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THE PRICE OF MONEY IN A PURE EXCHANGE MONETARY ECONOMY

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"Shew me the tribute money. And they brought unto him a penny.
And he saith unto them, Whose is this image and superscription?
And they say unto him, Caesar's.
Then saith he unto them, Render therefore unto Caesar the things which are Caesar's."
Matt 22:19-21

I. The Role of Money and Its Value in Exchange

Money is peculiar among commodities in that its usefulness depends on its price. In the terms of the classical economists value in use varies directly with value in exchange. In this essay the term "price of money" will mean for money precisely what the "price" of any other commodity means for that commodity. Price is a number which, taken in ratio with another such number indicates a rate of exchange between two commodities. If $p^m > 0$ is the price of money and p^n is the price of good n, then p^n/p^m is the number of units of money which must be traded (spent) on the market in order to acquire (buy) one unit of good n. This usage is at variance with two standard approaches. It is sometimes noted that money is the unique commodity performing the function of numeraire. Thus its "price" is set identically equal to unity. Alternatively, the term "price of money" is often used to denote the rental rate of money, that is, the rate of interest [6]. Neither of these last two usages enters here.

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It would not upset the theory of value if water or diamonds had a price of zero. But the theory of money depends on money having a positive price. Unfortunately for this theory it is far from clear that the price should be positive [4, 8, 9, 10]. This follows after all since modern money --debt instruments rather than items decorative or useful in themselves-- generally consists of useless pieces of paper or accounting units whose only use is to eventually be exchanged for some non-zero quantity of other goods. Money is accepted because it is accepted. But if the price of money were zero then for even arbitrarily large amounts of money one could buy precisely nothing. If money were not accepted then it would not be accepted because it would not be accepted. Thus when the price of money is zero there will be no unsatisfied demand for money. Hence, there is an equilibrium in which the price of money is zero. This is argued more completely below.

Further, it is distressingly easy to find economies in which zero is the only price of money consistent with equilibrium. Consider an economy over time with a finite horizon. With only one exception [2] this is the only treatment of time in which we can show that there exists an equilibrium. Near the terminal period the economy will be imbued with a Weltuntergangstimmung. Clearly there is no point in having a positive money holding at the end of the last period; at any positive price of money consumers will seek to trade money for goods to be consumed before the end of the world. But no one with any sense will accept money during the last period in exchange for goods. You can't take it with you. So the price of money in the last period will be zero. Money is useless during the last period.

But then in the next to last period traders should be wary of accepting money. Since the price of money is zero in the last period there is no point in getting stuck with any money at the end of the next to last period. So the price of money will be zero in the next to last period as well. But this argument can regress indefinitely so that the price of money is zero in all periods. We have argued then, that in any discrete time finite horizon model in equilibrium the price of money will be zero in all periods. This is the argument of [9] and of Theorem 4 below.

What to do then? Not only is it conceivable that there be an equilibrium where the price of money is zero. In the only model in which we can show there to be an equilibrium, zero is the only price of money consistent with equilibrium. But in order to write any meaningful monetary theory, the price of money must be positive. Prospects look bleak indeed for the integration of money and value theory.

Part of the resolution of the difficulty is to note that it is somewhat contrived. Though finite horizons are convenient to work with we don't really believe in the end of the world occurring at a definite future date. So it is not too unreasonable to impose terminal conditions on money holdings--vaguely analogous to terminal capital stock constraints in finite horizon growth models--to eliminate depletion of money balances in the terminal period. As is shown in Theorem 5 below, an appropriate depletion constraint can insure that in equilibrium the price of money need not be zero. Unfortunately there is still an equilibrium where the price of money is zero. The argument is symmetric after all: if money is accepted because it is accepted, then money is not accepted if it is not accepted.

Consider a trader who has an apple which he wishes to exchange for an orange. Suppose the prices of the two commodities are equal and their relative prices are unaffected by changes in the price of money. Our trader takes his apple to market, trades it for money at current prices, trades the money for an orange. If the price of money falls, he will require more money to make the same trade. As long as the price of money remains positive, the money required for this transaction varies inversely with the price of money. Let p^a denote the price of an apple (assumed equal to the price of an orange) and p^m denote the price of money. Then the money demanded to execute the transaction is p^a/p^m.

What happens when the price of money is zero? At that point money is merely worthless paper. A hundred pieces of worthless paper buy the same number of oranges as one piece which buys the same as no piece. When $p^m = 0$ the trader does equally well with money holdings anywhere in the non-negative reals $[0, +\infty)$. Thus the "money required" correspondence is

$$m(p^m) = \begin{cases} \dfrac{p^a}{p^m} & \text{for } p^m > 0 \\ [0, +\infty) & \text{for } p = 0 . \end{cases}$$

The important implication here is that there is a point in the "money required" (demand) correspondence which tells us that when the price of money is zero, there need be no demand for money; $0 \in m(0)$. This suggests that at a price of zero the market for money will not experience the excess demand required to raise the price.

But when the price of money (the value in exchange of money) is zero there can be no monetary exchange. Our trader will be forced either to content himself with consuming an apple rather than an orange, or he will --an option not further considered in this essay--resort to barter.

How can we eliminate the possibility of the price of money being zero in equilibrium? In order to do this we must arrange that whenever the price of money is zero there will be a positive excess demand for money. Abba Lerner summarized an appropriate and realistic technique for achieving this [6]:

> The basic condition for [money's] effectiveness is that it should be generally acceptable....The modern state can make anything it chooses generally acceptable as money and thus establish its value quite apart from any connection, even of the most formal kind, with gold or with backing of any kind. It is true that a simple declaration that such and such is money will not do, even if backed by the most convincing constitutional evidence of the state's absolute sovereignty. But if the state is willing to accept the proposed money in payment of taxes and other obligations to itself the trick is done. Everyone who has obligations to the state will be willing to accept the pieces of paper with which he can settle the obligations, and all other people will be willing to accept these pieces of paper because they know that the taxpayers, etc., will be willing to accept them in turn.

Taxes can be used to create a demand for money independent of its usefulness as a medium of exchange, thereby ensuring that its price will not fall to zero.

In the example above, we can use taxes to put a floor on the trader's demand for money, even at a price of zero, so that when the price of money is zero, there will be sufficient excess demand to drive the price up. In this instance the new "money required" correspondence may look like

$$m^*(p^m) = \begin{cases} \dfrac{p^a}{p^m} & \text{for } p^m > 0 \\ [\tau, +\infty) & \text{for } p = 0 \end{cases}$$

where $\tau > 0$ is a tax payment. Appropriately chosen taxes, τ, will ensure that there is no equilibrium with the price of money equal to zero.

