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1. Introduction

The objective function (maximand) most often adopted in the literature on optimal economic growth is a discounted sum of future utility flows, of the form

$$(1.1) \qquad \sum_{t=1}^{\infty} \alpha^{t-1} u(y_t) , \qquad 0 < \alpha < 1 ,$$

if we assume a discrete time variable t . Here $u(y)$ is a concave "single-period-utility" function of a consumption flow y (scalar or vector), y_t the flow in period t , and α a discount factor applied to utility flows. This function is then maximized subject to a specified initial capital stock z_1 and a given technology and resource base. If assumed constant over time,

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technological constraints may be expressed by

$$(z_t,\ y_t,\ w_t,\ z_{t+1}) \in \mathcal{T} \qquad \text{for all } t ,$$

where w_t is a scalar or vector resource input in period t, and the choice of $\mathcal{T}$ as a closed convex cone expresses the assumption of constant returns to scale. Constant resource constraints may be expressed by the inequalities*

$$0 \leqq w_t \leqq w \qquad \text{for all } t .$$

In this paper we are not concerned with finding the optimal growth path corresponding to any initial capital stock that is given a priori. We limit ourselves to the search for a capital stock $\hat{z}$ that is <u>invariant</u> under the optimization described. That is, we seek a value $z_1 = \hat{z}$ for the initial stock that gives rise to an optimal program $(\hat{y}_t,\ \hat{z}_t)$ such that $\hat{y}_t = \hat{y}$, $\hat{z}_t = \hat{z}$ for all t.

The simple case in which y_t, z_t are scalar and

$$\mathcal{T} = \{(z_t,\ y_t,\ z_{t+1}) \geqq 0 \mid y_t + z_{t+1} \leqq f(z_t)\}$$

has been extensively explored in the literature. (Here scarcity of the single resource, labor say, is expressed indirectly by the strict concavity of the

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* $a \geqq b$ for vectors a, b, denotes $a_i \geqq b_i$ for all i,
$a \geq b$ denotes $a \geqq b$ and $a \neq b$,
$a > b$ denotes $a_i > b_i$ for all i.

increasing production function f with $f(0) = 0$.) In this case (see, for instance, Koopmans [1965], where continuous time is used) $(\hat{y}, \hat{z})$ exists* and is unique, and also represents the limit

$$\lim_{t\to\infty} (\hat{y}_t, \hat{z}_t) \tag{1.2}$$

for the unique optimal path that starts from any other positive initial capital stock z_1 .

Sutherland [1967, 1970] has studied the more general case where y_t , z_t are vectors. He uses a single-period utility function $u^*(z_t, z_{t+1})$ that can be related to our $u(y_t)$ by

$$u^*(z_t, z_{t+1}) = \sup_{\substack{0 \leqq w_t \leqq w \\ (z_t, y_t, w_t, z_{t+1}) \in \mathcal{J}}} u(y_t) .$$

He finds that the problem of finding an invariant optimal capital stock is equivalent to a fixed-point problem, and uses Kakutani's fixed-point theorem to prove the existence of a solution. The solution is no longer necessarily unique, even if $u^*(z_t, z_{t+1})$ is strictly concave.

Sutherland also concludes to the near-equivalence of the problem of finding an invariant optimal capital stock to a single-period problem to be described below. The present paper utilizes that equivalence for an experiment in computing such a stock, or rather an approximately invariant stock. Both the equivalence proof and the computation are carried through in terms

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*With $\hat{z} > 0$, provided $f'(0) > \alpha^{-1} > \lim_{z\to\infty} f'(z)$.

of a specific model developed, in ignorance of Sutherland's work, by the second author (see Koopmans [1971]). This model uses a von Neumann type technology of capital transformation, in which resources and consumption goods have been incorporated. The method of computation we use is a member of the class of algorithms for computing an approximately fixed point of a continuous mapping, recently developed by Scarf and Hansen (Scarf [1967a, b, c, 1969], Hansen [1968, 1969], Scarf [forthcoming], see also Kuhn [1968, 1969] and Eaves [1970]). The present method was developed and applied by the first author. The equivalence proof was formulated by the second author.

We shall not be concerned here with the straightforward extension of the problem to equal exponential rates of exogenous growth applicable to all resources and to the consuming population.

The starred sections 4*, 7* are devoted to proofs and other technical observations, and can be skipped without affecting the reading of later sections.

2. The Model

2.1. Technology and resource constraints. In line with our emphasis on computability, we consider a von Neumann type technology with L capital goods, M resources other than capital goods, K consumption goods, and I productive processes that are constant over time, and each of which is defined by a unit activity.

The following constraints connect the capital input vector $z_t \equiv (z_{t1}, z_{t2}, \ldots, z_{tL})$ of the t^{th} period, the capital output vector z_{t+1} for that period, the resource availability vector $w \equiv (w_1, \ldots, w_M)$ which is assumed the same for all periods, and the consumption vector $y_t \equiv (y_{t1}, \ldots, y_{tK})$ for the t^{th} period, with the vector of activity levels $x_t \equiv (x_{t1}, \ldots, x_{tI})$ applicable to each process, $i = 1, \ldots, I$, for that period. All these vectors are nonnegative column* vectors.

$$
(2.1)\left\{\begin{array}{lll}
(2.1A) & \text{(capital input)} & -Ax_t \geqq -z_t\ , \\
(2.1B) & \text{(capital output)} & Bx_t \geqq z_{t+1}\ , \\
(2.1C) & \text{(resources)} & -Cx_t \geqq -w \\
(2.1D) & \text{(consumption)} & Dx_t \geqq y_t
\end{array}\right\}\quad t = 1,\ 2,\ \ldots\ .
$$

The constant coefficients of inputs and outputs of the various commodities that characterize the various processes have been assembled in matrices A , B , C , D of orders (L, I) , (L, I) , (M, I) , (K, I) , respectively, and are required to satisfy the following sign rules:

$$
(2.2)\qquad a_{\ell i},\ b_{\ell i},\ c_{mi},\ d_{ki} \geqq 0 \qquad \text{for all } \ell,\ m,\ k,\ i,
$$

$$
(2.3)\left\{\begin{array}{lll}
(2.3\ \text{row}) & \sum_i a_{\ell i},\ \sum_i b_{\ell i},\ \sum_i c_{mi},\ \sum_i d_{ki} > 0 & \text{for all } \ell,\ m,\ k, \\
 & & \\
(2.3\ \text{col}) & \sum_\ell a_{\ell i},\ \sum_m c_{mi},\ \sum_\ell b_{\ell i} + \sum_k d_{ki} > 0 & \text{for all } i\ .
\end{array}\right.
$$

The row constraint in (2.3) says that each capital good and each resource is an input to at least one process, and each capital good and each consumption good is an output of at least one process. The column constraint says that each process requires the input of at least one capital good and at least one resource, and produces at least one good, capital or consumption.

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*The notation (, ...,) for a vector will be used without making a distinction between column and row vectors. The symbol $\equiv$ denotes equality by definition.

A capital stock $z \geqq 0$ is called <u>reproduceable</u> if (2.1A, B) permit $z_1 = z_2 = z$. Clearly, the null stock $z = 0$ is reproduceable. To assure the existence of a positive reproduceable capital stock, we impose on the technology the <u>viability condition</u>

$$\text{(2.4)} \qquad Bx > Ax > 0 \quad \text{for some} \quad x \geq 0 \ .$$

We write $>$ rather than $\geqq$ in the first inequality in order that some positive reproduceable capital stock shall have some capacity to spare for the production of consumption goods, and, if desired, for further expansion of the capital stock itself.

For sufficiently small activity levels, this interpretation of the viability condition is independent of the resource constraints (2.1C), because we require that the resource availability vector w be positive,

$$\text{(2.5)} \qquad w_m > 0 \quad \text{for all} \quad m \ .$$

Therefore, the set of vectors $x \equiv (x_{\cdot 1}, x_{\cdot 2}, \ldots, x_{\cdot I}) \geqq 0$ satisfying just (2.1C) contains a "chip" of the form

$$\text{(2.6)} \qquad \mathcal{C}(\epsilon) \equiv \{x \mid x \geqq 0, \ \sum_{i=1}^{I} x_{\cdot i} \leqq \epsilon\} \ , \quad \epsilon > 0 \ ,$$

within which points satisfying the linear-homogeneous condition (2.4) can also be found.

Depreciation of capital can be represented, as in the original presentation by von Neumann [1937, 1945], by treating capital goods in different states of wear as different capital goods. In that interpretation, the statement that a process "produces" a capital good also applies to somewhat worn capital goods provided in that case we interpret "produces" as meaning "releases from use."

A z-feasible path, $z \geqq 0$, is now defined as a sequence $\{(x_t, y_t, z_t), t = 1, 2, \ldots\}$ satisfying (2.1) for all t and such that $z_1 = z$. The sign restrictions (2.2), (2.3) on the elements of C and (2.5) on those of w then imply boundedness of the set of all z-feasible paths,

$$(2.7)\quad 0 \leqq x_t \leqq \bar{x}\ , \quad 0 \leqq y_t \leqq D\bar{x} \equiv \bar{y}\ , \quad 0 \leqq z_t \leqq B\bar{x} \equiv \bar{z}\ , \quad t = 1, 2, \ldots,$$
$$\text{for some } \bar{x} > 0 .$$

Here we have imposed on the initial capital stock $z = z_1$ the same constraint, $0 \leqq z_1 \leqq \bar{z}$, found to apply to all z_t with $t > 1$, thus making the bounds (2.7) uniform for all z_t considered. Since the null-path, $x_t = y_t = 0$, $t = 1, 2, \ldots,$, $z_t = 0$, $t = 2, \ldots$, satisfies (2.1) for any $z_1 \geqq 0$, the set of z-feasible paths is also nonempty for any $z \geqq 0$.

2.2. The objective function. We adopt an objective function of the form* (1.1). We require that the single-period utility function $u(y)$, where $y \equiv (y_{.1}, y_{.2}, \ldots, y_{.k})$ is a consumption flow (column) vector, is defined for all $y \geqq 0$, is concave, and is continuously differentiable

*For a discussion deriving this form from postulates about a preference ordering on the space of consumption programs $(y_1, y_2, \ldots)$ see Koopmans [1972].

and increasing with regard to each component $y_{\cdot j}$ of y . We shall avoid a specification, often made in optimal growth models, that instead of assuming differentiability in the origin $y = 0$ requires that $\frac{\partial u}{\partial y_{\cdot k}} \to \infty$ for all k as $y \to 0$, and permits $u(y) \to -\infty$ as well. We shall argue at the end of section 3 below that the simplification bought by avoiding that specification causes no loss of generality in the present context.

The assumption that $u(y)$ is increasing in each y_k implies global nonsaturation with regard to all consumption goods. Therefore, no disposal of any consumption good ever takes place in an optimal path, and we can eliminate y by defining a utility function in terms of the activity vector x ,

$$v(x) \equiv u(Dx) \ , \quad x \geqq 0 \ . \tag{2.8}$$

Then $v(x)$ is again concave, and $v_i'(x) \equiv \frac{\partial v}{\partial x_{\cdot i}} \geqq 0$ for all i and all $x \geqq 0$. The row vector of these derivatives will be denoted $v'(x) \equiv (v_1'(x), \ \ldots, \ v_I'(x))$.

2.3. Invariant capital stock. We now formally state two problems which in combination define the topic of this paper.

$P_\infty(z,\alpha)$: Given a reproduceable initial capital stock z and a discount factor α with $0 < \alpha < 1$, maximize $\sum_{t=1}^{\infty} \alpha^{t-1} v(x_t)$ on the set of z-feasible paths.

A path $(\hat{x}_t, \hat{z}_t)$ that solves this problem is called z-optimal.

$P_\infty(\alpha)$: Choose a value $z = \hat{z}$ of the initial capital stock in $P_\infty(z,\alpha)$ such that there exists a constant path $(x_t, z_t) = (\hat{x}, \hat{z})$, $t = 1, 2, \ldots,$ that solves $P_\infty(\hat{z},\alpha)$.

