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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART VI

THE RATE OF INTEREST, NONCOOPERATIVE EQUILIBRIUM AND BANKRUPTCY

Lloyd Shapley and Martin Shubik

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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART VI

THE RATE OF INTEREST, NONCOOPERATIVE EQUILIBRIUM AND BANKRUPTCY*

by

Lloyd Shapley and Martin Shubik

1. INTRODUCTION

In this paper a simple two stage example is examined in detail to illustrate the many problems of modeling that exist in attempting to construct a multistage strategic model of a trading economy with fiat money which also has a money market for borrowing and lending.

As even the simplest model appears to call for a great amount of detail considerable stress is laid upon a series of simplifications.

We assume that there are two types of traders. There are n traders of each type. For simplicity we assume that there is only one consumer commodity and fiat money.

Traders of different types are differentiated by their utility functions and endowments. Endowments include money and quantities of the consumer commodity. There is only one type of money and it probably is useful to think of it as having a physical existence in some form such as "blue chips."

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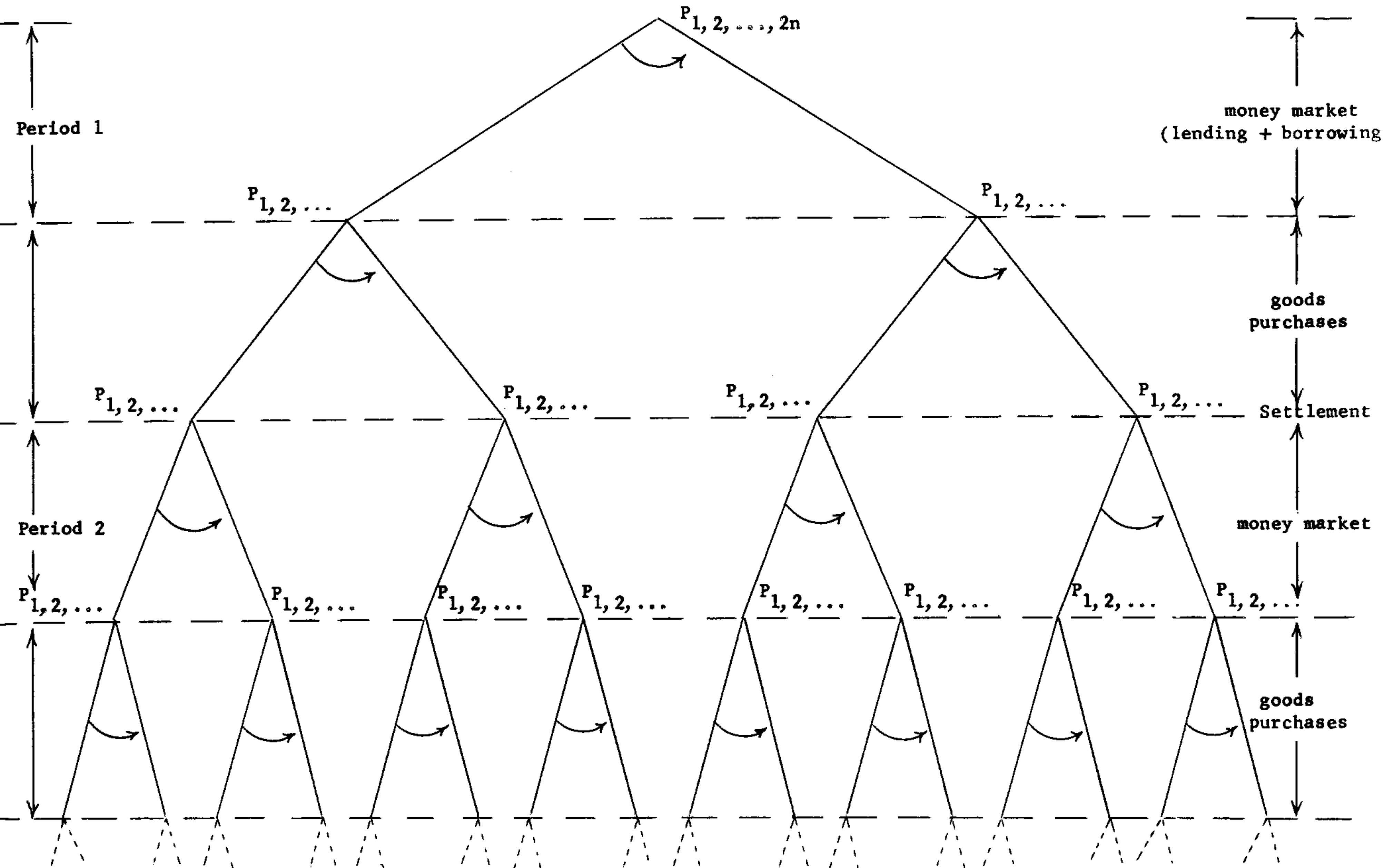
During the first period the ownership of the supply of consumer good is $(0, B)$ and during the second period ownership is given by $(A, 0)$. These signify that in the first period the traders of the first type own no consumer good, while traders of the second type own B units each. Similarly in the second period traders of the first type own A each and traders of the second type own nothing. It is assumed that the consumer good cannot be inventoried; it lasts only for one period. Money on the other hand is durable and can be stored at no cost. In the example the initial distribution of money is given by $(M, 0)$.

A simple artifact in modeling is employed which enables us to construct a nontrivial example using only one consumer good. We force the monetization of all trade. This introduces a cash flow problem and has a reasonable interpretation.

We imagine that an "owner" of resources during a period does not obtain them directly. They are deposited at the market which sells them. The original owner must buy back any goods he wishes to consume. He receives the money income from the sale of the goods he owns at the start of the next period, or he is paid ahead of sale by the marketing organization.

In a modern economy with individuals trading in many commodities, with the exception of labor-leisure it appears to be a reasonable approximation to assume that most economic activities go through the markets. Even the modern farmer may buy his milk. The one commodity modeled here may be regarded as a composite or aggregate. By making this simplification the allocation and pricing problem during each period is avoided however the basic contrast between "the real sector" and the monetary allocation remains.



$P_{1, 2, \ldots, 2n}$
Period 1
money market
(lending + borrowing)
$P_{1, 2, \ldots}$
goods
purchases
Settlement
Period 2
money market
goods
purchases

FIGURE 1

2. THE EXTENSIVE FORM OF THE TWO STAGE MARKET

The model we envision calls for two moves by each trader as well as a certain amount of bookkeeping which we assume is supplied by the market.

A detailed description of information conditions and money market conditions together with a justification of why a two stage model is an adequate model is called for. The last point is examined first. A more fully acceptable model would have a possibly infinite sequence of periods thus money would always be a store of value in the sense that there would always be a store of value in the sense that there would always be "a next trading period." If the model is of finite length then the money at the end of the last period is of no further worth. However even with two periods if all trade must be carried through with money its value will be supported until the end of the second period.

