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COMPETITIVE EQUILIBRIUM AND GAME THEORY SOLUTIONS:

PART I. THE CORE AND VALUE

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COMPETITIVE EQUILIBRIUM AND GAME THEORY SOLUTIONS:

PART I. THE CORE AND VALUE

by

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I. ON GENERAL EQUILIBRIUM MARKET MODELS

A general equilibrium model of an economy is a closed and total model of the economic units and their interactions. From Walras, to the formulation given by G. Debreu, the models have been presented at a high level of abstraction with a lack of "realism" in the sense that the role of government, financial and noneconomic institutions are not considered. In particular the actors in a general equilibrium model are usually only of one or two types. They are respectively traders, or consumers and producers. The consumers are portrayed as maximizing individual utility, while the producers are involved in attempting to maximize "profit" (see G. Debreu, 1959). These profits are paid out to the owners of the firms, thus accounts always balance and the "money" in the model is only a unit of account with an overall balance of zero.

The central result of general equilibrium analysis has been the analysis of the conditions for which a price system exists and has the property that each individual looking only at the array of prices can maximize his own utility or profit with the overall outcome being Pareto Optimal to the society as a whole. In other words, the price system provides for a decentralization of decisionmaking, to the level of the economic unit, that is efficient and balances budget conditions and the physical flows of goods.

All of the observations made above can be simply illustrated for an economy consisting of only two traders by means of the Edgeworth box diagram. This is shown in Figure 1. Prior to discussing the diagram however, a further note on the nature of the economic abstractions made in the study of general equilibrium models of markets is called for. It has been shown by T. Rader that if the usual conditions are assumed concerning the nature of consumer preferences and production processes, then an economy with production has associated with it an equivalent economic model involving only trading among the participants (see T. Rader, 1964). Essentially the effect of the production processes can be accounted for by adjusting the commodity space and the preferences of the economic agents. This result is useful inasmuch as any result that can be established for a trading model of an economy will also hold true for a model that introduces production explicitly.

Trading models are somewhat easier to analyze than models involving both trade and production. They call for the consideration of only one type of economic agent, the trader, who buys and sells to his own account.

There are many economic problems and processes for which the dis-

inction between producer and consumer is of central importance. The existence and properties of the price system given the type of assumptions made by Walras, Debreu and others (see K.J. Arrow and G. Debreu, 1954), is not one of these. Economic theorists have been somewhat cavalier in their treatment of production and ownership. Undoubtedly much work is needed to make general equilibrium models more realistic by providing for better and more detailed models of ownership as well as production models involving set-up costs and other "more realistic" features. However these pose other problems that we do not intend to deal with in this paper. Thus for the purposes at hand without much loss of generality, making use of Rader's result we may confine our remarks to trading models.

The simplest of trading models is that used by Edgeworth for expository purposes, involving two traders, trading in two commodities. Call the traders, Trader I and Trader II; call the commodities X and Y. Let the (variable) amount of commodity X be denoted by x and the amount of Y by y . Let the initial holdings of Trader I be denoted by $(0, b)$ and of Trader II by $(a, 0)$. Let the point (x, y) denote the allocation to Trader I hence $(a-x, b-y)$ specifies that to Trader II. Let the utility functions of the traders be respectively φ and ψ . For any point in the box Trader I will have a value of $\varphi(x, y)$ and Trader II of $\psi(a-x, b-y)$. The no trade point is T with coordinates $(0, b)$ and $(a, 0)$ respectively. The no asset points are O and O' for Traders I and II respectively. Trader I improves his welfare when he moves up and to the right in the Edgeworth box. Trader II does better moving down and to the left.

The family of curves $\varphi = \text{constant}$, appear as convex, whereas $\psi = \text{constant}$, appear as concave in the diagram.

The assumptions made in the various proofs of the existence of a price system (or a competitive equilibrium) are that the preferences of the players for different bundles of commodities can be portrayed by convex sets, or in terms of the usual textbook illustrations employing indifference curves, the indifference curves are concave when viewed from the north east.

The original work in the existence of a competitive equilibrium made use not only of convexity but also of properties of continuity and differentiability of indifference curves. Each of these conditions has specific economic meaning. The more modern approach, as exemplified by the work of Debreu (1959) has been able to dispense with most of these restrictions. The mathematical details of the work do not concern us immediately in this paper, however other results related to the existence of a price system do depend upon extra properties. These will be noted in Section II.

Market Games and General Equilibrium

In recent years there have been a series of applications of the theory of games to the study of markets (see L.S. Shapley and M. Shubik, 1969). They have also been addressed to an understanding of the price system. However the approach of these studies has been far different from the study of decentralization and efficiency that characterizes the study of competitive equilibrium.