A solution $\hat{z}$ of $P_\infty(\alpha)$ is then what we have already called an invariant optimal capital stock. By definition, such a capital stock must be reproduceable. We shall use the expression "solution of $P_\infty(\alpha)$ " also for the pair $(\hat{x}, \hat{z})$.

Obviously, $P_\infty(\alpha)$ has for all α one trivial and uninteresting solution, namely, $\hat{z} = 0$, $\hat{x} = 0$, the null solution.

3. An Equivalent One-Period Problem

In Section 4 we shall prove the near-equivalence between the problem of finding an invariant capital stock $\hat{z}$ with an associated activity vector $\hat{x}$, and a one-period problem now to be defined, likewise in two steps. The first step is to define the problem

$P(z)$: Given some reproduceable z , maximize $v(x)$ subject to

$$
(3.1)\quad \begin{cases}
(3.1A) & -Ax \geqq -z\ , & q_1 \\
(3.1B) & Bx \geqq z\ , & q_2 \\
(3.1C) & -Cx \geqq -w\ , & r \\
(3.1D) & x \geqq 0\ . &
\end{cases}
$$

This problem arbitrarily prescribes a reproduceable beginning-of-period capital stock z , and imposes the stationarity requirement that the same capital stock be reproduced at the end of the single period under consideration.

By our assumptions the constraint set (3.1) is nonempty, closed and bounded, the maximand continuous. Hence, for every reproduceable z there exists a solution $\hat{x} = \hat{x}(z)$ of $P(z)$. Since $v(x)$ is concave, by the Kuhn-Tucker theorem [1950, Sec. 3] any feasible vector $\hat{x}$ is a solution if and only if it has associated with it vectors q_1, q_2, r of dual variables corresponding to the constraints (3.1A, B, C), respectively, that satisfy the conditions

$$(3.2) \begin{cases} (3.2\geqq) & q_1,\ q_2,\ r \geqq 0\ , \\ (3.2\leqq) & v'(\hat{x}) - q_1A + q_2B - rC \leqq 0\ , \\ (3.2=) & v'(\hat{x})\hat{x} - q_1z + q_2z - rw = 0\ . \end{cases}$$

The components of q_1, q_2, r can be interpreted as shadow prices, expressed in terms of "marginal utility productivity," of the initial and terminal capital goods and of the resource flows, respectively. Since the timing of availability of these goods differs as between inputs and outputs it should be clarified that these valuations are all defined as "present values" as of the same "present" point in time, say, the beginning of the period. Then (3.2$\leqq$) says that the marginal shadow profits, associated with small increases in the level of each activity when starting from $x = \hat{x}$, are all nonpositive. The scalar condition (3.2=), taken together with (3.2$\geqq$), (3.2$\leqq$) and (3.1), then says in a concise implicit way that (i) marginal profits of all processes in use at $\hat{x}$ (i.e., with $\hat{x}_{\cdot i} > 0$) are zero, and (ii) there is at $\hat{x}$ no disposal of any capital good or resource that has a positive shadow price (see also Tucker [1957]). If in addition the q_1 , q_2 , r are uniquely determined by (3.2), they represent derivatives of the maximal attainable utility $v(\hat{x})$ with respect to net increases in the initial capital availabilities $z_1 = z$, decreases in the terminal capital requirements $z_2 = z$, and increases in the resource availabilities w . Finally, independently of the uniqueness of q_1 , q_2 , r , an opportunity to barter, for immediate or future delivery, any positive or negative amounts of initial capital goods or resources (delivered prior to production) at relative prices q_1^* , r^* , and of terminal capital or consumption goods (delivered after production) at relative "present" prices q_2^* , p^* , will not make attainment of a higher utility level possible, if and only if $v'(\hat{x}) = p^*D$, q_1^* , q_2^* , r^* satisfy the "dual constraints" (3.2) (see Koopmans [1951], Theorem 5.11, p. 93).

The second step is to define the problem

$P(\alpha)$: <u>Given a discount factor</u> α <u>with</u> $0<\alpha<1$, <u>choose a reproduceable value</u> $z = \hat{z}$ <u>of the capital vector</u> z <u>in</u> $P(z)$ <u>in such a way that there exists a solution</u> $\hat{x} \equiv \hat{x}(\hat{z})$ <u>of</u> $P(\hat{z})$ <u>with associated dual variables</u> q_1 , q_2 , r <u>which, besides</u> (3.2), <u>also satisfy</u>

$$q_2 = \alpha q_1 . \tag{3.3}$$

This condition requires that the shadow prices ("present values") of all capital goods diminish, when one changes availability from the beginning to the end of the single period considered, in the same proportion α , which in turn equals the given discount factor. If we again refer also to the pair $(\hat{x}, \hat{z})$ as a solution of $P(\alpha)$, we can now state a theorem that more fully sets out conditions under which the equivalence of $P(\alpha)$ and $P_\infty(\alpha)$ has been ascertained.

This theorem utilizes a slightly strengthened concept of reproduceability. A capital stock z is called <u>more-than-reproduceable</u> if there exist vectors z', z'' satisfying $0 \leqq z' \leqq z \leqq z''$ with $z' < z''$, and a z'-feasible path z_t of finite length τ with terminal capital stock $z_{\tau+1} \geqq z''$. This condition specifies some slack in the reproduceability of all components of z without prescribing whether the slack for any particular component can be achieved in the initial or terminal capital stock, and without placing an a priori limit on the number of steps needed to achieve the slack. It rules out the case $z = 0$, and some other cases where z is in the boundary of the reproduceable set. We give more information on these cases in Section 4^*.5 below, from which we infer that the condition of more-than-reproduceability, applied to an invariant optimal capital stock, covers all cases of practical interest in applications.

<u>Theorem 1</u> (Equivalence) <u>For a pair $(\hat{x}, \hat{z})$ to solve $P_\infty(\alpha)$ it is sufficient that $(\hat{x}, \hat{z})$ solve $P(\alpha)$. The latter condition is also necessary if $\hat{z}$ is more-than-reproduceable.</u>

Note that this Theorem allows us without further conditions to compute as many solutions of $P_\infty(\alpha)$ as we can compute of $P(\alpha)$. The theorem also tells us that all solutions of $P_\infty(\alpha)$ in which $\hat{z}$ is more-than-reproduceable occur as solutions of $P(\alpha)$.

The proof of Theorem 1 is given in Section 4^*. In the proof of necessity we first single out the instructive special case in which (3.2) with $z = \hat{z}$ by itself suffices to determine the prices q_1, q_2 uniquely.

Next we show that a certain boundedness condition on the prices associated with the constant path $(x_t, z_t) = (\hat{x}, \hat{z})$ over any finite period guarantees the necessity asserted in the Theorem. Finally we show that this boundedness condition is met if $\hat{z}$ is more-than-reproduceable.

Theorem 1 specializes two theorems stated by Sutherland [1967, pp. 15, 24 and 1970, pp. 587, 588] to the von Neumann type technology with provision for resources and for consumption. This specialization reveals interesting connections between the properties of the solution set of $P(\alpha)$ as a function of α and known properties of the strict von Neumann model, obtained from the present model by omitting (2.1C), (2.1D) from the constraint set. We return to these connections in Appendix B. In particular, Lemma 4 of Appendix B shows that the null solution of $P_\infty(\alpha)$, valid for all α, does not solve $P(\alpha)$ for some values of α.

Finally, we note a property of $P(\alpha)$ that enlarges the range of cases to which the present model applies. Let $(\hat{x}, \hat{z})$ solve $P(\alpha)$, and let (q_1, q_2, r) be a price vector satisfying (3.2) and (3.3). Now consider a different problem $P^*(\alpha)$, obtained from $P(\alpha)$ by substituting for $u(y)$, $v(x)$, functions

$$u^*(y) = \Phi(u(y)) , \qquad v^*(x) = \Phi(v(x)) ,$$

obtained by an increasing and differentiable transformation $\Phi(\ )$ of the single-period utility scale that preserves the concavity of $u(y)$, hence of $v(x)$. Then, unless Φ is linear, the preference ordering on the space of consumption programs is no longer the same. However, the set of solutions $(\hat{x}, \hat{z})$ of $P(\alpha)$ remains unchanged. This is so because the vector $v'(\hat{x})$ is changed only proportionally,

$$v^{*\prime}(\hat{x}) = \phi' \cdot v'(\hat{x}) , \qquad \text{where} \quad \phi' = \phi'(v(\hat{x})) > 0 .$$

and (3.2), (3.3) are satisfied again if we change the price vector correspondingly,

$$(q_1^*, q_2^*, r^*) = \phi' \cdot (q_1, q_2, r) .$$

For this reason there is for our present purpose no loss of generality in the assumption that $u(y)$ is differentiable also in the origin $y = 0$. The case referred to in Section 2.2, where $\lim_{y \to 0} u_k'(y) = \infty$, can be generated from the present case by an appropriate transformation of the scale for single-period utility.

4*. Proof and Further Evaluation

4*.1. Sufficiency. Assume first that $\hat{z}$ solves $P(\alpha)$. Then there also exists a solution $\hat{x}$ of $P(\hat{z})$. Therefore, all the results cited in Section 3 apply if we substitute in (3.1) and (3.2)

(4.1) $$z = \hat{z} , \quad q_2 = \alpha q_1 \equiv \alpha q , \quad \text{say.}$$

Now let (x_t, z_t), $t = 1, 2, \ldots,$ be any $\hat{z}$-feasible path, and consider a segment $(x_1, \ldots, x_T)$ of the path for x_t. Then, by multiplying (3.2$\leqq$) by $\sum_{t=1}^{T} \alpha^{t-1} x_t$, subtracting (3.2=) multiplied by $\sum_{t=1}^{T} \alpha^{t-1}$, using (4.1) and rearranging terms we obtain

$$(4.2)\quad \begin{aligned} &q[(-Ax_1 + \hat{z}) + \sum_{t=2}^{T} \alpha^{t-1}(-Ax_t + Bx_{t-1}) + \alpha^T(Bx_T - \hat{z})] + \\ &+ r \sum_{t=1}^{T} \alpha^{t-1}(-Cx_t + w) \leqq -v'(\hat{x}) \sum_{t=1}^{T} \alpha^{t-1}(x_t - \hat{x}) \leqq - \sum_{t=1}^{T} \alpha^{t-1}(v(x_t) - v(\hat{x})) , \end{aligned}$$

the second inequality again being due to the concavity of $v(x)$. By (3.1), (4.1), all expressions in parentheses in the left hand member are nonpositive, except possibly the one occurring in the term $q\alpha^T(Bx_T - \hat{z})$, which by (2.2) tends to zero as $T \to \infty$, since $0 < \alpha < 1$. Therefore

$$(4.3)\qquad \sum_{t=1}^{\infty} \alpha^{t-1} v(x_t) \leqq \sum_{t=1}^{\infty} \alpha^{t-1} v(\hat{x}) ,$$

and the constant path $(\hat{x}_t, \hat{z}_t) = (\hat{x}, \hat{z})$ for all t solves $P_\infty(\hat{z}, \alpha)$. Hence $\hat{z}$ solves $P_\infty(\alpha)$.

4*.2. Necessity if shadow prices are unique. Assume next that $\hat{z}$ solves $P_\infty(\alpha)$, and let the constant path $(\hat{x}_t, \hat{z}_t) = (\hat{x}, \hat{z})$ be $\hat{z}$-optimal. Then, in particular, for any T_1 , T_2 with $T_1 \leqq T_2$, the path segment $\hat{x}_t = \hat{x}$, $t = T_1, \ldots, T_2$, solves

${}_{T_1}P_{T_2}(\hat{z}, \alpha)$: Maximize $\sum_{t=T_1}^{T_2} \alpha^{t-1} v(x_t)$ subject to

$$(4.4)\qquad \left\{ \begin{array}{ll} -Ax_{T_1} \geqq -\hat{z} , & q_{T_1} \\ Bx_{t-1} - Ax_t \geqq 0 , \quad t = T_1+1, \ldots, T_2 , & q_t \\ Bx_{T_2} \geqq \hat{z} , & q_{T_2+1} \\ x_t \geqq 0 , \quad -Cx_t \geqq -w , \quad t = T_1, \ldots, T_2 . & r_t \end{array} \right.$$

This is so for $T_1 = 1$ because imposing the additional constraints $(x_t, z_t) = (\hat{x}, \hat{z})$, $t = T_2+1, T_2+2, \ldots,$ in the statement of $P_\infty(\hat{z}, \alpha)$ does not constrain the solution $(\hat{x}_t, \hat{z}_t) = (\hat{x}, \hat{z})$, $t = 1, 2, \ldots$. By multiplying the maximand by α^{T_1-1} one sees readily that the same statement holds for any other segment $t = T_1, \ldots, T_2$, including those with $T_1 < 1$.