II. Trade in a Monetary Economy

The central fact of a monetary economy is that goods are not traded for goods. Goods are traded (sold) for money and money is traded for (buys) goods. The task then is to write out a general equilibrium model which embodies this aspect of a monetary economy. A far more fundamental task--one I will not attempt here--is to discover conditions which will ensure the establishment of a monetary economy. In actual economies, we observe that the use of money is almost universal, but this does not stem from the impossibility or illegality of barter transactions. To the casual empiricist this would seem to be simply because monetary exchange is more efficient or more convenient than barter.* And certainly, when an individual chooses monetary over barter exchange it is on these grounds. But from the viewpoint of the theorist, nothing here is explained by assuming the obvious --that money is used because it is useful--any more than one "explains" the law of gravity by assuming that unsupported objects are likely to fall. What is called for is a general equilibrium model which includes both barter

*Professor Sontheimer's significant study [10] unfortunately makes use of just this sort of half explanation.

and monetary exchange as special cases where those properties of monetary exchange which cause traders to choose it over barter develop as the result of more fundamental considerations. Let's proceed to the more modest task of modelling the monetary economy. There are N real goods; money is the $N+1^{st}$ good. The real goods will be denoted $n = 1, 2, \ldots, N$. The $N+1^{st}$ good, money, is denoted m. Traders are elements of the set T. Each t in T has an endowment $\bar{x}^t > 0$, and a continuous utility function $u_t(x_t)$ on possible consumptions (elements of the nonnegative orthant of R^N). u_t is supposed to be semi-strictly quasi-concave, and to fulfill strong monotonicity. That is $x^1 \geq x^2$ implies $u_t(x^1) > u_t(x^2)$ and $\{y \; u_t(y) \geq u_t(x)\}$ is convex for all x. Further, one distinguishes between buying, β, and selling, α, transactions. An individual's trades are characterized by what goods he buys and what goods he sells. Trader t's trade will be represented by $y^t \in E^{2(N+1)}$. y^t has an entry $y^{t\delta n}$ for each of the $N+1$ commodities n, and each of the two possibilities, $\delta = \alpha, \beta$, selling or buying. $y^{t\alpha}$ is trader t's selling transactions. $y^{t\alpha}$ is composed of flows of goods from t to the market and flows of money from the market to t.

(1) $$y^{t\alpha n} \geq 0, \quad n = 1, \ldots, N$$

(1m) $$y^{t\alpha m} \leq 0.$$

$y^{t\beta}$ is t's buying transactions. $y^{t\beta}$ is composed of flows of goods from the market to t and flows of money from t to the market.

(2) $$y^{t\beta n} \geq 0, \quad \text{all } n = 1, \ldots, N,$$

(2m) $$y^{t\beta m} \leq 0.$$

y^t is the vector of both t's buying and selling transactions.

$$y^t = (y^{t\alpha}, y^{t\beta}) .$$

At the end of trade, trader t's holdings will be subject to nonnegativity constraints. t can't sell what he didn't have to start with and didn't acquire in trade.

$$(3) \qquad \bar{x}^{tn} + y^{t\beta n} - y^{t\alpha m} \geq 0 , \quad \text{for} \quad n = 1, \ldots, N .$$

Price vectors will be elements of P, the unit simplex in E^{N+1}. Let $P = \{p | p \in E^{N+1}, p \geq 0, \sum_{n=1}^{N+1} p_n = 1\}$. Prices are the same for buying and selling. If one wished to extend the analysis to a model with transactions costs it should not be difficult to let buying and selling prices differ merely by doubling the dimensionality of the price space.

A recurring problem in a finite horizon model with money is the tendency for money to be dumped on the market at the end of trade, forcing its price to 0. To prevent this, I will consider several possible constraints on depletion of money balances. To start, consider

$$(D.1) \qquad \bar{x}^{tm} - y^{t\alpha m} + y^{t\beta m} \geq 0 .$$

(D.1) is the constraint that trader t can't spend more money than his receipts from sales plus his initial endowment. (D.1) is merely the extension to money of the nonnegativity constraint on final holdings that has been enunciated for real goods above. It will be seen below that (D.1) is not a sufficiently strong constraint to prevent the price of money from going

to zero. An alternative constraint is (D.2) $\bar{x}^{tm} - y^{t\Omega m} + y^{t\beta m} = \bar{x}^{tm}$, or equivalently (D.2') $y^{t\beta m} - y^{t\Omega m} = 0$. (D.2) amounts to saying that the trader cannot spend more money than he takes in, independent of his initial endowment of money. This would make sense if the present model were one element in a sequence of economies over time, taken to be unchanging in the course of time, that is, a steady state. An alternative interpretion of (D.2) is that it makes the role of money in this model that of a medium of exchange rather than a store of value. This follows inasmuch as (D.2) eliminates money's store of wealth function by requiring that the store not be drawn down. Under (D.2) t's initial endowment of money will enhance t's liquidity but his consumption will depend on the value of his real endowment not on his initial money holdings.

A third form of depletion restriction I will consider is one where the constraint varies with prevailing prices.

$$(D.3) \quad \bar{x}^{tm} - y^{t\Omega m} + y^{t\beta m} \geq \sigma_t(p) ,$$

where $\sigma_t(p)$ is a continuous function of prices, there may be a different $\sigma_t(p)$ for each $t \in T$. One may interpret this restriction in terms of a sequence economy as a saving requirement depending on prices. Note that if $\sigma_t(p) \neq \bar{x}^{tm}$ then, in equilibrium, there is a change in money holdings among traders from the start to the end of trade. It will appear below that $\sigma_t(p)$ is closely related to the tax functions $\theta_t(p)$ to be developed in the latter part of this essay.

Having defined the constraints to which trades may be subject we can now define trader t's possible trade set Y_t. Y_t will of course depend

on which of the three money balance depletion constraints we decide to impose. In addition to one of the depletion constraints all trades will be subject to the constraints (1), (1m), (2), (2m), (3) discussed above. Thus, if we decide to require (D.1) trader t^i's possible trades will be the set

$$Y_t^1 = \{y | y \in E^{2(N+1)}, \quad y \text{ fulfills (1), (1m), (2), (2m), (3) and (D.1)}\} .$$

Under (D.2) trader t^i's possible trades are

$$Y_t^2 = \{y | y \in E^{2(N+1)}, \quad y \text{ fulfills (1), (1m), (2), (2m), (3) and (D.2)}\} .$$

Similarly

$$Y_t^3 = \{y | y \in E^{2(N+1)}, \quad y \text{ fulfills (1), (1m), (2), (2m), (3) and (D.3)}\} .$$

Budget constraints apply separately to buying and selling transactions. In the standard general equilibrium model, of course, each trader faces only one budget constraint. There the budget constraint says that the trader can receive from the market goods whose value at market prices does not exceed the value of the goods the trader supplies to the market at those prices. The twofold budget constraint here reflects the two requirements that the trader must supply to the market commodities equal in value to his money receipts from the market at market prices, and that the trader must pay to the market money equal in value at market prices to the goods he receives from the market. The constraints then are

(4.α) $$p \cdot y^{t\alpha} = 0 ,$$

(4.β) $$p \cdot y^{t\beta} = 0 .$$

Given the depletion constraint and prices we can now write t's trading opportunity set. This set consists of those trades consistent with the chosen money depletion constraint, the budget constraints at prevailing prices and the other requirements above--(1), (1m), (2), (2m), (3). Thus, t's trading opportunity set under depletion constraint (D.i), i = 1, 2, 3, is

$$\eta_t^i(p) = \{y | y \in Y_t^i , \quad y \text{ fulfills } (4.\alpha) \text{ and } (4.\beta) \text{ at } p\} .$$

It is from $\eta_t^i(p)$ that trader t will choose what trade to make. If he chooses $y \in \eta_t^i(p)$ then his consumption bundle consists of his original endowment plus his net trade. Consumption is $w^t = \bar{x}^t - y^\alpha + y^\beta$. The trader gets no satisfaction from money; utility varies only with the first N elements of w^t. Under depletion constraint (D.i) t's choice correspondence then is $\gamma_t^i(p) = \{y | y \in \eta_t^i(p) , \quad w^t = \bar{x}^t - y^\alpha + y^\beta$ maximizes $u(w)$ subject to $y \in \eta_t^i(p)\}$.

