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NONSTANDARD EXCHANGE ECONOMIES

Donald J. Brown and Abraham Robinson

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ERRATA FOR CFDP NO. 308

p. 11, l. 8 should read: " $\zeta_t = \zeta_f$, where f is a continuous function of"

p. 17, last l. should read: "It is obvious that $Y(t) \in {}^*\Omega_n$ for ..."

p. 18, l. 7 add: " , and $\mu(\overline{r}_i + I(t)) \cap \mu(X(t)) = \emptyset$. "

p. 18, l. 19 should read: "(3) For every $t \in R$, $\overline{p} \cdot Y(t) \not\geq \overline{p} \cdot I(t)$ "

p. 19, l. 5 should read: " ... $\not\geq \frac{1}{m} \sum_{t \in R} \overline{p} \cdot \sum_{t \in R} I(t)$ "

p. 22, l. 10 should read: " ... Also $\lim_{n \to \infty} \frac{1}{|\mathcal{E}_n|}$ ...."

p. 22, l. 15 should read: " $\lim_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{S}_n} I_n(t) - \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{S}_n} Y_n(t) = \overline{0}$. "

p. 23, l. 5 delete: "and $\liminf_{n \to \infty} \overline{p}_n > 0$. "

p. 24, last l. should read: " $< \delta \wedge \left| \frac{1}{m} \sum_{t \in \mathcal{S}_m} Y_m(t) - \frac{1}{m} \sum_{t \in \mathcal{S}_m} I_m(t) \right| < \delta \wedge (\forall t \in \mathcal{S}_m)$ ... "

p. 25, l. 5 should read: " ... such that $\lim_{m \to \infty} \frac{1}{|\mathcal{E}_m|}$ ... "

p. 25, l. 7 should read: " $\lim_{m \to \infty} \frac{1}{|\mathcal{E}_m|}$ ... , $\liminf_{m \to \infty} |\mathcal{S}_m| / |\mathcal{E}_m|$ ..."

p. 25, l. 8 should read: "... $\liminf_{m \to \infty} |Y_m(t_m) - X_m(t_m)| > 0$, "

p. 25, l. 14 should read: " $\mu(Y_m(t)) \cap \mu(X_m(t)) = 0$ ...."

p. 26, l. 11 should read: " $m \in N$, ..."

p. 26, last l. should read: " ... , $\lim_{n \to \infty} \overline{p}_m \cdot I_m(t_m)$ ..."

p. 28, l. 16 should read: " $n_{j-1} + 1$ through n_j ... $j = 1, 2, \ldots$ . ..."

p. 28, l. 19 should read: " ... $\lim_{n \to \infty} \overline{p}_n \cdot I_n(t_n) - \overline{p}_n \cdot \overline{y}_n \geq 0$, "

p. 28, l. 21 should read: " ... , $\lim_{n \to \infty} \overline{p}_n \cdot I_n(t_n) - \overline{p}_n \cdot \overline{y}_n$, "

NONSTANDARD EXCHANGE ECONOMIES

by

Donald J. Brown* and Abraham Robinson**

I. Introduction

An exchange economy consists of a set of traders each of whom is characterized by an initial endowment and a preference relation. In addition, one usually assumes that the set of traders is finite. But in order to state theorems precisely concerning the asymptotic or limiting properties of the core--such theorems will be called limit theorems--economies have been studied which have an infinite number of traders. For instance there are the denumerable economies of Debreu-Scarf [3] and the nondenumerable economies of Aumann [1].

The concepts of interest, here the core and competitive equilibrium, can be defined even in infinite economies. Hence one has the option of taking an imprecise statement about the core, such as Edgeworth's conjecture that "the core approaches the set of competitive allocations as the number of traders increases" and translating it into a theorem, T^0, about the core of an infinite economy or expressing it as a limit theorem, T, about the cores of large but finite economies.

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*Cowles Foundation for Research in Economics at Yale University.

**Mathematics Department, Yale University.

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Of course, these two approaches need not be incompatible in that one might be able to define a limiting process such that a statement T about the asymptotic behavior of a family of large but finite economies is reflected in a statement T' about a certain infinite economy, in the sense that T is true if and only if T' is true. In this context Edgeworth's conjecture gives rise to two theorems, T and T'.