By the Kuhn-Tucker theorem there now exists for each period $[T_1, T_2]$ a vector of the form

$$(q_{T_1}, r_{T_1}, q_{T_1+1}, \ldots, q_{T_2}, r_{T_2}, q_{T_2+1}) \tag{4.5}$$

satisfying the dual constraints

$$\left\{\begin{aligned} q_t, r_t, q_{t+1} &\geqq 0 \\ \alpha^{t-1}v'(\hat{x}) - q_tA - r_tC + q_{t+1}B &\leqq 0 \\ \alpha^{t-1}v'(\hat{x})\hat{x} - q_t\hat{z} - r_tw + q_{t+1}\hat{z} &= 0 \end{aligned}\right\} \; t = T_1, \ldots, T_2 . \tag{4.6}$$

Consider first the case $T_1 = 1$, $T_2 = 2$. If (3.2) determines q_1, q_2 uniquely, then the two subsets of the nonhomogeneous linear conditions (4.6) labeled $t = 1$ and $t = 2$, taken separately, uniquely determine (q_1, q_2) and (q_2, q_3), respectively, on the basis of the only nonhomogeneous terms, the given vectors $\alpha^{t-1}v'(\hat{x})$ of marginal utilities for $t = 1, 2$, respectively. It follows that $q_2 = \alpha q_1$, hence $(\hat{x}, \hat{z})$ solves $P(\alpha)$.

$4^*.3.$ Necessity if prices referred to delivery time are uniformly bounded. We now need to use (4.6) for $T_1 = 1$ and all $T_2 \equiv T > 1$. It will help to transform to prices quoted for payment at time of delivery,

$$(4.7) \qquad (\bar{q}_t, \bar{r}_t) \equiv \alpha^{-t+1}(q_t, r_t), \quad t = 1, 2, \ldots,$$

in terms of which (4.6) becomes

$$(4.8) \qquad \left\{ \begin{array}{r} \bar{q}_t, \bar{r}_t, \bar{q}_{t+1} \geqq 0 \\ v'(\hat{x}) \quad - \bar{q}_t A - \bar{r}_t C + \alpha \bar{q}_{t+1} B \leqq 0 \\ v'(\hat{x})\hat{x} - \bar{q}_t \hat{z} - \bar{r}_t w + \alpha \bar{q}_{t+1} \hat{z} = 0 \end{array} \right\} \quad t = 1, \ldots, T.$$

We note that the subsystems of (4.8) for specific values of t are all of the same form, defining congruent closed sets in the spaces of (q_t, r_t, q_{t+1}) for $t = 1, \ldots, T$, respectively.

Assume now that there exists for each $T \geqq 1$ a solution

$$(4.9) \qquad (\bar{q}_1^T, \bar{r}_1^T, \bar{q}_2^T, \ldots, \bar{q}_T^T, \bar{r}_T^T, \bar{q}_{T+1}^T)$$

of (4.8) such that the component vectors of all these solutions are bounded uniformly in t and T,

$$(4.10) \quad \sum_{\ell} (\bar{q}_{t\ell}^T + \bar{q}_{t+1,\ell}^T) + \sum_m \bar{r}_{tm} \leqq P < \infty \quad \text{for} \quad 1 \leqq t \leqq T, \quad T = 1, 2, \ldots .$$

Then the averages

$$(\bar{\bar{q}}_1^T, \bar{\bar{r}}_1^T, \bar{\bar{q}}_2^T) \equiv \frac{1}{T} \sum_{t=1}^{T} (\bar{q}_t^T, \bar{r}_t^T, \bar{q}_{t+1}^T)$$

taken successively in the solutions (4.9) all satisfy (4.8) and (4.10) for $t = 1$, hence belong to a closed and bounded set, and a subsequence of these averages converges to a vector $(\bar{q}_1^*, \bar{r}_1^*, \bar{q}_2^*)$ satisfying (4.8) with $t = 1$. Finally, since

$$\lim_{T\to\infty} (\bar{q}_2^T - \bar{q}_1^T) = \lim_{T\to\infty} \frac{1}{T} (q_{T+1}^T - q_1^T) = 0 ,$$

we have $\bar{q}_2^* = \bar{q}_1^*$. By (4.7), the limit vector transforms back into a vector (q_1, r_1, q_2) satisfying (3.2) with $z = \hat{z}$, and $q_2 = \alpha q_1$.

4*.4. Proof that prices at time of delivery can be bounded uniformly. We first recall an inequality from concave programming under linear constraints. Let $V(X)$ be concave, differentiable, and defined for $X \geqq 0$, X a column vector. Let E be a fixed matrix, e a column vector which we may vary, both of orders such that

$$(4.11) \qquad X \geqq 0 , \qquad EX \geqq e ,$$

represents a constraint set for the maximization of $V(X)$. We say that X is e-feasible if it satisfies (4.11), e-optimal if it maximizes $V(X)$ under those constraints. The maximum attained,

$$\hat{V}(e) \equiv V(\hat{X}(e)) \equiv \max\{V(X) | X \text{ is e-feasible}\}$$

is then defined on the set $\mathcal{E}$ of all e for which (4.11) is satisfied by at least one X. With any e-optimal vector $\hat{X} \equiv \hat{X}(e)$ for any $e \in \mathcal{E}$, the Kuhn-Tucker theorem associates a price vector Q such that

$$Q \geqq 0 \,, \qquad V'(\hat{X}) + QE \leqq 0 \,, \quad V'(\hat{X})\hat{X} + Qe = 0 \,.$$

Finally, for any $e' \in \mathcal{E}$, let X' denote any e'-feasible vector. Then, using the concavity of $V(X)$,

$$\text{(4.12)} \quad -\Delta \equiv V(X') - V(\hat{X}) \leqq V'(\hat{X})(X' - \hat{X}) \leqq -Q(EX' - e) \leqq Q(e - e') \,.$$

We now return to (4.6), choosing $T_1 = 1-\tau$, $T_2 \equiv T+\tau$, $T \geqq 1$, and specifying

$$\text{(4.13)} \qquad X \equiv (x_{1-\tau}, \ldots, x_{T+\tau}) \,, \qquad V(X) \equiv \sum_{t=1-\tau}^{T+\tau} \alpha^{t-1} v(x_t) \,,$$

$$\text{(4.14)} \quad E \equiv \begin{array}{c} (x_{1-\tau})(x_{2-\tau}) \cdots (x_0)(x_1) \cdots (x_{T+\tau}) \\ \begin{bmatrix} -A & & & & & & \\ -C & & & & & & \\ B & -A & & & & & \\ & -C & & & & & \\ & B & -A & & & & \\ & & \vdots & \vdots & & & \\ & & & B & -A & & \\ & & & & -C & & \\ & & & & B & -A & \\ & & & & & \vdots & \\ & & & & & B & -A \\ & & & & & & -C \\ & & & & & & B \end{bmatrix} \end{array} , \quad e \equiv \begin{bmatrix} -\hat{z} \\ -w \\ 0 \\ -w \\ 0 \\ \vdots \\ 0 \\ -w \\ 0 \\ \vdots \\ 0 \\ -w \\ \hat{z} \end{bmatrix} , \quad e' \equiv e + \begin{bmatrix} \hat{z}-z' \\ 0 \\ 0 \\ 0 \\ \\ \vdots \\ z''-\hat{z} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here τ , z' , z'' are the quantities appearing in the definition of the more-than-reproduceability of $\hat{z}$. Therefore there exists a path segment

$\{x'_t \mid t = 1-\tau, \ldots, 0\}$ satisfying those of the constraints (4.11), with e' replacing e, that refer to the time periods it covers, and delivering a capital stock $Bx'_0 \geqq z'' \geqq \hat{z}$ at the beginning of the period $t = 1$. By extending this path with the constant segment $\{x'_t = \hat{x} \mid t=1, \ldots, T+\tau\}$ we obtain an e'-feasible path. Hence $e' \in \mathcal{E}$, and (4.12) applies if for $\hat{X}$ we choose the e-optimal constant path $x_t = \hat{x}$, $t = 1-\tau, \ldots, T+\tau$. Therefore

$$-\Delta \leqq -q_{1-\tau}(\hat{z} - z') - q_1(z'' - \hat{z}) ,$$

and, since $z' \leqq \hat{z} \leqq z''$ and $q_t \geqq 0$, we have

$$\Delta \geqq q_{1-\tau,\ell}(\hat{z}_\ell - z'_\ell) , \qquad \Delta \geqq q_{1\ell}(z''_\ell - \hat{z}_\ell) , \qquad \ell = 1, \ldots, L .$$

Let $\mathcal{L}'$, $\mathcal{L}''$ denote the subsets of $\mathcal{L} \equiv \{1, \ldots, L\}$ in which $\hat{z}_\ell - z'_\ell$ and $z''_\ell - \hat{z}_\ell$ are positive, respectively. Then each ℓ belongs to at least one of $\mathcal{L}'$, $\mathcal{L}''$, and, if we let

$$\Delta' \equiv \Delta[\min_{\ell \in \mathcal{L}} (\hat{z}_\ell - z'_\ell)]^{-1} , \qquad \Delta'' \equiv \Delta[\min_{\ell \in \mathcal{L}''} (z''_\ell - \hat{z}_\ell)]^{-1} ,$$

we have

$$q_{1-\tau,\ell} \leqq \Delta' \text{ if } \ell \in \mathcal{L}' , \qquad q_{1,\ell} \leqq \Delta'' \text{ if } \ell \in \mathcal{L}'' .$$

For each $t = 2, \ldots, T+\tau+1$, we repeat the above reasoning with e' changed to

$$e'_t \equiv e + (\overset{1-\tau}{0}, \overset{2-\tau}{0}, 0, \ldots, 0, \overset{t-\tau}{\hat{z}-z'}, 0, \ldots, 0, \overset{t}{z''-\hat{z}}, 0, \ldots, 0, \overset{T+\tau+1}{0}) .$$

The only difference is that the segment of the e'_t-feasible path X'_t on which X'_t differs from the e-optimal path $\hat{X}$ is deferred by one period, without other change, whenever t increases by one. Because of the form (4.13) of $V(X)$, this leads to

$$\left.\begin{aligned} q_{t-\tau,\ell} &\leqq \alpha^{t-1}\Delta' , \quad \ell \in \mathcal{L}' \\ q_{t,\ell} &\leqq \alpha^{t-1}\Delta'' , \quad \ell \in \mathcal{L}'' \end{aligned}\right\} \quad t = 1, \ldots, T+\tau+1 .$$

Note that the ranges of both $t-\tau$ and t, as t ranges from 1 to $T+\tau+1$, include the values $t = 1, \ldots, T+1$. Therefore, returning by (4.7) to prices timed at delivery, we have

$$0 \leqq \bar{q}_{t,\ell} \leqq \max\{\alpha^{\tau}\Delta', \Delta''\} < \infty , \quad \ell \in \mathcal{L}, \quad 1 \leqq t \leqq T+1 ,$$

showing that in any sequence of solutions of (4.8) with $T = 1, 2, \ldots$, $\bar{q}^T_t$ is bounded uniformly in t and T, where $1 \leqq t \leqq T+1$. Since $w > 0$, the equality in (4.8) then implies that $\bar{r}^T_t$ is likewise uniformly bounded.

4*.5. Evaluation. In conclusion, we prove a lemma indicating that the requirement that a reproduceable capital stock z be more-than-reproduceable is not very restrictive. We shall call a capital stock $z \geqq 0$ *prolific** if either $z > 0$, or $z \not> 0$ and there exists a feasible path leading from the initial stock z to a stock $z' > 0$ in a finite number of periods. Clearly the null stock is not prolific. Also, if z is prolific

*Sutherland uses the term "sufficient" for this concept.

so is λz for all $\lambda > 0$. Therefore, the resource constraints (2.1C)--refer also to (2.6)--do not enter into whether a given z is prolific.