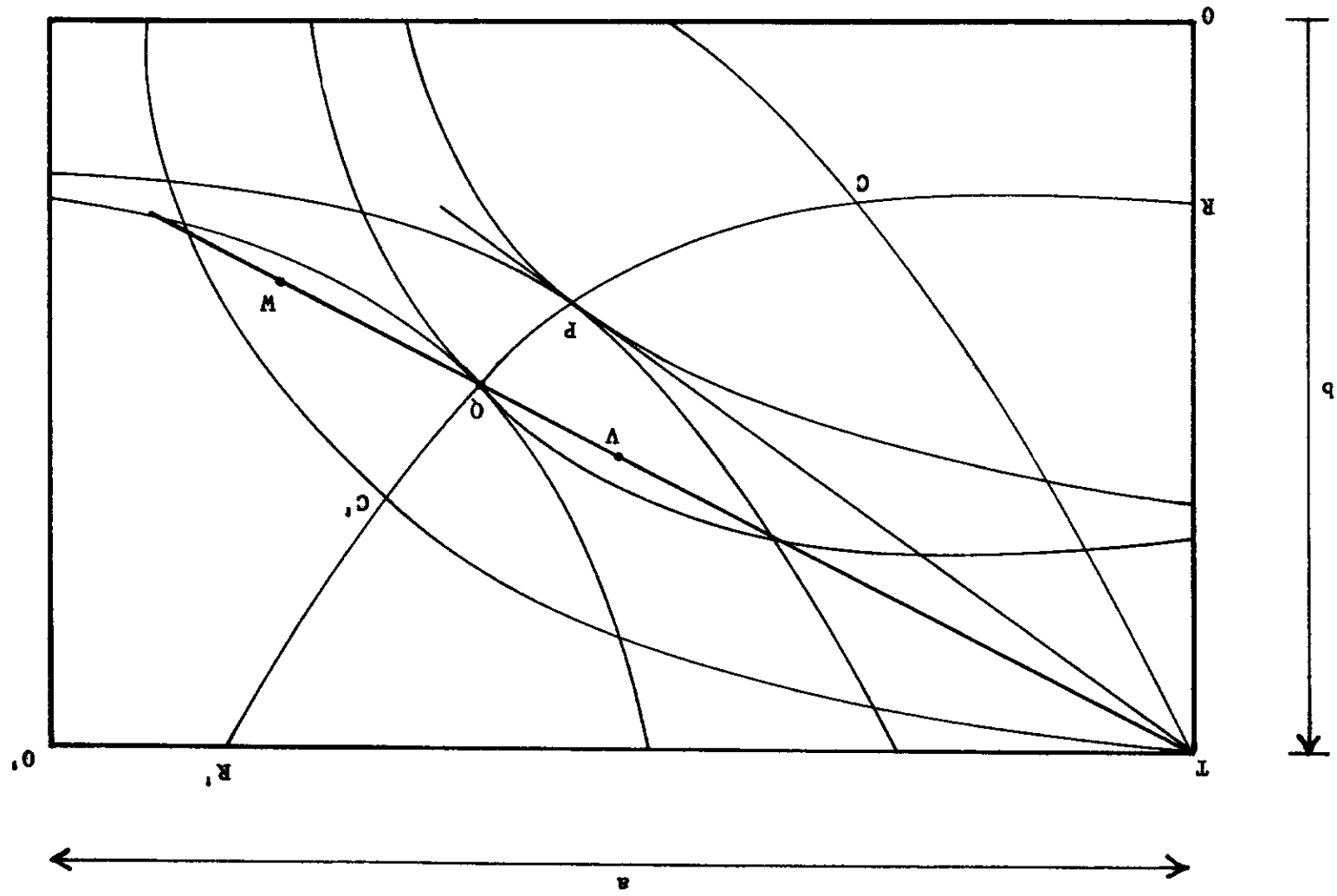
The theory of games is addressed to the investigation of situations involving potential conflict or cooperation with virtually any underlying structure. In order to apply it to economic problems the structure of the games or models to be studied needs to be limited to reflect the economic conditions. A class of games called market games (see L.S. Shapley and M. Shubik, 1969) may be defined so that they are based precisely upon the same assumptions used in studies of the competitive equilibrium and price system.

Much of the analysis of the theory of games has been directed to investigating the power of groups in coalition. Market games reflect this concern in an extremely natural manner. It can be shown that any market game with n players can be regarded as equivalent to a trading economy with n traders each with an initial endowment of goods. Any coalition of traders can obtain for themselves precisely the outcomes that can be achieved by regarding them as a special subeconomy where the traders choose to trade only among themselves.

II. MASS MARKETS WITH MANY INDIVIDUALS

The existence of a price system that could be used for decentralization of economically efficient decisionmaking is independent of the number of traders in the economy. The Edgeworth box shown in Figure 1 illustrates the price system that would prevail in this two person trading economy. It is given by the line TP . Suppose that we were to consider a market with four, rather than two traders. In particular suppose that two of the

FIGURE 1



traders were identically like Trader I in the previous market and the other two were identical with Trader II.

We can still use the Edgeworth box diagram to portray outcomes which treat players of the same type in the same way, even though we now have more than one trader of each type. In order to describe all of the possible distributions of the goods, however, we would need an Edgeworth box in four dimensions for this four person market. Similarly, if there were n traders of each type, for full generality, an Edgeworth box of $2n$ dimensions would be called for.

It is easy to see immediately that a price system will treat identical players identically (where identical means the same preferences and endowments). Let us call the market where there are n traders of each type, the n^{th} replication of the market where there is a single player of each type. We may denote the latter by $(1,1)$ and the former by (n,n) . Figure 1 shows the equal treatment outcomes for players of the same type for any replication of the original two person market. In particular the line TP shows that the price system will not be changed as the number of traders becomes larger in the manner described.

In general, as long as the structural conditions hold with regard to individual preferences and production conditions, then regardless of the number of traders, the number of types of traders or the number of commodities being traded, a price system will always exist. If some of the structural conditions are not satisfied then the price system may not exist, or at least if prices can be found that clear all markets they will not necessarily be efficient.

The type of structural conditions that destroy the price system are well known. They include increasing returns to scale in production; external economies or diseconomies in production or consumption; set up costs; indivisibilities, etc. We do not discuss them further in this paper.

The existence of the appropriate market structural conditions alone is not sufficient to guarantee that a price system will emerge in an actual market. They only guarantee that the existence of an efficient price system is logically consistent with the structure of tastes and technology.

The proofs of the existence of a competitive equilibrium in a market, and hence an efficient price system are all based upon a very special and extremely limiting behavioral assumption. This is the assumption that individual traders (producers or consumers) act as price takers. Suppose that we assume that prices are quoted for all commodities. Now we assume that any trader looks upon these prices as given and maximizes his welfare taking these prices as part of his environment. If the market has the appropriate structure then we can prove that there exists at least one set of prices which is efficient and balances all trades.

It is extremely important to observe that if we impose a price-taking behavior on all traders and assume the existence of some sort of prices, then the proof of the existence of efficient prices in no way depends upon the number of traders.

There is no a priori reason why traders should behave as price takers. A market may satisfy all of the structural conditions for the existences of a competitive equilibrium, but will fail to achieve it because of "failure" in behavior. In particular this is easily illustrated in Figure 1 and was

an early result of Edgeworth. If there are only two traders in a market and they are not forced to act as price-takers, then from economic considerations alone we may suspect that the final trade may lie anywhere along the curve connecting C and C' . This curve CC' is known as the contract curve. It lies on the Pareto optimal surface (which is fully described by $ORCPC'R'O'$). The bounds C and C' are determined by consideration of individual rationality. C lies on the indifference curve TC where the coordinates of T specify the initial holdings of both traders. No trader need accept an outcome that yields less satisfaction than he can obtain without trading hence Trader I will not accept an outcome less favorable than those on TC and Trader II will not accept an outcome less favorable than those on TC' . Consideration of joint rationality or efficiency or societal welfare make it reasonable to pick a Pareto optimal outcome. Consideration of individual rationality limits these outcomes to CC' .

