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A GENERAL EQUILIBRIUM CALCULATION OF THE EFFECTS OF DIFFERENTIAL TAXATION OF INCOME FROM CAPITAL IN THE U.S.

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A GENERAL EQUILIBRIUM CALCULATION OF THE EFFECTS OF DIFFERENTIAL TAXATION OF INCOME FROM CAPITAL IN THE U.S.*

by

John B. Shoven** and John Whalley**

1. Introduction

Economists have long been trying to measure the effects associated with various economic distortions from the perfectly competitive model. Both the efficiency costs and the incidence of market imperfections have received considerable attention. However, empirical estimates of the consequences of distortions (e.g. the corporate income tax) have been few, and those which have been presented have been subjected to criticism regarding the simplicity of the underlying model of the economy. Some modeling of the effects of either the imposition or the removal of the relevant distortion is necessary since in most cases it is impossible to observe the economy both in its presence and absence.

The model most commonly used in recent years for analyses of this type is the familiar static two sector--two factors of production general equilibrium model originally developed by Meade [1955] and Johnson [1956]

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for the study of international trade. Typically the economy is assumed to be perfectly competitive except for the single distortion being considered. Factors of production are assumed perfectly mobile between sectors but fixed in aggregate supply, and individuals are taken to have identical homothetic preference functions. Using this basic model, several studies have been presented (see, for example, Harberger [1962], [1966] and Johnson and Mieszkowski [1970]) which compare the economy with and without the relevant distortion.

In general, there has been some dissatisfaction with the use of this model for this type of analysis, although no superior alternative has been suggested. Among the criticisms of this approach have been:

(1) It is based on local analysis and approximations and as such is not well suited for the study of such distortions as the approximately fifty percent corporate income tax or the estimated fifteen percent wage differential between unionized and non-unionized labor.

(2) The level of aggregation (two sectors, two factors) is too severe to capture the major impact of the market imperfection.

(3) The assumption of fixed factor endowments is not realistic. This is particularly true when labor or capital return distortions are being considered.

(4) Analyzing one distortion at a time can be misleading since the effect of two simultaneous distortions need not even approximate the sum of their individual effects.

Given these and other criticisms, one would have expected alternative formulations of the effects of these distortions to have been presented. This has, however, not been the case.

The purposes of this paper are twofold. The first and primary goal is to illustrate that with some modification an algorithmic approach due to Scarf ([1967a], [1967b], [1972]) for the computation of general equilibrium prices is applicable to problems of the sort described above. The formulation is not based on differential calculus, requires no linearity assumptions, and is not as restrictive in either the number of sectors or the number of factors of production.* Several distinct consumers or consuming classes can be specified with differing tastes and initial endowments. It is possible to consider the effects of several distortions simultaneously, and the approach readily lends itself to dynamic extensions. Even in the static case the labor supply is free to vary in response to market distortions. In dynamic versions the capital stock could be augmented or depreciated through time.

The second purpose of this paper is to present a simple example of the technique by analyzing both the incidence and the efficiency costs associated with the differential taxation of income from capital in the U.S. economy. For purposes of simplicity of exposition and due to the specific characteristics of the problem being considered, several potential features of the approach will not be used in this application. However, the method

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*The computation time rises rapidly with the number of factors plus the number of sectors. At the current stage of development, this sum can easily be as large as twenty without becoming prohibitively expensive.

of their inclusion will be made clear. This particular application seems appropriate for several reasons. Primary among them are the sheer size of the distortion, the fact that its effects have long been debated, and that it has been extensively studied by Harberger [1959, 1962, 1966], Rosenburg [1969] and Mieszkowski [1967] using the earlier technique mentioned above.

2. The Algorithm

The algorithm utilized in this paper is due to Scarf* with some modification by T. Hansen [1968]. Its purpose is to compute an equilibrium price vector, the components of which are the prices of all outputs and inputs, which has the property that for commodities with a non-zero price supply equals demand and for free commodities supply is greater than or equal to demand. In addition, profit at this price vector is less than or equal to zero for all possible production techniques, being equal to zero for those techniques which are utilized. Perfect competition is assumed (i.e. both producers and consumers behave as price takers) as is constant returns to scale in production. The information required for the execution of the algorithm is (1) a description of the technological production possibilities through a listing of feasible activities, (2) market demand functions which are continuous and homogeneous of degree zero in prices, and (3) the economy's initial stock of commodities.

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*For the reader unacquainted with the method, the article by Scarf in Essays in Honor of Irving Fisher and his 1969 AER article offer lucid presentations. A more extensive elaboration will soon be available in his forthcoming book, The Computation of Economic Equilibria. The exposition presented here is necessarily brief and sketchy but may be adequate for the reader primarily interested in the effects of distortionary taxation of capital income.

Total market demand is the sum of individual demands, each of which may be derived from utility maximization subject to a budget constraint. In this case, of course, information regarding the preferences and initial holdings of individuals is required. It is necessary for the working of the algorithm that the market demand functions satisfy Walras' Law. This, however, is guaranteed if the demand functions are derived from individual maximization of utility subject to a budget constraint. To show this, let x_{ij} be the demand for commodity i by individual j and let w_{ij} be his corresponding initial endowment of the ith commodity. Individual j's budget constraint is simply

$$\sum_{i=1}^{n} P_i x_{ij} = \sum_{i=1}^{n} P_i w_{ij} \tag{2.1}$$

where there are n commodities (both outputs and inputs) and P_i is the price of the ith good. Summing these budget constraints over the J individuals we get

$$\sum_{j=1}^{J} \sum_{i=1}^{n} P_i x_{ij} = \sum_{j=1}^{J} \sum_{i=1}^{n} P_i w_{ij} , \tag{2.2}$$

which can be written as

$$\sum_{i=1}^{n} P_i (X_i - W_i) = 0 \tag{2.3}$$

where

$$X_i = \sum_{j=1}^{J} x_{ij} = \text{market demand for commodity } i$$

and

$$W_i = \sum_{j=1}^{J} w_{ij} = \text{total initial endowment of commodity } i \, .$$

Equation (2.3) is Walras' Law, which states that the value of market excess demands is zero.

Production is described by an activity analysis matrix

$$(2.4) \qquad A = \begin{bmatrix} -1 & 0 & \cdots & 0 & a_{1,n+1} & \cdots & a_{1,m} \\ 0 & -1 & \cdots & 0 & a_{2,n+1} & & a_{2,m} \\ \vdots & \vdots & & \vdots & & & \vdots \\ 0 & 0 & \cdots & -1 & a_{n,n+1} & \cdots & a_{n,m} \end{bmatrix}$$

in which each column represents a feasible activity. Outputs are given positive and inputs negative coefficients, and each activity can be operated at any non-negative level. The first n columns indicate the feasibility of free disposal of each commodity. It is assumed that the set of non-negative vectors y which satisfy $Ay + W \geq 0$ is bounded. This can be interpreted as implying that the production possibility frontier is finite in all dimensions.

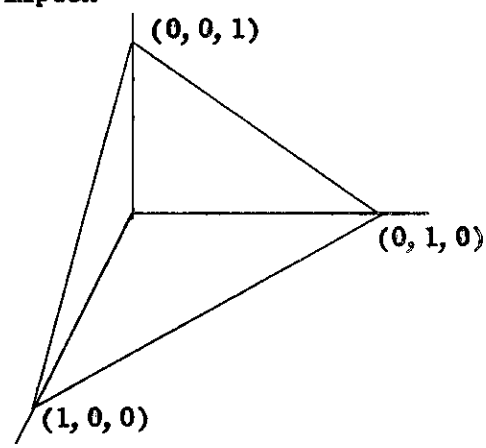
In this notation a competitive equilibrium is defined by a price vector P^* and a vector of activity levels y^* such that

$$(2.5) \qquad \begin{aligned} &(1) \quad X(P^*) = W + Ay^* \, , \quad \text{and also} \\ &(2) \quad \sum_{i=1}^{n} P_i a_{ij} \leq 0 \quad \text{for all } j \, , \quad \text{with equality if } y_j^* > 0 \, . \end{aligned}$$

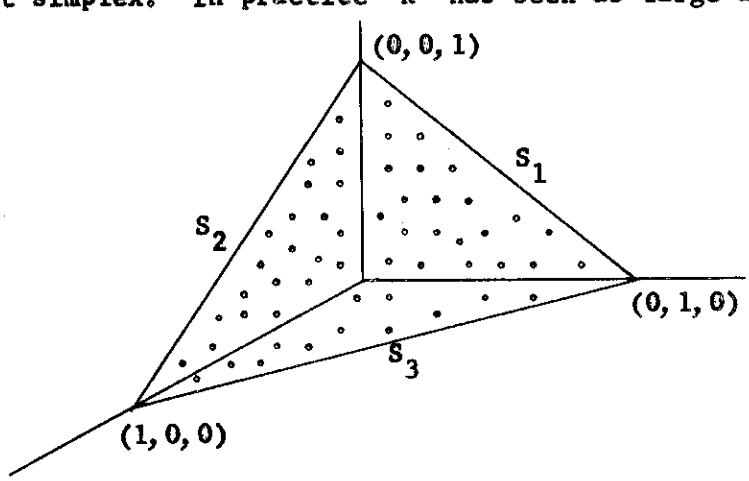
X and W are the vectors of market demands and initial holdings, respectively. The proof of existence of such an equilibrium has been thoroughly investigated (see, for example, McKenzie [1959]).

Since the demand functions are homogeneous of degree zero in terms of prices, we may arbitrarily normalize prices to sum to unity. That is, we will consider prices which are on the unit simplex

$$\sum_{i=1}^{n} P_i = 1 \qquad P_i \geq 0 \tag{2.6}$$



The algorithm is essentially a search procedure on this unit simplex for an approximate equilibrium price vector. The objective is to find a P^* and associated y^* which approximately meet the two above conditions. For this purpose a fine grid of price vectors, $P^{n+1}, \ldots, P^k$, is created on the unit simplex. In practice k has been as large as 10^{30}. The



vectors $P^1, \ldots, P^n$, which represent the sides of the simplex $s_1, \ldots, s_n$, are also created.