We could support the value of money at the end of a series of trades of finite length by merely attaching a special value function to the end of the utility function. This is methodologically sound in the sense that it may be regarded as a way for truncating an infinite process. The utility of the money at the end of a finite play may stand for an assigned worth of the future markets excluded from the model. Where the worth is a function of the amount of money in the trader's possession at the end of the last period.

If we wish to describe the strategic possibilities of the traders it is necessary to specify in detail the information conditions and the process of trade. Figure 1 shows the game tree. The details of this diagram are explained below.

Figure 1 is a modification of the usual game tree diagram. It is drawn in such a manner that it can represent any number of traders in this type of game. The labels at each node indicate that all of the 2n players are required to move simultaneously. This means that they are not informed of each other's actions until after they have all moved. The first move involves the borrowing and lending of money. The precise mechanism of this process is described later. After borrowing and lending has taken place, all traders are informed of the state of the economy and all simultaneously decide upon the amount of money they will spend on the consumer good. The first period ends, then the second period starts with the owners of resources receiving their incomes and with a repayment of debts. These acts are automatic and do not appear as moves in the game tree. The arrows between the branches of the tree indicate that there is an infinite number of alternatives from which to choose. The amounts offered for purchasing, borrowing or lending may be regarded as continuous quantities. For example the branch to the far left in the move calling for the purchase of goods stands for the situation in which all traders have decided to offer no money for goods.

2.1. The Money Market

There are several different ways in which the mechanism of a money market can be modeled. We can conceive of face-to-face arrangements between pairs of borrowers and lenders with the pledging of assets, the taking out of insurance and so forth. These are all real and important phenomena and need a separate discussion to do them justice. Here the simplest form of highly anonymous money market is constructed.

We assume that there exists a banking mechanism which merely acts as a broker or provides a market for present and (one-period) future money. The banking system is a fiction in the sense that it is costless to operate and has no strategies of its own. It merely carries out the paper work and gathering and distribution of funds. There is no banker's money. In this society people deal with "blue chips," a lender gives these chips to the bank which in turn gives them to the borrower.

A lender decides upon an amount of money he wishes to lend and gives this amount of money to the bank or money market. He does not know for certain what it will yield. A borrower decides upon how much one-period future money he will commit himself to pay back in return for a loan this period. He too does not know what the rates will be until all borrowers and lenders have acted. The bank rations the loans in proportion to the bids of the borrowers. In the subsequent period the bank (which we may envision as holding the nonnegotiable promissory notes of the borrowers) collects the money due and pays it out to the lenders in proportion to their loans.

Although it is assumed that the borrowers are utterly honest (i.e. there is no moral hazard in the sense that they do not lie or break the rules of the game); it is possible that a borrower will not have sufficient money to honor his pledge. This means that in order to completely define the rules of operation of this economy we must specify bankruptcy rules. The bankruptcy rules are discussed in Section 3.

3. A TWO-PERIOD GAME WITH A MONEY MARKET

In this section a formal description of the model to be analyzed and solved in Section 4 is presented.

Let M_t and N_t be the money holdings of a trader of type 1 and 2 respectively at the start of the t^{th} time period.

$$M_1 = M > 0 \quad \text{and} \quad N_1 = 0 .$$

There are n traders of each type. Traders of each type have utility functions of the form

$$(1) \quad \begin{aligned} \phi_i &= \varphi(q_{i,1}) + \rho_1 \varphi(q_{i,2}) \qquad \text{for } i = 1, 2, \ldots, n \\ \Psi_j &= \psi(r_{j,1}) + \rho_2 \psi(r_{j,2}) \qquad \text{for } j = 1, 2, \ldots, n \end{aligned}$$

where $\varphi' > 0$, $\varphi'' \leq 0$, $\psi' > 0$, $\psi'' \leq 0$ and $q_{i,t}$ and $r_{j,t}$ are the amounts of the consumer good received by traders i and j respectively in the t^{th} period.

We may regard equations (1) as showing the actual utility functions in a two-stage market or we may consider them to represent a partial disaggregation and partial solution to a set of dynamic maximization problems of indefinite length which could be expressed as:

$$(2) \quad \begin{aligned} \phi_i &= \varphi(q_{i,1}) + \rho_1 V_i(M_{1,2}, \ldots, M_{n,2}, N_{1,2}, \ldots, N_{n,2}) \quad i = 1, \ldots, n \\ \Psi_j &= \psi(r_{j,1}) + \rho_2 W_j(M_{1,2}, \ldots, M_{n,2}, N_{1,2}, \ldots, N_{n,2}) \quad j = 1, \ldots, n . \end{aligned}$$

In the second term of each equation in (2) we have an induced utility for money which depends both on the holdings of the individual and the holdings of others. This reflects the pecuniary externalities involved in using markets and money.[1/]

The owner of a resource at time t is paid in money at the start of time $t+1$ after his goods have been sold in the market.

In order to minimize the need for notation we make use of the special properties of the example. As is argued in Section 5 this does not influence the generality of the results.

If there is any borrowing or lending in this market it must take place during the first period with traders of the first type lending or borrowing and traders of the second type borrowing as they have nothing to lend.* During the second period no loans will be made as money is worthless at the end of the second period.** Furthermore all individuals will spend all they have during the second period bidding for real resources.

If we limit our attention to solutions in which traders of the identical type behave in the same manner then either no money market functions or all traders of the first type are lenders during the first period.***

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*We rule out lending money or selling goods that the individual does not possess. Thus Daniel Drew's addage does not apply: "He who sells what isn't his'n. Must pay up or go to prison."

**In order to rid ourselves of this "unreal condition" we need to solve the model as a dynamic program of indefinite length.

***It is well known that in many games symmetry among a group of players does not guarantee symmetric treatment in a solution however we will search for equilibria which reflect the symmetries.

u_i = the amount lent by the i^{th} trader of the first type during period 1.

v_j = the amount pledged by the j^{th} trader of the second type during period 1 to be repaid in period 2.

x_i = the amount bid by the i^{th} trader of the first type to purchase goods in period 1.

y_j = the amount bid by the j^{th} trader of the second type to purchase goods in period 1.

$$u = \sum_{i=1}^{n} u_i \ , \quad v = \sum_{j=1}^{n} v_j \ , \quad x = \sum_{i=1}^{n} x_i \quad \text{and} \quad y = \sum_{j=1}^{n} y_j \ .$$

The moves and the relevant accounting can now be described:

(3) $M_1 = M \ , \quad N_1 = 0$ (individual initial money holdings) .

3.1. Credit Line Constraints

For lenders and borrowers the following conditions hold:

(4) $$0 \leq u_i \leq M_i \quad \text{and} \quad 0 \leq v_j \leq C_j \ .$$

These state that an individual cannot lend a negative amount and cannot lend more than he has. A borrower is constrained from pledging a negative amount, whereas on the upper side he is limited in his borrowing by a credit line C_j . This is introduced as an extra parameter in the model as it is of considerable interest in the investigation of noncooperative equilibria and the role of bankruptcy.