Although as economists we are primarily interested in the truth of T, we shall establish T by proving T' and by showing that T is a consequence of a general metamathematical argument. Because of this argument, it is not surprising that an area of mathematical logic, model theory, provided an appropriate framework for defining the limiting process described above.

The fundamental result that we will need from model theory is the existence of a particular kind of extension of the real numbers, R, called the non-standard numbers and denoted $^*R$. $^*R$ has all the formal properties of R in that any mathematical property of R which can be described in some given language, such as the first order predicate calculus, can be translated into a mathematical property of $^*R$. Moreover the sentence, φ, expressing the fact that this property holds for R is true if and only if its interpretation in $^*R$, $^*\varphi$, is a true sentence about $^*R$. In addition, $^*R$ contains nonzero infinitesimals, i.e. nonzero numbers whose absolute value is less than any positive real number, and infinite integers, i.e. integers which are greater than every real integer; more particularly it is shown in [7] that every sequence $\{a_n\}_{n=1}^{\infty}$ can be extended to a nonstandard sequence $\{a_n\}_{n=1}^{\infty}$, where n now ranges over all the natural numbers and infinite integers, and that a_n converges to a in the standard δ, ϵ sense if and only if for every infinite integer n, $a_n - a$ is an infinitesimal.

It is the existence of infinitesimals and infinite integers that permits us to formulate precisely the notion of perfect competition, the economic concept which underlies the Edgeworth conjecture.

The operational definition of perfect competition is that the actions of any finite set of traders in terms of their willingness to buy or sell at the competitive prices should have a negligible effect on these prices. Clearly this cannot be the case for any finite economy, but perfect competition is a meaningful concept for an economy having ω traders, where ω is an infinite integer, and where the endowment for each trader relative to that of the whole economy is an infinitesimal.

After defining a nonstandard exchange economy and the notions of the core and competitive equilibrium in such an economy, we will prove that Edgeworth's conjecture is true in the following sense:

<u>Theorem (1)</u>. If $\mathcal{E}$ is a nonstandard exchange economy, then the allocation X is in the core of $\mathcal{E}$ if and only if there exists a price vector, $\bar{p}$, such that $\langle\bar{p}, X\rangle$ is a competitive equilibrium.

We then define competitive equilibrium and core allocations for a family of large but finite exchange economies, $\mathcal{G}$. Allocations, price vectors, and preference relations for $\mathcal{G}$ consist of sequences of allocations, price vectors, and preference relations belonging to the economies which make up $\mathcal{G}$.

We then show that:

Theorem (2). The allocation X is in the core of $\mathcal{G}$ if and only if there exists a price vector, $\bar{p}$, such that $\langle \bar{p}, X \rangle$ is a competitive equilibrium for $\mathcal{G}$.

Theorem (1) and Theorem (2) correspond to T' , and T discussed earlier. Theorem (2) is another formulation of Edgeworth's conjecture, but in terms of the relationship between core allocations and equilibria of large but finite economies.

II. Definitions and Assumptions

The basic mathematical notions that we will need will come from an area of model theory which is termed nonstandard analysis. An informal introduction to these ideas follows. The interested reader may consult [6], [7] and [8] for details.

Let R be the field of real numbers. We consider the properties of R in a higher-order language L , which includes symbols for all individuals (that is, numbers) of R , all subsets of R , all relations of two, three, ..., variables between individuals of R , all functions from individuals to individuals of R , and, more generally, all relations and functions of finite type (for example, functions from sets of numbers into relations between numbers) that can be defined beginning with the numbers of R . Let K be the set of all sentences formulated in L which hold (are true) in R then there exists a structure $^*R$ with the following properties.

II.1. $^*R$ is a model of K "in Henkin's sense." That is all sentences of K hold in $^*R$, provided, however, that we interpret all quantifiers other than those referring to individuals in a nonstandard fashion, as follows. Within the class of all entities of any given type other than 0 (the type of individuals) there is distinguished a certain subclass of entities called internal. And, for such a type, the quantifiers "for all x " and "there exists an x " are to be interpreted as "for all internal x ", "there exists an internal x " (of the given type). R can be injected into $^*R$, and so $^*R$ may, and will, be regarded as an extension of R .

II.2. Every <u>concurrent</u> binary relation S(x,y) in R possesses <u>a bound</u> in $^*R$.