The algorithm always works with a subset of n of the price vectors in the list $P^1, \ldots, P^k$ which are referred to as a primitive set and which are "close to each other" in a particular sense. Let a_i be the smallest* i^{th} component among an arbitrary collection of n vectors, $P^{j_1}, \ldots, P^{j_n}$, being considered. If there is no price vector P^j among the entire list $P^1, \ldots, P^k$ for which

$$P_i^j > a_i \quad \text{for all} \quad i = 1, \ldots, n, \tag{2.7}$$

then the vectors $P^{j_1}, \ldots, P^{j_n}$ form a primitive set. Thus, a primitive set can be thought of as an $n \times n$ matrix, PS, each column of which refers to a price vector and whose price columns have the above property. A three commodity example is

$$PS = \begin{pmatrix} 24/100 & 25/100 & 25/100 \\ 47/100 & 46/100 & 47/100 \\ 29/100 & 29/100 & 28/100 \end{pmatrix} . \tag{2.8}$$

Each price vector P^i in the list $P^1, \ldots, P^k$ is associated with a specific commodity vector b^i by the following rules:

(1) If any of the elements of P^i are zero, the associated vector contains a 1 in place of the first zero price and 0's elsewhere.

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*In practice the list of price vectors $P^{n+1}, \ldots, P^k$ often consists of all price vectors which can be expressed as $(m_1/D, m_2/D, \ldots, m_n/D)$, m_i and D being non-negative integers with $\sum_{i=1}^{n} m_i = D$. Numeric ties between two components can be broken in any of several consistent ways such as lexicographic ordering.

(2) The profitability of each activity in the technology matrix A is evaluated at prices P^i. Denoting the activity with the largest profit as a^* and its profit as π^*, then

(i) if $\pi^* > 0$, $b^i = -a^*$

(ii) if $\pi^* \leq 0$, $b^i = X(P^i)$.

The corresponding column is either a slack vector (the negative of the free disposal activity), the negative of an activity vector, or the column of market demands. This gives us a B matrix whose columns correspond to the price vectors $P^1, \ldots, P^k$,

$$(2.9) \qquad B = \begin{array}{c} \begin{array}{ccccccc} P^1 & P^2 & & P^n & P^{n+1} & & P^k \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & b_{1,n+1} & \cdots & b_{1,k} \\ 0 & 1 & \cdots & 0 & b_{2,n+1} & \cdots & b_{2,k} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & b_{n,n+1} & \cdots & b_{n,k} \end{bmatrix} \end{array}.$$

The main theorem upon which the algorithm is based is the following:

<u>Theorem</u>. There exists a primitive set $P^{j_1}, \ldots, P^{j_n}$ such that

$$By = W$$

has a non-negative solution where $y_j = 0$ for $j \neq j_1, \ldots, j_n$.

For those familiar with the notion of a feasible basis from linear programming, this theorem can be restated even more briefly.

<u>Theorem</u>. There exists a primitive set $P^{j_1}, \ldots, P^{j_n}$ such that the columns $j_1, \ldots, j_n$ form a feasible basis for $By = W$.

For its proof see Scarf [1967a] or [1972]. Also it is made clear in these references exactly in what sense the primitive set whose corresponding B columns form a feasible basis for $By = W$ defines an approximate competitive equilibrium. It is shown that at least one of the corresponding columns is a demand column (which indicates that at that particular price vector no production activity makes a profit) while others are the negative of activity columns (indicating that those activities earn positive profits). Since the price vectors of a primitive set are "close together," the profit of all utilized activities must be close to zero, with all unutilized activities having lower profits. The equations $By = W$ can be written as

$$(2.10) \qquad -\sum_j a_{i\ell} Z_j + \sum_j X_i(P^j) y_j = W_i \quad \text{for} \quad i = 1, \ldots, n$$

where those columns which are the negative of activities have been separated out and their corresponding weights are now indicated as Z's. In the two Scarf references he shows that the sum of the weights corresponding to demand columns goes to unity as the grid size approaches infinity. Thus, in the limit (2.10) becomes

$$X_i(P) = W_i + \sum a_{i\ell} Z_j \quad \text{for} \quad i = 1, \ldots, n .$$

This, however, is the equilibrium condition that supply equals demand for all commodities with positive prices. Note that the only way the disposal of commodity j can occur is if $P_j \approx 0$.

The algorithm searches in a systematic way for this primitive set whose corresponding B columns are a feasible basis for $By = W$, starting with a primitive set in one of the corners of the unit simplex and moving across the simplex surface according to fixed rules. The potential usefulness of the algorithm is due to the fact that empirically it finds a primitive set which approximates a competitive equilibrium very rapidly. While in many examples run there have been astronomical numbers of primitive sets defined on the unit simplex, rarely does the algorithm examine more than 1000 of them before terminating with the desired approximation.

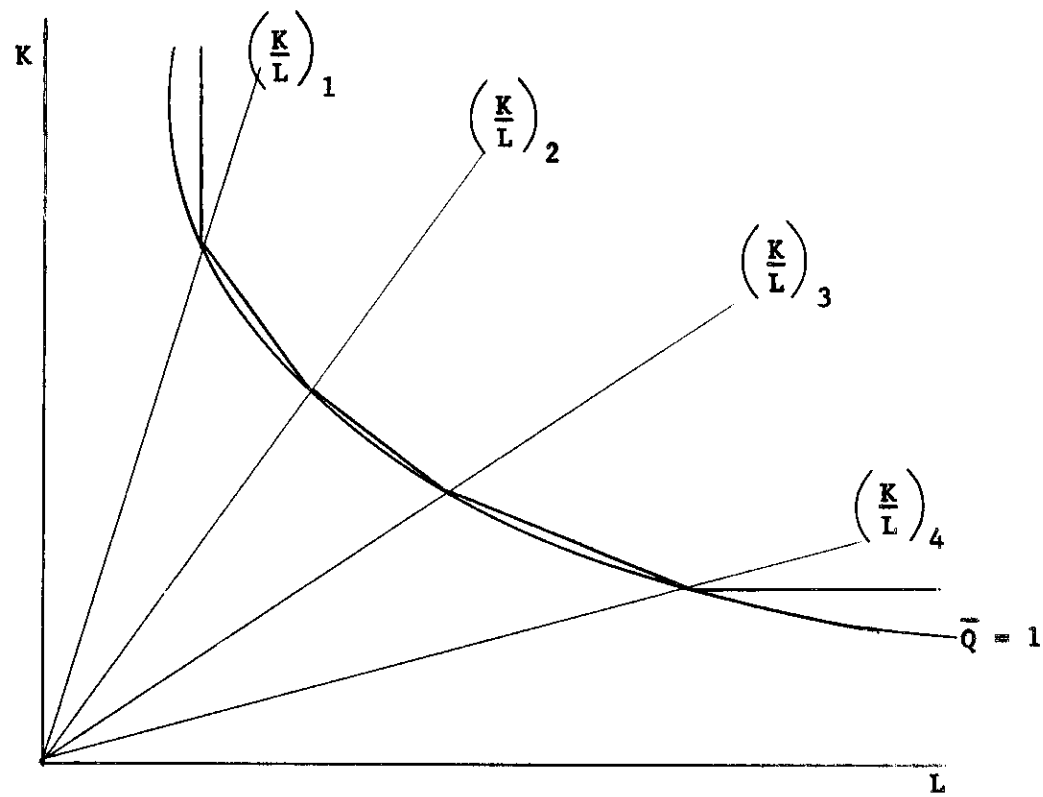
3. Incorporation of the Tax

Probably the clearest way to illustrate the incorporation of a partial factor tax is to appeal to the example of this paper concerning the differential taxation of income from capital. In that example we have two sectors, the "heavily taxed" sector (predominately corporate) and the "lightly taxed" sector (agriculture, housing, crude oil and gas). The sectorial definitions correspond with those used by Harberger ([1959], [1962] and [1966]) in his analysis of the distortion. Each sector's production possibilities are characterized by a constant elasticity of substitution (CES) production function

$$(3.1) \qquad Q_i = [\alpha_i L_i^{-\rho_i} + (1 - \alpha_i) K_i^{-\rho_i}]^{-1/\rho_i} \qquad i = 1, 2$$

with two inputs, labor and capital services. By evaluating (3.1) for several different values of L_i with a fixed K_i value (or vice versa) given α_i

and ρ_i, one could approximate the production function with a relatively small number of activities. One could then normalize the resultant output and inputs by dividing each of them by the output value. The resulting activities will approximate a unit isoquant as shown below.



The capital term in (3.1) must be interpreted as the decrease in the capital stock due to wear and tear (i.e. the flow of capital services).

We therefore have four commodities: (1) "heavily taxed" or "corporate" output, (2) "lightly taxed" or "noncorporate" output, (3) labor, and (4) capital services. The problem is how to incorporate a tax on the return to capital in the "corporate" sector. One method, which might suggest itself at first, seems not to be the way to proceed. That would be where one subtracts from the output of each corporate sector activity some multiple of the amount

of capital which it uses. One problem with this method is how to handle the revenue thus generated. It would be possible to have the government sell its expropriated corporate output and then use the funds for redistribution or to satisfy demands generated from a "governmental utility function." This method lacks some of the advantages of the alternative to be described and would involve a more substantial revision of the algorithm. If the government simply redistributes income, it seems that a far simpler way of modeling its role is to consider what might be termed "corporate capital tickets" as an additional commodity. The firms in the heavily taxed sector would be required to purchase one ticket for every unit of capital which they employ. There is, of course, some problem in determining what defines a physical unit of capital in an empirical sense, but this problem is inherent in models which are as aggregative as this.

The distribution of the income from the sale of these tickets (i.e. the tax revenue) can be handled conceptually without creating a government sector. Each consumer may be endowed with a given share of the total number of tickets which can be sold in an organized market. An individual's share of the proceeds equals his share of the tickets. If the revenue is to be equally distributed, then everyone is endowed with an equal number of tickets. These tickets do not enter directly into the individual's utility function and hence are all sold.

As an example of the incorporation of taxes into the activity matrix which must be specified for the working of the algorithm, consider the following simplified illustration. Suppose there are two sectors each of which is adequately described by a Cobb-Douglas production function of the

form

$$(3.2) \qquad Q_i = A_i K_i^{\alpha_i} L_i^{1-\alpha_i} \; ; \quad 0 < \alpha_i < 1 \; ; \quad i = 1, 2 \; .$$

For the parameter values $A_1 = 1.0$, $A_2 = 2.0$, $\alpha_1 = .5$, $\alpha_2 = .25$ we may approximate each of the unit isoquants by three linear facets whose vertices represent capital-labor ratios of 1 and 1/2. The A matrix is given by

$$(3.3) \quad A = \begin{array}{c} \left[\begin{array}{cccccccc} -1. & 0 & 0 & 0 & 1. & 1. & 0 & 0 \\ 0 & -1. & 0 & 0 & 0 & 0 & 0 & 1. \\ 0 & 0 & -1. & 0 & -1. & -1.414 & -.5 & -.842 \\ 0 & 0 & 0 & -1. & -1. & -.707 & -.5 & -.421 \end{array}\right] \end{array} \begin{array}{l} \text{output 1} \\ \text{output 2} \\ \text{labor} \\ \text{capital} \end{array} .$$

Taxes are incorporated by introducing the additional commodity, tickets, and by requiring the purchase of one ticket per unit of "corporate" capital used. Thus, if sector one is the corporate sector, the activity matrix in the presence of the tax becomes

$$(3.4) \quad A' = \left[\begin{array}{ccccccccc} -1. & 0 & 0 & 0 & 0 & 1. & 1. & 0 & 0 \\ 0 & -1. & 0 & 0 & 0 & 0 & 0 & 1. & 1. \\ 0 & 0 & -1. & 0 & 0 & -1. & -1.414 & -.5 & -.842 \\ 0 & 0 & 0 & -1. & 0 & -1. & -.707 & -.5 & -.421 \\ 0 & 0 & 0 & 0 & -1. & -1. & -.707 & 0 & 0 \end{array}\right] \begin{array}{l} \text{output 1} \\ \text{output 2} \\ \text{labor} \\ \text{capital} \\ \text{tickets} \end{array} .$$

Equilibrium prices can thus be determined by the algorithm both with and without tickets (i.e. taxes), and measures of the efficiency loss and incidence may be computed.