For some p, $\eta_t^i(p)$ may not be bounded so $\gamma_t^i(p)$ may be empty. As a technical convenience I introduce a bound on the sets under consideration. This is an artificial constraint; it will not be binding in equilibrium. If $x \in E^{2(N+1)}$, let $Ax = x^+ + x^-$. That is Ax is the vector each of whose components is the absolute value of the corresponding component of x. As the bound considers

$$\psi = \{x | x \in E^{2(N+1)} , \quad Ax \leq (2 \sum_{j \in T} \bar{x}^j \times 2 \sum_{j \in T} \bar{x}^j)\} .$$

Thus, limiting consideration to elements of Ψ amounts to restricting consideration to trades involving less than twice the economy's endowment. Not a particularly severe restriction.

Let $\hat{\eta}_t^i(p) = \eta_t^i(p) \cap \Psi$. $\hat{\eta}_t^i(p)$ is trader t's truncated (by Ψ) trading opportunity set at prices p with depletion constraint (D.i). When t chooses a trade in $\hat{\eta}_t^i(p)$, we should choose an element of

$$\hat{\gamma}_t^i(p) = \{y | y \in \hat{\eta}_t^i(p), \quad w^t = \bar{x}^t - y^\alpha + y^\beta \text{ maximizes } u_t(w) \text{ subject to } y \in \hat{\eta}_t^i(p)\}.$$

Clearly $\hat{\gamma}_t^i(p)$ is the counterpart of $\gamma_t^i(p)$ in the case where choice is constrained to be within Ψ. In equilibrium the constraint will not be binding so $\hat{\gamma}_t^i(p)$ and $\gamma_t^i(p)$ will have at least one point in common.

It is fairly easy to see that the function $\sigma_t(p)$ in (D.3), if not suitably restricted, can make the analysis vacuous. For example, if $\sigma_t(p)$ required trader t to amass a large final money balance at unfavorable commodity prices--in particular if the value of the required money balance is greater than the market value of the trader's endowment--then it might be impossible simultaneously to satisfy (1)-(4), (D.3). In such a case $\eta_t^3(p)$ would be empty. To avoid this I adopt the following:

Restriction on $\sigma_t(p)$:

For all $p \in P$,

$$p^m \sigma_t(p) < p \bar{x}^t$$

and

$$\sigma_t(p) < 2 \sum_{j \in T} \bar{x}^j.$$

The restriction says that under (D.3) no trader will be required to accumulate money holdings greater in value than his original endowment, nor will he be required to accumulate money holdings twice as large as the stock of money in the whole economy. The two parts of the restriction serve to ensure non-emptiness of $\eta_t^3(p)$ and $\hat{\eta}_t^3(p)$ respectively.

Lemmas 1 and 2 establish technical characteristics of η, $\hat{\eta}$, γ, and $\hat{\gamma}$. While of interest in themselves, these properties do not enter essentially in the analysis below. Their proofs and some other lemmas concerning η, $\hat{\eta}$, γ and $\hat{\gamma}$ are in the appendix.

Lemma 1: $\eta_t^i(p)$, $\hat{\eta}_t^i(p)$ are convex and nonnull for each i, p, and $\hat{\eta}_t^i$ is continuous in p at all $p \in P$ such that $p^m > 0$.

Lemma 2: For each i, $\hat{\gamma}_t^i(p)$ is nonnull, convex for all p. $\hat{\gamma}_t^i(p)$ is upper semicontinuous at all $p \in P$ such that $p^m > 0$.

The restriction of lemmas 1 and 2 to regions where $p^m > 0$ reflects a fundamental technical problem in this field. Consider $p^o \in P$ with $p^{om} = 0$ and for some commodity n $p^{on} = 0$. Then at prices p^o a typical trader can buy arbitrarily large quantities of good n. But consider $p^\nu \to p^o$ such that $p^{\nu n} = kp^{\nu m}$. For any ν, if $y^\nu \in \hat{\eta}_t^i(p^\nu)$, $y^{\nu n} \leq \frac{2}{k} \sum_{j \in T} \bar{x}^{jm}$. But there may be $y^o \in \hat{\eta}_t^i(p^o)$ with $y^{on} > \frac{2}{k} \sum_{j \in T} \bar{x}^{jm}$. This is why $\hat{\eta}_t^i(p)$, trader t's budget set, will not in general be lower semicontinuous in a neighborhood where $p^m = 0$.

This is vaguely analagous to aspects of Hahn's [5]. He also found that budget sets may fail to be lower semicontinuous in some areas of the price space.

III. Equilibrium in the Monetary Economy

$$\text{Let} \quad Z = \{x \mid x \in E^{(N+1)} \ , \quad Ax \leq 2|T| \sum_{j \in T} \bar{x}^j \} \ ,$$

$$\bar{\Omega}^i = \underset{j \in T}{X} Y_j^i \ , \qquad i = 1, 2, 3 \ .$$

$$\Omega = \underset{|T|}{X} \psi \ .$$

Conceivable attempted trades in the economy are elements of $\bar{\Omega}^i$. Such a point lists for each trader j an element of his possible trade set Y_j^i under depletion restriction (D.i). Ω is an artificially bounded version of $\bar{\Omega}$. On the basis of attempted trades we can compute excess demands. Let

$$\zeta : \Omega \to Z \ .$$

$$\zeta(x) = \sum_{j \in T} x^{j\beta} - \sum_{j \in T} x^{j\alpha} \ .$$

Thus for a proposed group of transactions $x \in \Omega$, the excess demand, $\zeta(x)$, is the amount sought for purchase, $\sum_{j \in T} x^{j\beta}$, less the amount traders seek to supply, $\sum_{j \in T} x^{j\alpha}$.

Definition: Let $p^* \in P$, $y^* \in \Omega$, $z^* \in Z$. (p^*, y^*, z^*) is an equilibrium for the economy under depletion restriction (D.i) if for all $t \in T$ (i) $y^{*t} \in \eta_t^i(p^*)$, for all $t \in T$ (ii) $w^t = \bar{x}^t - y^{*t\beta}$ maximizes $u_t(w)$ for all $y \in \eta_t^i(p^*)$, and (iii) $z^* = \zeta(y^*)$ and $z^* \leq 0$.