We observe that the outcome P resulting from the use of an efficient price system is contained within the contract curve, but it is by no means that only outcome consistent with efficiency and individual rationality.

Edgeworth suggested that the contract curve be regarded as the economic solution to trading in a market with just two traders. It is the economic solution in the sense that economic considerations alone provide us here with only the guidelines of individual rationality and social efficiency. These suffice only to limit the acceptable outcomes to CC' . More information is needed to be able to predict or prescribe a single specific outcome. Imposing the behavior condition that the traders must act as price-takers is sufficient. In this instance the outcome predicted is P .

Game Theoretic Solutions

There are many different concepts of solution which have been studied in the theory of games. Each solution concept may be regarded as specifying a different set of behavioral constraints on the players. In particular there are three major types of solutions:

- (1) Cooperative solutions
 - (2) Noncooperative solutions
- and (3) Mechanistic solutions.

All cooperative solutions employ the assumption that the players are at least interested in achieving a Pareto optimal outcome and will hence cooperate up to a point. The Edgeworth contract curve provides an example of one type of cooperative solution known as the core.

The noncooperative solutions are primarily concerned with individualistic, strategic behavior without bargaining or cooperation. Examples familiar to economists are Cournot's (1897) solution to the duopoly problem, or Chamberlin's (1950) solution for equilibrium in a market with monopolistic competition. Both of these are examples of a noncooperative equilibrium point (see J.F. Nash, Jr.). There are also other noncooperative solutions. In particular the maximum solution is of considerable importance in the study of certain military problems involving duels, search and allocation (see M. Dresher). However, in spite of considerable misinterpretation of the role or importance of the maximum solution it is not regarded as being of particular interest to economists or to behavioral scientists in general, as its useful application is to two-person zero-sum games. A two person zero-sum game is one with two players in which there is pure opposition. The winnings of both sum to zero and there is no economic creation of wealth.

A mechanistic solution is one in which the strategic freedom of the individuals is totally removed. The competitive equilibrium, as it is usually described is a mechanistic decision. We assume that the individuals must accept the market prices and that they must maximize their individual welfare. All elements of a game of strategy have been removed. Even a wheat farmer has the choice of withholding his wheat from the market and risking having it rot or getting a better price. However in the successive embellishments of the Walrasian system, individuals are not even given enough strategic freedom to commit blunders. An interpretation of the price system in terms of a decentralized single enterprise helps to illustrate this feature. The prices may be regarded as the signal sent from the central agency. By the rules or laws of the country or the corporation, district managers are required to act as price-takers maximizing goals or utility functions which have been set for them.

There are many different game theoretic solution concepts fitting into the three categories noted above. In this paper we are primarily concerned with the solutions known as the core and the value which will be defined in Section III. However we also note and contrast the noncooperative solutions are considered, the main one being the Bargaining Set.

Our intent is to apply each of these solutions, each of which represents a different behavioristic viewpoint to the same market structure. The structure to which we limit this investigation is given by a market game, i.e. it has precisely the conditions necessary for the existence of an efficient price system.

III. THE CORE, THE VALUE AND THE NONCOOPERATIVE EQUILIBRIUM POINT

There are three game theory solution concepts which in a special form are directly related to prior approaches by economic theorists to study distribution under situations involving bargaining, negotiations and settlement and to study distribution under monopolistic or oligopolistic competition.

The three game theory solutions are known as:

(1) The core (see L.S. Shapley and M. Shubik, 1969)

(2) The value (see L.S. Shapley)

and (3) The noncooperative equilibrium (see J.F. Nash, Jr.).

They are directly related to previous economic theory as follows:

Edgeworth's contract curve is a special instance of the general concept of the core and is also an early example of the literature on recontracting.

Zeuthen's (1930) bargaining model is a precursor of the value solution developed (in several variants) by J.F. Nash (1953), L.S. Shapley (1953), J. Harsanyi (1959), R. Selten (1964) and others.

The Nash noncooperative equilibrium (see J.F. Nash, Jr.) may be regarded as a generalization of the Cournot equilibrium point solution for an oligopolistic market. This solution is also that presented (in a special form) by Chamberlin (1950) and many others in the study of oligopolistic behavior.

The core reflects the power of groups, or countervailing power. An outcome is in the core if the final distribution of resources is not only consistent with individual rationality and is Pareto optimal (societal

rationality), but also is consistent with group rationality, in the sense that no group (or subset) within society as a whole could obtain more than they are being offered if they decided to stop cooperating with the others and to trade only among themselves. It is easy to show that sometimes there is no outcome which satisfies the core conditions (see L.S. Shapley and M. Shubik, 1967). However it has been proved that for market games with convex preferences there are always trades which satisfy the core conditions, and trading according to a competitive price system will yield an outcome in the core (see G. Debreu and H. Scarf).

The value may be regarded as a solution which stresses symmetry and "fair division." Harsanyi (1959) has approached the value by attempting to define "the rational expectations" of the individuals in a bargain. Others have suggested various axiom systems, all of which make use of axioms concerning symmetry or equal treatment for equal individuals. The variants of the value solution, in general,¹ prescribe or predict a unique pattern of the distribution of wealth in a market. Given that we accepted the concept of "fairness" implies in the value solution, then we may regard it as a prescription stating what individuals should contribute to society and what they should obtain from society if all are to be treated fairly.

The noncooperative equilibrium is suggestive of an outcome resulting from the inner-directed individual use of individual power. Each firm acts without cooperation or collusion with the others. There is no guarantee that the outcome will be Pareto optimal, and in general it is not.