The effective tax rate (being the ratio of the ticket price to the price of capital) cannot be directly imposed using this method but is determined by the aggregate number of tickets, which also imposes the amount of capital in the taxed sector in the presence of the tax, and by the production and preference function parameters. If the model is adequate for the analysis of the problem, the computed tax rate will correspond to the observed rate with all other parameters given reasonable values.

4. Continuous Production Functions*

When analyzing the effects of a market distortion, it is important to be able to measure such things as small changes in the input ratios of the various sectors. This requires either a very long list of feasible activities or an alteration in the algorithm to incorporate continuous production functions.

The optimal (i.e. cost minimizing) input ratio can be derived as an analytic function of input prices for CES production functions. Thus, for any price vector an optimal activity can be generated for each sector. In computing the corresponding commodity vector for a particular price vector, only the profitability of these optimal activities need be considered rather than a long list of feasible activities. In a two sector example, two cost minimizing unit production activities can be generated for any particular price vector. If the most profitable of these has a positive profit, then the B column associated with that price vector is the negative of the

*While the other sections are a product of joint work, the material of this section and the appendix concerning the termination routine was developed by John Shoven with valuable guidance from Herbert Scarf, and for the appendix, reference to T. Hansen [1968].

most profitable activity vector. Otherwise, the corresponding column is the column of demands evaluated at the price vector. If the price vector has a zero component, the same rule prevails as before.

The derivation of the optimal activities for any price vector proceeds as follows. Let P_K = price of capital services, P_L = price of labor, and P_Q = the output price for a particular sector. It is assumed that the technological production possibilities faced by managers may be adequately described by a CES production function of the form

$$Q = [\alpha L^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho} . \tag{4.1}$$

The cost minimizing inputs for a given output (say, unity) may be found by the method of Lagrange multipliers. Form the function

$$Z = P_K K + P_L L + \lambda\{1 - [\alpha L^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho}\} . \tag{4.2}$$

The first order conditions for a saddle point are

$$\frac{\partial Z}{\partial K} = P_K - \lambda(1-\alpha)K^{-\rho-1}[\alpha L^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho-1} \geq 0 \quad (= \text{if} \quad K > 0)$$

$$\frac{\partial Z}{\partial L} = P_L - \lambda\alpha L^{-\rho-1}[\alpha L^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho-1} \geq 0 \quad (= \text{if} \quad L > 0) \tag{4.3}$$

$$\frac{\partial Z}{\partial \lambda} = 1 - [\alpha L^{-\rho} + (1-\alpha)K^{-\rho}]^{-1/\rho} \leq 0 \quad (= \text{if} \quad \lambda > 0) .$$

Since P_L, P_K, and the two marginal products are positive, all three of the relations (4.3) hold with strict equality. Transferring the second term of each of the first two equations to the right hand side and dividing

the second by the first gives

$$\frac{P_L}{P_K} = \frac{\alpha}{1-\alpha}\left(\frac{L}{K}\right)^{-\rho-1} \quad . \tag{4.4}$$

Solving for the optimal labor-capital ratio as a function of P_K and P_L, one gets

$$\frac{L}{K} = \left[\frac{(1-\alpha)P_L}{\alpha P_K}\right]^{-1/\rho+1} = \left[\frac{\alpha P_K}{(1-\alpha)P_L}\right]^{1/\rho+1} \quad . \tag{4.5}$$

The third of the first order conditions (4.3) permits the determination of the labor and capital inputs required for a unit output. Writing it as

$$\alpha L^{-\rho} + (1-\alpha)K^{-\rho} = 1 \tag{4.6}$$

and dividing both sides by $K^{-\rho}$ gives

$$\alpha\left(\frac{L}{K}\right)^{-\rho} + 1-\alpha = K^{\rho} \ . \tag{4.7}$$

Substituting the previous result for (L/K) from (4.5), we obtain

$$K^{\rho} = \alpha\left[\frac{\alpha P_K}{(1-\alpha)P_L}\right]^{-\rho/\rho+1} + (1-\alpha) \tag{4.8}$$

or

$$K = \left[\alpha\left[\frac{(1-\alpha)P_L}{\alpha P_K}\right]^{\rho/\rho+1} + (1-\alpha)\right]^{1/\rho} \quad . \tag{4.9}$$

Similarly, the solution for the labor input per unit of output may be determined as

$$(4.10) \qquad L = \left[(1-\alpha) \left[\frac{\alpha P_K}{(1-\alpha)P_L} \right]^{\rho/\rho+1} + \alpha \right]^{1/\rho} .$$

Given usual grid sizes on the unit price simplex, this method permits the consideration of as many as 10^{30} possible activities for each sector. The final approximation to an equilibrium price vector is further refined by a series of termination linear programs as described in the appendix in such a manner that the number of possible activities is limitless. That is, with the method described in this section in conjunction with the termination routine, truly continuous production functions can be handled.

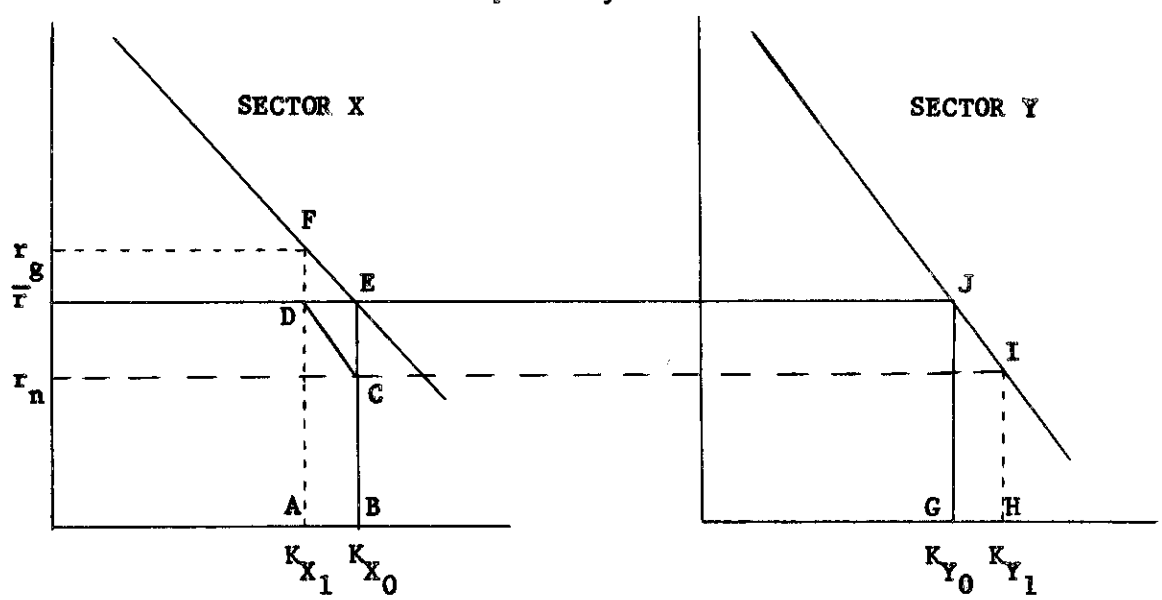
5. Harberger's Analysis of the Taxation of Income from Capital

In recent years considerable attention has focused upon the distortions introduced into the operation of the price system in the U.S. economy by the differential tax rates applied to income from capital originating in various sectors of economic activity. Prominent in the literature have been three articles by Harberger, [1959], [1962], [1966], in which he makes an empirical distinction between a heavily and lightly taxed sector. These sectors are sometimes referred to as the corporate and non-corporate sectors due to the major role played by the corporation income tax in causing the differential rates, although his sectoral division does not exactly correspond to the legal distinction between incorporated and unincorporated enterprises. He assumes that each sector employs two factors, capital services

and labor, in the production of homogeneous outputs.

In order to estimate the efficiency loss due to the differential taxation of the return to capital, Harberger applies a form of welfare analysis in the tradition of Marshallian consumer surplus. His conclusion (Harberger [1966]) is that the efficiency loss from factor misallocations was in the range of 1.75-3.5 billion dollars a year for the period 1953-1959. In an earlier paper (Harberger [1962]) which focuses mainly on the incidence effects of the corporation income tax he concludes that for "plausible" values of production and demand function parameters capital bears the entire burden of the tax in that its gross share (and, therefore, also labor's share) are the same both in the presence and the absence of the tax.

In the computation of the first of these results, it is assumed that the marginal products of capital schedules for each sector are linear as drawn below. Output units are chosen so that both commodity prices are unity, and it is assumed that all prices other than the price of capital are unaffected by the presence of the differential taxation. Given these assumptions, the changes in the capital allocation can be used to generate a measure of the social waste imposed by the distortion. In the absence



of any taxes, capital will allocate itself in a market economy such that the rate of return $\bar{r}$ is equal for the two sectors and the capital endowment will be fully employed. Upon the imposition of a tax on capital income in sector X, the gross rate of return r_g in that sector must be such that the net rate of return r_n is equalized across the sectors and capital is again fully employed. The difference between r_g and r_n is, by definition, the tax T per unit of capital utilized in sector X.