Lemma 1. For a reproduceable capital stock z to be more-than-reproduceable, it is necessary that z be prolific. If that condition is met, either of the following additional conditions is sufficient:

(i) There exists $x \geq 0$ with $Bx \geqq z > Ax$, $Cx \leqq w$.

(ii) There exists $x \geq 0$ with $Bx \geqq z \geqq Ax$, $Cx < w$.

The necessary condition is obvious.

The sufficient condition (i) requires some slack in reproduceability but not necessarily in resource use. The sufficiency is obvious, because the one-period feasible path given by $x_1 = x$ leads from $z_1 = z' \equiv Ax$ to $z_2 = z$, where $z' < z$. The interpretation is that z can be produced from a smaller z' by devoting all resources, or as much as proves necessary, to capital formation.

Condition (ii) places the slack in resource use, leaving a margin of the scarce resources for the production of consumption goods. To prove its sufficiency we set aside $(1-\epsilon)z$, where $0 < \epsilon < 1$, to be reproduced through the process vector $(1-\epsilon)x$ as often as necessary. We use ϵz to produce $\epsilon z'' > 0$ after a nonnegative number of periods, choosing ϵ small enough to stay within the available resource margin $w - (1-\epsilon)Cx$ throughout the path. If necessary, we diminish $\epsilon z''$ by some disposal to obtain $\epsilon z' > 0$ where z' satisfies the viability condition

$$Bx' > z' = Ax' > 0 \quad \text{for some} \quad x' \geq 0 .$$

We then have

$$\rho \equiv \min_{\ell} \frac{(Bx')_\ell}{(Ax')_\ell} > 1 ,$$

and it is technologically possible to increase $\epsilon z'$ by a factor ρ as often as needed (using disposal as needed), with a claim on resources given by

$$\epsilon \rho^n Cx'$$

in the n^{th} round. However, the number of rounds required to attain the target

$$\epsilon \rho^n z' > \epsilon z , \qquad \text{hence} \quad z'' = (1-\epsilon)z + \epsilon \rho^n z' > z ,$$

is independent of ϵ. Therefore, the claim on resources can be kept within the positive initial resource margin $w - Cx$ until the target is attained, by choosing ϵ small enough.

Obviously, there is a great deal of overlap between cases (i) and (ii). Presumably, one could find a more general third sufficient condition allowing the required slack to be distributed in a coordinated manner between resource availability and reproduceability. However, the lemma goes far enough to indicate that there are at most two ways (that may occur separately or perhaps in combination) in which a reproduceable capital stock z can fail to be more-than-reproduceable. In one case, z would have a zero component or subvector which, given the technology and the other components of z, cannot be made positive. In the other, z would have a component or subvector which can be reproduced but not increased under the resource constraints, and also cannot be reached starting from a lower level. Both cases lack relevance for any applications in which a model with constant technology and constant resource availabilities constitutes a tolerable simplification. In the former case, what is the point of putting capital goods in the model which historically could not, even with present technology, have been produced from an earlier initial state in which they were absent? Likewise, in the latter case, how could the capital goods in question have originally been brought up to the levels of which the reproduceability is resource-constrained, if present technology and resource constraints do not make this possible? We conclude that, in applications of the present model, it would not hurt us if we should be unable to compute some weird solutions of $P_\infty(\alpha)$ that are not solutions of $P(\alpha)$.

5. A Combinatorial Theorem Due to Scarf

For our purposes, the importance of the Equivalence Theorem resides in that $P(\alpha)$ is a much more suitable starting point for the computation of an approximately invariant capital stock than $P_\infty(\alpha)$. At first sight, $P(\alpha)$ looks very similar to an ordinary nonlinear programming problem. There is one crucial characteristic of the problem, however, that distinguishes it from all ordinary non-linear programming problems. This is the requirement that the vectors of Lagrangean multipliers q_1 and q_2 have to satisfy the additional linear constraint $q_2 = \alpha q_1$. This places the problem in the category of fixed point problems for a continuous mapping. One of us [Koopmans, 1971] has indicated two alternative, equivalent and still more condensed formulations of $P(\alpha)$ that explicitly have the form of fixed point problems, one in the space of activity levels, the other in the space of prices. The former of these equivalent formulations also provides a convenient basis for a proof of the existence of a solution of $P(\alpha)$.

In the present section we state a combinatorial theorem due to Scarf [1967a, b], which has provided the starting point for the algorithms for computing an approximate competitive equilibrium mentioned in Section 1. Section 6 describes the application of this theorem to the computation of an approximately invariant optimal capital stock. As a byproduct, this application also provides a direct and constructive proof for the existence of an invariant optimal capital stock.

The combinatorial theorem is expressed in terms of the concept of a <u>primitive set</u> of vectors selected from a larger set. In order to review that concept, let $\Pi = (\pi^1, \ldots, \pi^J)$ be a collection of (column) vectors in s-dimensional Euclidean space. The first s vectors of Π are assumed to have the form

$$(5.1)\begin{cases}\pi^1 = (0,\ M_1,\ \ldots,\ M_1)\\ \pi^2 = (M_2,\ 0,\ M_2,\ \ldots,\ M_2)\\ \vdots\\ \pi^s = (M_s,\ M_s,\ \ldots,\ 0)\end{cases} \qquad \text{where } M_1 > M_2 > \ldots > M_s > 1 \ .$$

and will be called the <u>slack</u> vectors. The remaining, <u>nonslack</u>, vectors lie on the unit simplex, i.e.

$$(5.2)\quad \text{for all } j > s\ ,\quad \sum_{i=1}^{s} \pi_i^j = 1\ , \quad \text{and} \quad 0 \leqq \pi_i^j \leqq 1 \quad \text{for all } i\ .$$

Finally, we make the nondegeneracy assumption that no two distinct vectors in Π have the same i^{th} coordinate for any i .

Before defining a primitive set, we define and denote the vector minimum for any set $\Phi \equiv \{\varphi^j | j = 1, \ldots, s\}$ of s vectors in an s-dimensional vector space by

$$\min\{\varphi^1, \ldots, \varphi^s\} \equiv (\underline{\varphi}_1, \ldots, \underline{\varphi}_s) \text{ , where } \underline{\varphi}_i \equiv \min\{\varphi_i^j | j = 1, \ldots, s\} \text{ , } i = 1, \ldots, s \text{ .}$$

A set $\Pi \equiv \{\pi^{j_1}, \ldots, \pi^{j_s}\}$ of s distinct vectors of Π is now defined to be a primitive set if there is no vector π^j in Π for which

$$\pi^j > \min\{\pi^{j_1}, \ldots, \pi^{j_s}\} \text{ .} \tag{5.3}$$

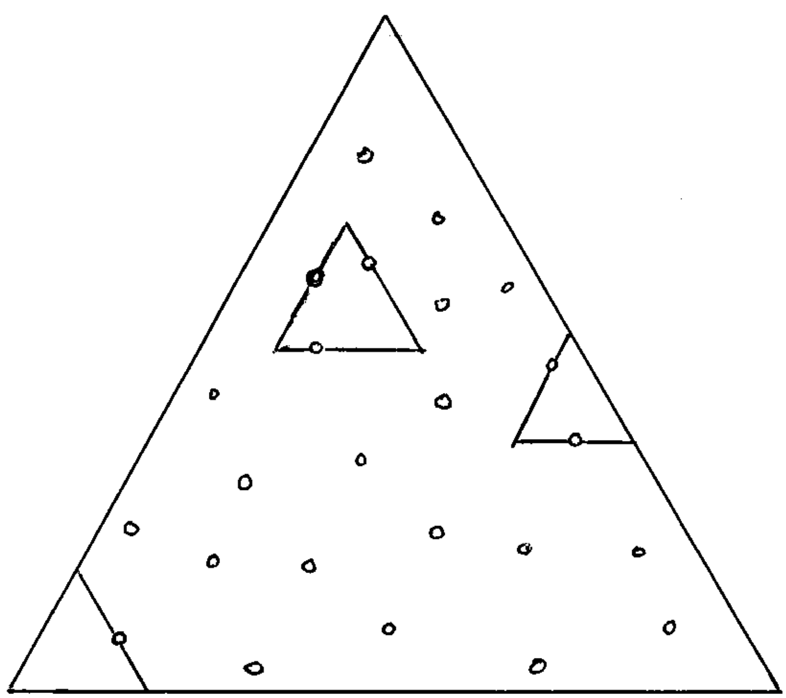

FIGURE 1

Figure 1 illustrates this concept. The large triangle encloses the simplex (5.2) in the space of $s = 3$ dimensions. The nonslack vectors π^j, $j > 3$ are represented by dots, the three slack vectors π^j, $j = 1, 2, 3$, by the sides of the large triangle on which $\pi^j_j = 0$. The three small triangles identify primitive sets with zero, one and two slack vectors, respectively. Each side of the triangle identifying a primitive set must contain a different one of the vectors π^j (if j is nonslack) or be contained in the large side representing π^j (if j is slack). The condition (5.3) specifies that the small triangle shall contain no nonslack point π^j in its interior. (Because of the nondegeneracy assumption, it can have no other such point in its boundary.) For further discussion, see Scarf [1967b].

Finally, we recall from linear programming the definition of a feasible basis for a set of linear constraints on a set of nonnegative variables. Given a matrix F of order (s, J), $s < J$ and a column vector g of order s, a submatrix F^* consisting of s linearly independent columns $f^{j_1}, \ldots, f^{j_s}$ of F is called a <u>feasible</u> <u>basis</u> for the system of equations $F\eta = g$ if that system has a solution η in which $\eta_j \geqq 0$ for $j = j_1, \ldots, j_s$, and $\eta_j = 0$ for all other j.

The combinatorial theorem (Scarf [1967a], Theorem 2) may now be stated:

<u>Theorem 2</u>. (Combinatorial Theorem) <u>Let</u> Π <u>be a matrix of order</u> (s, J) <u>as described above, and let</u>

$$F \equiv \begin{bmatrix} 1 & 0 & \cdots\cdots & 0 & f_1^{s+1} & \cdots & f_1^J \\ 0 & 1 & \cdots\cdots & 0 & f_2^{s+1} & \cdots & f_2^J \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 1 & f_s^{s+1} & \cdots & f_s^J \end{bmatrix}$$

be a matrix and $g \equiv (g_1, \ldots, g_s)$ a non-negative column vector such that the set of nonnegative vectors η satisfying $F\eta = g$ is bounded. Then there exists a primitive set $\pi^{j_1}, \ldots, \pi^{j_s}$, so that the corresponding columns $f^{j_1}, \ldots, f^{j_s}$ of F form a feasible basis for $F\eta = g$.

The proof of the above theorem also provides us with an algorithm for finding a primitive set whose associated columns in F form a feasible basis for $F\eta = g$.

6. Design for an Algorithm for Approximate Solution of $P(\alpha)$

In this section we shall show how the combinatorial theorem stated in the preceding section may be used to compute an approximate solution of $P(\alpha)$. We use the expression "approximate solution" rather than "approximation to a solution" since the algorithm is designed to compute vectors x, z, q_1, $q_2 = \alpha q_1$, r satisfying to a high degree of accuracy the conditions (3.2), (3.3) for a solution of $P(\alpha)$ (with x substituted for $\hat{x}$).

Since time subscripts on the vector x are no longer needed, we shall change the notation for its components to $x \equiv (\xi_1, \ldots, \xi_I)$.