The relation S(x,y) , of any type is called <u>concurrent</u> if for any $a_1, \ldots, a_n$, $n \geq 1$, for which there exists $b_1, \ldots, b_n$ such that $S(a_1, b_1), \ldots, S(a_n, b_n)$ hold in R , there also exists a b such that $S(a_1, b), \ldots, S(a_n, b)$ hold in R . And the entity b_s in $^*R$ is called <u>a bound</u> for S(x,y) if $S(a, b_s)$ holds in $^*R$ for every a for which there exists a , b such that S(a,b) holds in R .

II.3. For any mapping $f(y)$ from a set A (of any type) in R into the extension $^*B$ of a set B in R, there exists an internal mapping $\varphi(y)$ from $^*A$, the extension of A, into $^*B$ which coincides with $f(y)$ on A.

Any structure $^*R$ which satisfies II.1, II.2, and II.3 is called a comprehensive enlargement of R. In particular $^*R$ can be constructed as an ultrapower of R. (See [4] and [5] for the notions of an ultrapower.) It is shown in [7] that $^*R$ has the following properties:

(i) The set of individuals of $^*R$ is an ordered non-archimedean field. In this paper we shall call all elements of $^*R$ nonstandard numbers.

(ii) R, the real numbers, is a proper ordered subfield of the nonstandard numbers. Elements of R will be called standard numbers. Observe that according to our present usage every standard number is a nonstandard number.

(iii) There exist nonstandard numbers, in particular nonstandard integers which are greater than every standard number. These numbers are called infinite nonstandard numbers and infinite integers respectively.

(iv) There exist numbers in $^*R$ which are in absolute value less than any positive real number, in fact they, except zero, are just the reciprocals of the infinite nonstandard numbers. These numbers are called infinitesimals.

(v) The system of internal entities in $^*R$ has the following property. If S is an internal set of relations, then all elements of S are internal. More generally, if S is an internal n-ary relation, $n \geq 1$ and the n-tuple $(S_1, \ldots, S_n)$ satisfies (belongs to) S then $S_1, \ldots, S_n$ are internal.

A nonstandard number is said to be finite if in absolute value it is less than some standard number. If r is a finite nonstandard number, then there exists a unique standard number called the standard part of r, denoted by $^\circ r$, such that $r - {^\circ r}$ is an infinitesimal. Hence there are three kinds of nonstandard numbers: (i) the infinitesimals which in absolute value are less than every standard number, (ii) the finite nonstandard numbers which in absolute value are less than some standard number, and (iii) the infinite nonstandard numbers which are greater than every standard number. Note that according to this taxonomy every infinitesimal is a finite nonstandard number.

We will denote the n-dimensional vector space over $^*R$ by $^*R_n$. A commodity bundle $\bar{x}$ is a point in the nonnegative orthant $^*\Omega_n$ of $^*R_n$. Let $^*\rho$ be the nonstandard extension of the Euclidean metric ρ, then a set of points $S(\bar{x},r)$ in $^*R_n$ will be called an S-ball if there exists $\bar{x} \in {^*R_n}$ and a positive standard number r such that $S(\bar{x},r) = \{\bar{y} | {^*\rho}(\bar{x},\bar{y}) < r\}$. It is shown in [7] that the S-balls may serve as a basis for a topology in $^*R_n$. This topology will be called the S-topology.

The monad of $\bar{x}$, $\mu(\bar{x})$, is the set of points whose distance from $\bar{x}$ is an infinitesimal. If $\bar{y} \in \mu(\bar{x})$, we shall write $\bar{x} \simeq \bar{y}$. $\bar{x} > \bar{y}$,

$\bar{x} \geq \bar{y}$, and $\bar{x} \geqq \bar{y}$ will have their conventional meanings. $\bar{x} \gtrsim \bar{y}$ means that $\bar{x}$ is greater than $\bar{y}$ or $\bar{y}$ exceeds $\bar{x}$ by at most an infinitesimal amount. $\bar{x} \not\gtrsim \bar{y}$ means that $\bar{x}$ is greater than $\bar{y}$ by a finite amount.