Referring to the above graphs, the area ABEF has the interpretation of the loss in output in sector X when K_X decreases from K_{X0} to K_{X1} upon the imposition of the tax. GHIJ, analogously, is the increase in output in sector Y. Since we know that capital is fully employed both in the presence and absence of the tax, it must be true that $K_{X0} - K_{X1} = K_{Y1} - K_{Y0}$. The area FECD represents the social loss of the tax in the consumer surplus sense (it is simply ABEF - GHIJ) and is given by

$$(5.1) \quad \frac{1}{2}(r_g - \bar{r})(K_{X0} - K_{X1}) + \frac{1}{2}(\bar{r} - r_n)(K_{Y1} - K_{Y0}) = \frac{1}{2} T \, \Delta K_X .$$

The solution for ΔK_X is, in turn, computed by solving a system of equations corresponding to the description of the static two sector general equilibrium model due to Meade [1955] and Johnson [1956]. Demand for each product depends upon the level of consumer income and on relative prices. However, since Harberger makes the assumption that the government spends the tax revenue in the same manner as consumers would when faced with the existing prices, only relative commodity prices affect aggregate demand (this is a local approximation which ignores the income loss due to

the inefficiency of the taxation). Working then with the assumption that the quantity of X demanded depends only on P_X/P_Y, he differentiates this function obtaining

$$\frac{dX}{X} = E \cdot \frac{d(P_X/P_Y)}{(P_X/P_Y)} \tag{5.2}$$

where E is the price elasticity of demand for X. Given that $P_X \equiv P_Y \equiv 1$, a local approximation of (5.2) gives

$$\frac{dX}{X} = E(dP_X - dP_Y) \ . \tag{5.3}$$

The production function of sector X

$$X = F(K_X, L_X) \tag{5.4}$$

is assumed to be continuous, differentiable, and homogeneous of degree one. Taking a total derivative through (5.4) one gets

$$dX = \frac{\partial F(K_X, L_X)}{\partial K_X} \cdot dK_X + \frac{\partial F(K_X, L_X)}{\partial L_X} dL_X \ . \tag{5.5}$$

By dividing both sides by X, this can be written as

$$\frac{dX}{X} = \frac{\dfrac{\partial F(K_X, L_X)}{\partial K_X} \cdot K_X}{X} \cdot \frac{dK_X}{K_X} + \frac{\dfrac{\partial F(K_X, L_X)}{\partial L_X} \cdot L_X}{X} \cdot \frac{dL_X}{L_X} \tag{5.6}$$

or

$$\frac{dX}{X} = f_K \frac{dK_X}{K_X} + f_L \frac{dL_X}{L_X} \tag{5.7}$$

where f_K, f_L may be interpreted as the relative factor shares in sector X.

In sector Y we have by the definition of the elasticity of substitution between labor and capital in sector Y, S_Y, that

$$\frac{d(K_Y/L_Y)}{(K_Y/L_Y)} = S_Y \frac{d(P_K/P_L)}{(P_K/P_L)} . \tag{5.8}$$

A local approximation of (5.8) gives

$$\frac{dK_Y}{K_Y} - \frac{dL_Y}{L_Y} = S_Y(dP_K - dP_L) . \tag{5.9}$$

In the above expression (5.9) dP_K is the change in the price of capital relevant for production decisions in sector Y. That is, it is the change in the price of capital net of the tax. For sector X, the relevant change in the price of capital is the gross change, $dP_K + T$. Thus, the equation analogous to (5.9) for sector X is given by

$$\frac{dK_X}{K_X} - \frac{dL_X}{L_X} = S_X(dP_K + T - dP_L) . \tag{5.10}$$

The price of labor is taken to be the <u>numeraire</u>, the price in terms of which other prices are expressed, and, as such, is taken to be unity both in the presence and absence of the tax.

$$dP_L = 0 . \tag{5.11}$$

By the assumption of full employment of all factors, the relations

$$dK_Y = -dK_X \tag{5.12}$$

$$dL_Y = -dL_X \tag{5.13}$$

are obtained.

The production function of sector Y

$$Y = G(K_Y, L_Y) \tag{5.14}$$

is also assumed continuous, differentiable, and homogeneous of the first degree. These properties, along with competition in the factor markets, guarantee that factor payments just exhaust revenue, or

$$P_Y Y = P_L L_Y + P_K K_Y \ . \tag{5.15}$$

Taking a total derivative of each side of (5.15) and appealing to a local approximation gives

$$P_Y dY + Y dP_Y = P_L dL_Y + L_Y dP_L + P_K dK_Y + K_Y dP_K \ . \tag{5.16}$$

The equation analogous to (5.5) for sector Y is

$$dY = \frac{\partial G(K_Y, L_Y)}{\partial L_Y} \cdot dL_Y + \frac{\partial G(K_Y, L_Y)}{\partial K_Y} \cdot dK_Y \ . \tag{5.17}$$

Noting that competition implies that the marginal product of labor in Y is (P_L/P_Y) and that of capital is (P_K/P_Y), (5.17) may be written as

$$dY = \frac{P_L}{P_Y} dL_Y + \frac{P_K}{P_Y} dK_Y$$

or

$$P_Y dY = P_L dL_Y + P_K dK_Y \ . \tag{5.18}$$

Subtracting this result from (5.16) gives

$$YdP_Y = L_Y dP_L + K_Y dP_K \ . \tag{5.19}$$

Dividing both sides by Y and recalling that the initial prices of both factors and outputs are assumed to be unity, one gets

$$dP_Y = g_L dP_L + g_K dP_K \tag{5.20}$$

where g_L , g_K are the relative factor shares in sector Y . Performing a similar procedure for sector X results in the relation

$$dP_X = f_L dP_L + f_K(dP_K + T) \ . \tag{5.21}$$

Equations (5.21), (5.20), and (5.11) can be substituted into equation (5.3) giving

$$\frac{dX}{X} = E[f_K(dP_K + T) - g_K dP_K] \ . \tag{5.22}$$

By similarly substituting (5.11), (5.12), and (5.13) into (5.9) and (5.13) into (5.10) one obtains

$$\frac{K_X(-dK_X)}{K_Y K_X} - \frac{L_X(-dL_X)}{L_Y L_X} = S_Y dP_K \tag{5.23}$$

and

$$\frac{dK_X}{K_X} - \frac{dL_X}{L_X} = S_X(dP_K + T) \ . \tag{5.24}$$

By equating the right hand sides of equations (5.22) and (5.7) and rearranging terms in (5.23) and (5.24) the following system of three equations is derived:

$$\begin{aligned} Ef_K T &= E(g_K - f_K)dP_K + f_L \frac{dL_X}{L_X} + f_K \frac{dK_X}{K_X} \\ 0 &= S_Y \cdot dP_K - \frac{L_X}{L_Y}\frac{dL_X}{L_X} + \frac{K_X}{K_Y}\frac{dK_X}{K_X} \\ S_X T &= -S_X \cdot dP_K - \frac{dL_X}{L_X} + \frac{dK_X}{K_X} \end{aligned} \tag{5.25}$$

The solution for dK_X, which is required in order to evaluate the efficiency loss of the capital income taxation distortion as expressed in (5.1), can be achieved by applying Cramer's rule to (5.25). That is,

$$dK_X = K_X \cdot T \frac{\begin{vmatrix} E(g_K - f_K) & f_L & Ef_K \\ S_Y & -\frac{L_X}{L_Y} & 0 \\ -S_X & -1 & S_X \end{vmatrix}}{\begin{vmatrix} E(g_K - f_K) & f_L & f_K \\ S_Y & -\frac{L_X}{L_Y} & \frac{K_X}{K_Y} \\ -S_X & -1 & 1 \end{vmatrix}} \ , \tag{5.26}$$

or

$$(5.27) \qquad dK_X = K_X \cdot T \frac{-E\left[g_K S_X \frac{L_X}{L_Y} + f_K S_Y\right] - S_X S_Y f_L}{E(g_K - f_K)\left(\frac{K_X}{K_Y} - \frac{L_X}{L_Y}\right) - S_Y - S_X\left(\frac{f_L K_X}{K_Y} + \frac{f_K L_X}{L_Y}\right)} .$$

Similarly, the system of equations (5.25) can be solved for dP_K giving

$$(5.28) \qquad dP_K = \frac{E f_K\left(\frac{K_X}{K_Y} - \frac{L_X}{L_Y}\right) + S_X\left(\frac{f_L K_X}{K_Y} + \frac{f_K L_X}{L_Y}\right)}{E(g_K - f_K)\left(\frac{K_X}{K_Y} - \frac{L_X}{L_Y}\right) - S_Y - S_X\left(\frac{f_L K_X}{K_Y} + \frac{f_K L_X}{L_Y}\right)} \cdot T .$$

Harberger uses the solution to (5.28) to answer the question of the incidence of the distortion. If dP_K is equal to $-TK_X/(K_X + K_Y)$, capital is said to bear the full burden of the tax since the gross return to capital is the same both in the presence and absence of the tax. If $dP_K = 0$, capital and labor are said to bear the burden of the tax equally in the sense that relative shares are unchanged.

Before examining the empirical work which has been done for the purpose of evaluating (5.27) and (5.28) for the U.S. economy, several points should be made regarding the use of the model just presented. First, the point made in the introduction regarding the use of local approximations for the analysis of very large distortions certainly applied in this case. Second the assumptions of linear marginal product of capital schedules and constant returns to scale production functions are inconsistent, although, again, this would not be so bothersome if a "small" distortion were being considered. The linear marginal product schedules imply a term quadratic in capital services in the production functions.

Third, the explanation of the lack of an income term in the demand functions (that the government spends the tax proceeds in the same manner as the taxpayers would have done) is somewhat unsatisfactory. This ignores the efficiency loss of the tax and certainly leads to the conjecture that the set of demand functions is not derivable from utility maximization. A somewhat more realistic expression of the expenditure side of the tax is desirable. Fourth, Harberger looks at incidence as the effect of the distortion on the functional distribution of income. While this is of interest, in the U.S. economy capitalists work, laborers save, and both to a limited extent exercise a work leisure choice. Thus, at the least the incidence of both the taxation and the expenditure side on the personal distribution would be an additional interesting aspect of the distortion. One defense, of course, of not handling the expenditure side is to argue that what is being compared is the current tax situation to a situation in which capital is taxed in a non-distortionary manner (this is theoretically feasible due to the assumption of a fixed endowment of capital) with the same revenue yield.

In his 1959 article, Harberger, focusing solely upon the corporation income tax, asserts that a clear distinction can be drawn between the corporate and the non-corporate sector. Disaggregating fifty classes of output for 1953-1955 averaged data, the mean figure for total corporation income tax payments as a percentage of the total return to capital is observed as twenty-nine percent. With the exclusion of (1) farms; (2) agricultural services, forestry, and fisheries; (3) real estate; and (4) miscellaneous repair services, the average figure for the remaining forty-six sectors rises to forty-five percent, while for the excluded sectors the percentage

of their total return to capital going to corporation income tax payments is 1.2, 3.5, 2.3, and 3.6, respectively. The next smallest figure is 19.6% for crude petroleum and natural gas.