We shall make two additional stimpulations about the model stated in Section 2. First, because of (2.5), suitable choices of the unit levels of activity for all processes will insure that

$$(6.1) \qquad Cx \leqq w\ , \quad x \geqq 0 \ \text{ implies } \ \sum_{i=1}^{n} \xi_i \leqq \gamma < 1 \ \text{ for some } \ \gamma > 0 \ .$$

The second stipulation is indeed a new assumption, to be called the <u>growth capability condition</u>. It implies and strengthens the viability condition by introducing the discount factor into it:

$$(6.2) \qquad \text{there exists } \ \bar{x} > 0 \ \text{ such that } \ (A - \alpha B)\bar{x} < 0 \ .$$

This says that, disregarding consumption, the capital stock can be made to grow at a rate at least slightly higher than $\beta \equiv \alpha^{-1}$ per period, where of course $\beta > 1$. If we define

$$\bar{\beta} = \sup\{\beta \mid (A - \beta^{-1}B)x < 0 \ \text{ for some } \ x \geqq 0\}\ ,$$

then the viability condition (2.4) assures us that $\bar{\beta}^{-1} < 1$, hence that there is a nonempty open interval

$$(6.3) \qquad \bar{\beta}^{-1} < \alpha < 1$$

on which α satisfies both (1.1) and (6.2).

The growth capability condition derives additional plausibility from a study of the connections between the solution set of $P(\alpha)$ and the characteristics of the strict von Neumann model obtained if we delete all consumption goods and all resources from the present model. Some results of this kind are assembled in Appendix II. In any case, the growth capability condition simplifies matters by rendering services analogous to those of the constraint qualification in ordinary nonlinear programming.

Because of (2.6), we can also find an $\bar{x}$ satisfying both (6.2) and

$$C\bar{x} \leqq w\ , \quad \text{hence} \quad \sum_{i=1}^{I} \bar{\xi}_i \leqq \gamma < 1 \tag{6.4}$$

by (6.1). This helps because the concept of a primitive set will now be applied to the space of vectors x of activity levels ξ_i . In order to overcome the difficulty that the activity levels bear no natural relation to the simplex $\{x | \xi_i \geqq 0,\ \xi_1 + \ldots + \xi_I = 1\}$ we introduce an additional coordinate

$$\xi_0 = 1 - \sum_{i=1}^{I} \xi_i\ , \tag{6.5}$$

which by (6.4) is positive for all feasible vectors x . We can therefore now define primitive sets in relation to the simplex of dimensionality $n+1$,

$$\Sigma \equiv \{\xi \equiv (\xi_0,\ \xi_1,\ \ldots,\ \xi_I) | \xi_i \geqq 0 \quad \text{for} \quad i = 0,\ \ldots,\ n\ , \quad \sum_{i=0}^{I} \xi_i = 1\}\ . \tag{6.6}$$

Thus ξ differs from x only by the presence of an additional component ξ_0 . A primitive set will consequently consist of $I+1$ vectors with $I+1$ components each. A collection $\Xi \equiv \{\xi^0,\ \xi^1,\ \ldots,\ \xi^J\}$ of $(I+1)$-dimensional vectors is selected, where $J > I$. The $I+1$ slack vectors are given by

$$
(6.7)\qquad \left.\begin{aligned}
\xi^0 &= (0,\ M_0,\ M_0,\ \ldots,\ M_0)\\
\xi^1 &= (M_1,\ 0,\ M_1,\ \ldots,\ M_1)\\
&\ \vdots\\
\xi^I &= (M_I,\ M_I,\ M_I,\ \ldots,\ 0)
\end{aligned}\right\}\ \text{where}\quad M_0 > M_1 > \ldots > M_I > 1\ ,
$$

while the $J-I$ nonslack vectors are selected so as to have a rather even distribution over Σ, and again such that no two vectors have identical i^{th} coordinates for any i.

Each vector ξ^j in Ξ is associated with the corresponding column f^j in a matrix F, as follows:

$$
(6.8)\qquad F \equiv
\begin{array}{c}
\begin{array}{ccccccccccc}
\xi^0 & \xi^1 & \xi^2 & & \xi^I & \xi^{I+1} & & \xi^j & & \xi^J
\end{array}\\
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & f_0^{I+1} & \ldots & f_0^j & \ldots & f_0^J\\
0 & 1 & 0 & \ldots & 0 & f_1^{I+1} & \ldots & f_1^j & \ldots & f_1^J\\
0 & 0 & 1 & \ldots & 0 & f_2^{I+1} & \ldots & f_2^j & \ldots & f_2^J\\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots\\
0 & 0 & 0 & \ldots & 1 & f_I^{I+1} & \ldots & f_I^j & \ldots & f_I^J
\end{bmatrix}
\end{array}
$$

We define the columns f^j, $j = 0, \ldots, J$, of F as functions of the $\xi^j \equiv (\xi_0^j, x^j)$ by considering an exhaustive succession of cases, to be labeled 0_j, 1, 2_ℓ, 3_m where $j = 0, \ldots, I$, $\ell = 1, \ldots, L$, and $m = 1, \ldots, M$. The case that applies to a particular ξ^j with $j > I$ is determined on the basis of which, if any, of the following set of $L+M$ constraints are violated by x^j,

$$(A - B)x \leqq 0 ,$$
$$Cx \leqq w .$$

These constraints have been obtained from the primal constraint set of $P(\alpha)$ by elimination of $\hat{z}$.

0_j. As indicated, for $j \leqq I$, $f^j \equiv e^j$ is the unit vector with $e^j_j = 1$. All remaining cases refer to $j > I$.

1. If $(A - B)x^j \leqq 0$ and $Cx^j \leqq w$, then with ξ^j we associate

$$f^j = (1, \quad v_1'(x^j)+1, \quad \ldots, \quad v_I'(x^j)+1) . \tag{6.9}$$

2_ℓ. If one or more of the constraints $(A - B)x \leqq 0$ are violated for $x = x^j$, and if the first of these constraints to be violated is the ℓ^{th} , say, then with ξ^j we associate

$$f^j = (1, \quad -a_{\ell 1}+\alpha b_{\ell 1}+1, \quad \ldots, \quad -a_{\ell I}+\alpha b_{\ell I}+1) . \tag{6.10}$$

It is this stipulation that introduces the parameter α characterizing $P(\alpha)$ into the algorithm.

3_m. If $(A - B)x^j \leqq 0$, but one or more of the constraints $Cx \leqq w$ are violated for $x = x^j$, the first that is violated being the m^{th} , then with ξ^j we associate

$$f^j = (1, \quad -c_{m1}+1, \quad \ldots, \quad -c_{mI}+1) . \tag{6.11}$$

Finally, we select $g \equiv (1, \ldots, 1)$, and note that the constraint set $\mathcal{H} \equiv \{\eta | F\eta = g, \ \eta \geqq 0\}$ is bounded because $\eta \in \mathcal{H}$ implies

$$(6.12)\qquad 0 \leqq \eta_j\,,\quad j = 0,\ \dots,\ J\ ;\quad \eta_0 + \sum_{j=I+1}^{J} \eta_j = 1\ ;$$

$$\text{and}\quad \eta_i = 1 - \sum_{j=I+1}^{J} f_i^j \eta_j\,,\quad i = 1,\ \dots,\ I\ .$$

Theorem 1 consequently applies, and there exists a primitive set $\Xi^* \equiv \{\xi^{j_0}, \dots, \xi^{j_I}\}$ whose associated columns form a feasible basis $F^* \equiv \{f^{j_0}, \dots, f^{j_I}\}$ for $F\eta = g$. Hence

$$(6.13)\qquad \begin{bmatrix} f_0^{j_0} \\ f_1^{j_0} \\ \vdots \\ f_I^{j_0} \end{bmatrix} \eta_{j_0} + \dots + \begin{bmatrix} f_0^{j_I} \\ f_1^{j_I} \\ \vdots \\ f_I^{j_I} \end{bmatrix} \eta_{j_I} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

with the f^{j_i}, $i = 0, \dots, I$, linearly independent and with $\eta_{j_i} \geqq 0$ for all i.

In the next section we show

<u>Theorem 3</u>. (Convergence) *For any sequence* $\{\Xi(n) \mid n = 1, 2, \dots\}$ *of matrices* Ξ *of which the nonslack vectors* ξ^j, $j = I+1, \dots, J_n$, *are scattered with indefinitely increasing and rather uniform density over* Σ, *there exists a subsequence* $\{\Xi(n_h) \mid h = 1, 2, \dots\}$ *with a corresponding sequence* $\{\Xi^*(n_h)\}$ *of primitive sets of which all nonslack vectors converge to a vector* $\hat{\xi} \equiv (\hat{\xi}_0, \hat{x})$. *Moreover, if for any such vector* $\hat{x}$ *we define* $\hat{z} \equiv A\hat{x}$, *then* $(\hat{x}, \hat{z})$ *is a solution of* $P(\alpha)$.

Clearly, Theorem 3 also establishes the existence of a solution $(\hat{x}, \hat{z})$ to $P(\alpha)$ under the assumptions made.

It is not difficult to construct examples with only a few commodities and processes, for which more than one solution exists. In that case, the solution obtained may depend on the way in which the algorithm is started up.

Now that the construction of the matrix F has been fully specified in terms of particulars of the present problem, the steps carried out for any given pair (Ξ, F) to find a primitive set satisfying (6.13) can be laid out according to any version of the general procedure for problems of this kind. For the published versions, see Scarf [1967a, b] and Hansen [1968].

7*. Proof of Convergence of the Algorithm

We proceed from the relation (6.13) characterizing the pair (Ξ^*, F^*) of a primitive set Ξ^* and an associated feasible basis F^* for the equation $F\eta = g$.

Note first that the index set $\{j_i | i = 0, \ldots, I\}$ labeling the terms of (6.13) partitions in a unique way into the $I + 2 + L + M$ cases 0_j , 1 , 2_ℓ , 3_m describing the choice of f^{j_i} (each case assignment depending on ξ^{j_i} only). Observe also that the set $\Xi^0 \equiv \{\xi^0, \ldots, \xi^I\}$ is not a primitive set, and hence each primitive set Ξ^* contains at least one vector ξ^j with $j > I$.

We note for later use that the definitions of the columns of F^* allow us to write (6.13) in the form

$$\eta_0 + \sum_{i\in\mathcal{I}} \lambda_i + \sum_{\ell\in\mathcal{L}} \epsilon_\ell + \sum_{m\in\mathcal{M}} \delta_m = 1 \tag{7.1}$$

$$\eta_j + \sum_{i\in\mathcal{I}} \lambda_i(1 + v_j'(x^{j_i})) + \sum_{\ell\in\mathcal{L}} \epsilon_\ell(1 - a_{\ell j} + \alpha b_{\ell j}) + \sum_{m\in\mathcal{M}} \delta_m(1 - c_{mj}) = 1 \,, \tag{7.2}$$

$$j = 1, \ldots, I \,, \quad \text{where } \eta_j \equiv 0 \text{ for } j \notin \mathcal{J} .$$

Here $a_{\ell j}$, $b_{\ell j}$, c_{mj} $(j = 1, \ldots, I)$ are as given in (3.1), and $\mathcal{J}$, $\mathcal{I}$ $\mathcal{L}$, $\mathcal{M}$, are index sets corresponding to those of the cases 0_j, 1, 2_ℓ, 3_m, respectively, in the definition of F that have found their way into F^*. Each of the I+1 coefficients η_{j_i} in (6.13) recurs in (7.1) and (7.2) as one of the η_j (for $j_i \leqq I$), or one of the λ_i, ϵ_ℓ, δ_m (for $j_i > I$). We now have

$$(7.3)\begin{cases} (7.3a) & \eta_j, \lambda_i, \epsilon_\ell, \delta_m \geqq 0 \text{ for } j \in \mathcal{J}, \; i \in \mathcal{I}, \; \ell \in \mathcal{L}, \; m \in \mathcal{M}, \\ (7.3b) & j \notin \mathcal{J} \text{ for at least one } j \in \{0, 1, \ldots, I\}, \\ (7.3c) & 1 - \eta_0 = \sum_{i\in\mathcal{I}} \lambda_i + \sum_{\ell\in\mathcal{L}} \epsilon_\ell + \sum_{m\in\mathcal{M}} \delta_m > 0 . \end{cases}$$

The equality in (7.3c) arises from (7.1). The inequality arises from (7.3b) and (7.2). Finally, the number of elements in

$$\mathcal{F} \equiv \mathcal{J} \cup \mathcal{I} \cup \mathcal{L} \cup \mathcal{M} \tag{7.4}$$

is precisely I+1 .

