed to take place between one trader and another, in pairs. If we adopt the convention that a trader can be a member of only one pair at a time, it requires τ periods for each trader to form a pair with every other trader, once and only once, where

$$\tau = \begin{cases} J-1 & \text{if } J \text{ is even} \\ J & \text{if } J \text{ is odd.} \end{cases}$$

*For a vector $x = (x_1, \dots, x_N)$, $[x]^- \equiv (\min(x_1, 0), \dots, \min(x_N, 0))$ and $[x]^+ \equiv (\max(x_1, 0), \dots, \max(x_N, 0))$.

In the analysis below, we take the order in which pairs meet to be arbitrary. The order of meetings is described by $\{\pi^t\}$, $t = 1, \dots, \tau$, a sequence of permutations of the set I so that:

- (1) $\pi^t(i) = j$ if and only if $\pi^t(j) = i$, all $i \in I$ and $t = 1, \dots, \tau$.
- (2) $\pi^t(i) \neq i$, all i and t if J is even, and $\pi^t(i) = i$ for exactly one t , each $i \in I$, if J is odd.
- (3) $\pi^t(i) \neq \pi^s(i)$, all $i \in I$ and $s \neq t$.

These three conditions guarantee that for all $i, j \in I$, there is precisely one t , $1 \leq t \leq \tau$, such that $\pi^t(i) = j$. Call such a sequence, $\{\pi^t\}$, a round. Any such sequence will do in that none of our results depend on more than (1), (2) and (3).

At the start of the t^{th} period, trader i 's holdings will be represented by w_i^t with $w_i^1 = b_i$. The change in i 's holdings between t and $t+1$, $a_i^t = w_i^{t+1} - w_i^t$, is the trade i performs in period t . The matrix of trades in t is $A^t = \| \| a_{in}^t \| \| = \| \| w_{in}^{t+1} \| \| - \| \| w_{in}^t \| \| = W^{t+1} - W^t$. Trader i 's hitherto unsatisfied excess demands on entering period t are $v_i^t = v_i^1 - \sum_{T=1}^{T=t-1} a_i^T$, with $v_i^1 = z_i$.

Let $\pi^t(i) = j$. Consider the meeting and trade between i and j . Each brings his holdings, w_i^t and w_j^t , to the pair. We have already denoted i 's net receipts from trade as $a_i^t \in \mathbb{R}^N$. Positive entries indicate goods going from j to i and negative entries, goods going from i to j . After trading, i 's holdings will be $w_i^{t+1} = w_i^t + a_i^t$ and j 's will be $w_j^{t+1} = w_j^t + a_j^t$. We place the following three restrictions on a_i^t and a_j^t :

(A.1) Non-negativity of holdings,

$$w_i^t + a_i^t \geq 0, \quad w_j^t + a_j^t \geq 0.$$

(A.2) Conservation of commodities,

$$a_i^t = -a_j^t.$$

(A.3) Quid pro quo,

$$p \cdot a_i^t = 0 = p \cdot a_j^t.$$

Should trades fulfill (A) for all $i, j \in I$ and $t = 1, \dots, T$, we shall say that the sequence of trades is admissible.

The non-negativity requirement, (A.1), says that a trader can at no time have a negative holding of any commodity. A trader cannot deliver to his partner more of a commodity than he currently holds. This may be interpreted as a prohibition on the issue of I.O.U.'s.*

The conservation condition, (A.2), says that in the process of trade, commodities are neither created nor destroyed. Goods delivered are received and vice versa.

The quid pro quo condition, (A.3), requires that in the trade between i and j each delivers to the other goods of equal value. Full payment is made for value received where goods are evaluated at equilibrium prices.

Conditions (A.1) and (A.2) are feasibility restrictions defining bilateral exchange. The origins of (A.3) are behavioral; if A^t were proposed such that $p \cdot a_i^t < 0$ for some i , then i would refuse to trade.**

*In a model including futures markets (A.1) does not prevent a trader from selling futures contracts for goods which do not yet exist, but to which he has title.

**For a rationalization of this assumption in this model, see [8].

Given prices, an order of meetings for the pairs of traders, and an admissible sequence of trades, the outcome can be described as the resulting allocation of goods among traders. At the end of one round the outcome is $W^{\tau+1} = B + \sum_{t=1}^{\tau} A^t$. If prices are equilibrium prices, then we would hope that $W^{\tau+1} = D$, or equivalently, that

$$(E) \quad \sum_{t=1}^{t=\tau} A^t = Z.$$

If trades satisfy (E), we will say that full execution of the competitive equilibrium has been achieved in one round.

For example, let the order of meetings in a four-person economy be $(\overline{12}, \overline{34})$, $(\overline{14}, \overline{23})$, and $(\overline{13}, \overline{24})$ in periods 1, 2, and 3, respectively, where \overline{ij} indicates that i and j form a pair. Then if $p = (1, 1, 1, 1)$ and initial excess demands and endowments are given by

$$Z = \begin{array}{|c|c|c|c|} \hline 1 & -1 & 0 & 0 \\ \hline 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 1 & -1 \\ \hline -1 & 0 & 0 & 1 \\ \hline \end{array}, \quad B = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}$$

trades fulfilling (A) and (E) may be found. If the last period is dropped, either (A) or (E) may be satisfied, but not both.

Will (E) be fulfilled without violating (A)? To state whether this will be achieved, we specify

- (1) what information traders have at each trading opportunity, and
- (2) what is the relation between excess demands and initial endowments.

A trading rule is a function that tells each pair what trade to make. The inputs for the decision are not only what they have on hand-- which defines what they can do--but what they know. Indeed, what they have on hand is just a part of what they know. There is a substantial variety of alternative assumptions and out of these we choose to examine two extreme cases. Define a trading rule as a function

$$\rho(w_i^t, w_j^t | (\cdot)) = (a_i^t, a_j^t)$$

where $\pi^t(i) = j$ and (\cdot) is the information, beyond their current holdings, available to the pair.

A trading rule is said to be decentralized if it can be written as

$$(D) \quad \rho(w_i^t, w_j^t | (v_i^t, v_i^{t-1}, \dots, v_i^1; v_j^t, v_j^{t-1}, \dots, v_j^1))$$

so that the decision as to what i and j trade depends only on their current and past excess demands and holdings and does not depend on the names of the traders. (D) prohibits knowledge of other traders' trading histories and demands except what may be deduced from i and j 's trade with them. In practice, the decentralized rules we will use require even less information, usually only (v_i^t, v_j^t) .

A trading rule is centralized if it cannot be written to satisfy

(D). It will suffice to represent a centralized rule by

$$(C) \quad \rho(w_i^t, w_j^t | (v_1^t, v_1^{t-1}, \dots, v_1^1; v_2^t, \dots; v_J^t, v_J^{t-1}, \dots, v_J^1)) .$$

Here, each pair knows all other pairs' current excess demands and trading histories. With the assumption that the sequence of trading partners is

fixed in advance, it is shown below that (C) provides enough information to guarantee full execution.

The notion of decentralization advanced here is a suitable extension of that concept to the problem under consideration. An economic arrangement is generally described as decentralized if it involves individual agents making decisions based on a fairly small body of universally communicated information (e.g., prices) and on information which the agents themselves may be supposed to possess (e.g., individual tastes and technologies, in this case, the pair's trading history and unsatisfied demands). (D) applies these restrictions to trading rules for pairs of agents.*

From the point of view of the Walras-Arrow-Debreu general equilibrium theory whether B is equal to or greater than B_z is irrelevant. In terms of our problem, whether endowments are minimally sufficient or, on the contrary, admit a great deal of slack, is essential. From this distinction the role of the medium of exchange arises. It is our contention that the role of money as a medium of exchange consists in allowing full execution to be achieved in minimal time by a decentralized rule, whereas, in the absence of money, full execution requires more time, or a centralized rule, or sufficient quantities of non-money commodities.** These contentions are embodied in the following results.

*According to (D), traders know or remember only part of the trading histories they may previously have known and used to make trading decisions. A rule making full use of trading history data (precluded by (D)) would allow traders to make their trading decisions not only on the basis of their previous trades, their current partner's previous trades, but as well on their previous partners' previous trades, ..., etc. With this information each trader would be able to make more precise estimates of the probable excess demands of future partners and this would certainly allow the traders to come closer to full execution. But the record-keeping that such a rule would require, as well as its complexity, appears to be so great as to be unfeasible, or at least very costly.

**The role of inventories of non-money commodities was brought to our attention by R. W. Clower who pointed out Theorem 3 below.

THEOREM 1. There is a trading rule which for all (p, Z, B) satisfying (U) satisfies (A), (C), and (E).

THEOREM 2. There is no trading rule which for all (p, Z, B) satisfying (U) satisfies (A), (D), and (E).

THEOREM 3. Let (p, Z, B) satisfy (U), further let there be a trader $j \in I$ with $b_j \geq \sum_{i \neq j} [z_i]^+$. Then there is a trading rule which satisfies (A), (D), and (E).

THEOREM 4. Let (p, Z, B) satisfy (U), further let there be a commodity m , $1 \leq m \leq N$, so that $p_m b_{im} \geq p \cdot [z_i]^+$ for all $i \in I$. Then there is a trading rule which satisfies (A), (D), and (E).

Theorems 1 and 2 demonstrate the trade-off between full execution and limited information. Together, they say that although there exists a rule which makes (A) and (E) compatible for every competitive equilibrium (p, Z, B) , that rule must be centralized.