Limiting Behavior and Replicated Games

When we apply the solution concepts noted above and other solutions concepts to a specific game, for example a four person trading market, all solutions in general, predict different outcomes. On occasion the outcomes predicted by one solution may lie within or overlap with the solutions predicted by another. In such instances this implies that there are two different behavioral views of player action which are consistent with certain outcomes.

In particular when these solutions are applied to a market game the core will generally be an area on the Pareto optimal surface. The value will be a point on the Pareto optimal surface (not necessarily in the core). The competitive equilibrium will be a point on the Pareto optimal surface within the core and the noncooperative equilibrium (if it can be appropriately described for a market game)² will be a point below the Pareto optimal surface.

Underlying the popular writings on the virtues of competition is the idea that numbers do matter for the emergence of competition and an efficient price system. Somehow it is to be expected that when the number of competitors or traders becomes large the behavioral as well as the structural conditions should be right for the emergence of efficient market prices.

We are faced with the problem of translating this vague feeling about the role of numbers in the determination of a price system into a more precise formulation which can be analyzed. This can be done by considering replicated market games, as was originally suggested in Edgeworth's (1881)

work. In the first part of Section II an example of the replication of the Edgeworth bilateral monopoly was given. More generally suppose that we have a market with k types of traders where there are r_1 traders of the first type, r_2 of the second type and so forth. A trader type is defined by both the individual's utility function and by his initial holdings. Two individuals are the same type if both their preferences and initial holdings are the same. The associated n^{th} replication of this market would have nr_1 traders of type 1, nr_2 of type 2 and so forth.

It is, of course, hard to imagine an economy with exactly twice as many butchers, bakers and candlestick makers, each with the same preferences as their colleagues and each with the same initial endowments. However, at least this type of model provides us with a well defined meaning for the phrase "more competitors."

It is a matter of overcoming a series of technical problems in modeling and in mathematical formulation to be able to consider different types of successively larger, but associated games. However they have been formulated so that replication does not call for the strict and relatively unrealistic duplication noted above (see G. Debreu, 1969). Actually there are many models that can be constructed to show different features of additional numbers of competitors in a market. For example in his study of monopolistic competition Chamberlin considered a group of firms each selling a symmetrically differentiated product. He did not specify what would happen if the number of firms were to double in the market by having the new firms "cut in half" or having the market expand and new (but similar entrants) come in. Under either of these circumstances do we expect that the cross-

elasticities among the competing but differentiated products would remain the same or would change with more competitors? Both assumptions are logically consistent, however they imply extremely different limiting behavior among the competitors (see M. Shubik).

In this paper, when we refer to a replicated market we limit ourselves to a game in which the types of traders remain constant, but the number of traders of each type is multiplied. The results we wish to discuss hold substantially in the same form for more sophisticated forms of replication.

The key question we wish to ask is what happens to the solutions calculated by applying the different solution concepts to the successively larger replications of a market game? In particular can any regularity be observed? Is there a simple relationship between a specific solution applied to a game and that solution applied to larger related games?

As the number of replications grow extremely large can we talk about a limiting behavior, in the sense that the solution predicted for the n^{th} replication is extremely close to the solution for the $n+1^{\text{st}}$ replication?

The answer to all of these questions is yes. Furthermore, the limiting behavior of these extremely different solution concepts is such that they all, each in a different way, approach the competitive equilibrium. In other words, as the market becomes large a price system emerges. The emergence of the price system is predicted by many different solution concepts which give considerably different solutions when the number of traders is small.

Solutions: Sensitivity, Stability and Uniqueness

Some solution concepts predict a more or less unique outcome; some do not. In particular, in Figure 1 the competitive equilibrium, or the efficient price system was drawn as unique, whereas the contract curve or the core consisted of a set of possible outcomes among which no specific outcome is prescribed by the solution theory.

In general the price system is not necessarily unique. Even in an Edgeworth box there may be more than one set of prices which yield efficient trades. These of course will have extremely different welfare consequences to the traders. Nevertheless the odds are good that the price system will either be unique or contain sufficiently few alternatives that the "resolution power" of this solution will be great.³

Although the odds are good that the price system is unique, in general the core (or generalized contract curve) will contain a whole area of possible outcomes. The value solution, however, is also unique.

When considering the usefulness or applicability of different solution concepts, beyond checking for uniqueness or the resolving power of the solution, two other features are of considerable importance. They are the sensitivity and stability properties of the solution. In particular suppose that the parameters of the market model were changed slightly (for example a slight change in the initial endowments of the traders) would the change in the solution be smoothly related to the change in conditions, or is it possible for the solution to "jump around"? In order to test stability we need some sort of dynamic concept of an adjustment mechanism to describe behavior after a small displacement from the solution takes

place. The various game theory solutions call for concepts of stability considerably different from those treated in physics. The usual way in which the question of stability is treated in the study of the competitive equilibrium is as in Newtonian physics; however in human affairs stability may be of a far more sophisticated variety, involving both foresight and the coordinated motion of groups rather than individual "particles."

In this article we do not discuss sensitivity analysis and stability further. However we believe that it is of importance for the economist to consider different behavioral approaches to the same problem and to use criteria such as uniqueness, stability and sensitivity to judge the worth of the different solution concepts.

Convergence of the Core: An Example

Without going into mathematical detail it is possible to demonstrate the shrinking of the core towards the competitive equilibrium on the Edgeworth box shown in Figure 1. The line TVQW shows a price ray which cuts the contract curve at the point Q. This is not a competitive equilibrium point as can be seen by observing that the indifference curves, tangent to each other at Q, are cut by the line TVQW.

When there are only two traders the point Q is definitely well within the contract curve. We may now interpret the Edgeworth box somewhat differently, using it to represent symmetric distributions of wealth in markets in which there are n traders on each side. In total there are na units of the first good and nb units of the second. Any coalition with even numbers of traders of each type, say s of each type will have

at its disposal sa and sb units of the goods. If players of the same type each obtain the same amount, then the commodity bundle received by the typical player of the first type can be represented by (x, y) and of the second type by $(a-x, b-y)$. This corresponds to a point in the standard Edgeworth box. If, however, we wish to portray the amounts that two representative traders receive if they are members of a coalition with uneven numbers of both types of traders, say s of the first type and r of the second, then we must use two points in the box. If traders of the first type obtain (x, y) , then traders of the second type obtain $(s(a-x)/r, s(b-y)/r)$ where $0 \leq x \leq a$ and $0 \leq y \leq b$. If there are sufficiently many players it will be possible to select an r and an s so that traders of type 1 obtain an amount shown at W and traders of type 2 obtain an amount shown at V . Here the segments TV and TW are in the ratio of r to s . But these give the traders of each type more than they obtain at Q , hence the coalition of composition or "profile" (r, s) can block the consideration of the outcome Q .