In actuality the limits on credit lines are usually extremely important and depend upon the reputation of the borrower, the amount of money available, the assets pledged as security against default and many other factors. The model here scarcely reflects these features of the credit structure.

(5) $M''_{i,1} = M - u_i$, $N''_{j,1} = \frac{v_j}{v} u$ (money holdings after the money market)*

$M''_{i,t}$ = the money holdings of trader i of type 1 after the money market in period t , but before the purchase of goods.

$N''_{j,t}$ = the money holdings of trader j of type 2 after the money market in period t , but before the purchase of goods.

(6) $$0 \leq x_i \leq M - u_i \; , \quad 0 \leq y_j \leq \frac{v_j}{v} u$$ (market bids).

The inequalities in (6) are the budget constraints on the expenditures on goods during the first period

(7) $$q_{i,1} = \frac{nBx_i}{x+y} \; , \quad r_{j,1} = \frac{nBy_j}{x+y}$$ (purchase of goods)

$q_{i,1}$ = the amount of the consumer good purchased by the i^{th} trader of type 1 in the first period.

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*There is a difficulty encountered if $v = 0$. We may assign the value of 0 to v_j/v in this case. This also holds for expressions involving $1/u$ and $1/x+y$.

(8) $M_{i,2} = M - u_i - x_i$, $N_{j,2} = \frac{v_j}{v} u - y_j + \frac{x+y}{n}$ (money holdings at start of period 2).

At the start of period 2 prior to the settlement of debts traders of type 1 hold their initial amount of money minus their loans and expenditures. This is shown in the first part of (8). Traders of the second type obtain the income from having each sold B units of the consumer good in the first period. Thus their holdings at the start of period 2 are shown by the second part of (8).

3.2. On Bankruptcy

After incomes have been received and before the new loan market takes place outstanding debts must be settled. Even though there is no exogenous uncertainty in this market and the individuals are informed in advance of the resource size and ownership there is strategic uncertainty so that individuals are not sure that pledges for repayment can be honored.

We do not have assets pledged in this model, or other parts of the elaborate structure of an actual credit system. How are we to handle inability to pay? We use a simple and straightforward rule which is "to settle for what you can get." Specifically stated the rule is as follows:

If an individual is unable to pay his pledge in full he pays whatever he has and his future (money) income is appropriated until his pledge has been fully paid. He is constrained from further borrowing until that time.

Several observations about this rule must be made. In most societies the rules used are less harsh in some aspects and harsher in other aspects than this rule. In particular an interest rate or further penalty may be extracted for failure to pay on time. However in this model we begin by using the simpler and less harsh rule. A further investigation is needed to study the optimality properties of different bankruptcy laws.

The appropriation of all income until the debt is paid is somewhat harsher than in actuality. Although from the viewpoint of history, debtor's prisons might have been harsher than this rule.

Another distinction among types of default comes in quite naturally in considering different levels of personal or general involvement. In this model there is an intermediary who aggregates loanable funds and debts. A specific debtor j pays his note to the bank along with other debtors. The bank in turn pays the lenders out of the total. There is no direct contract between lender i and debtor j. In this formulation the risk is spread among all equally. We could have modeled bilateral loans or preferred classes of lenders.

The j^{th} trader of type 2 pays back the amount:

$$W_j = \min[N_{j,2}, v_j] . \tag{9}$$

Thus the total amount paid is:

$$W = \sum_{j=1}^{n} W_j . \tag{10}$$

The money holdings of the traders in period 2 after the settlement of previous loans but before the recontracting of new loans are given by:

$$\text{(11)}\quad M'_{i,2} = M - u_i - x_i + \frac{u_i}{u} W \quad \text{and} \quad N'_{j,2} = \max[(N_{j,2} - v_j),\ 0]\ .$$

3.3. The Payoffs

We observe that the amounts $M'_{i,2}$ and $N'_{j,2}$ will be spent on the purchase of goods in the second period, hence the amounts obtained are given by:

$$\text{(12)}\qquad q_{i,2} = \frac{nAM'_{i,2}}{nM} \quad \text{and} \quad r_{j,2} = \frac{nAN'_{j,2}}{nM}\ .$$

Thus the payoffs to traders of type 1 given that they have selected u_i and x_i and traders of type 2 have selected v_j and y_j are:

$$\text{(13)}\qquad \varphi\left(\frac{nBx_i}{x+y}\right) + \rho_1\varphi\left(\frac{AM'_{i,2}}{M}\right) \qquad \text{for} \quad i = 1, \ldots, n$$

and for traders of type 2

$$\text{(14)}\qquad \psi\left(\frac{nBy_i}{x+y}\right) + \rho_2\psi\left(\frac{AN'_{j,2}}{M}\right) \qquad \text{for} \quad j = 1, \ldots, n\ .$$

4. CONCEPTS OF SOLUTION AND SOLUTIONS

In Section 3 a two stage market was set up as a many person game. No suggestion has been made as to how to solve this game. There are several alternatives which we may consider. We can follow the traditional approach of general equilibrium theory and ignore the strategic, dynamic and monetary aspects of markets. We can apply a game theoretic solution. Here we have the choice among many different approaches. We can consider cooperative or noncooperative theories. These include the core, the value, the noncooperative equilibrium and several others.[2/]

A behavioral or macroeconomic approach might be adopted. Such a view would enable us to abandon the unrealistic requirements on information that both the game theory and general equilibrium approaches call for.

In this section we consider the general equilibrium solution, and a variant of the noncooperative equilibrium. The relationship between the use of Markovian or state strategies, the behavioral approach and the role of expectations need to be considered but they will not be investigated in this paper.

4.1. The Competitive Equilibrium

Although it is not pointed out in the literature the general equilibrium analysis makes use of many of the same concepts employed in game theory to avoid a truly dynamic analysis of multistage problems. A tree diagram of essentially the same structure as that shown in Figure 1 is implicitly employed by Debreu[3/] and others.

The assumption that each individual knows all of his future endowments and prices is critical to being able to define the maximization. The game theory model in contrast does not have prices (as they result from behavior and thus are deduced from a solution concept, if they exist) but it requires that the individuals are informed about all "of the rules of the game." This amounts to a knowledge about the size or expected size of all future endowments and a knowledge of the preferences of the other traders.

By using the concept of an overall strategy or complete plan a general equilibrium multistage market or game can be regarded as giving a single stage decision problem. As the individual is assumed to be in a position to work out every contingency he need make only one decision at the beginning of time. This is to specify a general strategy or plan which covers all contingencies.

In the simple example presented in Section 3, if we assume the existence of complete trust costless competition and frictionless transactions then money is not needed and if we are not constrained to use money the cash flow conditions disappear and the general equilibrium conditions can be solved virtually by inspection.

Before doing so it is important to note a difficulty in modeling the float.* When money must be used a choice must be made in the model as to when an owner is paid for his goods. It makes a difference to his cash flow if he is paid at the start of his period of ownership or at the start of the subsequent period. Without money it is implicitly assumed that income from goods sold during any period is immediately usable during that period.