Theorems 3 and 4 demonstrate the trade-off between full execution and the presence of inventories. They say that if there is enough slack in initial endowments--either a trader whose endowments are sufficient to fulfill all others' excess demands (Theorem 3), or a commodity such that the value of each trader's holding of the commodity is at least equal to the value of his planned purchases (Theorem 4)--decentralized trading is compatible with full execution. In particular, the commodity m in Theorem 4 is regarded as money and it behaves as money in the trading rule used to prove that theorem.

III. Ideal Coordination in a Barter Economy

This section is devoted to a proof of Theorem 1. The strategy for the proof is:

- (1) to define a chain as an elementary configuration of excess demands,
- (2) to show that there is a centralized rule which achieves full execution of a chain in one round,
- (3) show that any equilibrium configuration of excess demands may be represented as the sum of a finite number of chains, and
- (4) apply the rule developed in (2) to the chains found in (3) and show that the trade consisting of the sum of the trades prescribed for each of the chains is admissible and achieves full execution.

A chain is a list of traders, commodities, prices, and quantities.

Each trader is supposed to have an excess demand for precisely one of the commodities and an excess supply for precisely one of them. There is precisely one trader with an excess supply and precisely one with an excess demand for each commodity in the chain. Supplies equal demands and all are of the same value at the prices quoted. Formally:

Let $\{i_1, i_2, \dots, i_s\}$ be a subset of the traders, $\{n_1, n_2, \dots, n_s\}$ a subset of the commodities, and let δ be a non-negative scalar. Denote by $[(i_1 n_1 i_2 n_2 \dots i_s n_s i_1), p, \delta]$ that configuration of excess demands where

- (1) i_1 has excess supply of n_1 for which i_2 has excess demand, i_2 has excess supply of n_2 for which $i_3 \dots$, and i_s has excess supply of n_s for which i_1 has excess demand, and
- (2) the value of the commodity each individual demands is equal at the prices, p , to the value of the commodity he supplies and is equal to δ .

Call $[(i_1 n_1 i_2 \dots i_s n_s i_1), p, \delta]$ a chain of length s and width δ . For an economy with J traders and N commodities, $s \leq \min(J, N)$. In case $\delta = 0$, we shall say the chain is empty.

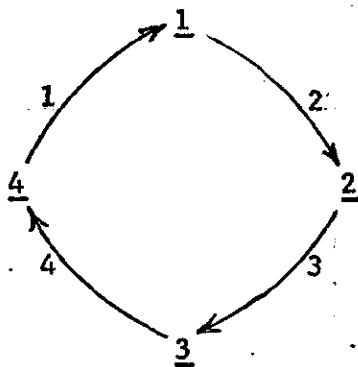
A matrix representation of the chain $[(i_1 n_1 i_2 \dots i_s n_s i_1), p, \delta]$ is a $J \times N$ matrix $E = \|e_{in}\|$ with

$$e_{in} = \begin{cases} -\delta/p^n, & \text{if } (i,n) = (i_r, n_r), \quad r = 1, \dots, s \\ \delta/p^n, & \text{if } (i,n) = (i_r, n_{r-1}), \quad r = 2, \dots, s \text{ or } (i,n) = (i_1, n_s) \\ 0, & \text{otherwise.} \end{cases}$$

A positive entry indicates demand and a negative entry indicates supply of a commodity. By construction,

- (1) there are at most two non-zero entries in each row and column,
- (2) the dot product of p with each row, e_i , is zero, and
- (3) all column sums are zero.

An example of a chain of length 4 and width 1 is given by the numerical illustration of Z in Section II, above. There, $i_r = r$, $r = 1, \dots, 4$ and $n_r = r+1$, $r = 1, 2, 3$ and $n_4 = 1$. This chain may be represented by the graph:



The underlined digits indicate traders, the others refer to commodities. The graph is read, "1 has an excess supply of 2 for which 2 has an excess demand, 2 has an excess supply of 3 for which 3 has an excess demand, 3 has an excess supply of 4 for which 4 has an excess demand, 4 has an excess supply of 1 for which 1 has an excess demand." Arrows point from suppliers to demanders.

It is convenient to have the following definition: i and j are common members of the chain $[(i_1 n_1 i_2 \dots i_s n_s i_1), p, \delta]$ if i and j belong to $\{i_1, \dots, i_s\}$ and $\delta > 0$. A pair of traders are members of the same chain only if the chain is not empty and a chain is not empty if it has at least two members.

As a criterion for full execution, we shall state the obvious:

LEMMA 1. A chain of length one is empty.

A $J \times N$ matrix, E , of excess demands can be represented as a sum of disjoint chains if there are matrices $\{E^k\}$ such that

$$(1) \quad E = \sum_k E^k,$$

(2) E^k represents a chain of width δ , all k , and

(3) for each n , there is at most one k such that $e_{in}^k \neq 0$;
for each i , there is at most one k such that $e_{in}^k \neq 0$.

When a matrix of excess demands can be represented as a sum of disjoint chains, the common members of each chain, E^k , form their own sub-economy and may ignore all other traders in achieving their demands. From the definition, there is:

LEMMA 2. If $\sum_k E^k$ is a sum of disjoint chains, then for any $i, j \in I$, i and j are common members of at most one chain, E^k .

We shall need one more building block before going on to construct the proof. An order of meetings was given for the various pairs of traders which tells in which of the τ periods a pair will form. It is essential to have a listing within periods. By the $t \cdot b^{\text{th}}$ pseudo-period we mean the b^{th} position in the list of pairs forming in the t^{th} period. Pseudo-period $t \cdot b$ precedes $t' \cdot b'$, denoted by $t \cdot b < t' \cdot b'$, if $t < t'$ or if $t = t'$ and $b < b'$ (the ordering of pseudo-periods is lexicographic).

To illustrate, suppose the order of meetings in a four-person economy is $(\overline{12}, \overline{34})$, $(\overline{13}, \overline{24})$ and $(\overline{14}, \overline{23})$ in periods 1, 2, and 3, respectively. Let pseudo-period 1.1 indicate when the pair $\overline{12}$ forms, 1.2 the pair $\overline{34}$, ..., and 3.2 the pair $\overline{23}$.

Let $E^{t \cdot b}$ be the matrix of excess demands at the start of pseudo-period $t \cdot b$ and suppose $E^{t \cdot b}$ can be represented as a sum of disjoint chains, $\sum_k E^k$.

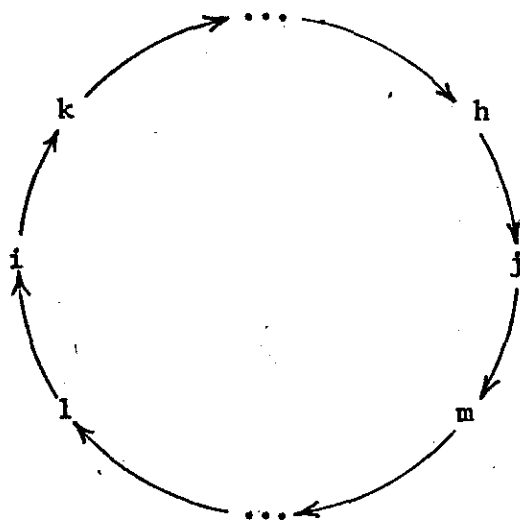
Trading rule α :* Let $A^{t \cdot b} = \|a_{hn}^{t \cdot b}\|$ be the $J \times N$ matrix describing trade between i and j , the trading partners at $t \cdot b$. $A^{t \cdot b}$ is prescribed by

- (1) $\|a_{hn}^{t \cdot b}\| = 0$, if there is no chain, E^k , for which i and j are common members, and
- (2) if i and j are common members of the chain E^k ,

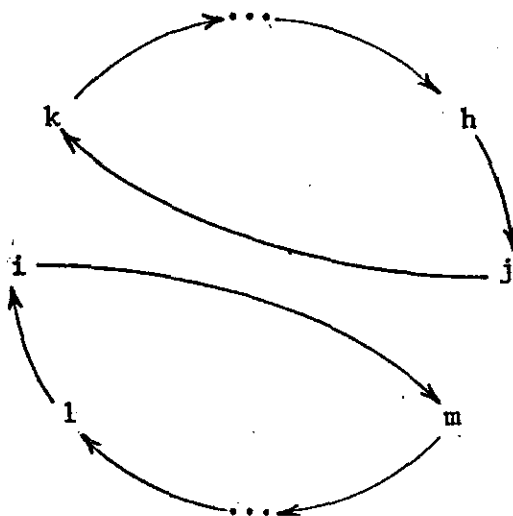
*A graph theoretic treatment of this rule and some of the other subjects of this essay may be found in [1].

$$\|a_{hn}^{t \cdot b}\| = \begin{cases} 0 & , \text{ if } h \neq i \text{ or } j \\ [e_{jn}^k]^- - [e_{in}^k]^- & , \text{ if } h = j \\ [e_{in}^k]^- - [e_{jn}^k]^- & , \text{ if } h = i . \end{cases}$$

Trading rule α says that i and j will not trade unless they are members of the same chain, in which case they will interchange their excess supplies. Representing the rule graphically, consider the chain



i and j meet, exchange excess supplies, and we are left with



two smaller disjoint chains.

Rule α is defined only when the matrix $E^{t \cdot b}$ can be represented as a sum of disjoint chains. The next result shows that the rule can be continued beyond $t \cdot b$.

LEMMA 3. If $E^{t \cdot b}$ can be represented as a sum of disjoint chains and $A^{t \cdot b}$ satisfies trading rule α , then

(1) $E^{t \cdot b} - A^{t \cdot b}$ can be represented as a sum of disjoint chains,

and

(2) if i and j are trading partners at $t \cdot b$, i and j are not common members of any of the disjoint chains representing $E^{t \cdot b} - A^{t \cdot b}$.