The corporation income tax is a tax on the return to equity capital in the corporate sector. In analyzing the effects of this tax, Harberger assumes that its removal would cause no change in the industrial division between the corporate and non-corporate forms of business. In addition, he assumes that the presence of the tax does not affect a sector's debt-equity ratio. These assumptions, which are also necessary for the formulation presented in this paper, are, of course, open to question. One argument in support of the first of these assumptions is the fact that there are few large firms which are closely held and which therefore could easily unincorporate. A partial defense of the second is made possible by appealing to data presented by Tambini [1966]. His data shows that over the period 1925-1955 the debt-equity ratio of the corporate sector was relatively stable.

In his 1966 paper, focusing on all taxation of income from capital, Harberger produces the table which is here reproduced as Table 1. Columns (1) and (2) are drawn from a disaggregated study of Rosenburg [1966]. Column (3) is meant to reflect the impact of the personal income tax on income from capital. Appealing to columns (4) and (5), Harberger notes that total taxes on net income in the "corporate" and "non-corporate" sectors average respectively 45 percent and 168 percent. Thus, the taxation of income from capital in the United States during this period may be approximated by a general tax of 45 percent on all net income from capital and an 85 percent surtax on the net income from capital originating in the heavily taxed sector (1.45 x 1.85 = 2.68).

TABLE 1

Taxes on Income From Capital, By Major Sectors
(annual averages, 1953-1959, in millions of dollars)

	(1) Total Income* from Capital	(2) Property and Corp. Income Taxes	(3) Other Tax Adjustments	(4) Total Tax on Income from Capital	(5) Net Income from Capital
"Non-Corporate" Sector	26,873	6,639	1,724	8,363	18,510
Agriculture	7,481	1,302	927[a]	2,229	5,252
Housing	18,429	5,140	797[b]	5,937	12,492
Crude Oil and Gas	963	197	---[c]	197	766
"Corporate" Sector	53,339	22,907	9,945[d]	32,852	19,547
TOTAL	79,272	29,546	11,669	41,215	38,057

*"Income" (Rosenburg [1969], page 125) is defined as income from capital for non-financial industry and includes

(1) Corporate sector net income before corporate profits tax liability and property tax payments.
(2) For the unincorporated sector, the portion of the total income of the unincorporated enterprise that is a return on equity capital, plus property tax payments.
(3) Net monetary interest paid by businesses on borrowed capital in the form of debt obligations.
(4) Net rent paid by an industry to persons for the use of physical capital.
(5) Net realized capital gains by the corporate sector that are considered as income to an industry.

[a]Assumes a 15% effective tax on income from capital in agriculture after payment of property and corporate income taxes.

[b]Assumes that 70% of income from capital in the housing sector is generated by owner-occupied housing, on which no personal income tax liability is incurred. It is assumed that the remaining 30% of capital income from housing is subjected to a 20% income tax rate after the deduction of property and corporate income taxes incurred.

[c]Assumes personal tax offsets on account of oil depletion allowances and similar privileges offsets any taxes on dividends and capital gains in this sector.

[d]Assumes a 50% dividend distribution rate, and a "typical" effective tax rate of 40% on dividend income.

Upon close examination of the Rosenburg averaged data for the period 1953-59, the proper division between the heavily taxed sector and the lightly taxed sector is less than clear-cut. Taking corporate income tax plus property tax as a percentage of total return to capital (that is, the column (2) entry as a percentage of the column (1) entry) for the industries Harberger includes in the lightly taxed sector gives the following:

Farms	17.19
Agricultural services, forestry and fisheries	28.37
Crude petroleum and gas	20.35
Real estate	27.89

The corresponding figures for some industries included in the heavily taxed sector are

Lumber and wood products	28.69
Petroleum and coal products	25.43
Personal services	27.75
Business services not elsewhere classified	25.10

It would seem that on the basis of the Rosenburg data alternative formulations of the "heavy" and "light" taxed sectors could be justified. If the four sectors above are transferred from the heavily taxed sector, the effective surtax rate increases from 85 percent to 92.1 percent. The additional transfer of the total trade sector yields a surtax rate of 118.4%. These figures were calculated using Harberger's assumptions regarding the effects of the personal income tax on the return to capital for these sectors.

They seem to indicate that reasonable redefinitions of the two sectors would significantly change Harberger's estimate of the efficiency cost of the distortion.

Several other comments regarding the data of Table 1 seem pertinent. First, while the figures concerning the corporation income tax by industry may be viewed as reliable, being derived from corporate income tax returns, Rosenburg encounters severe difficulties in the determination of both the total return to capital and the property tax payments by sector. Adequate data on some components of the total return to capital such as unrealized capital gains and capital gains in the unincorporated sector are unavailable. An apportionment of the income of a proprietor of an unincorporated enterprise between return to capital and return to labor must, of necessity, be somewhat arbitrary. These and other problems force Rosenburg's data for column (1) to be only a good approximation of the desired information. Determining the proper assignment of property tax liability by industry is also very difficult and poses a more serious problem. Aggregate property payments averaged 11.24 billion dollars per year for this period, and, as such, were of the same order of magnitude as the corporate income tax payments, which averaged 18.306 billion dollars annually. A number of procedures which lessen the reliability of the assignment are forced upon Rosenburg due to the lack of more appropriate data. Use is made of 1957 census data on assessed valuations to compute property tax revenues by types of property, but the assumption is required that statewide general property tax rates exist. Moreover, for the majority of property tax estimates it is assumed that the average property tax rate for 1957 is applicable to either the

entire 1953-1959 period or that it can be used for the 1953-1957 period with a simple adjustment of the 1961 census estimate providing the data for 1958 and 1959. In that property tax revenues increased from 9.4 billion dollars a year in 1953 to 15.0 billion in 1959 and to 18.0 billion in 1961, the assumption of an unchanging rate structure is questionable. One further problem with the property tax is that it is a tax on the value of an asset rather than the flow of returns generated by that asset. Thus, for two assets with similar values at a point in time but with different lifetimes and return streams (assumed constant for the life of the asset), the one with the longer life but lower annual return will bear a higher tax rate on the income from capital. In addition, to the extent that further property tax payments are capitalized into the asset values, the tax may not fall fully on returns to capital generated by these assets.

While columns (1) and (2) of Table 1 are derived from detailed statistical considerations of the available data, column (3) is not. The effects of the personal income tax treatment of the income from capital of the various sectors is an important determinant of the distortions in the tax treatment of income from capital, and, therefore, the arbitrariness in the derivation of column (3) is of some concern. Certainly both Harberger and Rosenburg are aware of most, if not all, of these qualifications concerning their data. Many of their assumptions are required by the paucity of the desired information.

Returning now to Harberger's evaluation of equation (6.27), he takes as a unit of capital that amount which generates one dollar of net income. With this definition, column (5) of Table 1 states that there were on average

18,510 million units of capital in the "non-corporate" sector during the 1953-1959 period, while the corresponding figure for the "corporate" sector was 19,547 million units. That is, $K_Y = 18.51$ billion; $K_X = 19.547$ billion. Likewise, his definition of a unit of labor is that amount which generates one dollar of return to labor (thus, the amount of labor in a sector is the same as the wage bill for that sector). With this definition, Harberger states that the labor allocation for the 1953-1959 period approximated 200 billion units in the heavily taxed sector and 20 billion units for the lightly taxed sector. Thus, $L_Y = 20$ billion units; $L_X = 200$ billion units. Noting that the gross return to capital in the "corporate" sector was about 50 billion dollars out of a total revenue product of approximately 250 billion dollars annually, he takes f_K, capital's share in the "corporate" sector, as .2, and, correspondingly, $f_L = .8$. In the lightly taxed sector the gross return to capital is about 27 billion dollars per year out of a total revenue product of some 50 billion dollars. Thus, $g_K = .54$.

Given that T is taken to be the surtax rate (that is, .85) the only additional parametric values needed in order to evaluate the change in the capital allocation given by (6.27) are E, S_X, and S_Y. In his 1962 article, Harberger assumes that the demand elasticity for the "corporate" sector is -1/7, while the elasticity of demand for the "non-corporate" sector is -6/7. The figure of $E_X = -1/7$ is also used in his 1966 paper. These figures are derived by using expenditure shares and a value of unity for a term V, referred to as the elasticity of substitution between X and Y.

$$V = \frac{\partial X}{\partial P_X} \frac{P_X}{X} \left[\frac{P_Y Y}{P_X X + P_Y Y} \right] .$$

The conclusion that the "non-corporate" sector's (agriculture, housing, and crude oil and gas) products are six times as price elastic as all other products on average seems counter-intuitive at best. A much more acceptable assumption (which Harberger makes in his 1959 article) seems to be to take as unity the price elasticity of demand for all products.

Many different estimates have been made of the elasticities of substitution between labor and capital for various sectors, often concluding with contradictory results. Most of the work based on cross section data suggests that for most two-digit manufacturing industries S is not significantly different than one (see, for example, Solow [1964] and Minasian [1961]). On the other hand, time series studies yield estimates significantly less than one (see Lucas [1969]). Given this situation, Harberger looks at several combinations of S_X and S_Y.

Using the above data and parameter values, Harberger generates the following table of results concerning the efficiency loss question (Harberger [1966]):

S_X	S_Y	ΔK_X (billions of units)	$-\frac{1}{2} T \Delta K_X$ ($ billions)
-1.	-1.	-6.9	2.9
-1.	-.05	-5.9	2.5
-.05	-1.	-5.2	2.2
-.05	-.05	-4.8	2.0
-1.	0.	-4.7	2.0
-.05	0.	-3.9	1.7

In addition, he makes some calculations for the assumptions that $V = 1/2$, and therefore $E = -1/14$. From this, the conclusion is drawn that the efficiency loss due to the differential taxation of income from capital was in the range 1.75-3.5 billion dollars per year for the period 1953-1959. Using similar data and parameter considerations, Harberger finds that dP_K from equation (5.28) approximates $-TK_X/(K_X + K_Y)$ and, therefore, that capital bears the full burden of the tax.

6. Results

Recomputations of the efficiency loss and the incidence effects of the differential taxation of income from capital have been performed using the algorithmic approach described above. This method allows a complete comparison of the equilibria both in the presence and the absence of the tax. Wherever feasible, Harberger's numerical values have been adopted in order to give a reasonable correspondence between the two approaches. Thus, to a limited extent, the validity of the type of local assumptions he uses may be gauged for this particular example. However, as will be made clear, we deviate from his assumptions in several respects. Perhaps most importantly, two classes of consumers are considered. This permits to some degree the evaluation of usage effects (the differential impact of the tax across individuals due to diverse tastes) of the tax as well as its effect on the personal distribution of income. In several of the examples presented here, the individuals have the ability to exercise a work-leisure choice. This, too, offers a greater generality than allowed for by Harberger.