The nonmonetary model best related to the model in Section 3 is not one with assets in each period of $(0, B)$ then $(A, 0)$ but one with assets in the first period of $(A, 0)$ and $(0, B)$ in the second. This reflects the advantage to a trader of the first type who in the monetary model begins with his income M.

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*A more detailed discussion of this is given elsewhere.[4]

Suppose that:

$$\phi_i = q_{i,1} + q_{i,2} \quad \text{and}$$

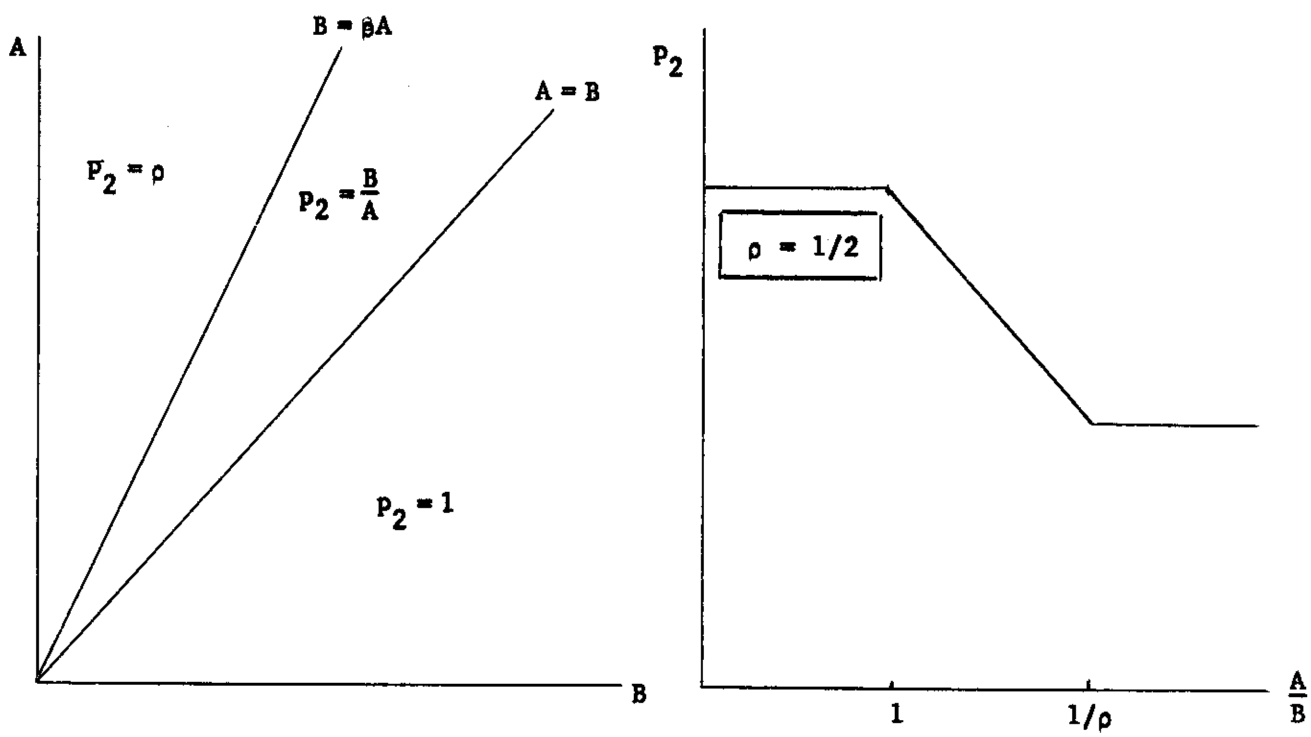
(15)

$$\Psi_j = r_{j,1} + \rho r_{j,2} \quad \text{where} \quad 0 \leq \rho \leq 1 \,.$$

Let the price in the first period be $p_1 = 1$ and in the second be p_2. By inspection we observe that the solutions are:

$1 \geq A/B = p_2 \geq \rho \qquad q_{i,1} = 0 \,, \quad q_{i,2} = B \,, \quad r_{j,1} = A \quad \text{and} \quad r_{j,2} = 0 \,.$

If $A/B > 1$ then $p_2 = 1$ and if $A/B < \rho$, $p_2 = \rho$. Figure 2 illustrates the solutions:



(a) FIGURE 2 (b)

4.2. The Noncooperative Equilibrium

A strategy is a general plan which covers all contingencies. Thus for this two stage game a general strategy by a trader of type 1 consists of a number u_i and a function $x_i(u_1, \ldots, u_n, v_1, \ldots, v_n)$. Similarly a strategy for a trader of type 2 consists of v_j and a function $y_j(u_1, u_2, \ldots, u_n, v_1, \ldots, v_n)$.

The noncooperative equilibrium will be given by the simultaneous solution of:

$$\max_{u_i,\, x_i}\left[\varphi\left(\frac{nBx_i}{x+y}\right) + \rho_1\varphi\left(\frac{AM'_{i,2}}{M}\right)\right] \quad \text{for } i = 1, \ldots, n \tag{16}$$

$$\max_{v_j,\, y_j}\left[\psi\left(\frac{nBy_j}{x+y}\right) + \rho_2\psi\left(\frac{AN'_{j,2}}{M}\right)\right] \quad \text{for } j = 1, \ldots, n$$

where the x_i and y_j are functions picked at the same time as the u_i and v_j.

Rather than consider the general class of all strategies which may include many complex threat and signalling conditions we limit ourselves to a specific type of equilibrium called a perfect equilibrium. A perfect equilibrium point[5/] is one which is not only an equilibrium point in the whole game but is also in equilibrium in every subgame. Here in the example this means that for strategies forming a perfect equilibrium if the u_i and v_j were already specified so that the game has progressed to the second level of the game tree in Figure 1 we may consider a new game starting at a point at this level and require that x_i and y_j be selected so that they form an equilibrium pair in this subgame.

The above observations indicate that we may solve for a strong equilibrium point by first maximizing the expressions in (17) and (18) with respect to x_i and y_j. We may then solve for the x_i and y_j in terms of u_i and v_j. Substituting these values back in we maximize with respect to u_i and v_j.

$$\text{(17)} \quad \max_{u_i} \left[\max_{x_i} \left[\varphi\left(\frac{nBx_i}{x+y}\right) + \rho_1 \varphi\left(\frac{AM'_{i,2}}{M}\right)\right]\right] \quad \text{for } i = 1, \ldots, n$$

and

$$\text{(18)} \quad \max_{v_j} \left[\max_{y_j} \left[\psi\left(\frac{nBy_j}{x+y}\right) + \rho_2 \psi\left(\frac{AN'_{j,2}}{M}\right)\right]\right] \quad \text{for } j = 1, \ldots, n .$$

If we can solve these equations for an equilibrium point we will obtain a price system and a price for future money or a short term rate of interest. We may then examine the behavior of the equilibrium as $n \to \infty$ to see what relationship this bears to the competitive equilibrium.