Proof. Let $E^{t \cdot b} = \sum_k E^k$. By hypothesis and Lemma 2 there is at most one E^k such that the rows i and j are both non-zero. If there is no E^k , rule α dictates no trade and (1) and (2) obviously hold.

As the other alternative, suppose rows i and j are non-zero for E^1 and E^1 is the matrix representation of the chain $[(i_1 n_1 i_2 \dots i_s n_s i_1), p, \delta]$, where $\delta > 0$ and $i = i_1$, $j = i_r$, and $1 < r \leq s$. The lemma will be proved if we can show that there are matrices E' and E'' with

$$(*) \quad E^{t \cdot b} - A^{t \cdot b} = \sum_{k \neq 1} E^k + E' + E'',$$

where E' is the matrix representation of the chain $[(i_1 n_1 i_{r+1} \dots i_s n_s i_1), p, \delta]$ and E'' is the matrix representation of $[(i_1 n_1 i_2 \dots i_{r-1} n_{r-1} i_r), p, \delta']$ and $\delta' = \delta'' = \delta$ if $2 < r < s$, $\delta' = \delta$ and $\delta'' = 0$ if $r = 2$, and $\delta'' = \delta$ and $\delta' = 0$ if $r = s$. This formally describes the fact that

trade between members of a chain, according to α , does not affect the relations among traders in other disjoint chains but does affect the members of the given chain (of length s), breaking it up into two chains of length $r-1$ and $s-(r-1)$, having no members in common.

To construct the rows of $E' = (e'_1, \dots, e'_j)$, let

$$e'_i = \begin{cases} 0, & \text{if } i \notin \{i_1, i_{r+1}, \dots, i_s\} \\ e_i^1, & \text{if } i \in \{i_{r+1}, \dots, i_s\} \\ e_{i_1}^1 + [e_{i_r}^1]^- - [e_{i_1}^1]^- , & \text{if } i = i_1 . \end{cases}$$

To construct the rows of $E'' = (e''_1, \dots, e''_j)$, let

$$e''_i = \begin{cases} 0, & \text{if } i \notin \{i_2, \dots, i_r\} \\ e_i^1, & \text{if } i \in \{i_2, \dots, i_{r-1}\} \\ e_{i_r}^1 + [e_{i_1}^1]^- - [e_{i_r}^1]^- , & \text{if } i = i_r . \end{cases}$$

According to rule α , the only non-zero rows of $A^{t \cdot b}$ are i_1 and i_r with

$$a_{i_1}^{t \cdot b} = [e_{i_1}^1]^- - [e_{i_r}^1]^- \quad \text{and} \quad a_{i_r}^{t \cdot b} = [e_{i_r}^1]^- - [e_{i_1}^1]^-$$

and it is easily verified that equation (*) holds.

Suppose $1 < r \leq s-1$. Then,

$$\begin{aligned} e'_{i_1} &= e_{i_1}^1 + [e_{i_r}^1]^- - [e_{i_1}^1]^- \\ &= [e_{i_1}^1]^+ + [e_{i_r}^1]^- , \end{aligned}$$

i.e., i has the same positive excess demands as in the chain E^1 but his excess supplies are now those of i_r . The only non-zero rows of E' are $\{i_1, i_{r+1}, \dots, i_s\}$ and, except for row i , whose change has been noted, these rows are identical to the corresponding rows of E^1 . Therefore, E' is the matrix representation of the chain

$$[(i_1^n i_{r+1} \dots i_s^n i_1), p, \delta'] \text{ with } \delta' = \delta.$$

Suppose $r = s$. Then, the only non-zero row of E' is i_1 , and, since the original chain had i_s giving to i_1 ,

$$e_{i_1}^1 = e_{i_1}^1 + [e_{i_s}^1]^- - [e_{i_1}^1]^- = 0.$$

Therefore, E' is the matrix representation of the chain

$$[(i_1^n i_{r+1} \dots i_s^n i_1), p, \delta'] \text{ with } \delta' = 0.$$

A similar argument shows that E'' is the matrix representation of the chain $[(i_1^n i_2 \dots i_{r-1}^n i_r), p, \delta'']$ with $\delta'' = \delta$ if $2 < r \leq s$ and $\delta'' = 0$ if $r = 2$. This means that equation (*) and all of its qualifying conditions have been satisfied and the Lemma is proved. Q.E.D.

We are now ready to prove:

LEMMA 4. Let E be the matrix representation of a chain. Let B fulfill $E \geq -B$. Then trading rule α applied to E will guarantee (A) and (E). (E is regarded here as a matrix of excess demands, B as a matrix of endowments).

Proof. (A.2) and (A.3) follow immediately from the description of α : trade occurs only if a pair of traders are common members of a chain in which case they interchange their excess supplies (fulfilling (A.2)) each of which has value δ (fulfilling (A.3)). To establish (A.1), note that it is true by hypothesis that $E^1 = E^{1 \cdot 1} \geq -B = -W^1$ and by definition $E^2 = \sum_b (E^{1 \cdot b} - A^{1 \cdot b}) = E^1 - A^1 \geq -(W^1 + A^1) = -W^2$. Trading rule α permits $a_{in}^1 < 0$ only if $e_{in}^1 < 0$ and then $a_{in}^1 \geq e_{in}^1 \geq -w_{in}^1$. Therefore, $W^1 + A^1 \geq 0$. Since $E^2 \geq -W^2$ and, by Lemma 3, E^2 is a chain, the same argument shows that trades according to α will guarantee $W^2 + A^2 \geq 0$ and $W^t + A^t \geq 0$ for all t .

To show (E), we shall proceed by induction on the length of the chain. If the length is one, the desired conclusion is immediate. (See Lemma 1.) Suppose it holds for all chains whose lengths are $\leq s-1$. We show it is true for any chain of length s .

Let $t \cdot b$ be the first pseudo-period in which any of the members of the chain $[(i_1 n_1 i_2 \dots i_s n_s i_1), p, \delta]$ meet and let $\pi^{t \cdot b}(i_r) = i_q$ where $r \neq q$ and $1 \leq r, q \leq s$. All possibilities of i_q and i_r 's relation to one another at this first meeting may be divided into the following:

- (a) $r = q-1, 1 \leq r \leq s-1$; or, $r = 1$ and $q = s$; or
- (b) $r \neq q-1, 1 \leq r \leq s-1$; or, $r = 1$ and $q \neq s$.

Application of rule α requires that whether (a) or (b) i_r and i_q will interchange excess supplies. If (a), i_r has an excess supply of the commodity for which i_q has an excess demand. Thus, the result of α will be to satisfy i 's excess demand and since trades satisfy (A) we are left with two chains by length $s-1$ and 1.

If (b), suppose $r > q$. By Lemma 3, the interchange of excess supplies means that i_r is now in the chain $[(i_r n_r i_{q+1} n_{q+1} \dots i_{r-1} n_{r-1} i_r), p, \delta]$ and

and i_q is in the chain $[(i_q^n i_{r+1} \dots i_s^n i_1 \dots i_{q-1}^n i_{q-1}^n i_q), p, \delta]$.
 These chains are disjoint and are of lengths $r-q$ and $s - (r-q-1)$, i.e.,
 the length of each is between 2 and $s-2$. A similar argument leads to
 a similar conclusion if $r < q$.

In either case (a) or (b) since $t \cdot b$ was the first pseudo-period
 in which any of the members of the original chain $\{i_1, \dots, i_s\}$ met and
 since $\{\Pi^{t \cdot b}\}$ allows all traders to meet, the members of the disjoint
 chains, each of whose lengths is $\leq s-1$, have yet to meet. Apply the
 induction hypothesis to obtain (E).

Q.E.D.

Not every matrix of equilibrium excess demands is a chain. However,
 every such matrix can be decomposed to a family of chains. That is,

THEOREM 5.* Let Z be a $J \times N$ matrix such that

(a) $Zp = 0$ for some $p > 0$, and

(b) $\sum_i z_{in} = 0$, $n = 1, \dots, N$.

Then there exist $J \times N$ matrices $\{Z^k\}$, $k = 1, \dots, K$, where $K \leq JN$,
which satisfy

- (1) Z^k is the matrix representation of a chain at prices p ,
all k ,
- (2) $\text{sign } z_{in} = \text{sign } z_{in}^k$, all k , and
- (3) $Z = \sum_k Z^k$.

*Although this theorem acts as a lemma in the proof of Theorem 1, it may be
 of independent interest. It calls attention to the interrelationships among
 traders so that one can begin to discuss issues such as: What is an effi-
 cient decomposition of the matrix of excess demands, Z , into chains?

Proof. Assume $Z \neq 0$, otherwise the theorem is immediate. Since (b) holds, there are traders, i_1 and i_2 , and a commodity n_1 such that $z_{i_1 n_1} < 0$ and $z_{i_2 n_1} > 0$. Since (a) holds, there must be a commodity n_2 such that $z_{i_2 n_2} < 0$. Continue in this manner to order traders and commodities such that $z_{i_s n_s} < 0$ and $z_{i_{s-1} n_{s-1}} > 0$ as long as $i_s \neq i_{r-1}$, $r = 1, \dots, s-1$ and/or $n_s \neq n_{r-1}$, $r = 2, \dots, s-1$. However, (a) and (b) insure that there is some $s \leq \min(J, N)$ with $i_s = i_r$ or $n_s = n_{r-1}$

(i) If $i_s = i_r$, then we have

$$p^{n_r} z_{i_r n_r} < 0, p^{n_r} z_{i_{r+1} n_r} > 0, \dots, p^{n_{s-1}} z_{i_{s-1} n_{s-1}} < 0$$

and

$$p^{n_{s-1}} z_{i_r n_{s-1}} > 0.$$

Let δ be the minimum of the absolute values of these amounts.