On the production side of the economy, each of the two sector's technological production possibilities is characterized by a CES production function such as

$$Q_i = [\alpha_i L_i^{-\rho_i} + (1 - \alpha_i)K_i^{-\rho_i}]^{-1/\rho_i} \quad . \tag{6.1}$$

Noting the discussion in the previous section of plausible estimates of the elasticity of substitution (in this case $S_i = 1/(1 + \rho_i)$), three cases were considered: two in line with Harberger, and a third considered to be a "best guess." These may be listed as

Case	S_X	S_Y
(1)	-1.	-1.
(2)	-1.	-.5
(3)	-.75	-.25

Rather than alter the program for the cases where $S_i = 1$ (and, therefore, $\rho_i = 0$) to handle Cobb-Douglas production functions, a CES formulation was used with $\rho_i = .01$. The α_i were determined by appealing to the relative shares of labor and capital in the two sectors in a manner which will be described later.

Two consumers are considered; the first heuristically represents the upper ten percent of the income recipients, while the second represents the lower ninety percent. The first is endowed with approximately 23 percent of the economy's labor (corresponding to the observed share of labor income going to the top decile of income receivers) and 40 percent of total stock of capital. This latter figure roughly corresponds with the share

It also follows that such demand functions have price and income elasticities of unity (the price elasticity is actually minus one, following our earlier convention). This, of course, deviates from Harberger's assumption (commented on in Section 6) of price elasticities of -1/7 in the "corporate" sector and -6/7 in the "non-corporate" sector.

Property (6.5) of the demand functions (6.2) permits one to impose the observed aggregate five to one expenditure ratio for the outputs of the heavily and lightly taxed sectors. Harberger's data reveals that approximately 250 billion dollars a year was spent on "corporate" products, while only 50 billion was spent on the output of the "non-corporate" sector. If each individual's tastes were such that

$$a_{1j} = 5a_{2j} \qquad j = 1,\ 2$$

where "corporate" product is labelled commodity 1 and the "non-corporate" product commodity 2, the expenditure ratios of the model would exactly correspond to the five to one figure. However, in the examples investigated here lower income individuals are taken to place a relatively higher weight on the output of the lightly taxed sector than higher income individuals. This is consistent with diminishing budget shares allocated to food expenditures (agricultural output) as income rises. The ratios a_{1j}/a_{2j} assumed are 7.00 for the higher income consumer and 4.25 for the lower income consumer.

The total endowments of the economy are assumed to correspond with Harberger's figures. That is, in the absence of a labor-leisure choice, the labor endowment is taken to be 220 billion units while the capital services

of capital income going to the top ten percent of the income receivers although it is much lower than the share of capital income going to the top ten percent of wealth holders (Projector and Weiss [1966]). Endowing the high income receivers with more than ten percent of the labor appeals to an equal endowment of labor in natural units but a disproportionate endowment in efficiency units.

Each consumer's demand functions are of the form

$$x_{ij} = \frac{a_{ij} I_j}{P_i} \qquad i = 1, \ldots, n \; ; \quad j = 1, 2 \tag{6.2}$$

where x_{ij} is consumer j's demand for commodity i, a_{ij} measures the intensity of his desire for commodity i, P_i is the price of the ith good, and I_j is individual j's income, given by

$$I_j = \sum_{i=1}^{n} P_i w_{ij} \tag{6.3}$$

where the w_{ij} are his initial holdings. These demand functions can be derived from Cobb Douglas utility functions of the form

$$U_j = X_1^{a_{1j}} X_2^{a_{2j}} \ldots X_n^{a_{nj}} = \prod_{i=1,n} X_i^{a_{ij}} \quad \text{where} \quad \sum_{i=1}^{n} a_{ij} = 1 \tag{6.4}$$

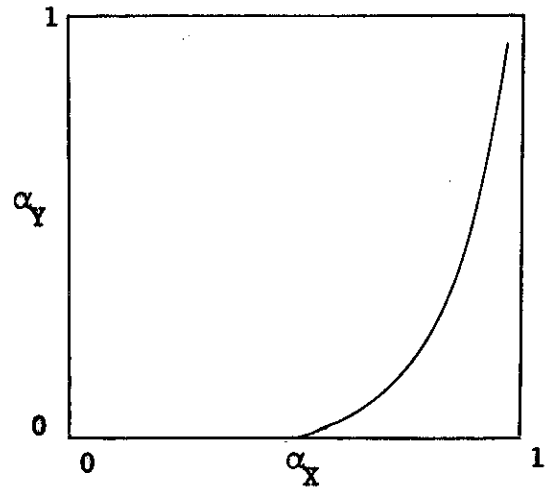
or any monotonic transformation of U_j. One convenient property of the demand functions (6.2) is that the share of individual j's income spent on commodity i equals his value of a_{ij}. That is,

$$a_{ij} = \frac{x_{ij} P_i}{I_j} \; . \tag{6.5}$$

endowment is assumed as 38 billion units. In the presence of a labor-leisure choice each individual's endowment of labor is increased by a factor of seven to four. This represents the possibility of working up to a seventy hour week instead of the more common forty hours. The demand for leisure is then imposed in such a way that each individual supplies the same amount of labor in the presence of the tax as in the fixed labor supply case. This artificial construction is then used to generate different solutions for cases where taxes are absent. In the presence of the tax, 19.5 billion units of the 38 billion units of capital are allocated to the "corporate" sector (from Table 1, Section 6). Thus, the initial endowment of "corporate capital tickets" is taken as 19.5 billion units. To take account of the redistributive element of the government expenditure side (operating through welfare payments, etc.) the higher income consumer (representing the top ten percent of the income receivers) is endowed with only five percent of the tickets, the remaining ninety-five percent going to the lower income consumer.

Each of the three sets of elasticity of substitution assumptions was examined with and without a labor-leisure choice, for a total of six cases. The procedural method for each case was to first consider the taxed or distortionary situation. As mentioned in Section 3, the tax rate cannot be directly imposed. This is also true for labor's relative share in the two sector. Harberger's data indicates that the observed surtax rate is 85 percent. He also states that the return to labor was approximately ten times the return to capital in the "corporate" sector ($200 billion vs. $20 billion) while the two returns were approximately equal in the "non-

corporate" sector ($20 billion for each) for the 1953-1959 period. The aggregate wage bill-net capital return ratio was 5.5. The parameters α_X and α_Y were chosen so as to match these observations "as closely as possible."* In each case many combinations of α_X, α_Y give rise to the desired 85 percent surtax rate. Shown below is a typical plot of all α_X, α_Y resulting in an 85 percent surtax rate. The point along this curve which



implies the most reasonable share data is presumed to give the appropriate set of α. In all the cases considered α_X and α_Y were determined such that the relative share data was within about twenty percent of the observed values. Once α_X and α_Y are determined, the taxes are removed (simply by eliminating tickets), and the before and after tax equilibria may be compared.

Table 2 contains a partial description of the before and after tax equilibria for the fixed labor supply case with $S_X = S_Y = 1$. When vectors are presented, the components refer to commodities in the order (1) "corporate" output, (2) "non-corporate" output, (3) labor, and (4) capital services.

*"as closely as possible" is loosely defined. Further econometric work in this area is necessary to develop more objective criteria.

The first row gives before and after tax prices, normalized to sum to unity. As one would expect, the price of "corporate" output increases while the price of the "non-corporate" output and the net price of capital services decrease with the imposition of the tax. This lowering of the price of the lightly taxed output benefits the lower income consumer relatively more than the higher income consumer, since he spends a larger fraction of his budget on these items. The second row, "labor normalized prices," gives the before and after tax prices, normalized such that the price of labor remains constant. This corresponds to Harberger's normalization. The factor substitution effects of the surtax are given in rows (4)-(7), while rows (8) and (9) show the change in aggregate output of the two sectors. Row (10) gives the aggregate relative share of labor, and, as such, is useful in determining the incidence of the distortion on the functional distribution of income. Rows (11) and (12) give the sectoral disaggregation of labor's relative share. The figure in row (13) represents the fraction of national income spent on the output of the heavily taxed sector and corresponds closely to Harberger's observation of 5/6. The individual relative share information presented in rows (16) and (17) is useful in estimating the impact of the surtax on the personal distribution of income.* Rows (18)-(21), on the other hand, provide statistics concerning the efficiency cost of the distortion. If we let X_1 be the total market demand

*It may be noted that welfare evaluations may be calculated both before and after the tax for the purpose of gaining further insight into the incidence questions. However, since the utility function assumed is determinate only up to a monotonic transformation, such welfare comparisons are to a degree arbitrary.

TABLE 2

SAMPLE RESULTS

Assumptions: $S_X = 1$ $S_Y = 1$ $E = 1$

Fixed Labor Supply

	Before Tax				After Tax			
1. Final Prices	.2525	.3605	.1425	.2445	.2989	.3471	.1610	.1930
2. Labor Normalized Prices	.2525	.3605	.1425	.2445	.2646	.3072	.1425	.1709
3. Tax Rate	0.0				84.9			
4. $(K/Q)_X$	.1890				.1536			
5. $(L/Q)_X$	1.4470				1.5159			
6. $(K/Q)_Y$	.6589				.8019			
7. $(L/Q)_Y$	1.3991				1.1941			
8. Total Demand for X	133.2131				126.9578			
9. Total Demand for Y	19.4679				23.0713			
10. $P_L L/P_K K$	3.37				4.83			
11. $P_L L_X/P_K K_X$	4.46				8.23			
12. $P_L L_Y/P_K K_Y$	1.24				1.24			
13. $P_X X/(P_X X + P_Y Y)$	.827				.826			
14. Individual 1's Demands	37.58	3.76	0.0	0.0	32.62	4.01	0.0	0.0
15. Individual 2's Demands	95.63	15.71	0.0	0.0	94.33	19.06	0.0	0.0
16. Individual 1's Rel.Share	.267				.243			
17. Individual 2's Rel.Share	.733				.757			
18. GNP before Tax Prices	40.648				40.368			
19. GNP after Tax Prices	46.574				45.955			

20. Laspeyres Real Income Index = .99311
21. Paasche Real Income Index = .98671
22. Shift Factor = .0108

for "corporate" output and X_2 the total market demand for "non-corporate" output, then row (18) evaluates GNP (i.e. $P_1X_1 + P_2X_2$) at before tax prices (row 1), while row (19) utilizes the after tax prices. Row (20) gives the Laspeyres real income index of the ratio of real income after tax to real income before tax, while row (21) contains the Paasche index for this ratio. While the true social loss due to the tax cannot be determined since we do not have or know the social utility function defined over aggregate outputs, it is easily shown (given "reasonable" assumptions) that the Laspeyres index, based on before tax or "initial" prices, provides a lower bound for welfare losses while the Paasche index, based on after tax or "final" prices, provides an upper bound. Row (22), the "shift factor," is calculated to provide further insight into the incidence of the tax. The definition of this term is

$$\text{(7.6)} \qquad \text{shift factor} = 1. + \frac{\Delta P_K K}{\text{govt. revenue}} = 1. + \frac{\Delta P_K K}{P_T T}$$

where ΔP_K is the change in the net price of capital, T is the total endowment of tickets, and P_T is the price of tickets. A shift factor of one implies that $\Delta P_K = 0$ and, given Harberger's definition, capital and labor may be said to bear the burden of the tax equally. If the shift factor is zero, the decrease in the total return to capital, $-\Delta P_K K$, is equal to the government revenue from the tax, and, in this sense, capital bears the full burden. A negative value of the shift factor would indicate that capital bears more than 100 percent of the burden of the tax.