We first consider the possibility that there may be an equilibrium in which no borrower defaults. If there is such an equilibrium this will simplify the structure of (17) and (18) as we may replace the W_j by v_j and can rewrite the payoff functions as:

$$\text{(19)} \qquad \varphi\left(\frac{nBx_i}{x+y}\right) + \rho_1 \varphi\left(\frac{A(M - u_i - x_i + \frac{u_i}{u} v)}{M}\right)$$

$$\text{(20)} \qquad \psi\left(\frac{nBy_j}{x+y}\right) + \rho_2 \psi\left(\frac{A(\frac{v_j}{v} u - y_j + \frac{x+y}{n} - v_j)}{M}\right)$$

specializing for the example:

$$(21) \qquad \phi_i = \frac{nBx_i}{x+y} + \frac{A}{M}\left[M - x_i - u_i + u_i\frac{v}{u}\right]$$

$$(22) \qquad \psi_j = \frac{nBy_j}{x+y} + \frac{\rho A}{M}\left[v_j\frac{u}{v} - y_j + \frac{x+y}{n} - v_j\right].$$

First we solve for x_i and y_j in terms of u_i and v_j. It can be seen immediately that $y_j = v_j\frac{u}{v}$ because the rate of interest cannot be less than zero (the lenders would carry rather than lend cash if it were). If the rate of interest is greater than or equal to zero the borrower will only borrow for immediate use. Hence:

$$(23) \qquad y_j = v_j\frac{u}{v}.$$

From (21) differentiating by x_i

$$(24) \qquad nB\left(\frac{x + y - x_i}{(x+y)^2}\right) = \frac{A}{M}.$$

Set $K = A/BM$ then from (24)

$$(25) \qquad K(x+y)^2 - n(x+y) + nx_i = 0.$$

Summing over $i = 1, 2, \ldots, n$ and dividing by n from (25) we obtain:

$$(26) \qquad K(x+y)^2 - n(x+y) + x = 0.$$

From (23) summing over $j = 1, \ldots, n$ we obtain $y = u$. Substituting this in (26) and solving for x in terms of u we have:

$$x = \frac{n-1}{2K} - u + \frac{1}{2K}\sqrt{(n-1)^2 + 4uK} \tag{27}$$

hence from (25) which shows that $x_i = x/n$ we have:

$$x_i = \frac{1}{2K}\left(\frac{n-1}{n}\right) - \frac{u}{n} + \frac{1}{2Kn}\sqrt{(n-1)^2 + 4uK} \ . \tag{28}$$

Substituting the values of x_i and y_j in (21) and (22) and differentiating these respectively by u_i and v_j, after a certain amount of manipulation we obtain:

$$\frac{2u_i}{\rho v_j}\left(\frac{n-1}{n}\right) = \left(\frac{n-1}{n}\right) + \sqrt{\left(\frac{n-1}{n}\right)^2 + \frac{4u_iK}{n}} \tag{29}$$

and

$$\left(\frac{n+1}{n}\right)\left(1\Big/\sqrt{(n-1)^2 + 4nu_iK}\right) = \frac{v_j}{u_i}\left(\frac{n-1}{n}\right) - \left(\frac{n-1}{n}\right) \ . \tag{30}$$

From (29) and (30) an equation for (v_j/u_i) may be found:

$$\left(\frac{v_j}{u_i}\right)^2 - \left[\frac{2}{\rho} + 1 - \frac{n+1}{(n-1)^2}\right]\frac{v_j}{u_i} + \frac{2}{\rho} = 0 \ . \tag{31}$$

Thus:

$$\left(\frac{v_j}{u_i}\right) = \frac{\frac{2}{\rho} + 1 - \frac{n+1}{(n-1)^2} - \sqrt{\left(\frac{2}{\rho} + 1 - \frac{(n+1)}{(n-1)^2}\right)^2 - \frac{8}{\rho}}}{2} \ . \tag{32}$$

This equation determines the money rate of interest.

From (32) it follows that the quantity within the radical must be positive for a real solution, thus:

(33) $$\rho \leq \frac{2}{\left(1 + \frac{\sqrt{n+1}}{n-1}\right)^2} \ .$$

Figure 3 illustrates the case distinctions for the critical ρ and variable n. We see immediately that in the limit ρ cannot be larger than 2. The knowledge that there are sufficiently many individuals in the market that a real solution is feasible does not guarantee that a money

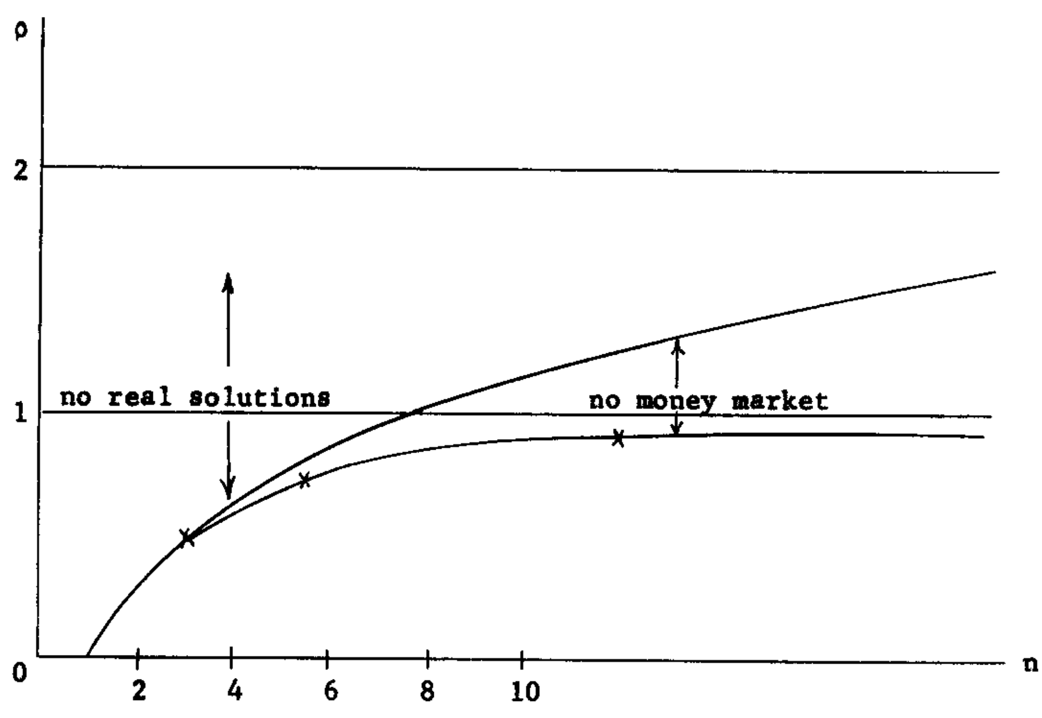


FIGURE 3

market will be active. For this to be the case we require that u_i be positive.