The <u>S-convex</u> <u>hull</u> of a set B is defined as the set of all finite standard convex combinations of elements of B , i.e., vectors of the form

$$\bar{y} = \alpha_1\bar{x}_1 + \dots + \alpha_n\bar{x}_n \text{ where } \bar{x}_i \in B\ , \quad \alpha_i \geq 0\ , \quad \sum_{i=1}^{n} \alpha_i = 1\ , \text{ the } \alpha_i$$

are standard numbers and n is a standard integer. B is defined as S-convex if B contains its S-convex hull.

A <u>nonstandard</u> <u>exchange</u> <u>economy</u> is defined as a pair of indexed sets, $\{\bar{x}_t\}_{t=1}^{\omega}$ and $\{ \succ_t \}_1^{\omega}$ where for all t , $\bar{x}_t \in {}^*\Omega_n$, $\succ_t \in {}^*\Omega \times {}^*\Omega_n$, and ω is an infinite (nonstandard) integer. $\bar{x}_t$ is to be interpreted as the initial endowment of the t^{th} trader and $\succ_t$ his preference relation. The nonstandard exchange economies which we will consider are assumed to have the following properties:

(i) The function indexing the initial endowments, $I(t)$, is internal.

(ii) $I(t)$ is standardly bounded, i.e. there exists a standard vector $\bar{r}_0$ such that for all t , $I(t) \leq \bar{r}_0$.

(iii) $\frac{1}{\omega}\sum_{1}^{\omega} I(t) \not\gtrsim \bar{0}$.

(iv) The relation, Q , where $Q = \{<t, \succ_t>| t \in T, \succ_t \subseteq {}^*\Omega_n \times {}^*\Omega_n\}$ is internal.

(α) $\succ_t$ is irreflexive

(β) If $\bar{x} \geq \bar{y}$ then $\bar{x} \succ_t \bar{y}$

(γ) If $\mu(\bar{x}) \cap \mu(\bar{y}) = \emptyset$ and $\bar{x} \succ_t \bar{y}$ then there exists a standard $\delta > 0$ such that if $\bar{z} \in S(x, \delta)$ then $\bar{z} \succ_t \bar{y}$.

Equivalently, if $\mu(\bar{x}) \cap \mu(\bar{y}) = \emptyset$ and $\bar{x} \succ_t \bar{y}$ and $\bar{z} \in \mu(\bar{x})$ then $\bar{z} \succ_t \bar{y}$.

An <u>assignment</u> is an internal function from T, the set of traders, into ${}^*\Omega_n$.

An <u>allocation or final allocation</u> is a standardly bounded assignment $Y(t)$ from the set of traders, $\{1, 2, \ldots, \omega\}$, into ${}^*\Omega_n$ such that $\frac{1}{\omega} \sum_{t=1}^{\omega} Y(t) \simeq \frac{1}{\omega} \sum_{t=1}^{\omega} I(t)$.

(iv) implies that for all internal $X, Y \in {}^*\Omega_n^T$, where T is the set of traders, that

(v) $\{t \mid X(t) \succ_t Y(t)\}$ is an internal set of traders.

A <u>coalition</u>, S, is defined as an internal set of traders. It is said to be <u>negligible</u> if $|S|/\omega \simeq 0$. Note that if S is negligible then for all allocations $X(t)$, $\frac{1}{\omega} \sum_{t \in S} X(t) \simeq \bar{0}$.

A coalition, S, is effective for an allocation Y if $\frac{1}{\omega} \sum_{t \in S} Y(t) \simeq \frac{1}{\omega} \sum_{t \in S} I(t)$.

An allocation Y <u>dominates</u> an allocation X via a coalition S if for all $t \in S$, $\mu(X(t)) \cap \mu(Y(t)) = \emptyset$ and $Y(t) \succ_t X(t)$, and S is effective for Y.

The <u>core</u> is defined as the set of all allocations which are not dominated via any non-negligible coalition.

A <u>price vector</u>, $\bar{p}$, is a finite nonstandard vector such that $\bar{p} \gneq \bar{0}$.

The t^{th} traders budget set, $B_{\bar{p}}(t)$, is $\{\bar{x} \in {}^*\Omega_n | \bar{p}\cdot\bar{x} \lesssim \bar{p}\cdot I(t)\}$.