Table 3 contains a summary of the results of the six cases examined in this study. The figures in parentheses are those of Hargerger for the

TABLE 3

SUMMARY OF RESULTS

(1) S_X	(2) S_Y	(3)	(4) Relative Share of Rich Before Tax/ After Tax	(5) $(K_X)_{B.T.}-(K_X)_{A.T.}$	(6) $(L_X)_{B.T.}-(K_X)_{A.T.}$	(7) Shift Factor	(8) ΔGNP B.T. Prices $ billion	(9) ΔGNP A.T. Prices $ billion	(10) Harberger's Est. of ΔGNP $ billion
1.0	1.0	Fixed L	.267 / .243	-5.673 (-6.9)	-.308	.0108	2.067	3.987	(2.9)
1.0	1.0	Labor/Leisure	.252 / .237	-5.618	-.257	.0072	2.103	3.984	
1.0	.5	Fixed L	.266 / .242	-4.716 (-5.9)	-2.429	-.1586	1.317	3.450	(2.5)
1.0	.5	Labor/Leisure	.252 / .236	-4.667	-1.442	-.1448	.285	2.349	
.75	.25	Fixed L	.270 / .243	-4.053	-3.747	-.0887	.966	3.519	
.75	.25	Labor/Leisure	.254 / .238	-3.990	-.3059	-.0806	.321	2.787	

two cases most comparable to those shown in his 1966 article. Column (4) indicates that the distortionary treatment of income from capital plus the redistributionary expenditure program assumed here reduces the relative share of the top ten percent of the income recipients by somewhat less than two percentage points. The results shown in column (5) indicate the capital shift due to the presence of the tax and, for the two comparable cases, our estimates of this transfer are somewhat smaller than Harberger's. The seventh column of Table 3 is of interest in that, with the exception of the $S_X = S_Y = 1$ cases, capital bears more than the full burden of the tax. This result contrasts strikingly with the conclusion of Krzyzaniak and Musgrave [1963] that the corporate income tax is more than one hundred percent shifted and, hence, capital bears none of the burden. The final three columns of the table contain estimates of the efficiency cost of the surtax. The first of these, column (8), is the Laspeyre index of the loss multiplied by 300 billion dollars, the approximate average national income for the 1953-1959 period. Column (9) is based on the Paasche index of real income loss which is likewise multiplied by 300 billion dollars. These two numbers for each case provide an upper and lower bound for the efficiency loss or the dead weight loss of the distortionary situation. For the two cases for which a comparison is possible, Harberger's point estimates are well within the two extreme values given here. However, it should be noted that in the $S_X = 1.0$, $S_Y = .5$ and in the $S_X = .75$, $S_Y = .25$ cases the addition of the possibility of a labor-leisure choice substantially reduces the loss estimates. Indeed, for the first of these two cases, even the loss estimate based on after tax prices is less than Harberger's estimate of 2.5 billion dollars for the corresponding case with a fixed labor supply.

7. Conclusion

The algorithmic method presented for evaluating the effects of economic distortions seems to offer a great deal of added generality and flexibility over previous methods. For the example presented in this paper, it has proven to be a practical, usable technique. In particular, it would seem that distortions which due to their size or multiplicity can only be inadequately dealt with by the techniques of differential calculus now can be more satisfactorily analyzed at an empirical level. Moreover, in the example presented, while a similarity of results between these general equilibrium calculations and earlier findings is obtained in the specific case of an elasticity of substitution in the neighborhood of unity in each sector, this similarity disappears when other formulations of the production side of the economy are considered. When the added generality of a labor-leisure choice is introduced, even more divergence from earlier calculations is obtained. It would seem that in those areas where policy judgments are to be made on the basis of calculations of distortionary impacts, major attention should be focused upon analyzing the effects with general equilibrium computation techniques such as presented here.

REFERENCES

Hansen, T., [1968]. "On the Approximation of a Competitive Equilibrium," Ph.D. Thesis, Yale University, 1968.

Harberger, A.C., [1959]. "The Corporation Income Tax: An Empirical Appraisal," in Tax Revision Compendium, House Committee on Ways and Means, 86 Congress 1 sess., Vol. 1, 1959, pp. 231-250.

_____, [1962]. "The Incidence of the Corporation Income Tax," Journal of Political Economy, Vol. 70, 1962, pp. 215-240.

_____, [1964]. "The Measurement of Waste," American Economic Review, Papers and Proceedings, Vol. 54, 1964, pp. 58-76.

_____, [1966]. "Efficiency Effects of Taxes on Income from Capital," in Effects of Corporation Income Tax, M. Krzyzaniak, ed., Detroit, 1966.

Johnson, H.G., [1956]. "General Equilibrium Analysis of Excise Taxes: Comment," American Economic Review, Vol. 46, 1956, pp. 151-156.

_____, [1966]. "Factor Market Distortions and the Shape of the Transformation Curve," Econometrica, Vol. 34, 1966, pp. 686-695.

_____ and P.M. Mieszkowski, [1970]. "The Effects of Unionization on the Distribution of Income: A General Equilibrium Approach," Quarterly Journal of Economics, Vol. 84, 1970.

Krzyzaniak, M. and R.A. Musgrave, [1963]. The Shifting of the Corporation Income Tax, Baltimore, Md., Johns Hopkins Press, 1963.

Lucas, R.E., Jr., [1969]. "Labor-Capital Substitution in U.S. Manufacturing," in The Taxation of Income from Capital, Arnold C. Harberger and Martin J. Bailey, eds., Washington, D.C., The Brookings Institution, 1969.

McKenzie, L.W., [1959]. "On the Existence of General Equilibrium for a Competitive Market," Vol. 27, No. 1, pp. 54-71.

Meade, J., [1955]. Mathematical Supplement to Trade and Welfare, Oxford, Oxford University Press, 1955.

Mieszkowski, P.M., [1967]. "On the Theory of Tax Incidence," Journal of Political Economy, Vol. 75, 1967, pp. 250-262.

Minasian, J.R., [1961]. "Elasticities of Substitution and Constant-Output Demand Curves for Labor," Journal of Political Economy, Vol. 69, 1961, pp. 261-270.

Projector, D.S. and G.S. Weiss, [1966]. Survey of Financial Characteristics of Consumers, Washington, D.C., Federal Reserve Technical Papers, 1966.

Rosenburg, L.G., [1969]. "Taxation of Income from Capital, by Industry Group," in The Taxation of Income from Capital, Arnold C. Harberger and Martin J. Bailey, eds., Washington, D.C., The Brookings Institution, 1969.

Scarf, H.E., [1967]. "On the Computation of Equilibrium Prices," Ten Economic Studies in the Tradition of Irving Fisher, New York, John Wiley and Sons, Inc., 1967.

_____, [1969]. "An Example of an Algorithm for Calculating General Equilibrium Prices," The American Economic Review, Vol. 59, No. 4, pp. 669-677.

_____, [1972]. The Computation of Economic Equilibria, forthcoming.

Solow, R.M., [1964]. "Capital, Labor, and Income in Manufacturing," in The Behavior of Income Shares, NBER, 1964, Princeton, Princeton University Press, 1964.

Tambini, L., [1969]. "Financial Policy and the Corporation Income Tax," in The Taxation of Income from Capital, Arnold C. Harberger and Martin J. Bailey, eds., Washington, D.C., The Brookings Institution, 1969.

APPENDIX

A. Final Termination Routine

The algorithm terminates with a final primitive set of n price vectors which are close to each other and approximate an equilibrium price vector. As a first point estimate of an equilibrium vector we can take the center (i.e. the mean) of the n primitive set price vectors. This first approximation may be improved upon by solving a series of linear programming problems.

Let the final primitive set of the algorithm be given by

$$PS^* = \begin{pmatrix} p_1^{*1} & \cdots & p_1^{*n} \\ \vdots & & \vdots \\ p_n^{*1} & & p_n^{*n} \end{pmatrix}, \tag{A.1}$$

where subscripts identify commodities and superscripts price vectors. Given this notation, the first approximation for a competitive equilibrium price vector is given by

$$\begin{aligned} P_1^* &= \sum_{j=1}^{n} p_1^{*j}/n \\ &\vdots \\ P_n^* &= \sum_{j=1}^{n} p_n^{*j}/n \,. \end{aligned} \tag{A.2}$$