As the number of traders of each type increases, more and more coalitions with different ratios of players can be formed so that it becomes possible to block points on CPC^1 arbitrarily close to P . P itself can never be blocked as the price ray is precisely tangent to both sets of indifference curves at that point. This establishes that a price system could emerge out of the interplay of the countervailing power of progressively larger and larger groups of individual economic agents.

The Value and Convergence of the Value

The value provides a prescription for distributing the gains from economic activity equitably to all in accordance with their contributions to the economy as well as in accordance with their wants.

The value takes into account the needs of the individual in the sense that the preferences of each are important in calculating the final distribution. It takes into account the contribution or the abilities of the individual in the sense that the marginal worth of each individual to every subset of individuals in society is calculated and averaged.

An extremely simple three person example is used to illustrate the type of calculation for the value. Consider an economy with one product which is evaluated by each individual in proportion to the quantity he possesses. Table 1 shows the amounts that all coalitions of the individuals A, B and C can obtain

TABLE 1

Coalition of A	obtains 1
Coalition of B	obtains 1
Coalition of C	obtains 1
Coalition of AB	obtains 2
Coalition of AC	obtains 3
Coalition of BC	obtains 4
Coalition of ABC	obtains 7

The marginal worth of individual A to each coalition is shown in Table 2:

TABLE 2

$$ABC - BC = 7 - 4 = 3$$

$$AB - B = 2 - 1 = 1$$

$$AC - C = 3 - 1 = 2$$

$$A - \emptyset = 1 - 0 = 1$$

We average the marginal worth of the individual applying equal weights to each size of coalition. This reflects every way we can account for an individual's contribution to the total product of society. This is shown in Table 3.

TABLE 3

\bar{ABC}	A forms a coalition first
\bar{ACB}	A forms a coalition first
\bar{BAC}	A enters a two person coalition
\bar{CAB}	A enters a two person coalition
\bar{CBA}	A joins to form a three person coalition
\bar{BCA}	A joins to form a three person coalition

$$\text{Average of marginal worth of A} = \frac{1}{6} (1 + 1 + 1 + 2 + 3 + 3) = \frac{11}{6}$$

$$\text{Average of marginal worth of B} = \frac{1}{6} (1 + 1 + 1 + 3 + 4 + 4) = \frac{14}{6}$$

$$\text{Average of marginal worth of C} = \frac{1}{6} (1 + 1 + 2 + 3 + 5 + 5) = \frac{17}{6}$$

$$\text{TOTAL} = 7$$

By averaging this way we have made an assumption of sociological neutrality. No way of combining is favored. In actuality institutions, social groups and individual characteristics play important roles in determining who is hired first or fired first (see Owen). "Fairness" or considerations of basic symmetry call for these differences to be excluded.

Returning to the example we see that the value allocations of $(11/6, 14/6, 17/6)$ when preferences are the same (as they are in this case) are more to the more economically productive individuals. The rewards are always tied in with the productivity. Thus if we wished to apply this to welfare theory it would have to be modified to account for obvious non-symmetries in most societies, such as initial social structure, age and health.

The example presented here was extremely simple. All individuals were assumed to have the same preferences and there was only one commodity being produced. The value can be defined for a general economy which includes any number of commodities and individuals each with different and noncomparable preferences (see J.C. Harsanyi, and L.S. Shapley and M. Shubik, 1969). The difficulties encountered in investigating the value for general large market economies are basically mathematical and computational and not conceptual.

Unlike the core, but like the competitive equilibrium, the value, in general, is a one point solution. It usually specifies exactly how the wealth of society should be distributed. When the players are few, the core suggests a range, as can be seen from the contract curve which is the core of a two-person trading game.

In general, when market games with few traders are examined, the distribution of resources suggested by the value has little or nothing in common with the resource distribution suggested by the competitive equilibrium price system. It is a remarkable fact that for a replicated market game, in other words, a mass market, the value approaches the competitive equilibrium. For mass markets the association of resources suggested by the value based upon considerations of fairness and symmetry will be the same as the allocation suggested by the price system, based on considerations of decentralization and efficiency.

The detailed discussion and the mathematical proof of these assertions are given elsewhere (see L.S. Shapley and M. Shubik, 1969). However heuristically, we may see the basis for the proof immediately. In an extremely large market economy, in order to calculate the imputation of resources assigned to the individual by the value, it is necessary to calculate his marginal worth to all coalitions. Most of the coalitions will be large and have a profile of members not too far different from the economy as a whole (i.e. most large coalitions will have no more than a few percent more or less, butchers and bakers than the economy as a whole). In a large coalition which is a reasonable sample of the economy as a whole the marginal worth of an extra individual is approximately the same as the marginal worth assigned to him by the price system. As the size of the economy is increased the averaging process is predominantly over coalitions in which the individual contributes his worth to the competitive economy.

In Section IV a numerical example is given which displays the behavior of the core, the value and the competitive equilibrium as the number of traders increase in a market.

IV. AN EXAMPLE

A simple two-sided Edgeworth market is examined for which all solutions can be determined explicitly, but in which there is not so much symmetry that the solutions are uninteresting.

Consider a market with n traders on each side. We assume that there are two goods in trade (no money and no credit granted). Let the initial holdings be

$$\begin{cases} (1, 0) & \text{(1st type),} \\ (0, 1) & \text{(2nd type),} \end{cases}$$

and let the (cardinal) utility functions be

$$(1) \quad \begin{cases} u_1(x, y) = \sqrt{xy} & \text{(1st type),} \\ u_2(x, y) = \sqrt{x^2 + y^2 + 23xy} & \text{(2nd type),} \end{cases}$$

The number 23 is only a convenience; all that really matters is that these functions are concave, homogeneous of degree 1, and symmetric in the two goods.