Then we may form the chain $[(i_r n_r i_{r+1} \dots i_{s-1} n_{s-1} i_r), p, \delta]$.

(ii) If $n_s = n_{r-1}$, then we have

$$p^{n_r} z_{i_r n_r} < 0, p^{n_r} z_{i_{r+1} n_r} > 0, \dots, p^{n_{r-1}} z_{i_{s-1} n_{r-1}} < 0$$

and

$$p^{n_{r-1}} z_{i_r n_{r-1}} > 0.$$

Let δ be the minimum of the absolute values of these amounts.

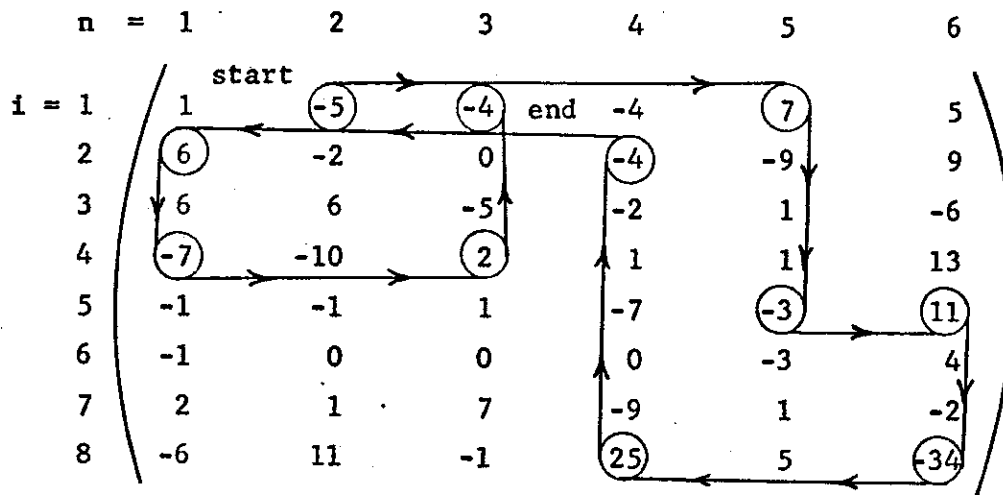
Then we may form the chain $[(i_r n_r i_{r+1} \dots i_{s-1} n_{r-1} i_s), p, \delta]$.

In either case let Z^1 be the matrix representation of the resulting chain. By construction, $\text{sign } z_{in} = z_{in}^1$. Further, $Z - Z^1$ satisfies (a) and (b) and has at least one fewer non-zero entry than Z . To find Z^2 apply to $Z - Z^1$ the procedure used to derive Z^1 . To find Z^l , apply the procedure to $Z - \sum_{k=1}^{l-1} Z^k$. This will guarantee that $\text{sign } z_{in} = \text{sign } z_{in}^l$ and that $Z - \sum_{k=1}^{l-1} Z^k$ has at least one fewer non-zero entry than $Z - \sum_{k=1}^{l-2} Z^k$. Since Z has at most JN non-zero entries, the procedure must terminate--i.e., $Z - \sum_{k=1}^K Z^k = 0$ --for some $K \leq JN$ and the theorem is proved. Q.E.D.

To illustrate the construction in Theorem 5, let $N = 6$, $J = 8$, $p = (1, 1, 1, 1, 1, 1)$. Consider the excess demand matrix

$$\begin{array}{r}
 n = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
 i = 1 \quad \left(\begin{array}{cccccc}
 1 & -5 & -4 & -4 & 7 & 5 \\
 2 & 6 & -2 & 0 & -4 & -9 \\
 3 & 6 & 6 & -5 & -2 & 1 \\
 4 & -7 & -10 & 2 & 1 & 1 \\
 5 & -1 & -1 & 1 & -7 & -3 \\
 6 & -1 & 0 & 0 & 0 & -3 \\
 7 & -2 & 1 & 7 & -9 & 1 \\
 8 & -6 & 11 & -1 & 25 & 5 \\
 \end{array} \right) \\
 \end{array}$$

Following the construction outlined in the proof, choose some non-zero element of the matrix as a starting point, 1,2 for instance. Circled elements indicate chosen by the construction. Directed line segments indicate its progression:



The process moves from the starting point, a negative element, to a positive element in the same row to a negative element in the latter's column to a positive element in the same row...until returning to some row or column previously used, in this case the starting row. The matrix of chosen elements then is

$$\begin{pmatrix} 0 & 0 & -4 & 0 & 7 & 0 \\ 6 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 0 & -34 \end{pmatrix}$$

The final step of the construction is to replace the chosen matrix with a matrix whose signs are the same but the value of whose entries at p are equal:

$$\begin{pmatrix} 0 & 0 & -2 & 0 & 2 & 0 \\ 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 \end{pmatrix}$$

This matrix is the chain Z^1 . $Z - Z^1$ has at least one more zero entry than has Z . In particular element 4,3 of z is non-zero, 4,3 of $Z - Z^1$ is zero.

Theorem 5 permits the extension of trading rule α to any equilibrium matrix of excess demands which may now be viewed as a sum of (not necessarily disjoint) chains. It has already been shown that for any chain Z^k there is a sequence of admissible trades, denoted by $\{A^{kt}\}$, $t = 1, \dots, \tau$ guaranteeing full execution.

Trading rule β : Let $\{A^{kt}\}$ be the trades prescribed by trading rule α for the chain Z^k and let $\sum_k Z^k = Z$ where Z^k and Z satisfy the conditions of Theorem 5. To execute the matrix Z , prescribe trades in period t to be

$$A^t = \sum_k A^{kt}.$$

THEOREM 1. Let (p, Z, B) satisfy (U). Trading rule β satisfies (A), (C), and (E).

Proof. By inspection the information contained in (C) allows β to be implemented. By Lemma 4 $Z^k = \sum_t A^{kt}$. By hypothesis and Theorem 5 $Z = \sum_k Z^k$. Therefore, $Z = \sum_t \sum_k A^{kt}$ and (E) is satisfied. To establish

(A), note that $Z \geq -B$ and $\text{sign } z_{in}^k = z_{in}$. From Lemma 4, it is known that for $\pi^t(i) = j$, trading rule α prescribes $a_i^{kt} = -a_j^{kt}$ and $p \cdot a_i^{kt} = 0$ for all t . Therefore $\sum_k a_i^{kt} = -\sum_k a_j^{kt}$ [(A.2)] and $p \cdot \sum_k a_i^{kt} = 0$ [(A.3)]. It only remains to show (A.1) to prove the theorem.

By hypothesis, Z^k is a chain, $\text{sign } z_{in} = \text{sign } z_{in}^k$ and $Z = \sum_k Z^k$. Since $z_{in} \geq -b_{in}$, there exists b_{in}^k such that $\sum_k b_{in}^k = b_{in}$ and $z_{in}^k \geq -b_{in}^k$. From Lemma 4, we know that the trading rule α applied to (p, Z^k, B^k) satisfies (A.1) and, therefore, so does trading rule β applied to $(p, \sum_k Z^k, \sum_k B^k)$. Q.E.D.

Theorem 1 asserts that a rather complicated procedure, trading rule β , can be used to achieve full execution of a competitive equilibrium in one round of trade in a barter economy, while fulfilling the conditions of quid pro quo, conservation and, non-negativity. The procedure consists of several steps:

(1) The matrix of excess demands is broken up into a family of chains. There may be as many as JN of these (JN is not really a least upper bound. That is more likely to be around $JN - 3$.) Some traders may be members of only a few chains. Others may be members of almost all chains. The breakup into chains is both complex and arbitrary. Without some central direction and notification there is no reason why a trader should have any idea to which chains he is assigned and who the other members are.

(2) An artificial ordering (pseudo-periods) is assigned within each meeting period. This is essential, arbitrary, and could--though it need not--vary with the chain.

(3) Trades are assigned on the basis of (1) and (2). At any meeting a pair engages in trade depending on all chains in that period of which both elements of the pair are members. These may be numerous, generating correspondingly complex trades. Assigned trades may include such non-obvious actions as passing up mutually beneficial trading opportunities (when traders meeting are not members of the same chain) and often exchanging for each other goods which neither party to the trade desires. These latter trades are made in anticipation of future trades for desired goods or future trades for goods which, in future trades, will be traded for desired goods, or future trades for goods which in future trades will be traded for goods which in future trades will be traded....

The trades a trader makes in a given period following trading rule β are based on a complex of the excess demands and supplies of his future trading partners and those of his future trading partners' trading partners, and so forth. In particular, the trade a pair makes when it meets cannot be determined merely from the excess demands and supplies of the elements of the pair. It depends on other traders' excess demands and supplies and what trades they will be making in succeeding periods. Because of this interdependence, trading rule β cannot be decentralized.

IV. The Impossibility of Decentralizing Exchange in a Barter Economy

In this section we shall demonstrate the impossibility of (A), (D) and (E) within a particular class of exchange economies and this will, of course, suffice to prove Theorem 2. It will require that we construct two economies which fulfill the following conditions: (i) under (D) a pair of traders will not be able to determine in which one of the economies they

are, and (ii) the necessary conditions for full execution in the two economies are disjoint for the pair of traders (no matter what the other pairs do).*

It is desirable to choose the "smallest" example for which (i) and (ii) obtain. Simple calculations show that there will have to be at least three commodities and three traders: and it is not difficult to show that the case of three traders is also too small, no matter what the number of commodities. We shall show that the following trading rule satisfies (A), (D) and (E) for all (p, Z, B) fulfilling (U) if $J \leq 3$.