This is a traditional definition of equilibrium. One has an equilibrium when the results of individual maximizations subject to constraint imply non-positive excess demand.

IV. The Price of Money

We now note a curious property of the monetary economy. When the value of exchange of money is zero, no one can seek to trade. On reflection this is quite reasonable in the context of the model. We have required that all trade take place using money as medium of exchange. If the price of money is zero then money is literally worthless paper. What will one sell for worthless paper? Nothing. What can one buy with worthless paper? Nothing. Thus we have

Lemma 3: Let $p \in P$, $p^m = 0$. Then if $y \in \eta_j^i(p)$, $p^n y^{\delta n} = 0$ for all $j \in T$, $i = 1, 2, 3$, $n = 1, \ldots, N$. The same holds for all $y \in \gamma_j^i(p)$.

Proof: $\gamma_j^i(p) \subset \eta_j^i(p)$. But $y \in \eta_j^i(p)$ implies y fulfills $(4.\alpha)$ and $(4.\beta)$ at p. Thus $p \cdot y^\alpha = 0$, $p \cdot y^\beta = 0$. Let y^c denote the N-dimensional vector consisting of the first N components (the real goods elements) of y.

$$p \cdot y^\delta = 0 \qquad \delta = \alpha, \beta .$$

$$p \cdot y^\delta = p^c \cdot y^{\delta c} + p^m \cdot y^{\delta m}, \quad p \geq 0 .$$

By (1), (1m), (2), (2m) we have $y^{\delta m} \leq 0$, $y^{\delta c} \geq 0$. But $p^m = 1$ implies $p^m \cdot y^{\delta m} = 0$. So

$$0 = p \cdot y^\delta = p^c \cdot y^{\delta c} + p^m \cdot y^{\delta m} = p^c \cdot y^{\delta c} .$$

So $p^n \cdot y^{\delta n} = 0$, $n = 1, \ldots, N$.

Q.E.D.

Lemma 4: Let $p \in P$, $p^m = 0$, $p^n > 0$ all $n = 1, \ldots, N$. Then for each $j \in T$, $y \in \eta_j^i(p)$ implies $y^c = 0$, $0 \in \gamma_j^1(p)$ $0 \in \eta_j^2(p)$ $0 \in \gamma_j^2(p)$. If $\sigma_j(p) \leq \bar{x}^{jm}$ in (D.3) then $0 \in \eta_j^3(p)$, $0 \in \gamma_j^3(p)$.

Proof: By Lemma 3 we have $y \in \eta_j^i(p)$ implies $p^n y^{\delta n} = 0$. But $p^n > 0$ for $n = 1, \ldots, N$ so $y^{\delta n} = 0$ for $n = 1, \ldots, N$. Thus $y^c = 0$. Under (D.1), (D.2), (D.3) with $\sigma_j(p) \leq \bar{x}^{jm}$ we can satisfy the depletion constraint with $y^m = 0$ thus, $0 \in \eta_j^i(p)$. Further since u_j does not vary with w^{jm}, j is indifferent among all elements of $\eta_j^i(p)$. Thus $0 \in \gamma_j^i(p)$.

Q.E.D.

This leads us to the fundamental quandary of this study. Lemma 4 tells us that if we announce a price vector p for the market such that the price of money is zero and the price of goods is positive then traders will demand and supply zero quantities of all goods. But if all individual demands and supplies are zero then excess demands and supplies are zero in all markets. The markets are in equilibrium. p is an equilibrium price vector.

But this is a very curious equilibrium. It is an equilibrium with no trade. This is not to say that there are no mutually beneficial trades conceivable between traders. Rather, because it is required in the monetary economy that trade take place through money, there are no effective demands or supplies when the price of money is zero.

I think there is legitimate question as to the significance of the equilibrium with zero price of money. Within the bounds of the model the

implication is explicit: no trade. An alternative interpretation--going outside the model--is that there is no monetary trade but that there is probably recourse to barter. I think this is an acceptable interpretation but I will not deal with it further here (see [10]). The implication of the structure of our demand functions is

Theorem 1: Under (D.1), (D.2), (D.3), with $\sigma_j(p^o) < \bar{x}^{jm}$ all $j \in T$, there is an equilibrium for the economy (p^o, y^o, z^o) such that $p^{om} = 0$, $y^o = 0$.

Proof: Let $p^{om} = 0$, $p^{on} > 0$, $n = 1, \ldots, N$. By Lemma 4, $0 \in \gamma_j^i(p^o)$. But $\zeta(0) = 0$ so $(p^o, 0, 0)$ is an equilibrium.

Q.E.D.

The equilibrium $(p^o, 0, 0)$ shown to exist in Theorem 1 is peculiar in that no trade takes place. There is only one case in which such an equilibrium can be Pareto efficient. It will be Pareto efficient if and only if the original endowment of goods, $\bar{x}^j$, is a Pareto efficient allocation. However, there are equilibria where the price of money is positive.

Theorem 2: Under (D.2) there is an equilibrium (p^o, y^o, z^o) for the economy with $p^{om} > 0$.

Proof: Let (p^*, y^*, z^*) be an equilibrium for the barter (Arrow-Debreu) economy. Choose p^{om} so that $1 > p^{om} > 0$. Let

$$p^o = ((1 - p^{om})p^*, p^{om}),$$

$$y^{ojo\alpha} = (y^{*j\alpha}, -(1 - p^{om})p^{*}.y^{*j\alpha})$$

$$y^{ojo\beta} = (y^{*j\beta}, -(1 - p^{om})p^{*}.y^{*j\beta})$$

Then $z^{o} = (z^{*}, 0)$ and since (p^{*}, y^{*}, z^{*}) is an equilibrium (p^{o}, y^{o}, z^{o}) is an equilibrium.

Q.E.D.

In an equilibrium with a positive price of money trade takes place unimpeded. Just as a competitive equilibrium is Pareto efficient in a barter economy [1], so a competitive equilibrium with a positive price of money is Pareto efficient.

Theorem 3: Let (p^{o}, y^{o}, z^{o}) be an equilibrium for the economy under (D.i). Let $1 > p^{om} > 0$. Then $w^{ot} = \bar{x}^{t} - y^{ot\alpha} + y^{ot\beta}$ is a Pareto efficient distribution of goods among $t \in T$.