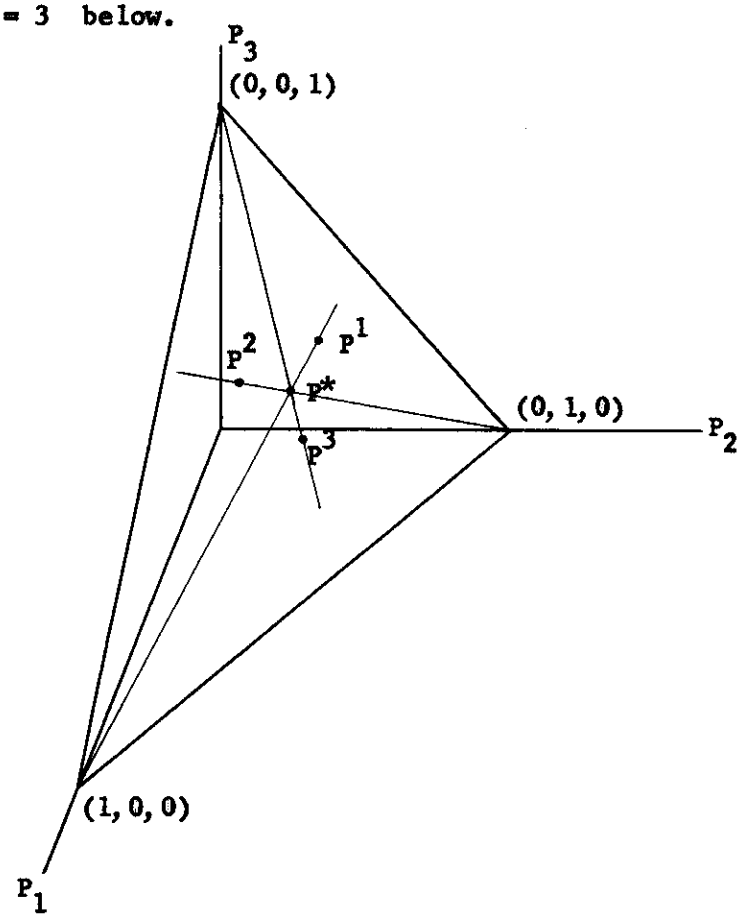
With P^* as a point estimate we generate a set of n price vectors surrounding it which together define the region of search for a better approximation. These are given by

$$
\text{(A.3)} \qquad
\begin{array}{lcl}
P_1^1 = \dfrac{P_1^* - \Delta P_1^*}{D_1} & \cdots & P_1^n = \dfrac{P_1^*}{D_n} \\
P_2^1 = \dfrac{P_2^*}{D_1} & \cdots & P_2^n = \dfrac{P_2^*}{D_n} \\
\vdots & & \vdots \\
P_n^1 = \dfrac{P_n^*}{D_1} & \cdots & P_n^n = \dfrac{P_n^* - \Delta P_n^*}{D_n}
\end{array}
$$

where Δ is a small number (typically, say, .02) and

$$
\text{(A.4)} \qquad D_j = 1 - \Delta P_j^* / \sum_{i=1}^{n} P_i^* \qquad j = 1, \ldots, n .
$$

P^i can be thought of as a vector arrived at by a movement along the ray connecting P^* and the ith vertex in a direction away from that vertex as is shown for $n = 3$ below.



Assuming an interior solution (i.e. no free commodities), we desire a price vector for which demand is equal (or very nearly so) to supply for all commodities and for which the profitability of each sector is zero when it utilizes its optimal technology. Let $f(P) = X(P) - S(P)$ be the vector of excess demands and let $\pi^i(P)$ be the per unit profit of the ith sector when it operates with cost minimizing input proportions. We shall make use of the linearization assumptions

$$\text{(A.5)} \qquad f\left(\sum_{j=1}^{n} \alpha_j P^j\right) \cong \sum_{j=1}^{n} \alpha_j f(P^j)$$

and

$$\text{(A.6)} \quad \pi^i\left(\sum_{j=1}^{n} \alpha_j P^j\right) \cong \sum_{j=1}^{n} \alpha_j \pi^i(P^j) \quad \text{for all } i \qquad \sum_{j=1}^{n} \alpha_j = 1 \qquad \alpha_j \geq -\delta \ .$$

That is, the excess demand vector for some weighted sum of the price vectors P^j is approximately the weighted sum of the excess demands at those prices, and similarly for the profitability of each sector.

The market demand functions $X(P)$ are assumed continuous and uniquely defined for all positive price vectors P. However, since we have assumed constant returns to scale (CRTS) production technology, the supply response $S(P)$ is less well defined. As is well known, there is a scale indeterminacy with CRTS when profits are zero. One possible supply response would give all of the capital services (a fixed factor) to the sector whose optimal activity is most profitable if that profit is non-negative. If none of the sectors can break even, there would be no production, and the supply,

$S(P)$, would simply be the vector of initial holdings, W . This provides such a discontinuous supply response that it and other similar formulations were found unworkable. Linearizing such a response is simply not a good approximation.

We know that the absolute value of the profitability of each sector is quite small in the vicinity of the final primitive set since at least one of the corresponding commodity (B) vectors is a column of demands (i.e. none of the sectors can make a non-negative profit) while at the same time others are the negative of activity vectors, indicating a positive profitability. Therefore, a reasonable supply response operates each sectors with its optimal technology and scales their activity levels so as to meet output demand as nearly as possible given the fixed supply of capital services. That is, given a price vector P , the vector of market demands $X(P)$ can be determined. If we let k_i = the optimal capital service input per unit of output in sector i , then the total requirement for capital services is

$$K^* = \sum_{i=1}^{nsect} X_i k_i \quad \text{where nsect = the number of sectors.} \tag{A.7}$$

Each sector i may be allocated capital services in the following manner:

$$K_i = \frac{X_i k_i}{K^*} K \tag{A.8}$$

where K is the economy's endowment of capital services. This then determines the scale of each sector and results in a continuous supply response which can reasonably be linearized. Certainly there are other supply responses which may work as well as this one.

In the supply response described above, all of the capital service available is used. That is, the net supply of capital services is zero. Since in this model only outputs and leisure give utility, the demand for capital services is also zero. The excess demand for capital services is by definition zero, and hence we are concerned only with the excess demands of the remaining n-1 commodities being small in absolute value. In the linear programming problems referred to earlier we also impose the constraints that the absolute value of the profitability of each sector be "small" in a sense which will be made clear.

The linear programming problem can be formulated as

$$
\begin{aligned}
&\min \ \epsilon \\
&\text{subject to} \quad \sum_{j=1}^{n} \alpha_j f_i(P^j) \leq \epsilon \qquad \text{for } i = 1, \ldots, n-1 \\
&\qquad\qquad \sum_{j=1}^{n} \alpha_j \pi^i(P^j) \leq \epsilon \qquad \text{for } i = 1, \text{nsect} \\
&\qquad\qquad \sum_{j=1}^{n} -\alpha_j \pi^i(P^j) \leq \epsilon \qquad \text{for } i = 1, \text{nsect} \\
&\qquad\qquad \sum_{j=1}^{n} \alpha_j = 1 , \quad \alpha_j \geq -\delta .
\end{aligned}
\tag{A.9}
$$

That is, the α_j , $j = 1, \ldots, n$, are desired which minimize the largest of the linearized excess demands for the n-1 commodities other than capital services and the absolute value of the per unit profitability of each sector. This can be transformed into a more manageable problem as will be shown.

Consider the first $n-1$ constraints. Add to each side $\delta \sum_{j=1}^{n} f_i(P^j)$ and then divide by $\epsilon+M$ where M is a positive constant. This gives

$$\text{(A.10)} \quad \sum_{j=1}^{n} \frac{(\alpha_j + \delta)}{\epsilon + M} f_i(P^j) \leq \frac{\epsilon + \delta \sum_{j=1}^{n} f_i(P^j)}{\epsilon + M} \qquad i = 1, \ldots, n-1 .$$

Let

$$\text{(A.11)} \qquad y_j = \frac{\alpha_j + \delta}{\epsilon + M}$$

and note that

$$\text{(A.12)} \qquad \sum_{j=1}^{n} y_j = \frac{1 + n\delta}{\epsilon + M} .$$

Since M, δ, and n are given parameters, maximizing Σy_j is equivalent to minimizing ϵ. The first $n-1$ constraints can be written as

$$\text{(A.13)} \quad \sum_{j=1}^{n} y_j f_i(P^j) \leq 1 + \frac{\delta \sum_{j=1}^{n} f_i(P^j) - M}{\epsilon + M} \qquad i = 1, \ldots, n-1 .$$

From (A.12) above we know that

$$\text{(A.14)} \qquad 1/\epsilon+M = \sum_{j=1}^{n} y_j/(1 + n\delta) ,$$

and thus (A.13) can be written as

$$\text{(A.15)} \quad \sum_{j=1}^{n} y_j f_i(P^j) \leq 1 + \left(\frac{\delta \sum_{j=1}^{n} f_i(P^j) - M}{1 + n\delta} \right) \sum_{j=1}^{n} y_j \qquad i = 1, \ldots, n-1 .$$

Let

$$C_i = \frac{\delta \sum_{j=1}^{n} f_i(P^j) - M}{1 + n\delta} \tag{A.16}$$

and subtract $C_i \sum_{j=1}^{n} y_j$ from each side of (A.15). The first n-1 constraints are then given by

$$\sum_{j=1}^{n} (f_i(P^j) - C_i)y_j \leq 1 \quad \text{for } i = 1, \ldots, n-1 . \tag{A.17}$$

The remaining four constraints can be similarly transformed giving

$$\sum_{j=1}^{n} (\pi^i(P^j) - D_i)y_j \leq 1 \quad \text{for } i = 1, \text{nsect} \tag{A.18}$$

and

$$\sum_{j=1}^{n} (-\pi^i(P^j) - E_i)y_j \leq 1 \quad \text{for } i = 1, \text{nsect} \tag{A.19}$$

where

$$D_i = \frac{\delta \sum_{j=1}^{n} \pi^i(P^j) - M}{1 + n\delta} \tag{A.20}$$

and

$$E_i = \frac{-\delta \sum_{j=1}^{n} \pi^i(P^j) - M}{1 + n\delta} . \tag{A.21}$$

With the above manipulations, the problem (A.9) has been transformed to

$$\max \sum_{j=1}^{n} y_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} (f_i(P^j) - C_i)y_j \leq 1 \quad \text{for } i = 1, \ldots, n-1$$

$$\text{(A.22)} \qquad \sum_{j=1}^{n} (\pi^i(P^j) - D_i)y_j \leq 1 \quad \text{for } i = 1, \text{nsect}$$

$$\sum_{j=1}^{n} (-\pi^i(P^j) - E_i)y_j \leq 1 \quad \text{for } i = 1, \text{nsect}$$

$$y_j \geq 0 \ .$$

Given a solution to this linear programming problem, the optimal α's and ϵ for the original one (A.9) are

$$\hat{\epsilon} = \frac{1 + n\delta}{\sum_{j=1}^{n} \hat{y}_j} - M$$

and

$$\hat{\alpha}_j = (\hat{\epsilon} + M)\hat{y}_j - \delta \qquad j = 1, \ldots, n \ .$$

The new point approximation of an equilibrium price vector is given by

$$P^* = \sum_{j=1}^{n} \hat{\alpha}_j P^j \ .$$

At P^* the excess demands and the profitabilities of the sectors are computed and if they are too large in absolute value a new linear programming problem is set up by defining a set of n price vectors P^j which center

around P^*. The area of search, defined by Δ in (A.3), is systematically reduced, and as this occurs, the linearity assumptions become more valid. Another estimate of an equilibrium price vector is generated and, again, the process may be repeated. The computational experience we have had indicates that $\hat{\epsilon}$, and, indeed, actual excess demands, rapidly converges towards zero.