Trading rule γ : For $\pi^t(i) = j$, let $a_i^t = x_i^t + y_i^t$ and $a_j^t = x_j^t + y_j^t$ where

$$(1) \quad x_{in}^t = -x_{jn}^t = \begin{cases} 0 & \text{if } v_{in}^t v_{jn}^t \geq 0 \\ \min(|v_{in}^t|, |v_{jn}^t|) & \text{if } v_{in}^t > 0 \text{ and } v_{jn}^t < 0 \\ -\min(|v_{in}^t|, |v_{jn}^t|) & \text{if } v_{in}^t < 0 \text{ and } v_{jn}^t > 0 \end{cases}$$

and $y_i^t = -y_j^t$ is constructed according to

$$(2) \quad \begin{aligned} [v_i^t - x_i^t]^- < y_i^t < 0 \text{ and } p \cdot (x_i^t + y_i^t) = 0, \text{ if } p \cdot x_i^t > 0 \\ -[v_j^t - x_j^t]^- > y_j^t > 0 \text{ and } p \cdot (x_j^t + y_j^t) = 0, \text{ if } p \cdot x_j^t < 0. \end{aligned}$$

Part (1) of rule γ says, for example, that if i has an excess demand of one unit for a commodity for which j has an excess supply of two units, j is to give one unit to i . Trades are made so as to reduce the partners' excess demands to the maximum extent consistent with the dictum "never change the sign of your excess demand."

*See [8], Proposition 5, for an impossibility result similar to Theorem 2.

In the likely event that $p \cdot x_i^t \neq 0$, say $p \cdot x_i^t > 0$, part (2) says that i may choose from among any commodities for which he still has an excess supply ($v_{in}^t - x_{in}^t < 0$) and give to j any bundle of those commodities whose value will allow him to maintain quid pro quo. The fact that payment is made in any of the selected set of bundles shows that the rule is compatible with (D) and also accounts for its failure to satisfy (E) when $J \geq 4$.*

An important consequence of rule γ is that if $y_{in}^t > 0$, then j may be required to more than fulfill his excess demand for commodity n in order to allow i to maintain quid pro-quo--i.e., j must be willing to convert an excess demand into an excess supply. To formalize the opposite, we shall say that a trading rule satisfies property (P) at date t , if for all i and n ,

$$(P) \quad \begin{aligned} v_{in}^t \geq v_{in}^{t+1} \geq 0, & \text{ if } v_{in}^t \geq 0 \\ v_{in}^t \leq v_{in}^{t+1} \leq 0, & \text{ if } v_{in}^t < 0. \end{aligned}$$

Property (P) specifies that for each commodity trade should not change the sign or increase the absolute value of any excess demand.**

The relation between γ and (P) is

*However, a modification of rule γ , which introduces a convention as to how to redistribute excess supplies, can be made such that (A), (D) and (E) appear to be compatible for all (p, Z, B) in (U) if $J \leq 4$ so that we shall have to admit economies having at least five traders.

**In an atemporal context trades satisfying (P) are called excess-demand-diminishing in [10]. Their properties in the present model of a trading economy are studied in [8].

LEMMA 5. For trading rule γ to satisfy property (P) at date t , it is necessary and sufficient that $a_i^t = x_i^t$ (i.e., $y_i^t = 0$), $i = 1, \dots, J$.

Proof. (Sufficiency) By inspection if $v_{in}^t \geq 0$, then $v_{in}^t \geq x_{in}^t \geq 0$ and $v_{in}^t \geq v_{in}^{t+1} = v_{in}^t - a_{in}^t = v_{in}^t - x_{in}^t \geq 0$. Similarly, if $v_{in}^t < 0$, $v_{in}^t \leq x_{in}^t \leq 0$ and $v_{in}^t \leq v_{in}^{t+1} = v_{in}^t - x_{in}^t \leq 0$.

(Necessity) If $a_i^t \neq x_i^t$, there is a $y_{in}^t \neq 0$, say $y_{in}^t < 0$. From (2) of rule γ , $y_{in}^t < 0$ implies $p \cdot x_i^t > 0$ and $v_{in}^t < x_{in}^t \leq 0$. Since $a_{jn}^t = -a_{in}^t = -(x_{in}^t + y_{in}^t)$. Then either $v_{jn}^t \leq 0$, in which case $v_{jn}^{t+1} = v_{jn}^t + (x_{in}^t + y_{in}^t) = v_{jn}^t + y_{in}^t < v_{jn}^t$; or $v_{jn}^t > 0$, in which case $-x_{in}^t = v_{jn}^t$ and $v_{jn}^{t+1} = v_{jn}^t + (x_{in}^t + y_{in}^t) < 0$. In both cases, (P) is contradicted.

Q.E.D.

Rule γ will be a basic ingredient in the proof of Theorem 2 because it is the essentially unique method of guaranteeing full execution in economies with three traders whenever endowments are minimally sufficient.*

LEMMA 6. For all (p, Z, B) fulfilling (U) such that initial endowments are minimally sufficient ($b_i = -[z_i]^T$, $i = 1, \dots, J$) and $z_i \neq 0$ for at most three traders, trading rule γ is necessary and sufficient for (A) and (E).

* From the remarks of Section II, especially Theorems 3 and 4, it is to be expected that a demonstration of Theorem 2 would involve minimally sufficient endowments. The closer the initial endowments are to being minimally sufficient, the smaller is the set of admissible sequences leading to full execution.

Proof. (Necessity) The result is trivial when all $z_i = 0$ and is readily verified for exactly two $z_i \neq 0$, in which case there is a perfect double coincidence of wants. When there are three non-zero vectors z_i , we can without loss of generality denote them by $i = 1, 2, 3$ and assume that they meet in the sequence $\overline{12}$, $\overline{13}$ and $\overline{23}$ in periods 1, 2, and 3, respectively.

For any trading pair \overline{ij} , the set of trades consistent with rule γ are defined by the restrictions (i) $a_i^t = -a_j^t$, (ii) $p \cdot a_i^t = 0$ and for all n , (iii) $(v_{in}^t - a_{in}^t)(v_{jn}^t - a_{jn}^t) \geq 0$. Condition (iii) says that after i and j perform trades according to γ , there is no commodity such that i has an unsatisfied excess demand for the commodity and j has an unrelieved excess supply. Since (i) and (ii) hold for any admissible trade, γ , is violated if and only if (iii) is not satisfied.

If (iii) is not satisfied by the pair $\overline{12}$ at $t = 1$, there must be at least one n such that (a) $v_{1n}^2 (= v_{1n}^1 - a_{1n}^1) > 0$ and $v_{2n}^2 < 0$ or (b) $v_{1n}^2 < 0$ and $v_{2n}^2 > 0$. Suppose (a); then by the hypothesis that endowments are minimally sufficient and $\sum_i v_{in}^t = 0$, $[v_{1n}^2]^+ = -[v_{2n}^2]^- - [v_{3n}^2]^- > -[v_{3n}^2]^- = -[v_{3n}^1]^- = b_{3n}$. Since $t = 2$ is i 's last opportunity to trade and 3 cannot possibly fulfill all of i 's excess demand for commodity n , (A) and (E) are contradicted. Supposing (b), a similar conclusion follows. This shows that γ is necessary at $t = 1$ and the same arguments may be applied to the trading pair $\overline{13}$ at $t = 2$ to show that if γ (condition (iii), above) is violated (A) and (E) will be contradicted. Finally, at $t = 3$, (E) implies $v_2^3 = -v_3^3$ which requires $a_2^3 = v_2^3 - v_3^3$, exactly what γ prescribes.

(Sufficiency) This may be constructed directly from the above.

However, since γ is necessary and by Theorem 1 there always exist trading rules satisfying (A) and (E), it must be sufficient. Q.E.D.

The informational restrictions imposed by (D) will be shown to imply that any pair of traders will not be able to determine whether they are or are not in a three-trader economy. That is, (D) requires that they perform the same trade independent of what economy they are in. When this is coupled with Lemma 5, the result is a sweeping limitation on the "degrees of freedom" in any successful, decentralized rule. If the rule is to work it must satisfy γ , and with the aid of Lemma 4, we shall see that except in special cases this will mean that it must violate (P). To prove Theorem 2, it only remains to construct economies where if (P) is violated, full execution is precluded.

Throughout the remainder of this section we shall consider only those economies (p, Z, B) satisfying (U) such that (1) there are five traders and four commodities, (2) initial endowments are minimally sufficient so that $b_i = -[z_i]^\top$, $i = 1, \dots, J$ and (3) equilibrium prices are the same for all commodities, e.g., $p = (1, 1, 1, 1)$. Under conditions (1)-(3), any economy fulfilling (U) can be completely described by a 5×4 matrix (of initial excess demands), Z , whose row and column sums are zero. Denote by \mathcal{U} the set of all such Z .

Also throughout, let the sequence of trading partners in a five-trader economy be $(\bar{12}, \bar{35}, \bar{4})$, $(\bar{13}, \bar{24}, \bar{5})$, $(\bar{14}, \bar{25}, \bar{3})$; $(\bar{1}, \bar{23}, \bar{45})$, and $(\bar{15}, \bar{34}, \bar{2})$ in periods 1-5 consecutively.