Proof: By [1] it is sufficient to show that w^{ot} maximizes $u_{t}(x)$ subject to $p^{oc}.w^{otc} \geq p^{oc}.x^{c}$. Suppose not. Then there is $r \in T$ so that for some x^{rc}

$$p^{oc}.x^{rc} \leq p^{oc}.w^{orc}, \quad u_{r}(x^{rc}) > u_{r}(w^{or}).$$

We will show that this implies that y^{or} is not a maximizing choice in $\eta_{r}^{i}(p^{o})$ and hence a contradiction of the hypothesis. Without loss of generality take $p^{oc}.x^{tc} = p^{oc}.w^{orc}$. Choose y^{r} so that

$$y^{rorc} = (x^{rc} - \bar{x}^{rc})^{-}$$

$$y^{r\alpha m} = -p^{oc}\cdot(x^{rc} - \bar{x}^{rc})^- \frac{1}{p^{om}}$$

$$y^{r\beta c} = (x^{rc} - \bar{x}^{rc})^+$$

$$y^{r\beta m} = -p^{oc}\cdot(x^{rc} - \bar{x}^{rc})^+ \frac{1}{p^{om}}$$

$y^r \in \eta_t^i(p^o)$ since $p^{oc}\cdot(x^{rc} - \bar{x}^{rc}) = p^{oc}\cdot x^{rc} - p^{oc}\bar{x}^{rc} = p^{oc}\cdot w^{orc} - p^{oc}\cdot\bar{x}^{rc} = 0$.

Thus (p^o, y^o, z^o) cannot be an equilibrium. The contradiction proves the theorem.

Q.E.D.

Lemma 5: Let $p^m > 0$, $p^n > 0$ for some $n = 1, \ldots, N$. Then let $y^i \in \gamma_j^i(p)$ for $i = 1, 2, 3$. Then

$$\bar{x}^{jm} - y^{1\alpha m} + y^{1\beta m} = 0,$$

$$\bar{x}^{jm} - y^{2\alpha m} + y^{2\beta m} = \bar{x}^{jm},$$

$$\bar{x}^{jm} - y^{3\alpha m} + y^{3\beta m} = \sigma_j(p).$$

Proof: Suppose (D.i) is overfulfilled by the amount s_j^i. Then there is $*y^i \in \eta_j^i(p)$ with $*y^{i\alpha n} = y^{i\alpha n}$, $*y^{i\beta m} < y^{i\beta m}$, $*y^{i\beta n} \geq y^{i\beta n}$ with the strict inequality holding for some n. Further by strong monotonicity of u, $*y^i$ is preferred to y^i so $y^i \notin \gamma_j^i(p)$ contrary to hypothesis. The contradiction proves the lemma.

Q.E.D.

The gist of Lemma 5 is that when the price of money is positive, the depletion restrictions (D.i) are binding. The mischief one gets into then when the (D.i) are insufficiently restrictive now arises. When the price of money is positive traders will deplete their money holdings to the point where the depletion constraint is binding. But if the constraint is not restrictive enough this will result in an excess supply of money on the market and an excess demand for goods--clearly a disequilibrium. If we do not make the depletion constraints more restrictive the sole alternative is to let the price of money become zero. This gives us an equilibrium of the sort in Theorem 2.

Theorem 4 In (D.3) let $\sum_{j\in T} \sigma_j(p) < \sum_{j\in T} \bar{x}^{jm}$ for all $p \in P$. Let (p_i, y_i, z_i) be an equilibrium for the economy under (D.i) $i = 1, 3$ $(i \neq 2)$. Then $p_i^m = 0$.

Proof: By insatiability of u_j, $p_i^n > 0$ some n. Suppose the theorem is false; $p_i^m > 0$. Then by Lemma 10

$$\sum_{j\in T} (\bar{x}^{jm} - y_i^{j\alpha m} + y_i^{j\beta m}) < \sum_{j\in T} \bar{x}^{jm}$$

which implies $\sum_{j\in T} y_i^{j\alpha m} > \sum_{j\in T} y_i^{j\beta m}$. But we have $0 = p_i \cdot y_i^{j\alpha} = p_i^c \cdot y_i^{j\alpha c} + p_i^m \cdot y_i^{j\alpha m}$ and $0 = p_i \cdot y_i^{j\beta} = p_i^c \cdot y_i^{j\beta c} + p_i^m \cdot y_i^{j\beta m}$. We have then

$$y_i^{j\beta m} = \frac{-p_i^c y_i^{j\beta c}}{p_i^m} \quad \text{and} \quad y_i^{j\alpha m} = \frac{-p_i^c y_i^{j\alpha c}}{p_i^m} .$$

By the inequality above then

$$\sum_{j\in T}\left(-\frac{p_i^c}{p_i^m}\right)\cdot y_i^{j\alpha c} > \sum_{j\in T}\left(-\frac{p_i^c}{p_i^m}\right)\cdot y_i^{j\beta c}$$

so for some $n = 1, \ldots, N$, $\sum_{j\in T} y_i^{j\alpha n} < \sum_{j\in T} y_i^{j\beta n}$. But $z_i^n = \sum_{j\in T} y_i^{j\beta n} - \sum_{j\in T} y_i^{j\alpha n} > 0$ and therefore (p_i, y_i, z_i) is not an equilibrium. The contradiction shows that $p_i^m = 0$.

Q.E.D.

Theorem 4 tells us that not only do we face the difficulty of Theorem 1 --that there exist equilibria with price of money equal to zero--but that in a broad class of cases, those where the money depletion constraints are not restrictive enough, the only equilibria are those where the price of money is zero. Such a situation could make life in a monetary economy downright awkward. Faced with a situation where the equilibrium price of a commodity is for whatever reason too low the economist's first impulse is to increase demand for the commodity. The impulse is sound. In this model the demand for money is based on the depletion constraint. As illustrated in Theorem 4, when that constraint does not require sufficiently large terminal money holdings the demand for money is not sufficiently great to lift the equilibrium price above zero. It will appear below that the converse holds--at least partially--when terminal money holdings are required to be sufficiently large, but not so great as to be impossible, the only equilibrium will be those with a positive price of money.

Thus,

Theorem 5: Let $\sigma_j(p)$ be $\sigma_j(p^m)$, a function of p^m only.

Let $\sigma_j(p)$ be such that there is $0 \leq b < 1$ so that for all $p \in P$ so that $0 \leq p^m \leq b$, $\sum_{j \in T} \sigma_j(p) > \sum_{j \in T} \bar{x}^{jm}$. Then there is an equilibrium for the economy under (D.3) and if (p^o, y^o, z^o) is such an equilibrium then $p^{om} > b$.

Proof: Note that the restriction on $\sigma_j(p)$ implies that if $p^m = 1$, $p^c = 0$ then $\sigma_j(p) < \bar{x}^{jm}$. Then by the intermediate value theorem there is p^{*m}, $b \leq p^{*m} < 1$ so that $\sum_{j \in T} \sigma_j(p^{*m}) = \sum_{j \in T} \bar{x}^{jm}$. Consider the barter (Arrow-Debreu) economy with prices on the simplex

$$S = \{p \mid p \in E^N, \ p \geq 0, \ \Sigma p^i = 1 - p^{*m}\}$$

and where traders, $t \in T$, choose x^{tc} to maximize utility subject to the budget constraint

$$p \cdot x^{tc} \leq p \cdot \bar{x}^{tc} + (p^{*m} \sigma_t(p^{*m}) - p^{*m} \bar{x}^{tm}) .$$

Then there is an equilibrium price vector $p^{*c} \in S$ in this economy [3]. Let

$$y^{t\alpha c} = (\bar{x}^{tc} - x^{tc})^+$$

$$y^{t\alpha m} = \frac{-1}{p^{*m}} p^{*c} \cdot y^{t\alpha c}$$

$$y^{t\beta c} = (\bar{x}^{tc} - x^{tc})^-$$

$$y^{t\beta m} = \frac{-1}{p^{*m}} p^{*} \cdot y^{t\beta c} .$$

Then $p^* = (p^{*c}, p^{*m})$ is an equilibrium price vector for the economy under (D.3). This gives existence.