$\bar{y}$ is said to be maximal in $B_{\bar{p}}(t)$ if $\bar{y} \in B_{\bar{p}}(t)$ and there does not exist an $\bar{x} \in B_{\bar{p}}(t)$ such that $\mu(\bar{x}) \cap \mu(\bar{y}) = \emptyset$ and $\bar{x} \succ_t \bar{y}$.

A competitive equilibrium is defined as a pair $\langle\bar{p}, X\rangle$, where $\bar{p}$ is a price vector and X an allocation such that $X(t)$ is maximal in $B_{\bar{p}}(t)$ for almost all the traders. That is, if $K = \{t | X(t)$ is maximal in $B_{\bar{p}}(t)\}$ then $|K|/\omega \simeq 1$.

III. Theorems

We shall first demonstrate that our definition of a nonstandard economy is consistent with the assumptions we wish to make concerning the initial endowments and preferences of the traders.

Let f be a continuous function of Ω_n into R and define the relation $\succ_t$ on Ω_n as $\bar{x} \succ_f \bar{y}$ if $f(\bar{x}) > f(\bar{y})$ for all $\bar{x}, \bar{y} \in \Omega_n$. Then it is shown in [7] that ${}^*f : D \to {}^*R$ is S-continuous where ${}^*f$ is the nonstandard extension of f and D a standardly bounded subset of ${}^*\Omega_n$. ${}^*f$ induces a preference relation $\succ_{*f}$, on ${}^*\Omega_n$ where for all $\bar{x}, \bar{y} \in {}^*\Omega_n$, $\bar{x} \succ_{*f} \bar{y}$ if ${}^*f(\bar{x}) > {}^*f(\bar{y})$. Note that $\succ_{*f} = {}^*(\succ_f)$.

It is obvious that $\succ_{*f}$ is irreflexive. Suppose $\bar{x} \succ_{*f} \bar{y}$ and $\mu(\bar{x}) \cap \mu(\bar{y}) = \emptyset$. Then there exists a standard $\delta > 0$ such that for all $\bar{w} \in S(\bar{x}, \delta)$, ${}^*f(\bar{w}) > {}^*f(\bar{y})$ or $\bar{w} \succ_{*f} \bar{y}$. Now suppose that for all $\bar{x}, \bar{y} \in \Omega_n$, if $x \geq y$ then $f(\bar{x}) > f(\bar{y})$. Thus by transfer for all $\bar{x}, \bar{y} \in {}^*\Omega_n$ if $\bar{x} \geq \bar{y}$ then ${}^*f(\bar{x}) > {}^*f(\bar{y})$, i.e. if $\bar{x} \geq \bar{y}$, then $\bar{x} \succ_{*f} \bar{y}$. Finally since $\succ_{*f} = {}^*(\succ_f)$, we see that $\succ_{*f}$ is internal.

An example of a function having all of the above properties is $f(\bar{x}) = x_1 + \dots + x_n$ for all $\bar{x} \in \Omega_n$. Hence $^*f(\bar{x}) = x_1 + \dots + x_n$ for all $\bar{x} \in {}^*\Omega_n$.

<u>Consistency Lemma</u>: Let $\mathcal{E}$ be a countably infinite standard economy which has the following properties for all traders, t, in $\mathcal{E}$:

(a) There exists a standard vector $\bar{r}_0$ such that $I(t) \leq \bar{r}_0$.

(b) $I(t) > \bar{0}$.

(c) $\succ_t = \succ_f$, where f is a uniformly continuous function of Ω_n into R and for all $\bar{x}, \bar{y} \in \Omega_n$, if $\bar{x} \geq \bar{y}$ then $f(\bar{x}) > f(\bar{y})$.

Then there exists a nonstandard exchange economy $\mathcal{H}$ satisfying the assumptions of Section II.

<u>Proof</u>: Let N be the nonnegative standard integers, then $I : N \to R_n$. Thus its nonstandard extension $^*I$ maps $^*N$ into $^*R_n$. $^*I$ restricted to $\{1, 2, \dots, \omega\}$ for any infinite integer ω is a function which satisfies assumptions (i), (ii), and (iii) in Section II. We then assign the same preference relation $\succ_{f^*}$ to each of the traders $1, 2, \dots, \omega$. This completes the proof.

In the analysis of finite standard economies, the existence of an equilibrium price vector is usually established by invoking a separating hyperplane theorem. Since our notions of convexity and topology are external notions, we cannot prove separating hyperplane theorems in $^*R_n$ by simple "transfer" from R_n.