Figure 2 is the "Edgeworth box" for this market. The origin O represents the allocation that gives $(0, 0)$ to type 2, and hence $(1, 1)$ to type 1. The opposite corner C^0 is the "origin" for type 1, and R is the initial or no-trade point. The segment CC^0 is Edgeworth's contract curve for the case of two traders (i.e., $n = 1$). The unique competitive allocation ω , which gives all traders $(1/2, 1/2)$, is represented by

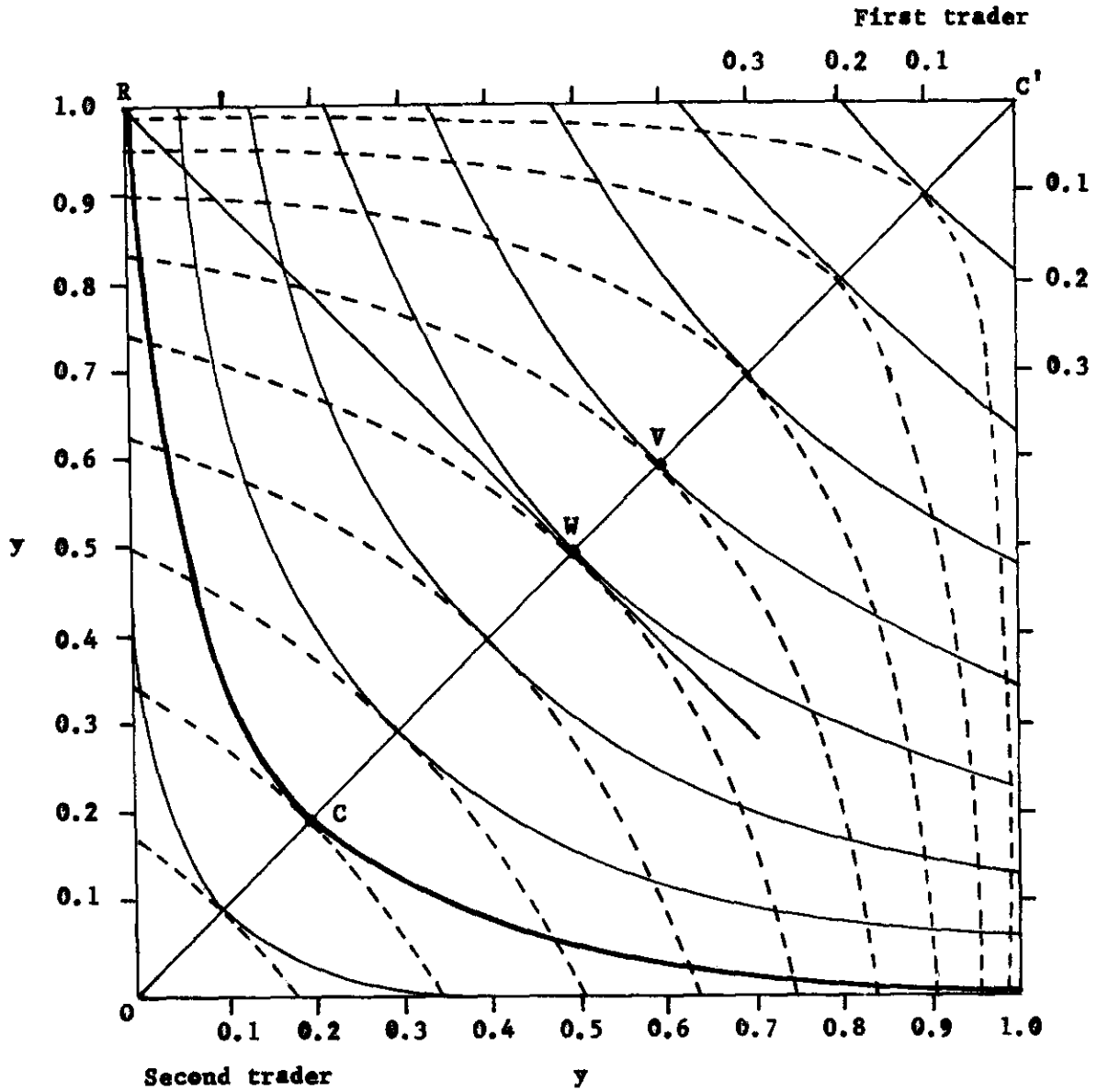


FIGURE 2
The Edgeworth Box

the point W ; this is independent of n . (The point V represents the value allocation for $n = 1$, which will be determined later.)

We emphasize that for $n > 1$, the Edgeworth box serves to represent only the symmetric allocations--those in which traders of the same type are treated alike.

The Core

We first discuss the core. For $n = 1$ the core comprises the whole contract curve CC^0 . More precisely, the core is the image of CC^0 in the utility space under the relationship (1), as shown later in Figure 3.

For $n > 1$, we first observe that all imputations in the core must be symmetric, since any nonsymmetric imputation can be improved upon by a two-man coalition consisting of one least-favored trader of each type. We may therefore transfer our attention from the $2n$ -dimensional space of all imputations to the two-dimensional subspace of symmetric imputations.

Because of the homogeneity of (1), the Pareto-optimal symmetric imputations lie along a straight line (OC^0 in Figure 3); its equation is

$$(2) \quad 5u_1 + u_2 = 5 .$$

The core, for each n , is a subset of this line. It remains to discover which points on the line can be improved upon by some coalition, and hence excluded from the core. The most efficient coalitions in this regard are those that have almost, but not quite, the same relative composition as the market as a whole.

To verify this, let $r(S)$ denote the ratio of first types to second types in an arbitrary coalition S , and let $(1-t, 5t)$, $0 \leq t \leq 1$, be an arbitrary point on OC^1 , representing the symmetric imputation α_t . Then a routine calculation reveals that in order for S to exclude α_t it is necessary and sufficient that $r(S)$ lie strictly between 1 and a certain critical ratio r_t , given by

$$r_t = \frac{t(t^2 - .04)}{(1-t)(t - t^2 - .04)}.$$

Note that $r_{1/2} = 1$, showing that the competitive imputation $\alpha_{1/2}$ cannot be excluded, since there is no number strictly between 1 and 1. However, any other α_t will be excluded by some coalition if n is sufficiently large.