Let $\mathcal{L} \subset \mathcal{U}$ be the class of economies representable by matrices of the form,

$$Z = \begin{array}{|c|c|c|c|} \hline a & 1-a & -b & b-1 \\ \hline 0 & 0 & e & -e \\ \hline -a & a-1 & b-e & 1-b+e \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$$

where $0 \leq a, b \leq 1$ and $b-e \geq 0$ if $e \geq 0$ and $1-b+e \geq 0$ if $e \leq 0$.*

PROPOSITION 1. If $Z (= V^1) \in \mathcal{L}$, a necessary condition for (A) and (E) is that $v_2^2 = 0$.

Proof. By Lemma 6, γ is necessary. By inspection, its application leads to $v_2^2 = 0$.

Let $\mathcal{M} \subset \mathcal{U}$ be the class of economies representable by matrices of the form,

$$Z = \begin{array}{|c|c|c|c|} \hline a & 1-a & -b & b-1 \\ \hline 0 & 0 & e & -e \\ \hline -c & c-1 & d & 1-d \\ \hline c-a & a-c & b-d-e & d+e-b \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$$

where $0 \leq a, b, c, d \leq 1$ and again $b-e \geq 0$ if $e \geq 0$ and $1-b+e \geq 0$ if $e \leq 0$.

*Traders 4 and 5 are "dummies" having no endowments and thus no demands. Their presence in this and the following constructions is a device for minimizing the difficulty in describing the set of admissible trades satisfying (E). When it is recognized that a dummy is a special case of a trader who has already fulfilled his excess demands before trade begins, it will be seen that the overall results would follow a fortiori if we replaced these dummies by "real" traders.

PROPOSITION 2. A necessary condition for (A), (D) and (E) to hold for all $Z \in \mathcal{U}$ is that if $Z \in \mathcal{M}$, trades must satisfy trading rule γ .

Proof. By inspection and the definition of (D), the pair $\bar{12}$ cannot determine at $t = 1$ whether $Z \in \mathcal{L}$ or \mathcal{M} . The pair must make the same trade in both economies. From Proposition 1, rule γ is necessary if $Z \in \mathcal{L}$. Therefore, if (A), (D) and (E) are to hold for all $Z \in \mathcal{U}$, the pair $\bar{12}$ must adopt it for all $Z \in \mathcal{L} \cup \mathcal{M}$. Since the other traders must be inactive, this shows that rule γ is necessary at $t = 1$. Following γ at $t = 1$, trader 2 enters period 2 with all excess demands and supplies fulfilled.

If at $t = 2$ we ignore the presence of trader 2 whose excess demands are zero, we may apply Lemma 5 to obtain the desired result. It only remains to show that if trader 2 does exchange at any date $t > 1$, (A) and (E) are contradicted. Trader 2 is precluded from exchange at $t = 3$ when he meets the dummy trader 5 and at $t = 5$ when he meets no one. Clearly, if his first non-zero trade is at $t = 4$, (A) and (E) are impossible, since he would then acquire an excess demand which could not relieve in the subsequent period of no trade. This leaves $a_2^2 \neq 0$ as the remaining possibility. There appear to be numerous ways of showing a contradiction. One is: assume e , $(a-c)$ and $(d+e-b)$ all greater than zero. Since $v_2^2 = 0$ and endowments are minimally sufficient $w_2^2 = (0, 0, e, 0)$. Thus, if $a_2^2 \neq 0$, 2 must give up some amount, δ , of commodity 3 to 4, his partner at $t = 2$. Since $w_4^2 = w_4^1 = (a-c, 0, d+e-b, 0)$, the only way for 4 to satisfy quid pro quo is to give up δ of commodity 1, so $0 < \delta \leq \min(e, a-c)$.

Now, $v_{11}^1 = a$ and by non-negativity $v_{11}^2 = v_{11}^1$ and $v_{11}^3 = v_{11}^4 = a-c$ and 1's last opportunity to exchange is at $t = 3$ when he meets trader 4.

However, because 4 gave up δ of commodity 1 at $t = 2$, $v_{11}^4 > a - c > w_{41}^4 = a - c - \delta$. So 1's demand for commodity 1 will remain unfulfilled.

Q.E.D.

Let $\eta \subset \mathcal{U}$ be the class of economies representable by matrices of the form,

$$Z = \begin{array}{|c|c|c|c|} \hline a & 1-a & -b & b-1 \\ \hline 1-a & a & b-1 & -b \\ \hline -c & c-1 & d & 1-d \\ \hline c-1 & -c & 1-d & d \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$$

where $0 \leq a, b, c, d \leq 1$.

Whether $Z \in \mathcal{M}$ or \mathcal{N} , $z_1 (= v_1^1)$ is defined by the parameters (a, b) . The possible changes in trader 1's excess demands and their implications are stated as

PROPOSITION 3. (i) For all $Z \in \mathcal{M} \cup \mathcal{N}$, if z_1 is defined by (a, b) and trades satisfy (A), then v_1^2 is defined by (a, b') , $0 \leq b' \leq 1$; and (ii) if (A), (D), and (E) are to hold for all $Z \in \mathcal{U}$, the trading pair $\overline{13}$ will not be able to determine at $t = 2$ whether $Z \in \mathcal{M}$ or \mathcal{N} .

Proof. (i) For trader 1, whether in \mathcal{M} or \mathcal{N} , at $t = 1$ he can only trade in commodities 3 and 4. If \mathcal{M} , then $a_1^1 = (0, 0, x, -x)$ where $\min(e, 0) \leq x \leq \max(e, 0)$. Since $v_1^2 = v_1^1 - a_1^1$, set $b' = b - x$ and obtain $0 \leq b' \leq 1$. If \mathcal{N} , then $a_1^1 = (0, 0, x, -x)$ where $-b \leq x \leq 1 - b$. Again, set $b' = b - x$ to get $0 \leq b' \leq 1$.

(ii) From Proposition 2, it has been shown that if $Z \in \mathcal{M}$, trader 1 must set $b' = b - e$. Therefore, if $Z \in \mathcal{N}$, and trader 1 changes from b to any \bar{b} , $0 \leq \bar{b} \leq 1$ there exists $Z \in \mathcal{M}$ with e such that $\bar{b} = b - e$. For trader 3, $a_3^1 = 0$ for all $Z \in \mathcal{M} \cup \mathcal{N}$. Since traders can only recall their previous trades and not their previous partners' excess demands, the observations made by 1 and 3 in \mathcal{M} or \mathcal{N} are indistinguishable. Q.E.D.

We have shown that the classes of economies \mathcal{M} and \mathcal{N} will be indistinguishable to $\bar{13}$ at $t = 2$ and that if $Z \in \mathcal{M}$, rule γ is necessary. We now show that if $Z \in \mathcal{N}$, trades must satisfy (P) at $t = 2$.

First, note that if $Z \in \mathcal{N}$ is defined by the parameters (a, b, c, d) , then since trade at $t = 1$ takes place only between traders 1 and 2 redistributing their excess supplies of commodities 3 and 4, V^2 will belong to the class \mathcal{N} with $w_i^2 = -[v_i^2]^-$ and will be defined by the parameters (a, b', c, d) . For any $V^2 \in \mathcal{N}$, we shall examine some necessary conditions for (A) and (E).

Let r_n be the absolute value of the amount of commodity n exchanged at $t = 2$ between $\bar{13}$ and let s_n be analogously defined for the pair $\bar{24}$. Trader 3 cannot give up more of commodity 1 than $w_{31}^2 = w_{31}^1 = c$. If $c > a$ and $c \geq r_1 > a$, trader 3 will have on hand at the start of the fourth period only $c - r_1$ of commodity 1 to meet trader 2's excess demand which, at the very least, is $1 - a - (1 - c) = c - a$. Since $c \geq r_1 > a$, and $t = 4$ is 2's last trading opportunity, (A) and (E) are contradicted. Therefore, $r_1 \leq \min(a, c)$ which implies that trader 1 should not take more of commodity 1 than will fulfill his excess demand. Reasoning along the same lines yields $s_2 \leq \min(a, c)$; $r_2, s_1 \leq \min(1 - a, 1 - c)$; $r_3, s_4 \leq \min(b', d)$; and

$r_4, s_3 \leq \min(1-b', 1-d)$. Therefore, we have

PROPOSITION 4. For any $Z \in \mathcal{N}$, a necessary condition for (A) and (E) is that trades satisfy (P) at $t = 2$.

Proof. The above inequalities on r_n and s_n prohibit a positive excess demand from being increased or converted into a negative. Also shown above was that if $Z \in \mathcal{N}$, endowments at $t = 2$ would be minimally sufficient for excess demands which makes it impossible by (A) for a negative excess demand to be increased (in absolute value) or converted into a positive. These conditions define (P). Q.E.D.

So far, it has been established that trades must satisfy rule γ in every period if $Z \in \mathcal{M}$ and trades must satisfy (P) at $t = 2$ if $Z \in \mathcal{N}$. It only remains to show that these two necessary conditions are disjoint. For all $Z \in \mathcal{M} \cup \mathcal{N}$, Proposition 3 says that at $t = 2$ the excess demands and minimally sufficient endowments of $\bar{13}$ will be of the same form. To this form, apply the results of Lemma 5 to exhibit the necessary and sufficient conditions under which rule γ and property (P) will overlap. They are,

$$(*) \quad \min(a, c) + \min(1-a, 1-c) = \min(b', d) + \min(1-b', 1-d)$$

which describes that coincidence where excess demands can be reduced to the maximum extent (satisfying part (1) of γ) without any commodities changing hands to maintain quid pro quo (not violating (P)). It is now shown that such an exact state cannot always obtain.