Let $s = v_i/u_i$ when this ratio is real. We know immediately that $s \geq 1$ (it pays lenders to hoard rather than lend otherwise). From equation (29) replacing u_i/v_j by $1/s$ we obtain:

$$u_i = \frac{(n-1)^2}{nK}\left[\frac{1-\rho s}{(\rho s)^2}\right] . \tag{34}$$

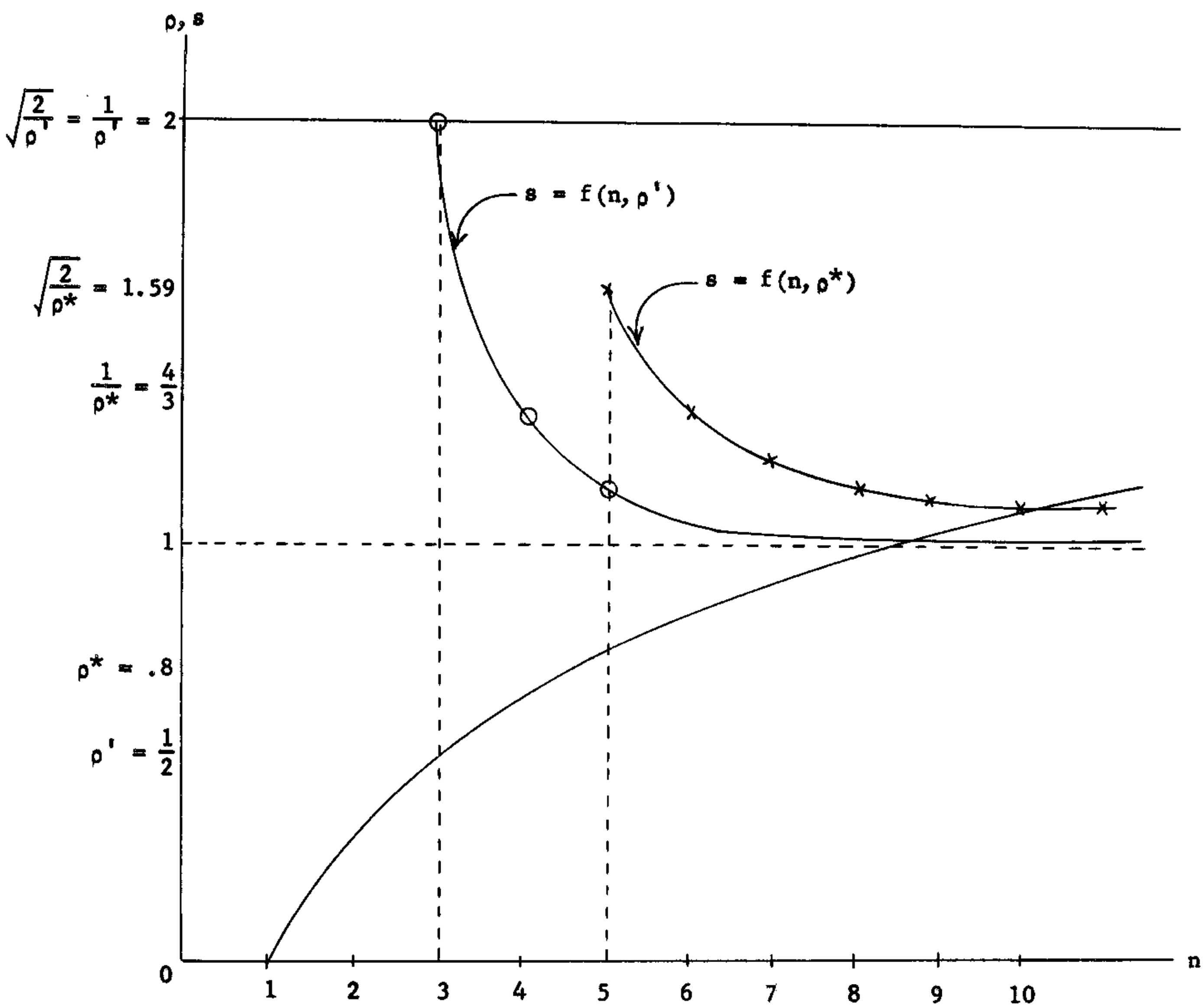


FIGURE 4

For any fixed ρ from (32) it can be seen that s decreases as n grows. From (34) u_i increases as s decreases. For any fixed ρ the money market will become active when $\rho s > 1$. As numbers increase lending may increase in a manner indicated in (34) until a bound is reached. This bound involves the money supply. Figure 4 shows s as a function of n for two values of ρ. We see that $1 \leq s \leq 1/\rho$ and that for the first real solution $s = \sqrt{2/\rho}$.

$$1 \leq s \leq 1/\rho .$$

Substituting in $1/\rho$ for v_j/u_i in (32) we may derive the condition for the boundary of an inactive money market:

(35) $$n = 1 + \frac{\rho}{2(1-\rho)} \left\{ 1 + \sqrt{1 + 8 \frac{(1-\rho)}{\rho}} \right\} .$$

We observe that the number of competitors required for an active money market is unbounded as $\rho \to 1$. The boundary for the inactive money market is illustrated in Figure 3.

4.3. Lending, Borrowing, Hoarding and the Supply of Money

Although the conditions for an active money market have been examined the supply of and demand for loanable funds has not yet been discussed. These will depend upon the amount of money M and the ratio of the amount of goods in the first and second periods.

The equilibrium we are examining first is one where boundary constraints are not binding. We proceed to specify the constraints and to interpret them in terms of lending, hoarding, spending and bankruptcy.

From (28) substituting in (34) for u_i we can obtain x_i in terms of ρ, s and n.

(36) $$x_i = \frac{(n-1)}{K(\rho s)^2}\left[\rho s - \frac{n-1}{n}\right] .$$

We could replace s by the expression (32) giving s in terms of ρ and n, however it is more convenient to consider the constraints in terms of ρ and s. Furthermore these have an immediate economic meaning. s is the one plus the money rate of interest whereas ρ is the marginal rate of substitution to traders of the second type between goods in the first and second periods.

It is important to stress that although there may be a tendency to regard ρ as the "natural discount rate;" this rate only applies to one type of trader and its important function in this model is in terms of the marginal rate of substitution. Different individuals may have different time preferences, but except in the linear case that is used in this example it is in general not fruitful to discuss time preference because, among other factors (such as uncertainty) marginal rate of substitution will vary with endowments.

From (36) we observe that spending by traders of the first type will drop to zero in the first period when:

(37) $$\rho s \leq \frac{n-1}{n} .$$

By substituting $s = (n-1)/n\rho$ in (31) and solving for ρ in terms of n we determine the bound on when spending is zero. This is:

(38) $$\rho = \left(\frac{n-1}{n}\right)^2 .$$

This curve is shown in Figure 3.