For each n , the coalitions of size $2n-1$ provide the best type-composition ratios available, namely $(n-1)/n$ and $n/(n-1)$. Setting

TABLE 4

Endpoints of the Core

n	C		C ¹	
1	0.800,	1.000	0.000,	5.000
2	0.577,	2.116	0.421,	2.897
3	0.546,	2.272	0.453,	2.733
4	0.533,	2.337	0.467,	2.665
5	0.525,	2.374	0.474,	2.638
10	0.512,	2.440	0.488,	2.560
as $n \rightarrow \infty$	$\frac{1}{2} + \frac{21}{184n}$,	$\frac{5}{2} - \frac{105}{184n}$	$\frac{1}{2} - \frac{21}{184n}$,	$\frac{5}{2} + \frac{105}{184n}$

r_t equal to these numbers in turn, and solving for t , gives us the endpoints of the core. We have done this numerically for several values of n , as shown in Table 4 and Figure 3.⁴ Asymptotically (last line of Table 4), the length of the core varies inversely with the size of the market.

The Value

We now turn to the value. The technique, as explained in detail in a separate publication by L.S. Shapley and M. Shubik (1969) as to introduce a set of "weights" on which to base hypothetical exchanges of utility among the traders. It can be shown that in this example the weights are in the ratio of 5:1 and the value of the game is given by

$$\begin{cases} \theta_i = \frac{1}{2} - \frac{1}{5(n+1)} & \text{(1st type),} \\ \theta_j = \frac{5}{2} + \frac{1}{n+1} & \text{(2nd type),} \end{cases}$$

Table 5 is intended for comparison with Table 4. In Figure 4 all three kinds of solutions are shown together, as functions of market size. We see that the value is always more favorable to traders of type 2 than the competitive solution, and that it begins inside the core but moves outside at $n = 3$ because of its somewhat slower rate of convergence. These comparisons should not be interpreted too broadly, however, as the value is essentially a cardinal-utility concept, while the other solutions are not.

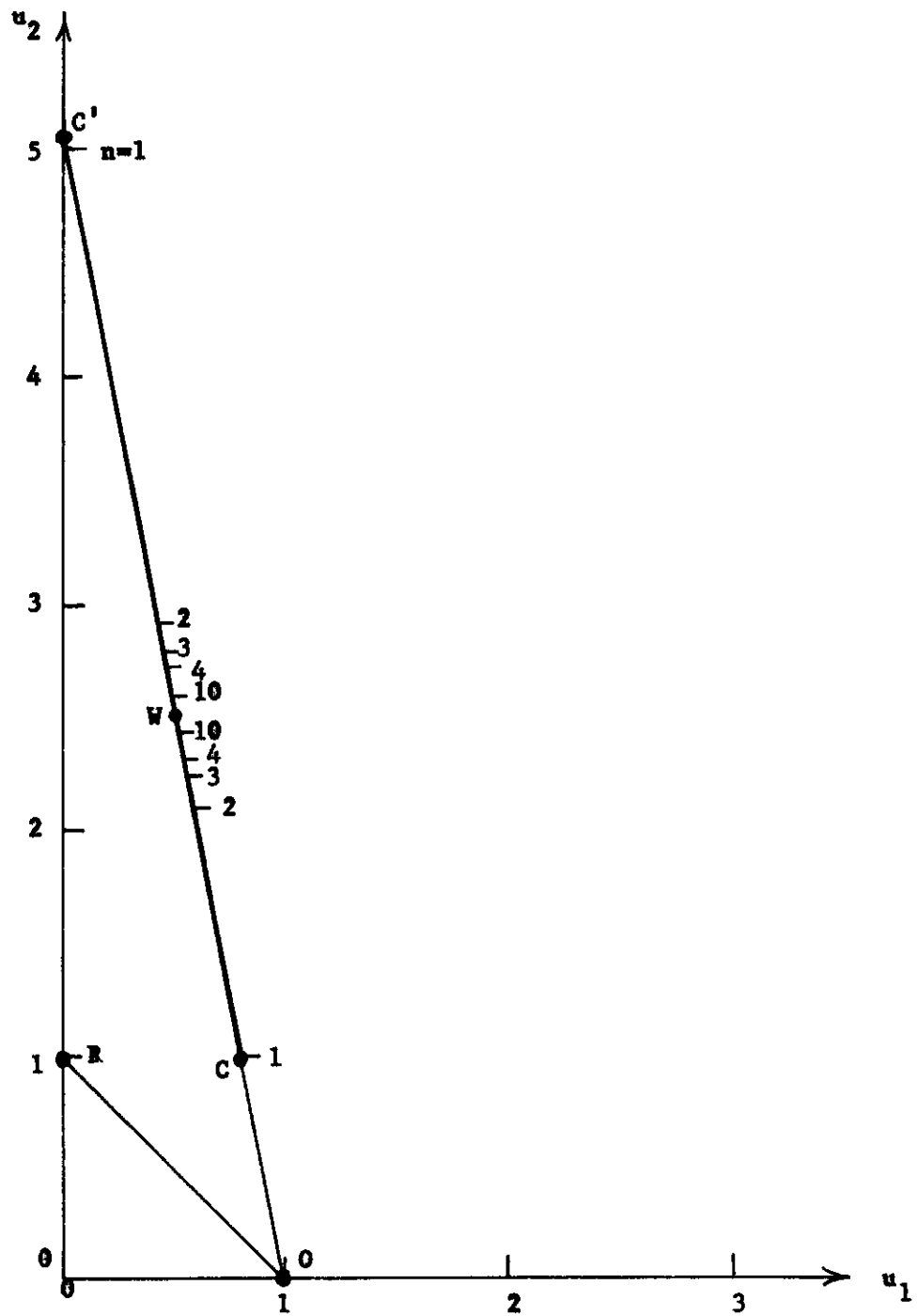


FIGURE 3

The Utility Space (Symmetric Payoffs only), Showing the Shrinking of the Core to the Competitive Payoff