A set D in ${}^*R_n$ is said to be <u>nearstandard</u> if every vector in D is finite. We shall establish a separation theorem for nearstandard S-convex sets.

<u>Lemma (1)</u>. If A is a subset of ${}^*R_n$ and $\overline{0} \notin$ S-Interior of A, S-Int(A), then for all $\overline{x} \in$ S-Int(A), $\mu(\overline{0}) \cap \mu(\overline{x}) = \emptyset$.

<u>Proof</u>: Suppose there exists an $\overline{x} \in$ S-Int(A) such that $\mu(\overline{0}) \cap \mu(\overline{x}) \neq \emptyset$, then ${}^*\rho(\overline{0},\overline{x}) \simeq 0$. But $\mu(\overline{x}) \subset S(\overline{x},r) \subset A$ for some standard r. Hence $S(\overline{0},r) \subset A$, a contradiction.

<u>Lemma (2)</u>. If A a subset of ${}^*R_n$ and $\overline{0} \notin$ S-Int(A), then $\overline{0} \notin$ standard part of S-Int(A), ${}^{\circ}$(S-Int(A)). ${}^{\circ}$(S-Int(A)) $= \{{}^{\circ}\overline{x} \mid \overline{x} \in$ S-Int(A), $\overline{x}$ finite$\}$.

<u>Proof</u>: If $\overline{0} \in {}^{\circ}$(S-Int(A)), then there exists an $\overline{x} \in$ S-Int(A) such that $\overline{0} \in \mu(\overline{x})$, i.e. $\mu(\overline{x}) = \mu(\overline{0})$. But this contradicts Lemma (1).

To show that the S-Interior of an S-convex set in ${}^*R_n$ is S-convex we will adapt Rockafellar's proof, [9], that the interior of a convex set in R_n is convex.

Let $\overline{S}(\overline{x},\delta)$ denoted the closed S-ball centered on $\overline{x}$ with radius δ and $B = \overline{S}(\overline{0},1)$, then $\overline{S}(\overline{x},\delta) = \overline{x} + \delta B$. The S-closure, S-cl(A), and S-Int(B) can then be expressed as S-cl(A) $= \bigcap\{A + \delta B \mid \delta > 0\}$ and S-Int(A) $= \{\overline{x} \mid$ there exists a δ, $\overline{x} + \delta B \subset A\}$.

<u>Lemma (3)</u>. Let A be a S-convex set in ${}^*R_n$. Let $\overline{x} \in$ S-Int(A) and $\overline{y} \in$ S-cl(A). Then $(1-\lambda)\overline{x} + \lambda y$ belongs to S-Int(A) for standard λ such that $0 \leq \lambda < 1$.

Proof: Suppose λ standard and $\epsilon[0,1)$, then we must show that $(1-\lambda)\bar{x} + \lambda\bar{y} + \delta B \subseteq A$ for some standard $\delta > 0$. We have $\bar{y} \in A + \delta B$ for all standard $\delta > 0$, because $\bar{y} \in S\text{-cl}(A)$. Thus for every standard $\delta > 0$, $(1-\lambda)\bar{x} + \lambda\bar{y} + \delta B \subset (1-\lambda)\bar{x} + \lambda(A + \delta B) + \delta B = (1-\lambda)[\bar{x} + \delta(1+\lambda)(1-\lambda)^{-1}B] + \lambda A$. But $\bar{x} + \delta(1+\lambda)(1-\lambda)^{-1}B \subset A$ for sufficiently small δ. Hence $(1-\lambda)\bar{x} + \lambda\bar{y} + \delta B \subset (1-\lambda)A + \lambda A = A$ for some standard $\delta > 0$.

Lemma (4). If A is an S-convex set in ${}^*R_n$, then S-Int(A) is S-convex.

Proof: Take $\bar{y}$ to be in S-Int(A) in Lemma (3).

Lemma (5). If A is an S-convex set in ${}^*R_n$, then ${}^{\circ}A$ is a convex set in R_n.