There remains to establish conditions on hoarding and bankruptcy. The amount of money hoarded by a trader of type 1 during the first period is given by $M - x_i - u_i$. This cannot be less than zero, hence for an interior solution we require:

(39) $$x_i + u_i \leq M .$$

Substituting for x_i and u_i from (36) and (34) and replacing K by A/BM we obtain:

(40) $$\rho s \geq \frac{B}{A}\left(\frac{n-1}{n}\right) .$$

Finally if bankruptcy is to be avoided we require that v_j should be no larger than $x_i + u_i$. Replacing v_j by su_i we have the inequality $x_i \geq (s-1)u_i$. Substituting for x_i and u_i we obtain:

(41) $$\rho s \geq 1 - \frac{\rho}{n-1} .$$

It may be observed that for a finite n it is possible to have a market in which all of the inequalities are satisfied, i.e. there is spending, lending, borrowing, hoarding and no bankruptcy. Such an example is shown in Figure 5 for $\rho = .8$ and $n = 7$ or $n = 8$. The condition on hoarding depends upon B/A . It coincides with the spending condition if $B/A = 1$.

As n becomes large one or more of the boundary conditions are encountered and the solution changes.

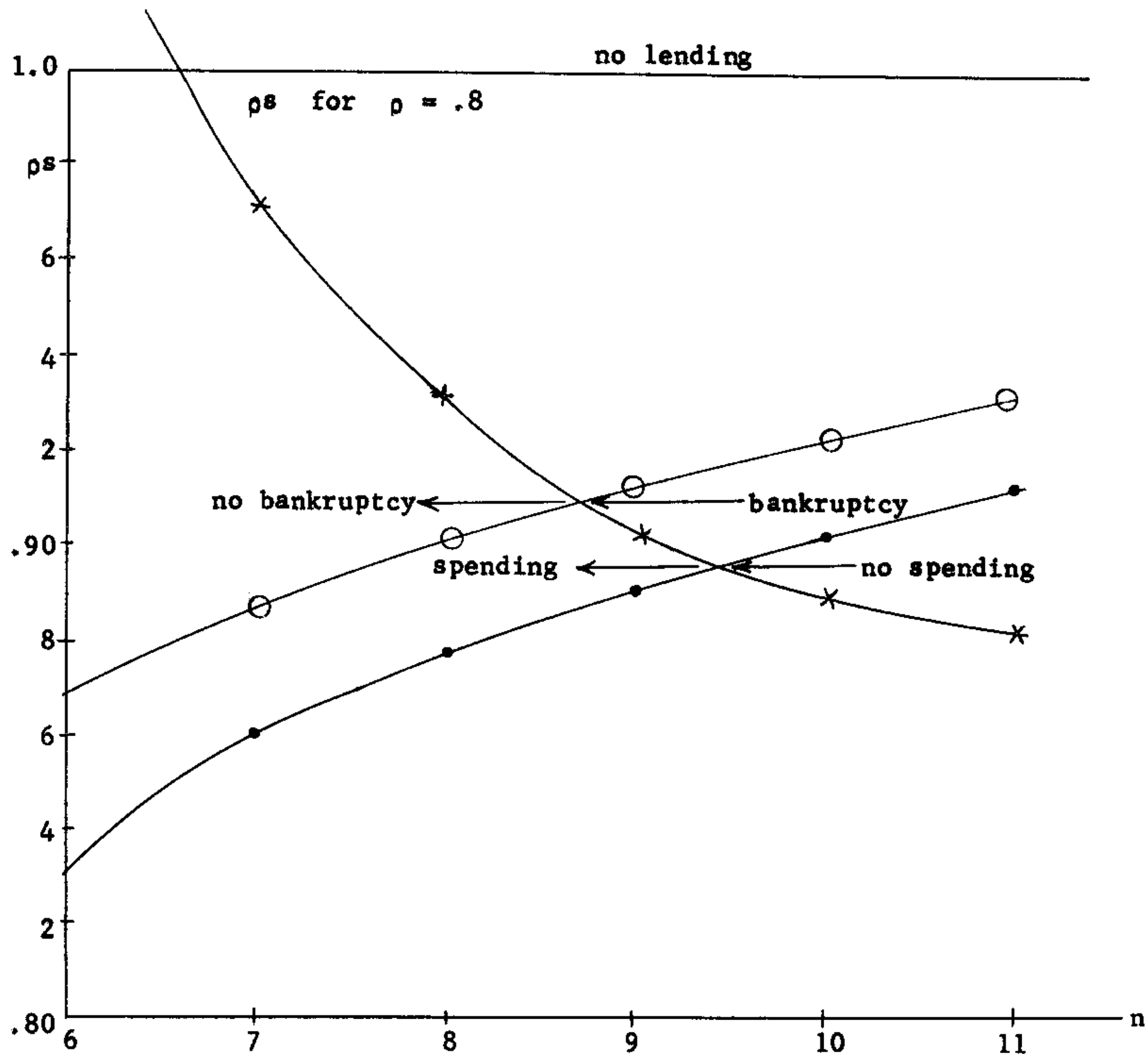


FIGURE 5

4.4. The Supply of Goods

We may expect that there will be three critical zones in the relative supply of goods in the economy. They are:

$$B/A \leq \rho$$

(42) $$\rho < B/A < 1 \quad \text{and}$$

$$B/A \geq 1 \ .$$

For a fixed ρ the solutions as n is increased are investigated for the three cases in (42)

Case 1. $B/A \leq \rho$

For n sufficiently large traders of type 1 will stop spending in the first period, as can be seen from (37). They will always lend and hoard as can be seen from (34) and (40). From (21) and (22) we obtain

$$\phi_i = \frac{A}{M}\left[M - u_i + u_i \frac{v}{u}\right]$$

(43)

$$\psi_j = \frac{nBy_j}{y} + \frac{\rho A}{M}\left[v_j \frac{u}{v} - y_j + \frac{y}{n} - v_j\right]$$

as $x_i = 0$. Also $y_j = v_j \frac{u}{v}$.

Differentiating the equations of (43) with respect to u_i and v_j respectively we obtain:

(45)
$$u_i = \left\{1 - \frac{1}{n}\right\}^2 \frac{1}{\rho K} \ , \quad v_j = \left\{1 - \frac{1}{n}\right\}\frac{1}{\rho K}$$

$$\text{and} \quad s = \frac{1}{1 - \frac{1}{n}} = \left(\frac{n}{n-1}\right) .$$

In the limit $u_i = v_j = \frac{B}{A}\frac{M}{\rho}$ and $s = 1$. The money rate of interest is zero.

Traders of the first type hoard $M\left(1 - \frac{B}{A\rho}\right)$ thus the price of the consumer good in the first period is $M/A\rho$ and in the second M/A .

<u>Case 2</u>. $\rho < B/A < 1$

For n sufficiently large traders of the first type do not spend thus case 1 apparently prevails. However on examining (45) and observing that $B/A = \rho + \epsilon$ where $0 < \epsilon < 1-\rho$, for n sufficiently large:

$$\left\{\frac{n-1}{n}\right\}\left(\frac{BM}{A\rho}\right) > M .$$

Specifically this is true when

$$\epsilon > \frac{\rho}{n-1} . \tag{46}$$

Thus we must set $u_i = M$. We have:

$$u_i = M \qquad v_j = \left\{1 - \frac{1}{n}\right\}\frac{BM}{A\rho} \tag{47}$$

$$\text{and} \quad s = \left(\frac{n-1}{n}\right)\left(\frac{B}{A\rho}\right) .$$

As the ratio of B/A changes s varies in the range

$$\left(\frac{n-1}{n}\right) < s < \left(\frac{n-1}{n}\right)\left(\frac{1}{\rho}\right) .$$

For large n the traders of the first type neither hoard nor spend in the first period. The prices of the goods are identical in each period and the money rate of interest varies from 0 to $1-\rho/\rho$.