If the number J of vectors in Ξ is large, and if the nonslack vectors ξ^j are distributed rather evenly throughout Σ, then, as Figure 1 suggests, the nonslack vectors in the primitive set $\xi^{j_0}, \xi^{j_1}, \ldots, \xi^{j_I}$ (those for which $j_i > I$) must be close to each other. Likewise, for any slack member ξ^j of the primitive set $(j \leqq I)$ we have $\xi^j_j = 0$, and hence $\xi^{j_i}_j$ must be close to zero for all nonslack members $(j_i > I)$. For otherwise, vectors $\xi^{j'}$ could be found in Ξ that have the relation (5.3) to Ξ^* that is excluded by the definition of a primitive set.

Consider now an infinite sequence $\{\Xi(n) | n = 1, 2, \ldots\}$ of sets $\Xi(n)$ in which the "density" of nonslack vectors ξ^j (those with $j > I$) increases indefinitely in all parts of Σ. From each $\Xi(n)$ we select a primitive set $\Xi^*(n)$ such that its associated submatrix $F^*(n)$ of $F(n)$, determined as before, satisfies the relations (6.13) and therefore (7.1), (7.2), (7.3). Without continually changing notation, we select a finite sequence of subsequences of $\{\Xi(n)\}$, each subsequence being selected from the preceding sequence or subsequence, such that the following conditions are successively met:

(i) the (nonempty) sets of the nonslack vectors $\xi^j(n)$, $j > I$, of $\Xi^*(n)$ converge, as $n \to \infty$, to some single vector $\hat{\xi} \equiv (\hat{\xi}_0, \hat{x})$, say, where $\hat{\xi} \in \Sigma$,

(ii) the set of vectors $v'(x^j(n))$ for j labeling the nonslack vectors $\xi^j(n)$ of $\Xi^*(n)$ converges to $v'(\hat{x})$,

(iii) the index sets $\mathcal{J}(n)$, $\mathcal{L}(n)$, $\mathcal{M}(n)$ specifying the types of columns of $F(n)$, corresponding to cases 0_j, 2_ℓ, 3_m, incorporated in $F^*(n)$, are independent of n (and will again be denoted $\mathcal{J}$, $\mathcal{L}$, $\mathcal{M}$),

(iv) the coefficients $\eta_j(n)$, $\epsilon_\ell(n)$, $\delta_m(n)$ occurring in the analogues of (7.1), (7.2) for each n converge to $\hat{\eta}_j$, $\hat{\epsilon}_\ell$, $\hat{\delta}_m$, respectively, where $j \in \mathcal{J}$, $\ell \in \mathcal{L}$, $m \in \mathcal{M}$. The sum $\sum_{i \in \mathcal{I}(n)} \lambda_i(n)$ of the coefficients $\lambda_i(n)$ converges to $\hat{\lambda}$.

The compactness sufficient for convergence under (i) follows from (6.6). Convergence under (ii) then follows from the continuity of $v'(x)$. The stabilization of the index sets in (iii) is made possible by the finiteness of the set $\mathcal{F}$ in (7.4) and of the larger set of $2I + 2 + L + M$ "cases" (column types of $F(n)$) from which $F^*(n)$ is selected. The compactness sufficient for convergence under (iv) follows from (7.1), (7.2), (7.3a).

We have defined and define

$$(7.5) \quad \hat{\lambda} \equiv \lim_{n\to\infty} \sum_{i \in \mathcal{I}(n)} \lambda_i(n) , \qquad \hat{\epsilon} \equiv \sum_{\ell \in \mathcal{L}} \hat{\epsilon}_\ell , \qquad \hat{\delta} \equiv \sum_{m \in \mathcal{M}} \hat{\delta}_m ,$$

with the understanding that, if any of the index sets $\mathcal{I}$, $\mathcal{L}$, $\mathcal{M}$ is empty, the corresponding sum $\hat{\lambda}$, $\hat{\epsilon}$, $\hat{\delta}$ equals zero. We now list some properties of $\hat{x}$, $\hat{\eta}_j$, $\hat{\lambda}_i$, $\hat{\epsilon}_\ell$, $\hat{\delta}_m$ that follow directly from the construction. From (7.3) we have, in the limit for $n \to \infty$,

$$(7.6) \begin{cases} (7.6a) \quad \hat{\eta}_j,\ \hat{\epsilon}_\ell,\ \hat{\delta}_m,\ \hat{\lambda},\ \hat{\epsilon},\ \hat{\delta} \geqq 0 \quad \text{for } j \in \mathcal{J},\ \ell \in \mathcal{L},\ m \in \mathcal{M}, \\ \qquad\qquad \text{where } \hat{\eta}_j = 0 \text{ if } j \notin \mathcal{J}, \\ (7.6b) \quad j \notin \mathcal{J} \text{ for at least one } j \in \{0, 1, \ldots, I\}, \\ (7.6c) \quad \hat{\eta}_0 + \hat{\lambda} + \hat{\epsilon} + \hat{\delta} = 1 . \end{cases}$$

If $j \in \mathcal{J}$, we have $\xi_j^j(n) = 0$ for all n, hence $\hat{\xi}_j = 0$, for the reason stated in the paragraph following (7.4). Therefore,

$$\text{(7.7)} \qquad \text{for } j = 0, \ldots, I, \quad \hat{\xi}_j > 0 \text{ implies } \hat{\eta}_j = 0 .$$

From the definitions of the columns of F,

$$\text{(7.8)} \qquad \hat{\lambda} > 0 \text{ implies } (A - B)\hat{x} \leqq 0 \text{ and } C\hat{x} \leqq w ,$$

$$\text{(7.9)} \qquad \ell \in \mathcal{L},\ \hat{\epsilon}_\ell > 0 \text{ imply } (a_\ell - b_\ell)\hat{x} \equiv \sum_{j=1}^{I} (a_{\ell j} - b_{\ell j})\hat{\xi}_j \geqq 0 ,$$

$$\text{(7.10)} \qquad m \in \mathcal{M},\ \hat{\delta}_m > 0 \text{ imply } c_m\hat{x} \equiv \sum_{j=1}^{I} c_{mj}\hat{\xi}_j \geqq w_m .$$

Before we can bring these relations to fruition, we must show that $\hat{\lambda} > 0$. To that end, we rewrite the limiting form of (7.2) as

$$\text{(7.11)} \quad \hat{\eta}_j + \hat{\lambda}(1 + v_j'(x)) + \sum_{\ell \in \mathcal{L}} \hat{\epsilon}_\ell (1 - a_{\ell j} + \alpha b_{\ell j}) + \sum_{m \in \mathcal{M}} \hat{\delta}_m (1 - c_{mj}) = 1 ,$$

$$j = 1, \ldots, I ,$$

and subtract (7.6c) from (7.11), obtaining

$$\text{(7.12)} \quad \hat{\eta}_j - \hat{\eta}_0 + \hat{\lambda} v_j'(\hat{x}) + \sum_{\ell \in \mathcal{L}} \hat{\epsilon}_\ell (-a_{\ell j} + \alpha b_{\ell j}) - \sum_{m \in \mathcal{M}} \hat{\delta}_m c_{mj} = 0 , \quad j = 1, \ldots, I .$$

Finally, let $\bar{x} \equiv (\bar{\xi}_1, \ldots, \bar{\xi}_I) \geq 0$ be a column vector satisfying (6.2) and (6.4), and write $\bar{\xi} \equiv (\bar{\xi}_0, \bar{x}) \in \Sigma$. Multiplying (7.12) by $\hat{\xi}_j - \bar{\xi}_j$ and summing for $j = 1, \ldots, I$ we have, since $\sum_{j=1}^{I} (\hat{\xi}_j - \bar{\xi}_j) \equiv \bar{\xi}_0 - \hat{\xi}_0$, rearranging terms,

$$(7.13)\quad \sum_{j=0}^{I} \hat{\eta}_j \hat{\xi}_j - \sum_{j=0}^{I} \hat{\eta}_j \bar{\xi}_j + \hat{\lambda} \sum_{j=1}^{I} v_j'(\hat{x})(\hat{\xi}_j - \bar{\xi}_j) + \sum_{\ell \in \mathcal{L}} \hat{\epsilon}_\ell \sum_{j=1}^{I} (-a_{\ell j} + \alpha b_{\ell j})\hat{\xi}_j$$

$$+ \sum_{\ell \in \mathcal{L}} \hat{\epsilon}_\ell \sum_{j=1}^{I} (a_{\ell j} - \alpha b_{\ell j})\bar{\xi}_j - \sum_{m \in \mathcal{M}} \hat{\delta}_m \sum_{j=1}^{I} c_{mj}(\hat{\xi}_j - \bar{\xi}_j) = 0 \ .$$

The first term vanishes by (7.7). The second, fourth, fifth and sixth terms are nonpositive, the fourth because, by (7.9),

$$\hat{\epsilon}_\ell > 0 \quad \text{implies} \quad \sum_{j=1}^{I} a_{\ell j}\hat{\xi}_j \geqq \sum_{j=1}^{I} b_{\ell j}\hat{\xi}_j \geqq \alpha \sum_{j=1}^{I} b_{\ell j}\hat{\xi}_j$$

the fifth by (6.2), the sixth by (7.10) and (6.4). In particular, the fifth term is negative if $\hat{\epsilon}_\ell > 0$ for some ℓ, the sixth if $\hat{\delta}_m > 0$ for some m. Therefore,

$$(7.14)\qquad \hat{\lambda} \sum_{j=1}^{I} v_j'(\hat{x})(\hat{\xi}_j - \bar{\xi}_j) \begin{bmatrix} > \\ \geqq \end{bmatrix} 0 \quad \text{if} \quad \hat{\epsilon} + \hat{\delta} \begin{bmatrix} > \\ = \end{bmatrix} 0 \ .$$

Now suppose $\hat{\lambda} = 0$. Then, by (7.14), $\hat{\epsilon} + \hat{\delta} = 0$, hence, by (7.6c), $\hat{\eta}_0 = 1$, and, by (7.11), $\hat{\eta}_j = 1$ for $j = 1, \ldots, I$, contradicting (7.6b), (7.6a). Therefore $\hat{\lambda} > 0$.

If follows by (7.8) that, if we define $\hat{z} \equiv A\hat{x}$, the pair $(\hat{x}, \hat{z})$ satisfies the constraints (3.1A, B, C) for $P(\alpha)$. In particular, from (7.9), (7.10),

$$(7.15)\quad \begin{cases} \hat{\varepsilon}_\ell > 0 \ \text{implies} \ \sum_{j=1}^{I} a_{\ell j}\hat{\xi}_j = z_\ell = \sum_{j=1}^{I} b_{\ell j}\hat{\xi}_j \,, \quad \ell = 1, \ldots, L \,, \\ \hat{\delta}_m > 0 \ \text{implies} \ \sum_{j=1}^{I} c_{mj}\hat{\xi}_j = w_m \,, \quad m = 1, \ldots, M \,, \end{cases}$$

because $\hat{\varepsilon}_\ell \equiv 0$ for $\ell \notin \mathcal{L}$, $\hat{\delta}_m \equiv 0$ for $m \notin \mathcal{M}$. Moreover, (3.1C) for $x = \hat{x}$ implies $\hat{\xi}_0 > 0$ by (7.1), hence, by (7.7) and (7.6c),

$$(7.16)\qquad \hat{\eta}_0 = 0 \,, \quad \hat{\lambda} + \hat{\varepsilon} + \hat{\delta} = 1 \,.$$

If, at last, we define row vectors q, r of orders L, M, respectively, with the components

$$(7.17)\qquad q_{\cdot\ell} \equiv \hat{\lambda}^{-1}\hat{\varepsilon}_\ell \,, \quad \ell = 1, \ldots, L \,, \quad r_{\cdot m} \equiv \hat{\lambda}^{-1}\hat{\delta}_m \,, \quad m = 1, \ldots, M \,,$$

then (7.12) and (7.16) yield the Kuhn-Tucker conditions (7.18$\leqq$), (7.7) and (7.15) the conditions (7.18=),

$$(7.18)\quad \begin{cases} (7.18\leqq) \qquad v'(\hat{x}) - qA + \alpha qB - rC \leqq 0 \,, \\ (7.18=) \qquad v'(\hat{x})\hat{x} - (1-\alpha)q\hat{z} - rw = 0 \,. \end{cases}$$

The proof of the Convergence Theorem 3 is thereby complete.