Proof: Suppose $\bar{x}, \bar{y} \in {}^{\circ}A$ and α a standard number such that $0 \leq \alpha \leq 1$. Then there exists infinitesimals $\bar{\epsilon}_1$, $\bar{\epsilon}_2$ such that $\bar{x} + \bar{\epsilon}_1$, $\bar{y} + \epsilon_2 \in A$. But A is S-convex which implies that $\alpha(\bar{x} + \bar{\epsilon}_1) + (1-\alpha)(\bar{y} + \bar{\epsilon}_2) \in A$. We can express $\alpha(\bar{x} + \bar{\epsilon}_1) + (1-\alpha)(\bar{y} + \bar{\epsilon}_2)$ as $\alpha x + (1-\alpha)\bar{y} + \bar{\epsilon}_3$, where $\bar{\epsilon}_3 = \alpha\bar{\epsilon}_1 + (1-\alpha)\bar{\epsilon}_2 \simeq \bar{0}$. Thus $\alpha\bar{x} + (1-\alpha)\bar{y} \in {}^{\circ}A$.

S-Separation Lemma: If A is a near standard S-convex set in ${}^*R_n$ and $\bar{0} \notin$ S-Int(A) then there exists a standard $\bar{p} \neq \bar{0}$ such that for all $\bar{y} \in$ S-Int(A), $\bar{p}\cdot\bar{y} \underset{\sim}{>} 0$.

Proof: By Lemma (4) S-Int(A) is S-convex. By Lemma (5) ${}^{\circ}$(S-Int(A)) is a convex set in R_n. Lemma (2) $\bar{0} \notin {}^{\circ}$(S-Int(A)). Hence there exists a $\bar{p} \in R_n$ such that $\bar{p} \neq \bar{0}$ and for all $\bar{x} \notin {}^{\circ}$(S-Int(A)), $\bar{p}\cdot\bar{x} \geq 0$.

See [2] for a proof of this result. Every $\bar{y} \in$ S-Int(A) can be expressed as $\bar{y} = \bar{x} + \bar{\epsilon}$ where $\bar{x} \in {}^{\circ}$(S-Int(A)) and $\bar{\epsilon} \simeq \bar{0}$. Thus for all $\bar{y} \in$ S-Int(A), we have that $\bar{p}\cdot\bar{y} = \bar{p}\cdot\bar{x} + \bar{p}\cdot\bar{\epsilon} \geq 0 + \bar{p}\cdot\bar{\epsilon} \simeq 0$, i.e. $p\cdot y \not\gtrsim 0$.

Let $\mathcal{E}$ be a nonstandard exchange economy where the set of traders is an internal set T and $|T| = \omega$, an infinite integer. A set of traders U is said to be full if U is internal and $|T-U|/\omega \simeq 0$. Let $X(t)$ be a fixed allocation in the core of $\mathcal{E}$, then $F_n(t) = \{\bar{x} \in {}^*\Omega_n | \bar{x} \succ_t X(t)$, $\bar{x}$ finite, $|\bar{x} - X(t)| \geq \frac{1}{n}\}$ and $G_n(t) = F_n(t) - I(t)$ and $G(t) = \bigcup_{n \in N} G_n(t)$. Let $\Delta(U)$ denote the S-convex hull of the union $\bigcup_{t \in U} G(t)$.

<u>Principal Lemma</u>: There exists a full set of traders U such that $\bar{0} \notin$ S-Int(Δ(U)).

<u>Proof</u>: For each finite $\bar{x} \in {}^*R_n$, let $G^{-1}(\bar{x})$ be the set of all traders t for which $G(t)$ contains $\bar{x}$. Then $G^{-1}(\bar{x}) = \bigcup_{n \in N} G_n^{-1}(\bar{x})$, where $G_n^{-1}(\bar{x}) = \{t | \bar{x} + I(t) \succ_t X(t),\ \bar{x} + I(t) \in {}^*\Omega_n,\ |\bar{x} + I(t) - X(t)| \geq \frac{1}{n}\}$. It follows from assumption (v) that for each n and every $\bar{x}$, $G_n^{-1}(\bar{x})$ is internal. Let M be the set of all those standard rational points $\bar{r} \in {}^*R_n$ (i.e. points with standard rational coordinates) such that for all $n \in N$, $G_n^{-1}(\bar{r})$ is negligible. Since the standard rational points can be put into a one-to-one correspondence with the standard integers, we can express M as $\{\bar{r}_i\}_{i=1}^{\infty}$, where i ranges over the standard integers. Let N be the set of standard integers. For each $\bar{r}_i$ there exists a $\nu_i \in {}^*N$ such that $G_n^{-1}(\bar{r}_i) \subseteq G_{\nu_i}^{-1}(\bar{r}_i)$, where $G_{\nu_i}^{-1}(\bar{r}_i)$ is internal and negligible. Let $B_n = \bigcup_{i=0}^{n} G_{\nu_i}^{-1}(\bar{r}_i)$, $n \in N$.