<u>Case 3</u>. $B/A \geq 1$

For large n traders of the first type will stop hoarding. Thus $x_i = M - u_i$ and $y_j = v_j \frac{u}{v}$. From (21) and (22) we obtain:

$$\phi_i = \frac{Bx_i}{M} + \frac{A}{M}\left[u_i \frac{v}{u}\right] \tag{48}$$

$$\psi_j = \frac{By_j}{M} + \frac{\rho A}{M}[M - v_j] \ .$$

From the form of (48) individual optimization calls for traders of type 2 to borrow up to the limit if s is less than or equal to $B/A\rho$. It is here that a limit or indebtedness is of importance. Referring to (4) we observe that if we set C as a limit then setting $v_j = C$ in (48) enables us to calculate that:

$$u_i = \left(\frac{n-1}{n}\right)\frac{AC}{B} \tag{49}$$

hence

$$\frac{v_j}{u_i} = \left(\frac{n}{n-1}\right)\frac{B}{A} \ . \tag{50}$$

We see that for n sufficiently large s is indeed less than or equal to $B/A\rho$ hence our assumption is confirmed.

In this model it is important to stress the critical role played by the credit limit. If $C > M$ then the borrowers may wish to borrow more than M, this would guarantee bankruptcy as their total indebtedness would exceed the money supply. Thus <u>ex post</u> and <u>ex ante</u> interest rates would not be the same. More modeling, especially concerning the information conditions is required for the bankruptcy case. In this simple model with symmetry and a fixed money supply of M it is easy to specify the bound $C = M$

If $C < M$ the credit restrictions may hurt traders of the second type.

5. CONCLUSIONS

5.1. The Money Supply and the Rate of Interest

In this model fiat money is intrinsically worthless however it is used as the means of trade and hence as a store of wealth.

The activity of a money market requires a minimum number of traders. This number depends on the parameters of the system as is shown in (34).

There are no futures markets in this system except the short term one period money market. The "game" is played in extensive form where each time the strategy of each trader is based upon the state he is in.

The ex ante rate of interest* is deduced in this model from the supply and demand of loanable funds. If r_t is the rate of interest in time t, then $s_t = v_t/u_t = 1 + r_t$ or $r_t = s_t - 1$. In the simple example solved in Section 4 the importance of the relative quantities of money can be seen immediately. Because of the limit on money supply, relative prices that would prevail in an economy with trading in futures cannot prevail here except by hoarding or changes in velocity. We have already seen that in a money economy without exogenous uncertainty hoarding may take place even as $n \to \infty$ if the quantity of money is fixed. If the amount of money is varied appropriately then there will be no hoarding.

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*When bankruptcy is possible ex post and ex ante rates must be distinguished.

The model above can be regarded as one with "merchant bankers." The traders both trade and lend or borrow a fixed amount of cash. A modern market economy has a commercial banking system to control the money supply.* How can one obtain a varying supply of money which eliminates public hoarding and results in a specific rate of interest?

The control of the money supply can be modeled in several ways. It is conjectured that two ways that are somewhat conceptually different may work. They both involve the use of a reserve banking system where there are two types of paper money and banking rules relating the size of loans that can be issued to the deposits of one type of money. The banking system can be modeled as a mechanism for the loanable funds market. Alternatively the bankers can be modeled as a special class of players. A sketch of one of these banking models has been presented elsewhere,[6/] however a detailed discussion will be given in another paper.

5.2. Further Problems and Generalizations

The next important extension of this type of model is to the case where there is exogenous uncertainty. Although the extension is laborious, conceptually it is not difficult for an economy of finite length, as a perfect equilibrium can be solved for by a "backwards solution" in the same way the example in Section 4 (or the recent work of Sobel[8/] may apply). Unlike the treatments of uncertainty in the nonmonetary models[9/] the number of markets does not proliferate but money now takes on insurance properties.

*For a brilliant discussion of the money market see Bagehot.[4/]

There is a modeling difficulty that is encountered when durable assets other than money are introduced. The convention that all goods are automatically deposited with the market for sale is no longer reasonable. A new move must be introduced in which the goods are divided into two piles: those which go to market and those which are inventoried. The presence of assets also influences credit conditions and bankruptcy rules.

The convention concerning the purchase of goods can not only be modified to permit inventorying but also to allow for a certain amount of barter or direct consumption by owners outside of the market. The modeling is relatively straightforward, but the increase in mathematical difficulties is considerable.

Although the emphasis in this paper has been on model building and the analysis has been carried out in terms of an exhaustive investigation of a simple example it is conjectured that the results are quite general. In particular it is conjectured that the following hold true.

Given the conditions usually assumed for the existence of a competitive equilibrium in a trading economy with T time periods, plus the assumption that trade takes place via money then we conjecture that:

1. The economy can be formulated as a noncooperative game and this game has at least one nontrivial perfect noncooperative equilibrium point.*
2. In the n-fold replication of the noncooperative game the perfect equilibrium points will approach limits. If there is a fixed money supply there will almost always be hoarding.

*The no trade point is a trivial equilibrium point.

3. There exists a series of quantities of money $M_1, M_2, \ldots, M_T$ such that if these amounts are available within a banking mechanism the set of the limit points of the perfect equilibrium points in the noncooperative game with the banking mechansim contains* the competitive equilibria of the related nonmonetary economy and the money rate of interest in the limit may be positive.
4. There exists a "money game" with bankers as specialized players or as a mechanism with two types of fiat money where the initial amount of the cash M is specified and a reserve ratio k sets the upper bound for the amount of credit money at kM such that, upon replication, the set of limit points of the perfect equilibria contain the competitive equilibrium points of the related nonmonetary economy.

In subsequent work these conjectures that have been sketched will be more precisely specified and hopefully proved.

It should be noted that the models have been for T time periods. Such an approximation appears to be reasonable and even though money is worthless at the end, the finite model is not particularly pathological. Nevertheless it would be highly desirable to study the behavior of the models as $T \to \infty$.

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*One cannot make a strict comparison as the dimensions of the models are different. Thus "contains" refers to the common dimensions of prices for real goods and distribution of real goods but excludes money.

Production has not been included in this model thus the results and conjectures apply only to trading games. Growth models require a separate discussion.

5.3. On Financial Institutions and Mathematical Economics

The general equilibrium analysis is institution-free and static. This analysis here is a first step towards dynamics and involves institutions as carriers of process. Bankruptcy, credit restrictions, commercial banks, merchant bankers, investment bankers and so forth must be a fundamental part of any satisfactory monetary theory at even the most abstract level.

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