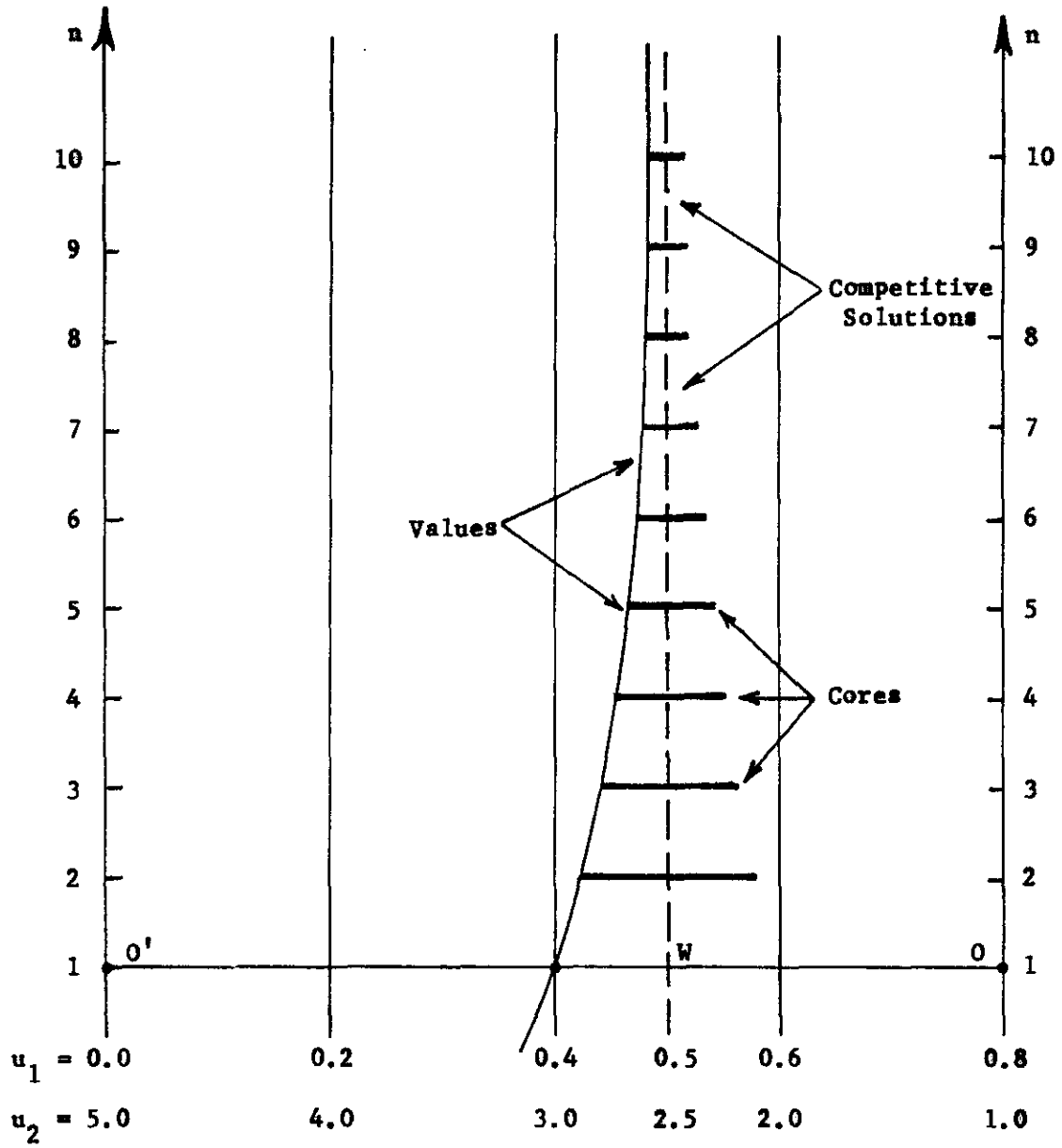


FIGURE 4
Convergence of Solutions

TABLE 5
The Value

n	V	
1	0.400,	3.000
2	0.433,	2.833
3	0.450,	2.750
4	0.460,	2.700
5	0.467,	2.667
10	0.482,	2.591
as $n \rightarrow \infty$	$\frac{1}{2} - \frac{1}{5n}$,	$\frac{5}{2} + \frac{1}{n}$

The Noncooperative Equilibrium

A demonstration (under somewhat restricted circumstances) of the convergence of the noncooperative equilibrium to the competitive price system has been given elsewhere at a relatively nonmathematical level (see L.S. Shapley and M. Shubik, 1967) and is not presented again here. The restriction in that model is due to some fundamental difficulties in modeling non-cooperative behavior in a closed market.

V. CONCLUSIONS

The price system in a mass market economy can be approached from many different viewpoints. The presence of many individuals, provided that

the basic technological structure of the economy is satisfactory, implies that many basically different motivational and social patterns will call for the same behavior, and that behavior is portrayed by a price system.

When externalities and other structural "market imperfections" are present, these different solutions will not all lead to a price system and they will differ from each other.

FOOTNOTES

1. There are some exceptions which are not of importance to the discussion here. Those interested in the technical details are referred elsewhere (see L.S. Shapley and M. Shubik, unpublished manuscript).
2. There are considerable difficulties in the application of the noncooperative solution to a model of a closed economy. They are discussed elsewhere (see L.S. Shapley and M. Shubik, unpublished manuscript).
3. Recently some advanced work by G. Debreu has cast light upon the problem of how to characterize and measure the possibility of the occurrence of more than one efficient price system.

Another feature of considerable importance in determining the existence of a unique price system is the presence of a side-payment commodity or "idealized money," such that a utility function representation of an individual's preferences can be found in the form:

$\varphi_i(x_1, x_2, x_3, \dots, x_m) + \lambda_i x_{m+1}$ where the x_{m+1} may be regarded as an idealized form of money with a constant marginal utility for each trader i , given by λ_i . If the income effect were sufficiently weak that it could be ignored then there would be a unique efficient price system.

4. We have used the same letters to denote allocation points in the (x, y) -space and the corresponding payoff points (imputations) in the (u_1, u_2) -space.

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