Suppose contrary to the theorem $p^{\circ m} \leq b$. Then we have

$$\bar{x}^{\circ jm} - y^{\circ j\alpha m} + y^{\circ j\beta m} \geq \sigma_j(p)$$

$$\sum_{j\in T} \bar{x}^{\circ jm} - \sum_{j\in T} y^{\circ j\alpha m} + \sum_{j\in T} y^{\circ j\beta m} \geq \sum_{j\in T} \sigma_j(p) > \sum_{j\in T} \bar{x}^{jm}$$

$$\sum_{j\in T} y^{\circ j\beta m} - \sum_{j\in T} y^{\circ j\alpha m} > 0 .$$

There is excess demand for m and hence $(p^\circ, y^\circ, z^\circ)$ is not an equilibrium. The contradiction proves the theorem.

Q.E.D.

Corollary to Theorem 5: Let $\sigma_j(p)$ be a continuous function of p such that for $p^m = 0$, $\sum_{j\in T} \sigma_j(p) > \sum_{j\in T} \bar{x}^{jm}$. Then there is an equilibrium for the economy under (D.3) and if $(p^\circ, y^\circ, z^\circ)$ is such an equilibrium then $p^{\circ m} > 0$.

Proof: Let $b = 0$ in Theorem 5.

Q.E.D.

Theorem 5 and its corollary tell us that if we can get traders to fulfill the right sort of terminal money holding constraint $\sigma_j(p)$, that the price of money will be positive. Why should trader j want to hold $\sigma_j(p)$ of paper when he could acquire goods instead? He shouldn't. The depletion constraint with by far the strongest claims to legitimacy and consistency with micro-economic theory is (D.1) which requires only that terminal money holdings be nonnegative.

V. The Economy with Taxation

However, the appropriate taxation has much the same effect. If the government presents trader j with a tax bill for $\sigma_j(p)$ payable in money this will create the same demand for money as the depletion constraint (D.3). The rigorous demonstration of this statement requires some additional structure: the tax system. Taxes are merely one more transaction for our traders. They are paid in money. Taxes are distinct from other money expenditures only in that they don't buy anything for the trader. The addition of tax payments to trader j's actions consists merely in the addition of one more component to the vector of his actions y. Thus $y = (y^\alpha, y^\beta, y^\tau)$, where y^α, y^β are $N+1$ dimensional and y^τ, the trader's tax payment, is a scalar. Denote the first $2(N+1)$ components of y, those dealing with market transactions by y^π, the final component, that dealing with tax payments is of course y^τ. It turns out that the distinction among the several depletion constraints is less significant in the economy with taxes so we will only require that terminal money holdings after spending and tax payments be nonnegative, that is,

$$(D^*)\bar{x}^{tm} - y^{t\alpha m} + y^{t\beta m} - y^{t\tau} \geq 0 \ .$$

Further, introduce trader t's required tax payment function (which may vary with prices) $\theta_t(p)$. Then another constraint on trades is that traders pay their taxes. That is,

(5) $$y^{t\tau} = \theta_t(p) \ .$$

What will be meant by an equilibrium in the monetary economy with taxation? Proposed prices, and transactions are said to be an equilibrium for the economy if for each trader (i) the transactions fulfill the sign conventions and commodity nonnegativity constraints (1), (1m), (2), (2m), (3); (ii) at the proposed prices transactions fulfill the price consistency constraints (4.α), (4.β); (iii) transactions fulfill the depletion constraint (D*) ; (iv) traders pay their taxes, that is, transactions fulfill (5); (v) the transactions are individually utility maximizing subject to (i), (ii), (iii), (iv), and (vi) excess demands are nonpositive. That is, we have an equilibrium for the economy with taxation if in addition to the other qualities of an equilibrium the point in question includes payment of taxes. This statement of the problem allows us to use a good deal of the structure developed for the economy without taxation. We can do this because of

Lemma 6: Let $y \in E^{2(N+1)+1}$. y fulfills (D*) and (5) if and only if y^{T} fulfills (D.3) with $\sigma_t(p) = \theta_t(p)$.

Proof: (D*) and (5) are $\bar{x}^{tm} - y^{t\alpha m} + y^{t\beta m} - y^{t\tau} \geq 0$, which hold if and only if $\bar{x}^{tm} - y^{t\alpha m} + y^{t\beta m} - \theta_t(p) \geq 0$ which holds if and only if

$$\bar{x}^{tm} - y^{t\alpha m} + y^{t\beta m} \geq \theta_t(p)$$

which holds if and only if y fulfills (D.3) with $\theta_t(p) = \sigma_t(p)$.

Q.E.D.

Lemma 6 establishes the equivalence of depletion constrant (D*) and the requirement that taxes be paid in the economy with taxation to the depletion constraint (D.3) in the economy without taxation. This equivalence allows us to apply the results achieved in the economy without taxation directly to the economy with taxation. We can do this by virtue of

Lemma 7: Let $y^{ot\tau} = \theta_t(p^o)$. (p^o, y^o, z^o) is an equilibrium for the economy with taxation if and only if $(p^o, y^{o\pi}, z^o)$ is an equilibrium for the economy without taxation under (D.3) with $\sigma_t(p) = \theta_t(p)$.

Proof: Let an asterisk denote functions for the economy with taxation. By Lemma 6, $\{y^{\pi} \mid y \in \eta_t^*(p^o)$ in the economy with taxation$\} = \eta_t^3(p^o)$. Neither taxes nor final money balances enter the utility function so maximizing elements of the opportunity set are the same (except for the addition of the tax component) for the two economies, that is $\gamma_t^*(p^o) = \gamma_t^3(p^o)$ and $y^{ot} \in \gamma_t^*(p^o)$ if and only if $y^{ot\pi} \in \gamma_t^3(p^o)$. The remaining conditions for equilibrium are identical in the two economies.

Q.E.D.

Lemma 7 makes life very convenient. It allows us to extend directly to the economy with taxation all the results previously derived for the economy without taxation under (D.3). Thus

VI. The Price of Money in the Economy with Taxation

Theorem 6: Let $\theta_j(p) \leq \bar{x}^{jm}$ for $p^m = 0$ for all $j \in T$. Then there is an equilibrium for the economy with taxation such that $p^{om} = 0$, $y^{o\pi} = 0$.

Proof: Theorem 1 and Lemma 7.

Q.E.D.

Theorem 6 notes that if taxes are not sufficiently high when the price of money is nil then zero is an equilibrium price of money and there is no trade in goods (though taxes will be paid) at equilibrium.