8. A Numerical Example

We consider an economy with 3 consumption goods, 2 capital goods, and 2 resources other than capital goods (e.g., skilled and unskilled labor). Each good can be produced by 2 alternative processes. Thus, of the total of 10 processes, processes 1 through 6 produce consumption goods.

The matrices A , B , C and D are given by

$$A = \begin{bmatrix} 2. & 2. & 2. & 2. & 2. & 2. & 2. & 2. & 2. & 2. \\ 3. & 3. & 2. & 2. & 1. & 1. & 1. & .5 & 1. & .5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 4. & 3. & 1.5 & 1.5 \\ 2.7 & 2.7 & 1.8 & 1.8 & .9 & .9 & .9 & .4 & 2. & 1.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. & 1. \\ .5 & 1.5 & 1.5 & .5 & .5 & 1.5 & 1.5 & .5 & .5 & 1.5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1. & 2.5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 2.5 & 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 2. & 3. & 0. & 0. & 0. & 0. \end{bmatrix}$$

and the resource availabilities by

$$w = (0,8,\ 0,8) .$$

The utility functions $u(y)$, $v(x)$ are

$$u(y) = y_{.1}^{0.2} y_{.2}^{0.2} y_{.3}^{0.2} , \quad \text{so} \quad v(x) = (\xi_1+2.5\xi_2)^{0.2}(2.5\xi_3+\xi_4)^{0.2}(2\xi_5+3\xi_6)^{0.2} ,$$

The nonslack vectors $\xi^{11}, \ldots, \xi^{J}$ in Ξ are selected to be all vectors of the form

$$\left(\frac{m_0}{100}, \frac{m_1}{100}, \ldots, \frac{m_{10}}{100}\right), \quad \sum_{i=1}^{10} m_i = 100 , \quad m_i \text{ strictly positive integers.}$$

This particular choice of the grid has substantial appeal since it provides a uniform density throughout the simplex, and need not be stored explicitly in the memory units of the computer. It does violate the non-degeneracy assumption required in the definition of primitive sets and in the description of the algorithm. However, in the presence of degeneracy the algorithm can readily be made to retain its validity by a suitable tie breaking rule. We have used a specific tie breaking procedure which has its origins in Hansen's thesis [1968], and which has other computational advantages besides the saving on memory use already mentioned. Its relationship to primitive sets is explained in Scarf (forthcoming).

The algorithm has been run for 3 different values of α, namely $\alpha = 0.7$, 0.8 and 0.9. The terminal primitive sets, the number of iterations for each α and the total computing time for the three α's taken together are shown in Table 1. A preliminary approximation x^* of $\hat{x}$ was obtained from the vectors ξ^{j_i}, $i = 0, \ldots, I$, of the terminal primitive set as follows. If ξ^{j_i} is a slack member, we write $\xi^*_{j_i} = 0$. The other components of x^* are arithmetic averages of the corresponding components of the nonslack vectors ξ^{j_i} in the set. A preliminary approximation q^*, r^* of the shadow prices is obtained by:

1. calculating the η_{j_i} of (6.13) and the sum $\lambda \equiv \sum_{i \in \mathcal{I}} \eta_{j_i}$ of those η_{j_i}, denoted λ_i in (7.1), that correspond to the partial derivatives of $v(x)$,
2. dividing the other η_{j_i}, denoted η_j, ϵ_ℓ, δ_m in (7.1), by λ.

If, for example, η_{j_i} for some i is equal to 0.2 and $\lambda = 0.4$ and if further $f^{j_i} = (1, -a_{\ell 1}+\alpha b_{\ell 1}+1, \ldots, -a_{\ell n}+\alpha b_{\ell n}+1)$, this means that a preliminary approximation for the ℓ^{th} component $q_{1\ell}$ of q_1 is given by $q_{1\ell} = 0.2/0.4 = 0.5$ and for that of q_2 by $q_{2\ell} = \alpha q_{1\ell} = 0.5\alpha$. If some ℓ does not occur in $\mathcal{L}$, we put $q_{1\ell} = q_{2\ell} = 0$, and similarly for r_m.

Finally, the function $v(x)$ was locally linearized and a set of specific linear programming problems were solved to yield the final approximation $\tilde{x}$, which is given in Table 2. Details of this computation are given in an Appendix.

Note that in this example, as α increases, utility attained, the optimal capital stock and, to the extent possible, resource use increase, and that such shifts in processes as occur all economize on resource use.

Table 1. Terminal Primitive Sets $\{\xi^{j_i} | i = 0, \ldots, 10\}$ for a Constructed Example.

Discount factor α = 0.7

Number of iterations introducing nonslack vectors	335					
Labels j_i of slack vectors ξ^{j_i}	1	4	5	8	9	
Components $\xi_{i'}^{j_i}$ of nonslack vectors ξ^{j_i}						
$i' = 0$	43	44	44	44	44	44
1	1	1	1	1	1	1
2	9	8	8	9	9	9
3	10	10	11	10	10	10
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	15	15	14	14	15	15
7	11	11	11	11	10	11
8	1	1	1	1	1	1
9	1	1	1	1	1	1
10	7	7	7	7	7	6

Discount factor α = 0.8

Number of iterations introducing nonslack vectors	420					
Labels j_i of slack vectors ξ^{j_i}	1	4	5	8	10	
Components $\xi_{i'}^{j_i}$ of nonslack vectors ξ^{j_i}						
$i' = 0$	38	39	39	39	39	39
1	1	1	1	1	1	1
2	10	9	10	10	10	10
3	12	12	11	12	12	12
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	15	15	15	14	15	15
7	12	12	12	12	11	12
8	1	1	1	1	1	1
9	8	8	8	8	8	7
10	1	1	1	1	1	1

Discount factor α = 0.9

Number of iterations introducing nonslack vectors	721						
Labels j_i of slack vectors ξ^{j_i}	1	4	6	10			
Components $\xi_{i'}^{j_i}$ of nonslack vectors ξ^{j_i}							
$i' = 0$	19	20	20	20	20	20	20
1	1	1	1	1	1	1	1
2	12	11	11	12	12	12	12
3	13	13	14	13	13	13	13
4	1	1	1	1	1	1	1
5	23	23	22	22	22	23	23
6	1	1	1	1	1	1	1
7	13	13	13	13	14	13	13
8	6	6	6	6	5	5	6
9	10	10	10	10	10	10	9
10	1	1	1	1	1	1	1

Computing Time: 14 minutes on IBM-1130, equivalent to about one minute on IBM 369-50

Table 2. Approximately Invariant Optimal Capital Stock

Discount factor	α	0.7	0.8	0.9
One-period utility attained	$v(\tilde{x})$	0.48855	0.52216	0.55935
Activity levels for producing consumption good 1	$\tilde{\xi}_1$	0.	0.	0.
	$\tilde{\xi}_2$	0.08615	0.10528	0.12621
2	$\tilde{\xi}_3$	0.11086	0.12581	0.13986
	$\tilde{\xi}_4$	0.	0.	0.
3	$\tilde{\xi}_5$	0.	0.	0.24817
	$\tilde{\xi}_6$	0.15544	0.15630	0.
capital good 1	$\tilde{\xi}_7$	0.10667	0.11789	0.13393
	$\tilde{\xi}_8$	0.	0.	0.04345
2	$\tilde{\xi}_9$	0.	0.08416	0.10839
	$\tilde{\xi}_{10}$	0.07423	0.	0.
Capital stock	$\tilde{z}_{\cdot 1} = (A\tilde{x})_1$	1.06667	1.17889	1.60000
	$\tilde{z}_{\cdot 2} = (A\tilde{x})_2$	0.77937	0.92581	1.17055
Unused resources	$(w - C\tilde{x})_1$	0.26667	0.21056	0.
	$(w - C\tilde{x})_2$	0.	0.	0.
Shadow prices, capital goods	$q_{\cdot 1}$	0.35921	0.33408	0.31705
	$q_{\cdot 2}$	0.68327	0.57821	0.45534
Resources	$r_{\cdot 1}$	0.	0.	0.02690
	$r_{\cdot 2}$	0.02304	0.15933	0.26258

Notes: 1. All subscripts refer to processes or commodities, none to time.

2. The numbers given satisfy $(A - B)\tilde{x} = 0$ and (7.18).

3. An expression is interpreted as 0 if it differs from 0 by less than 10^{-7} in absolute value.

APPENDIX I. Final Approximation Procedure

The purpose of this appendix is to describe the sequence of linear programming problems used in obtaining the final approximation $\tilde{x} \equiv (\tilde{\xi}_1, \ldots, \tilde{\xi}_I)$ given in Table 2.

Let $x^*(0)$ denote the preliminary approximation of $\hat{x}$ and $q^*(0) \equiv (q^*_1(0), \ldots, q^*_I(0))$, $\alpha q^*(0)$ and $r^*(0)$ the preliminary approximations of the shadow price vectors. Let $s = 1, \ldots, S$ label the programming problem within the sequence and let $x^*(s)$ denote the approximation of $\hat{x}$ resulting from the s^{th} programming problem. Finally let $q^*(s)$, $\alpha q^*(s)$ and $r^*(s)$ denote the associated approximations to the shadow price vectors resulting from the s^{th} programming problem.

Define the linear functions $V_i(x, q, r)$, $i = 1, \ldots, I$, as follows:

$$V_i(x,q,r) \equiv v_i'(x^*(s-1)) - \sum_{\ell=1}^{L} q_\ell(a_{\ell i} - \alpha b_{\ell i}) - \sum_{m=1}^{M} r_m c_{mi} + \sum_{j=1}^{I} v''_{ij}(x^*(s-1))(\xi_j - \xi_j^*(s-1)) ,$$

where the $v''_{ij}(x^*)$ denote second derivatives of $v(x)$ at x^*. The s^{th} programming problem is then given by:

Minimize u by choice of $x \equiv (\xi_1, \ldots, \xi_I)$, q, r, subject to $x, q, r \geqq 0$ and

$$|V_i(x,q,r)| \leq u \quad \text{for all } i \text{ with } \xi_i^*(s-1) > 0 ,$$

$$V_i(x,q,r) \leq 0 \quad \text{for all } i \text{ with } \xi_i^*(s-1) = 0 ,$$

$$\left|\sum_i (a_{\ell i} - b_{\ell i})\xi_i\right| \leq u \quad \text{for all} \quad \ell \quad \text{with} \quad q^*_{\cdot\ell}(s-1) > 0 \ ,$$

$$\sum_i (a_{\ell i} - b_{\ell i})\xi_i \leq 0 \quad \text{for all} \quad \ell \quad \text{with} \quad q^*_{\cdot\ell}(s-1) = 0 \ ,$$

$$\left|\sum_i c_{mi}\xi_i - w_m\right| \leq u \quad \text{for all} \quad m \quad \text{with} \quad r^*_{\cdot m}(s-1) > 0 \ ,$$

$$\sum_i c_{mi}\xi_i - w_m \leq 0 \quad \text{for all} \quad m \quad \text{with} \quad r^*_{\cdot m}(s-1) = 0 \ ,$$

$$|\xi_i - \xi^*_i(s-1)| \leq 0.1\ \xi^*_i(s-1) \quad \text{for all} \quad i \ ,$$

$$|q_{\cdot\ell} - q^*_{\cdot\ell}(s-1)| \leq 0.1\ q^*_{\cdot\ell}(s-1) \quad \text{for all} \quad j \ ,$$

$$|r_{\cdot m} - r^*_{\cdot m}(s-1)| \leq 0.1\ r^*_{\cdot m}(s-1) \quad \text{for all} \quad j \ .$$

where all summations extend from $i = 1$ to I .

The optimal solution of the s^{th} linear programming problem is denoted $x^*(s)$, $q^*(s)$, $r^*(s)$. Observe that if an activity is operated at a zero level in the preliminary approximation $x^*(0)$ it stays at a zero level through the sequence of linear programming problems. Similarly if a shadow price is 0 in the preliminary approximation it is not changed through the linear programming problems.