PROPOSITION 5. There exists no trading rule satisfying (A) such that for all $Z \in \mathcal{M} \cup \mathcal{N}$, the application of trading rule γ at $t = 2$ will exhibit property (P).

Proof. Suppose the contrary. Then, it must be true that for all $0 \leq a, b, c, d \leq 1$, there exists b' , $0 \leq b' \leq 1$, satisfying (*) above. But, for all b' , $0 \leq b' \leq 1$,

$$\min(b', d) + \min(1-b', 1-d) \geq \min(d, 1-d) ;$$

and, for any δ , $0 \leq \delta \leq 1$, there exist values of a and c , $0 \leq a, c \leq 1$, such that

$$\delta \geq \min(a, c) + \min(1-a, 1-c) .$$

Since a , c and d are independent, they may be chosen such that $\min(d, 1-d) > \delta$.

Q.E.D.

Propositions 1-5 are now combined to prove Theorem 2.

Proof. From Proposition 3, the trading pair $\overline{13}$ will not be able to determine at $t = 2$ whether $Z \in \mathcal{M}$ or \mathcal{N} . From Proposition 2, if \mathcal{M} , trading rule γ must be followed; and, from Proposition 4, if \mathcal{N} , the trading rule must satisfy (P). From Proposition 5, there will exist economies in both \mathcal{M} and \mathcal{N} in which the application of rule γ will fail to satisfy (P). Since (D) requires that the same trading decision must be made in all such cases, whichever of the mutually exclusive alternatives-- γ or (P)--is chosen, it will fail to achieve (A) and (E) in either \mathcal{M} or \mathcal{N} .

Q.E.D.

Bradley [1] has found an example showing that (A), (D) and (E) are inconsistent for $J \geq 5$ if the trading rule satisfies the additional assumption that it is insensitive to a relabelling of commodities--i.e., it is only amounts of commodities, not their names, which matter. Theorem 4 with its monetary trading rule, below, contradicts Bradley's assumption. The money commodity is definitely in an asymmetric position compared to all other commodities. This could be interpreted as showing the advantages of having at least one commonly recognizable commodity to be used by all traders in settling their accounts. However, suppose, as we have in (D), that all commodities are recognizable. Theorem 2 shows that even when they are, this is not sufficient to guarantee full execution.

In the proofs of Propositions 1-5, care has been taken to show that the results hold when the parameters satisfy inequality rather than equality conditions. This suggests that Theorem 2 holds for a positive fraction of all $Z \in \mathcal{U}$. It would be interesting to inquire whether the fraction of economies for which there exists no decentralized rule leading to full execution increases with the number of individuals and commodities.

V. Money and Commodity Inventories

The difficulties of barter exchange may be traced to an over-determinacy in the demand for commodities. They are wanted both for final consumption and as a means of payment. Without a double coincidence of wants, these two functions cannot be easily satisfied and each unit of a commodity supplied may have to serve both as a means of payment to one's present trading partner and as an exchange which will satisfy the demands of one's present trading partner's future partners, ..., etc.

Assuming that information beyond that given by (D) is simply not available, some slack in initial endowments is essential to guarantee full execution. When endowments are large, relative to what is minimally sufficient, there are trading rules permitting these two conflicting demands for commodities to be separated in a straightforward, decentralized manner.

The very presence of a medium of exchange implies a slackness--i.e., that $B > B_Z$. Imagine a money economy, with m the money commodity, in a position of equilibrium where traders have no desire to increase or decrease their money balances ($z_{im} = 0$). If endowments were minimally sufficient, this would imply $b_{im} = 0$, hardly a description of a money economy.

Call the difference between B and B_Z the quantities of inventories on hand. Although their levels may remain constant on the average, fluctuations in their levels from period to period should be regarded as a valuable input in the process of exchange. (Of course, inventories cannot fluctuate if their initial levels are zero.) This is the view taken by R. W. Clower [2], who has emphasized that it is not only inventories of money, but of all commodities, which facilitates exchange. In this section, Theorems 3 and 4 support this contention.

Suppose trader 1's initial endowments are so large that he may fulfill all other traders' excess demands-- $b_1 \geq \sum_{i \neq 1} [z_i]^+$. Consider

Trading rule σ : For $\pi^t(i) = j$, let

$$a_i^t = 0 = -a_j^t \quad \text{if } i, j \neq 1$$

$$a_i^t = v_i^t = -a_j^t \quad \text{if } i \neq j = 1.$$

It is clear that we have,

THEOREM 3. If $b_1 \geq \sum_{i \neq 1} [z_i]^+$, trading rule σ satisfies (A), (D) and (E).

Trading rule σ specifies that trader 1 act as a clearinghouse. He is the hub of commodity exchange. Other traders look to him and only to him, to purchase their demands and sell their supplies (cf. [10], lemma 1, though [10] lacks our nonnegativity constraint).

As an alternative to the central distributor scheme, suppose there is a commodity, m , such that the value of each trader's endowment of it is at least as large as the value of his desired purchases--

$p_m b_{im} \geq p \cdot [z_i]^+$, $i \in I$. Consider

Trading rule μ : For $\pi^t(i) = j$, let

$$a_i^t = x_i^t + y_i^t \quad \text{and} \quad a_j^t = x_j^t + y_j^t$$

where,

$$(1) \quad x_{in}^t = -x_{jn}^t = \begin{cases} 0 & \text{if } v_{in}^t v_{jn}^t \geq 0 \\ \min(|v_{in}^t|, |v_{jn}^t|) & \text{if } v_{in}^t > 0 \text{ and } v_{jn}^t < 0 \\ -\min(|v_{in}^t|, |v_{jn}^t|) & \text{if } v_{in}^t < 0 \text{ and } v_{jn}^t > 0 \end{cases}$$

and

$$(2) \quad y_{in}^t = -y_{jn}^t = \begin{cases} 0 & \text{if } n \neq m \\ q \text{ where } p \cdot x_i^t + p_m q = 0 & \text{if } n = m. \end{cases}$$

Trading rule μ assigns a unique, asymmetrical role to the commodity m . Our concept of the money commodity m , is well-defined. It applies to a specific item used in a specific way in a specific trading rule. Although condition (2) is identical to the conventional budget constraint, it is clear from (1) that when rule μ is followed supplies of other commodities, as compared to m , money, are not equally useful as payment for purchases.*

The rule says that when traders i and j meet they should make trades that relieve one of excess supply and reduce the other's excess demands. Any failure of quid pro quo should be made up in money. This rule is strikingly similar to the way trade actually takes place in monetary economies. Rule μ should be compared with rule γ of Section IV. Parts (1) of each rule are identical. Part (2) of γ says that payment can be made in any assortment of commodities in which the trader has an excess supply whereas part (2) of μ narrows the choice of means of payment to commodity m without, however, imposing the restriction that m be in excess supply before it is given up.

THEOREM 4. If $p_m b_{im} \geq p \cdot [z_i]^+$, trading rule μ satisfies (A), (D) and (E).

Proof. The idea of the proof is simple. If full execution is not achieved then there is some trader with an unfulfilled excess demand and some corresponding trader with an unrelieved excess supply. They must have violated the rule when they met or afterward. The rule requires only a knowledge of the pairs' excess demands and the convention that payment be made in commodity m . It satisfies (D).

*Cf. Clower [3]. Trading rule μ appears also in [10].

Ignoring the monetary exchanges, y_i^t , entirely, it will be shown that the rule consisting only of (1), above, satisfies (A.1), (A.2) and (E) and then it will be shown that (2) never contradicts the trades prescribed in (1).

According to (1), if $v_{in}^1 \geq 0$, $v_{in}^t \geq x_{in}^t \geq 0$ and if $v_{in}^1 < 0$, $v_{in}^t \leq x_{in}^t \leq 0$, all t . Since $v^t \geq -w^t$ (A.1) is satisfied. (A.2) holds by definition. There is also the implication that if

$$(3) \quad \begin{aligned} v_{in}^1 \geq 0, & \text{ then } v_{in}^t \geq v_{in}^{t'} \geq 0 \text{ and if} \\ v_{in}^1 < 0, & \text{ then } v_{in}^t \leq v_{in}^{t'} \leq 0 \end{aligned}$$

for $t' > t$.

Suppose $v_i^{\tau+1} \neq 0$. Then, there is at least one commodity, n , and two traders, i and j , such that $v_{in}^{\tau+1} > 0$ and $v_{jn}^{\tau+1} < 0$. By hypothesis, there is exactly one period, s , when $\pi^s(i) = j$. By (3), $v_{in}^s \geq v_{in}^{\tau+1}$ and $v_{jn}^s \leq v_{jn}^{\tau+1}$. If the traders followed (1), either $|v_{jn}^s| \geq |v_{in}^s|$, in which case $v_{in}^{s+1} = 0$ or $|v_{jn}^s| < |v_{in}^s|$, in which case $v_{jn}^{s+1} = 0$. They cannot both be non-zero, contradicting the assertion that $v_i^{\tau+1} \neq 0$.

By hypothesis

$$p_m b_{im} = p_m w_{im}^1 \geq p \cdot [z_i]^+ = p \cdot [v_i^1]^+$$

and from (3)

$$[v_i^1]^+ \geq \sum_{t=1}^{t=s} [x_i^t]^+, \quad s = 1, \dots, \tau.$$

Therefore

$$\begin{aligned}
 p_m w_{im}^s &= p_m w_{im}^1 - p \cdot \left(\sum_{t=1}^{t=s-1} x_i^t \right) \geq p_m w_{im}^1 - p \cdot \left(\sum_{t=1}^{t=s-1} [x_i^t]^+ \right) \\
 &\geq p_m w_{im}^1 - p \cdot [v_i^1]^+ \geq 0 .
 \end{aligned}$$

(A-3) follows from trading rule μ , part (2).