Theorem 7: Let (p^o, y^o, z^o) be an equilibrium for the monetary economy with taxation and let $p^{om} > 0$. Then $w^{ot} = \bar{x}^t - y^{ot\pi\alpha} + y^{ot\pi\beta}$ is a Pareto efficient distribution of goods among $t \in T$.

Proof: Theorem 3 and Lemma 7.

Q.E.D.

Theorem 7 tells us that if we can arrange taxes so as to keep the price of money positive, then an equilibrium will be efficient. Theorem 8 tells us explicitly how not to achieve this.

Theorem 8 Let $\sum_{t\in T} \theta_t(p) < \sum_{t\in T} \bar{x}^{tm}$ for all $p \in P$. Let (p^o, y^o, z^o) be an equilibrium for the economy with taxation. Then $p^{om} = 0$.

Proof: Theorem 4 and Lemma 7.

Q.E.D.

Theorem 8 tells us what class of tax functions are definitely inadequate to achieve a positive price of money. Theorem 9 finishes the job, telling us what taxes are adequate to insure positvity of p^m.

Theorem 9 : Let $\theta_t(p)$ be a continuous function of p such that:

(i) for each $t \in T$, all $p \in P$, $p^m\theta_t(p) \leq p\cdot\bar{x}^t$,

(ii) $\theta_t(p)$ is $\theta_t(p^m)$, a function of p^m alone,

and

(iii) there is $0 < b < 1$ so that for all p so that $0 \leq p^m \leq b$, $\sum_{j\in T} \theta_j(p) > \sum_{j\in T} \bar{x}^{tm}$ and for some $1 > p^m > b$, $\sum_{j\in T} \theta_j(p) \leq \sum_{j\in T} \bar{x}^{jm}$.

Then there is an equilibrium for the economy with taxation and if (p^o, y^o, z^o) is such an equilibrium then $p^{om} \geq 0$.

Proof: Theorem 5 and Lemma 7.

Q.E.D.

Corollary to Theorem 9 : Let $\theta_t(p)$ be a continuous function of p so that for $p^m = 0$, $\sum_{j\in T} \theta_j(p) > \sum_{j\in T} \bar{x}^{jm}$ and for some p, $1 > p^m > 0$, $\sum_{j\in T} \theta_j(p) \leq \sum_{j\in T} \bar{x}^{jm}$. Let θ_t fulfill (i), (ii) of Theorem 9 for all $t \in T$. Then

there is an equilibrium for the economy with taxation and if (p^o, y^o, z^o) is an equilibrium then $p^{om} > 0$.

Proof: Let $b = 0$ in Theorem 9 .

Q.E.D.

Theorem 9 tells us that a sufficiently exacting tax system puts a positive floor on the price of money. All one has to make sure is that the tax function will create an excess demand for money when the price of money is below the proposed floor but will allow supply to equal demand for money somewhere at or above the floor. The tax is payable in money and varies inversely with the price of money. Moreover, the tax is not a tax on any commercial transaction; it is not a tax on income nor is it an excise. Rather, it is a capitation, a head tax or poll tax. This can probably be generalized without difficulty.

There is a more felicitous interpretation of θ . We can consider $\theta_j(p^m)$ to be a tax on the money value of trader j's endowment. Let $\theta_j(p^m) = a \min [K_j/p^m, M]$ where $1 > a > 0$. M large, in particular $a|T|M > \sum_{j \in T} \bar{x}^{jm}$. Then θ_j will do the job in Theorem 9 . Nevertheless, a tax on income or sales will not suffice in Theorem 9 because when $p^m = 0$ no trader has positive income or sales so that any tax proportional to these variables will not create an excess demand for money at $p^m = 0$ and hence equilibrium will remain undisturbed.

VII. Summary

The structure of the problem and its solution are now fully articulated. In the pure exchange monetary economy we have the following statements:

(i) There exists an equilibrium (Theorems 1, 2, 5).

(ii) Under (D.1), (D.2), (D.3) with insufficiently restrictive constraints, there is an equilibrium with the price of money equal to zero (Theorem 1).

(iii) Indeed, under (D.1) and (D.3) insufficiently restrictive, the only equilibria are those with the price of money equal to zero (Theorem 4). Under (D.2) there are equilibria with positive price of money (Theorem 2).

(iv) Under (D.3) sufficiently restrictive there are equilibria and they all have positive price of money (Theorem 5).

(v) Similarly, a sufficiently exacting tax system will ensure that the price of money is positive in the economy with taxation (Theorem 9) but,

(vi) an insufficiently exacting tax system will have equilibria with price of money equal to zero, indeed, these may be the only equilibira (Theorems 6, 8).

(vii) In an economy with or without taxation, an equilibrium with positive price of money is Pareto efficient (Theorems 3, 7).

APPENDIX

Technical Properties of Budget Sets and Choice Functions in the Monetary Economy

Lemma 1: $\eta_t^i(p)$, $\hat{\eta}_t^i(p)$ are convex and nonnull for each i , p , and $\hat{\eta}_t^i(p)$ is continuous in p at all $p \in P$ such that $p^m > 0$.

Proof: Convexity is trivial. Continuity requires both upper and lower semicontinuity. Let $p^\nu \to p^o$, $y^\nu \in \eta_t^i(p^\nu)$, $y^\nu \to y^o$ then by continuity of $p.y^{t\delta}$ $\delta = \alpha, \beta$, and for $i \neq 3$, continuity of $\sigma_t(.p)$, $y^o \in \eta_t^i(p^o)$. This is upper semicontinuity. Let $p^\nu \to p^o$, $y^o \in \eta_t^i(p^o)$, $p^{om} > 0$. To show lower semicontinuity we need to find $y^\nu \in \hat{\eta}_t^i(p^\nu)$ so that $y^\nu \to y^o$. The construction below is written in terms of (D.3). To write the proof for (D.1), merely substitute 0 for $\sigma_t(p)$, for (D.2) substitute $\bar{x}^{tm}$.