The reader should have no difficulty in convincing himself that if the value of the objective function of the linear programming problem at the S^{th} iteration is 0 and in addition $x^*(S) = x^*(S-1)$ then the desired final approximation is obtained by setting $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{q}_1, \tilde{r}, \tilde{q}_2) \equiv (x^*(S), Dx^*(S), Ax^*(S), q^*(S), r^*(S), \alpha q^*(S))$.

For each of the problems described in Section 8 the linear programming problem above needed to be applied 3 times.

To illustrate the approximation the preliminary as well as the final approximation for $\alpha = 0.7$ is given in the table below.

Table 3. Preliminary and Final Approximation of an Invariant Optimal Capital Stock for the Case $\alpha = 0.7$.

		Preliminary (x^*)	Final ($\tilde{x}$)
One Period Utility Attained	$v(x)$	0.47520	0.48855
Activity levels for producing			
Consumption Good 1	ξ_1	0.	0.
	ξ_2	0.08666	0.08615
2	ξ_3	0.10166	0.11086
	ξ_4	0.	0.
3	ξ_5	0.	0.
	ξ_6	0.14666	0.15544
Capital Good 1	ξ_7	0.10833	0.10667
	ξ_8	0.	0.
2	ξ_9	0.	0.
	ξ_{10}	0.06833	0.07423
Capital Stock	$z_{\cdot 1} = (Ax)_1$	1.02333	1.06667
	$z_{\cdot 2} = (Ax)_2$	0.75250	0.77937
Unused Resourced	$(w - Cw)_1$	0.28834	0.26667
	$(w - Cw)_2$	0.03251	0.
Shadow Prices, Capital Goods	$q_{\cdot 1}$	0.36271	0.35921
	$q_{\cdot 2}$	0.68995	0.68327
Resources	$r_{\cdot 1}$	0.	0.
	$r_{\cdot 2}$	0.02326	0.02304

APPENDIX II. Connections with the von Neumann Model.

If we delete resources and consumption goods from the present model, we are left with a technological constraint set which, for a single period, takes the form

$$\text{(II.1)}\begin{cases}\text{(II.1A)} & -Ax \geqq -z_1 , & \text{(capital input)}\\ \text{(II.1B)} & Bx \geqq z_2 , & \text{(capital output)}\\ \text{(II.1x)} & x \geqq 0 . & \end{cases}$$

In this Appendix we trace some connections between the solution sets of $P(\alpha)$ and known properties of vectors x achieving fastest proportional capital growth under these constraints. We first summarize these properties.

A growth factor $\beta > 0$ is achievable if there exist vectors x, z_1, z_2 satisfying (II.1) with $z_2 \geqq \beta z_1 \geq 0$. Eliminating z_1, z_2, the problem of fastest growth is

$$\bar{P} : \underline{\text{Maximize}}\ \ \beta\ \ \underline{\text{subject to}}\ \ (B - \beta A)x \geqq 0\ \ \underline{\text{for some}}\ \ x \geq 0 .$$

If $\bar{\beta}$, $\bar{x}$ solve $\bar{P}$ then there exists a vector $\bar{q}$ such that

$$\text{(II.2)}\begin{cases}\text{(II.2a)} & (B - \beta A)x \geqq 0 , & x \geq 0 ,\\ \text{(II.2b)} & q(B - \beta A) \leqq 0 , & q \geq 0 ,\\ \text{(II.2c)} & q(B - \beta A)x = 0 , & \end{cases}$$

holds with $\beta = \bar{\beta}$, $x = \bar{x}$, $q = \bar{q}$. One can interpret $\bar{q}$ as a price vector $\bar{q}_t$ referred to time t of availability, which in the present model happens to be independent of t. Then $\rho = \bar{\beta} - 1$ becomes an interest rate to be used in adding $\rho\,\bar{q}\,A$ to the cost vector $\bar{q}\,A$ of capital inputs for the unit of activity of each process at the beginning of the period before subtracting it from the vector of $\bar{q}\,B$ of proceeds from capital outputs at the end of the period. The unit profit vector then is $\bar{q}(B - \bar{\beta}A)$, and (II.2b) says that no process yields a positive profit. Furthermore, the addition of (II.2c) to the other constraints forces a zero rate of profit on all processes in use $(\bar{x}_i > 0)$, and a zero price on all capital goods produced in excess, i.e., for which

$$(B\bar{x})_\ell > \bar{\beta}(A\bar{x})_\ell .$$

Dual to $\bar{P}$ is the problem

$\underline{P}$: Minimize β subject to $q(B-\beta A) \leqq 0$ for some $q \geq 0$.

If $\underline{\beta}$, $\underline{q}$ solve $\underline{P}$ then there exists a vector $\underline{x}$ such that (II.2) holds with $\beta = \underline{\beta}$, $x = \underline{x}$, $q = \underline{q}$.

Clearly, $\underline{\beta} \leqq \bar{\beta}$. One has $\underline{\beta} = \bar{\beta}$ if the model is regular, that is [Gale, 1956], if $B\bar{x} > 0$ for all $\bar{x}$ such that $\bar{\beta}$, $\bar{x}$ solve $\bar{P}$. This is the case, in particular, if the indefinitely continued production of each good in the model requires, either as a direct input, or (indirectly) as an input to the production of a direct or indirect input, the continued production of every good in the model.

For simplicity we state the connections between the solution set of $P(\alpha)$, the growth capability condition, and the critical growth rates $\bar{\beta}$, $\underline{\beta}$ solving $\bar{P}$, $\underline{P}$, respectively, only for the case $\bar{\beta} = \underline{\beta} \equiv \beta^*$, say.

Lemma 2. The growth capability condition (6.2) is satisfied if and only if $\alpha > \beta^{*-1}$.

Lemma 3. If $(\hat{x}, \hat{z})$ solves $P(\alpha)$ with $C\hat{x} < w$, then $\alpha \leqq \beta^{*-1}$.

Lemma 4. The solution set of $P(\alpha)$ contains (does not contain) the origin $(\hat{x}, \hat{z}) = (0, 0)$ if $\alpha < (>)\ \beta^{*-1}$.

The proofs are written so that a determined reader can lift from them generalizations of Lemmas 2, 3, 4 to the case where $\underline{\beta} < \overline{\beta}$, by ignoring all references to β^*.

Proof of Lemma 2: Let (6.2) hold. Then there exists $\overline{x} < 0$ such that $(B - \alpha^{-1}A)\overline{x} > 0$, hence $(B - (\alpha^{-1} + \epsilon)A)\overline{x} \geqq 0$ for some $\epsilon > 0$. Therefore (using $\overline{P}$) $\alpha^{-1} + \epsilon \leqq \overline{\beta}$ so $\alpha > \overline{\beta}^{-1} = \beta^{*-1}$.

Assume now that (6.2) does not hold. Then, for all $x > 0$ we must have $(B - \alpha^{-1}A)x \ngtr 0$. This means that the cone $\mathcal{B}^o(\alpha^{-1}) \equiv \{(B - \alpha^{-1}A)x | x > 0\}$ in L-dimensional space, spanned by the columns of $(B - \alpha^{-1}A)$ does not intersect the interior $\mathcal{Z}_L^o \equiv \{z \mid z > 0\}$ of the positive orthant of that space, and, by the separation theorem for convex sets, there exists a vector $q \neq 0$ with

$$q(B - \alpha^{-1}A)x \leqq 0 \quad \text{for all} \quad x > 0, \quad qz \geqq 0 \quad \text{for all} \quad z > 0.$$

Therefore $q(B - \alpha^{-1}A) \leqq 0$, $q \geq 0$, and, using $\underline{P}$, $\alpha^{-1} \geqq \underline{\beta}$, hence $\alpha \leqq \underline{\beta}^{-1}$. Therefore $\alpha > \underline{\beta}^{-1} = \beta^{*-1}$ implies (6.2).

Proof of Lemma 3: If $C\hat{x} < w$, where $(\hat{x}, \hat{z})$ solves $P(\alpha)$, (3.2=) implies $r = 0$. Then by (3.3), (3.2$\leqq$), $q(\alpha B - A) \leqq -v'(\hat{x}) \leq 0$ for some $q \geqq 0$, and, since $q = 0$ would not do, for some $q \geq 0$. Hence, using $\underline{P}$, $\alpha^{-1} \geqq \underline{\beta}$, and $\alpha \leqq \underline{\beta}^{-1} = \beta^{*-1}$.

Proof of Lemma 4: First let $\alpha > \beta^{*-1} = \underline{\beta}^{-1}$. Then Lemma 3 excludes $C\hat{x} < w$, and, in particular, $\hat{x} = 0$.

Next let $\alpha < \beta^{*-1} = \overline{\beta}^{-1}$, hence $\alpha^{-1} > \overline{\beta}$. Then, using $\overline{P}$, there exists no $x \geq 0$ with $(B - \alpha^{-1}A)x \geqq 0$, and, as in the proof of Lemma 2, the (this time closed) cone

$$\mathcal{B}(\alpha^{-1}) \equiv \{(B - \alpha^{-1}A)x \mid x \geqq 0\}$$

intersects the (closed) positive orthant $\mathcal{Z}_L \equiv \{z \mid z \geqq 0\}$ only in the origin. We use the theory of polar cones (we shall cite theorems from Gerstenhaber [1951]) to obtain a somewhat stronger separation statement for this case. If in Gerstenhaber's theorem 12(1)(e) we substitute our $\mathcal{Z}_L$ for his A, and our $-\mathcal{B}(\alpha^{-1})$ for his B, we obtain that the vector sum of the positive polar

$$\mathcal{Z}_L^+ \equiv \{q \mid qz \geqq 0 \quad \text{for all} \quad z \in \mathcal{Z}_L\} \equiv \{q \mid q \geqq 0\}$$

of $\mathcal{Z}_L$ and the negative polar

$$\mathcal{B}^-(\alpha^{-1}) \equiv \{q \mid qz \leqq 0 \quad \text{for all} \quad z \in \ (\alpha^{-1})\}$$

of $\mathcal{B}(\alpha^{-1})$ is the entire L-dimensional space, $\mathcal{R}_L$, say,

$$\mathcal{Z}_L^+ + \mathcal{B}^-(\alpha^{-1}) = \mathcal{R}_L \; .$$

Now $\mathcal{Z}_L^+$ is L-dimensional. To show that the same holds for $\mathcal{B}^-(\alpha^{-1})$ we use $\bar{\bar{P}}$ once more, writing $x = x' + x''$, to conclude that there exist no $x' \geq 0$, $x'' \geq 0$, such that

$$(B - \alpha^{-1}A)x' = -(B - \alpha^{-1}A)x'' \neq 0 \; .$$

Therefore $\mathcal{B}(\alpha^{-1})$ does not contain an entire line, hence, by theorem 12(2), $\mathcal{B}^-(\alpha^{-1})$ is L-dimensional.

Finally, substituting in Gerstenhaber's theorem 13 our $\mathcal{Z}_L^+$ for his A and our $\mathcal{B}^-(\alpha^{-1})$ for his B, we obtain that the intersection of the interiors of $\mathcal{Z}_L^+$ and $\mathcal{B}^-(\alpha^{-1})$ is not empty. Therefore, there exists q such that

$$q > 0 \, , \quad q(B - \alpha^{-1}A)x < 0 \quad \text{for all} \quad x \geq 0 \, ,$$

hence $q(B - \alpha^{-1}A) < 0$. This allows (3.2), (3.3) to be satisfied for $\hat{x} = 0$, $z = 0$, by choosing $r = 0$, $q_1 = \lambda q$, $q_2 = \alpha\lambda q$ with λ sufficiently large to have

$$v'(0) \leqq \lambda q(A - \alpha B) .$$

We believe that further results along these lines can be obtained if we impose additional restrictions on the signs of the elements of A , B , C , D , which are plausible for many applications.

In the example of Section 8, $\underline{\beta} = \overline{\beta} = \beta^* = 1.468$, $\alpha^* \equiv \beta^{*-1} = .6812$.

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