Q.E.D.

Between the two trading rules, σ and μ , and the slack conditions they impose, can we point to one as more efficient than the other? Within the confines of the model, we cannot. Nevertheless, it should be recognized that these qualifying conditions on initial endowments are fundamentally different. One rule requires that the slack be "real" and concentrated in a single agent; the other that the slack be in value, concentrated in a single commodity.

With trading rule σ , trader 1 acts as a clearinghouse for excess supplies and demands. But there is a crucial difficulty. Trader 1 cannot perform his function without having, to start, substantial quantities of his own commodities. There is no way of getting around this requirement if trades are to satisfy (A) and (E). With only a modest endowment to start, trader 1 could act as the unique redistributor by accepting all commodities offered for sale and filling purchase orders from previously received supplies. Clearly, if quid pro quo is to be maintained in every exchange, full execution will require many more periods than are contained on one round.

In trading rule μ , the amounts b_{im} are unimportant as long as they are positive and p_m is sufficiently large. These features are peculiar

to commodity m only. In trading rule μ , the proof of Theorem 4 shows that part (1) of the rule almost suffices to guarantee (A), (D) and (E). It fails only in not satisfying quid pro quo. The sole purpose of trade in commodity m is to establish a counting device to insure that the sum of additions to and subtractions from the value of one's holdings during the course of trade is zero. That the device is embodied in a tangible commodity is clearly inessential.*

VI. Conclusion

Jevons writes that difficulties of barter arise because of the absence of double coincidence of wants:**

The earliest form of exchange must have consisted in giving what was not wanted directly for that which was wanted. This simple traffic we call barter..., and distinguish it from sale and purchase in which one of the articles exchanged is intended to be held only for a short time until it is parted with in a second act of exchange. The object which thus temporarily intervenes in sale and purchase is money.

The first difficulty of barter is to find two persons whose disposable possessions mutually suit each others wants. There may be many people wanting, and many possessed those things wanted; but to allow an act of barter, there must be a double coincidence which will rarely happen. [5, p. 3]

Monetary exchange requires only single coincidence: a demander of a commodity encountering a supplier of the commodity and paying the supplier in money. Why should a trader refuse to exchange one excess supply for another? Why is double coincidence regarded as necessary for nonmonetary exchange?

*If it were, it is hard to see how we could have advanced from the abacus to the pencil-and-paper method of doing sums. See [8].

**Other classical writers including Smith and Mill make virtually the same argument.

Jevons was concerned with decentralizing the process of exchange. In the presence of double coincidence, barter exchange can take place in a fully decentralized way. Given double coincidence, achievement of an equilibrium distribution of goods requires traders to consult only their own and their current trading partner's excess supplies and demands and then trade so as to yield up their excess supplies and fulfill their excess demands. In the absence of double coincidence such a trading rule will achieve an inefficient allocation far from competitive equilibrium ([8], [10]). Monetary exchange, on the contrary, Jevons contends achieves equilibrium through decentralized trading.

Trading arrangements that achieve full execution (E) at equilibrium prices in an economy where trade is restricted by non-negativity, conservation, and quid pro quo (A) exist to the extent:

- (i) There is a centralized procedure that achieves (E) for an arbitrary economy (Theorem 1).
- (ii) It is not in general possible to find a decentralized procedure that achieves (E) for an arbitrary economy (Theorem 2).
- (iii) In a monetary economy there is a decentralized procedure that achieves (E) (Theorem 4).

The usefulness of money is that it allows the trading process to be decentralized.

In order to function as a medium of exchange, money must have a positive value in exchange, a positive price. It has been suggested by some that the major aim in the integration of monetary and value theory is to construct a model in which money has a positive equilibrium price. It seems fairly clear that such a conclusion cannot be obtained from the model of an exchange

economy without additional assumptions on the backing or utility of money (Cf. Hahn [4], Kurz [6], Marschak [7], Sontheimer [9], Starr [11]). The ad hoc nature of these constructs suggest that the problem of integrating monetary and value theory is not equivalent to the demonstration of a positive price for money.

This paper represents another tack. For a commodity with positive value, are there any conditions under which it could be usefully employed as a medium of exchange? With the aid of an explicit model of the trading process, we have been able to state this question more precisely and to obtain an affirmative answer.

Structure of the present model suggests approaches for further research. In actual economies, unlike the model, trade does not take place between each agent and every other agent but rather between each agent and a small group of other agents. A suitable next step is to investigate a model with retailers, wholesalers, and specialized trade. Further, trade in actual economies does not take place in a fixed order given in advance but rather in an order which is itself a matter of choice. In a model which allowed for choice of trading order one should be able to weaken the condition in Theorem 4 requiring large initial endowments of the monetary commodity.

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APPENDIX

AN ALTERNATIVE PROOF OF LEMMA 4

The following proof of Lemma 4 may make the workings of trading rule α a bit clearer. Starting from Lemmas 1, 2, 3, we add Lemma 7 and use these as the basis of an alternate proof of Lemma 4.

LEMMA 7. Let $E^{t \cdot b}$ be representable as a sum of disjoint chains. If i , and j are not members of the same chain in pseudo-period $t \cdot b$, and if trading rule α is applied, then i , j will not be members of the same chain at any $t' \cdot b' > t \cdot b$.

Proof. Notice that the lemma does not require i and j to be trading partners at $t \cdot b$. For this reason, it is sufficient to show that the lemma is true for the pseudo-period immediately following $t \cdot b$ --call it $t' \cdot b'$ --since the argument for this case can be continued.

Let $S_i \subseteq I$ be the set of traders who are common members of the chain to which i belongs at $t \cdot b$ and let T_i be the set of traders who are common members of the chain to which i belongs at $t' \cdot b'$. By definition, $i \notin S_i$ or T_i . Let S_j and T_j be similarly constructed for trader j .

The lemma will be proved if it can be shown that

- (1) T_i and/or T_j is the empty set, or
- (2) for T_i and T_j not empty, $T_i \cap T_j$ is empty.

Suppose S_i is empty; then, T_i is empty and (1) obtains. Similarly for S_j empty and T_j .

Suppose S_i and S_j are not empty. By hypothesis, S_i and S_j are disjoint. Let i_1 and i_2 be the pair which forms at $t \cdot b$. The following exhausts the possible cases when S_i and S_j are non-empty and disjoint:

- (a) $i_1, i_2 \in S_i$ or $i_1, i_2 \in S_j$,
- (b) $i_1 \in S_i$ and $i_2 \in I - S_i$ or $i_2 \in S_i$ and $i_1 \in I - S_i$, and
- (c) $i_1, i_2 \in S_i \cup S_j$.

In (a) suppose $i_1, i_2 \in S_i$. Lemma 3 shows that $T_i \subseteq S_i$ and $T_j = S_j$ and (2) follows. A similar argument holds for $i_1, i_2 \in S_j$.

In (b), if $i_1 \in S_i$ and $i_2 \in I - S_i$, trading rule α prohibits trade, so that $T_i = S_i$ and $T_j = S_j$ and (2) follows. Similarly for $i_2 \in S_i$ and $i_1 \in I - S_i$.

In (c), neither i_1 nor i_2 have any non-zero excess demands in common with i and j . Whether or not trade occurs, $S_i = T_i$ and $S_j = T_j$ so (2) follows again.

We are now ready to prove:

LEMMA 4. Let E be the matrix representation of a chain. Let B fulfill $E \geq -B$. Then trading rule α applied to E will guarantee (A) and (E). (E is regarded here as a matrix of excess demands, B as a matrix of endowments).

Proof. (A.2) and (A.3) follow immediately from the description of α : trade occurs only if a pair of traders are common members of a chain in which case they interchange their excess supplies (fulfilling (A.2)) each of which has value δ (fulfilling (A.3)). To establish (A.1), note that it is true by hypothesis that $E^1 = E^{1 \cdot 1} \geq -B = -W^1$ and by definition $E^2 = \sum_b (E^{1 \cdot b} - A^{1 \cdot b}) = E^1 - A^1 \geq -(W^1 + A^1) = -W^2$. Trading rule α permits $a_{in}^1 < 0$ only if $e_{in}^1 < 0$ and then $a_{in}^1 \geq e_{in}^1 \geq -w_{in}^1$. Therefore, $W^1 + A^1 \geq 0$. Since $E^2 \geq -W^2$ and, by Lemma 3, E^2 is a chain, the same argument shows that trades according to α will guarantee $W^2 + A^2 \geq 0$ and $W^t + A^t \geq 0$ for all t .

To show (E), suppose the contrary. Then, since Lemma 3 says that $E^{t \cdot b}$ is always a sum of disjoint chains, there are at least two traders, i and j , who are common members of a chain. From the definition of a round, there was exactly one pseudo-period, $t \cdot b$, in which i and j were trading partners. From Lemmas 2 and 3, i and j were common members of at most one chain, E^k , where $E^{t \cdot b} = \sum_k E^k$. Now, either

(1) they were common members of no chain or, by Lemma 3,

(2) after trade at $t \cdot b$ they were common members of no chain.

In both cases, Lemma 7 guarantees that they are not common members of any chain from the following period onward. This contradicts the assertion that i and j are members of the same chain. Therefore excess demands at the end of one round must be represented by a sum of chains, each of length one, and by Lemma 1, (E) must be fulfilled.