Consider $\tilde{y}^\nu$ defined as follows:

Let

$$\tilde{y}^{\nu o\alpha c} = \left(\min\left[1, \frac{2 \sum_{j \in T} \bar{x}^{jm}}{\frac{p^{\nu c}}{p^{\nu m}} \cdot y^{o\alpha c}}\right]\right) y^{o\alpha c}$$

$$\tilde{y}^{\nu o\alpha m} = -\frac{p^{\nu c}}{p^{\nu m}} \cdot \tilde{y}^{\nu o\alpha c}$$

$$\tilde{y}^{\nu\beta c} = \left(\min\left[1, \frac{y^{o\beta m}}{\frac{p^{\nu c}}{p^{\nu m}} \cdot y^{o\beta c}}\right]\right) y^{o\beta c}$$

$$\tilde{y}^{\nu\beta m} = -\frac{p^{\nu c}}{p^{\nu m}} \cdot \tilde{y}^{\nu\beta c}$$

$p^{\nu m} > 0$ for ν large so $\tilde{y}^{\nu}$ is well defined. $\tilde{y}^{\nu}$ fulfills (1), (1m), (2), (2m), (3), (4.α), (4.β) but $\tilde{y}^{\nu}$ may not fulfill (D.3) at p^{ν}. We can now use $\tilde{y}^{\nu}$ to construct y^{ν} which fulfills (D.3) as well. Let

$$y^{\nu c}(k) = \tilde{y}^{\nu c} + k(\bar{x}^{tc} - \tilde{y}^{\nu\alpha c}, -\tilde{y}^{\nu\beta c}) ,$$

$$y^{\nu\alpha m}(k) = \frac{p^c}{p^m} \cdot y^{\nu\alpha c}(k) ,$$

$$y^{\nu\beta m}(k) = \frac{p^c}{p^m} \cdot y^{\nu\beta c}(k) , \quad \text{for } 0 \leq k \leq 1 .$$

$y^{\nu}(k)$ fulfills (1), (1m), (2), (2m), (3), (4.α), (4.β) and by the restriction on $\sigma_t(p)$ there is k, $0 \leq k \leq 1$, so that $y^{\nu}(k)$ fulfills (D.3) at p^{ν}. Let k^{ν} be the smallest such k. Then by continuity of $\sigma_t(p)$ we have $\sigma_t(p^{\nu}) \to \sigma_t(p^o)$ implies $k^{\nu} \to 0$ so $y^{\nu}(k^{\nu}) \to \tilde{y}^{\nu} \to y^o$. Thus, letting $y^{\nu} = y^{\nu}(k^{\nu})$ we have $y^{\nu} \in \eta_t^i(p^{\nu})$, $y^{\nu} \to y^o$. This proves lower semicontinuity.

Under (D.1), (D.2) nonnullness is trivial since for all $p \in P$, $0 \in \eta_t^1(p)$, $0 \in \eta_t^2(p)$. For (D.3) note that if $p^m = 0$, $0 \in \eta_t^3(p)$ by the restriction on $\sigma_t(p)$. If $p^m > 0$, take $y^{\alpha c} = \bar{x}^{jc}$, $y^{\alpha m} = -\frac{1}{p^m} p^c \bar{x}^{jc}$, $y^{\beta} = 0$. y fulfills (1)-(4). $w^{tm} = \bar{x}^{tm} - y^{\alpha m} + y^{\beta m} = \bar{x}^{tm} + \frac{1}{p^m} p^c \bar{x}^{tc} = \frac{1}{p^m} p \cdot \bar{x}^t \geq \sigma_t(p)$ by the restiction on σ_t. Thus $y \in \eta_t^3(p)$ and $\eta_t^3(p)$ is nonnull. If $y \notin \hat{\eta}_t^3(p)$ then let $y' = y(-2 \sum_{j \in T} \bar{x}^{jm}/y^{\alpha m})$. y' fulfills

(1)-(4) being merely a scalar multiple of y which fulfills (1)-(4).

$$w^{tm} = \bar{x}^{tm} - \frac{2 \sum_{j \in T} \bar{x}^{jm}}{y^{\alpha m}} (y^{\alpha m} + y^{\beta m})$$

$$= \bar{x}^{tm} + 2 \sum_{j \in T} \bar{x}^{jm} > \sigma_t(p)$$

by the restiction on σ_t . $y' \in \hat{\eta}_t^3(p)$ since $y' \in \psi$, in particular, $y'^{\alpha m} \leq 2 \sum_{j \in T} \bar{x}^{jm}$.

Q.E.D.

Lemma 2: For each i , $\hat{\gamma}_t^i(p)$ is nonnull, convex, for all p . $\hat{\gamma}_t^i(p)$ is upper semicontinuous at all $p \in P$ such that $p^m > 0$.

Proof: Convexity follows from quasi-concavity of $u_t(x)$. Upper semicontinuity is shown by [3], Section 1.8.k(4). $\hat{\gamma}_t^i(p)$ is nonnull since $\hat{\eta}_t^i(p)$ is compact, u_t is continuous and hence achieves its maximum on $\hat{\eta}_t^i(p)$.

Q. E. D.

Lemma 8: Let $p \in P$, $y \in \gamma_t^i(p)$. Suppose for some $n^o = 1, \ldots, N$, $y^{\alpha n^o} > 0$, $y^{\beta n^o} > 0$. Then there is $y' \in \gamma_t^i(p)$ so that either $y'^{\alpha n^o} = 0$ or $y'^{\alpha n^o} = 0$ or both, and $y'^{\beta} - y'^{\alpha} = y^{\beta} - y^{\alpha}$.

Proof: Without loss of generality suppose $y^{\alpha n^o} \geq y^{\beta n^o}$. Then let $y'^{\alpha n^o} = y^{\alpha n^o} - y^{\beta n^o}$, $y'^{\beta n^o} = 0$, $y'^{\delta n} = y^{\delta n}$ for $\delta = \alpha, \beta$, $n \neq n^o$, $N+1$, $y'^{\delta N+1} = y^{\delta N+1} + p^{n^o} y^{\beta n^o}$. y satisfies (D.i) implies that y'

satisfies (D.i). $w' = \bar{x}^t - y'^{\alpha} + y'^{\beta} = \bar{x}^t - y^{\alpha} + y^{\beta} = w$ so that y maximizes $u_t(w)$ implies y' does also.

$$y^{\beta n^o} - y^{\alpha n^o} = (y^{\beta n^o} - y^{\beta n^o}) - (y^{\alpha n^o} - y^{\beta n^o})$$

$$= y'^{\beta n^o} - y'^{\alpha n^o} .$$

Q.E.D.

The implication of Lemma 8 is that there are no wash sales (simultaneous sales and purchases of the same commodity by a given trader). More precisely, whenever a wash sale is chosen it can be replaced by a transaction without wash sale. In the absence of Lemma 8 we would face the difficulty that transactions might be unbounded, consisting of simultaneous arbitrarily large purchases and sales by the same trader.

Lemma 9: Let $y^t \in \hat{\gamma}_t^i(p)$ for some $p \in P$, and suppose $y^t \in$ Interior of Ψ with the possible exception of the $N+1^{st}$ elements of $y^{t\alpha}$, $y^{t\beta}$. Then $y^t \in \gamma_t^i(p)$.

Proof: Suppose not. Then there is $y' \in \eta_t^i(p)$ so that $u_t(w') > u_t(w)$ where $w' = \bar{x}^t - y'^{\alpha} + y'^{\beta}$ and $w = \bar{x}^t - y^{t\alpha} + y^{t\beta}$. But for $\alpha = 0$ sufficiently small $\alpha y' + (1-\alpha)y^t \in \Psi$ and hence by semi-strict quasi concavity of u, $u_t(\alpha w' + (1-\alpha)w) > u_t(w)$. But then $y^t \notin \hat{\gamma}_t^i(p)$. The contradiction proves the lemma. The exception of $N+1^{st}$ elements is irrelevant since there is no utility for the $N+1^{st}$ good.

Q.E.D.

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