The mapping $f : N \to \mathcal{P}({}^*R_n)$, where $f(n) = B_n$, need not be internal. But since we are working in a comprehensive enlargement there exists a g

such that g is internal, $g : {}^*N \to \mathcal{P}({}^*R_n)$, $f(n) = g(n)$ for all $n \in Z$, there exists $m \in {}^*N$ such that $g(n) \subseteq g(n+1)$ for all $n \in {}^*N$, where $n \leq m$. In [7] the following theorem is established. Let $\{s_n\}$ be an internal sequence such that s_n is infinitesimal for all finite n. Then there exists an infinite integer ν such that s_n is infinitesimal for all $n < \nu$. The sequence $\{|g(n)|/\omega\}$ satisfies the hypothesis of this theorem, hence there exists an infinite integer ν such that B_n is negligible for all $n \leq \nu$. Define $U = T - B_\nu$, then U is full and $B_n \subseteq B_\nu$ for all standard n.

Suppose $\bar{0} \in$ S-Int($\Delta(U)$). Then there exists an $\bar{x} \gtrsim \bar{0}$ such that $-\bar{x} \in \Delta(U)$; by definition of $\Delta(U)$, $-\bar{x}$ is a S-convex combination of n points in $\bigcup_{t \in U} G(t_i)$, where n is finite. That is, there are traders $t_1, \ldots, t_n \in U$, (not necessarily distinct) points $\bar{x}_i \in G(t_i)$, and positive standard numbers $\beta_1, \ldots, \beta_n$ such that $\sum_{i=1}^{n} \beta_i \bar{x}_i = \bar{x} \lesssim 0$. We may assume that for all i, $\beta_i \bar{x}_i \gtrsim \bar{0}$. $\bar{x}_i \in G(t_i)$ implies that $\bar{x}_i + I(t_i) \succ_i X(t_i)$, and $\mu(\bar{x}_i + I(t_i)) \cap \mu(X(t_i)) = \emptyset$ hence by assumption (iv-γ), continuity of the preference relation, there exists an S-ball, $S(\bar{x}_i + I(t_i), \delta)$ where for all $\bar{w} \in S(\bar{x}_i + I(t_i), \delta)$, $\bar{w} \succ_{t_i} X(t_i)$. Consequently $S(\bar{x}_i + I(t_i), \delta) \subset F(t_i)$ which implies that $S(\bar{x}_i, \delta) \subset G(t_i)$. Hence for every standard $\theta > 0$ there exists a standard rational $\bar{r}_i \in G(t_i)$ such that ${}^\circ\rho^*(\bar{r}_i, \bar{x}_i) < \theta$. Hence in all cases there exists standard rational points $\bar{r}_i \in G(t_i)$ sufficiently close to the $\bar{x}_i$, and positive standard rational numbers γ_i sufficiently close to the $\bar{r}_i$ so that we still have $\sum_{i=1}^{n} \gamma_i \bar{r}_i \lesssim \bar{0}$ and $\sum_{i=1}^{n} \gamma_i = 1$.

Let $-\bar{r} = \sum_{i=1}^{n} \gamma_i \bar{r}_i$ and pick an arbitrary trader t_0 in U. Since $\bar{r} \gtrsim \bar{0}$, we have $\alpha\bar{r} + I(t_0) \gtrsim X(t_0)$ for sufficiently large positive finite rational α. Hence by the desirability assumption $\alpha r + I(t_0) \succ_{t_0} X(t_0)$, i.e. $\alpha\bar{r} \in G(t_0)$. Now set $\bar{r}_0 = \alpha\bar{r}$, $\alpha_0 = 1/(\alpha+1)$, $\alpha_i = \alpha\gamma_i/\alpha+1$ for $i = 1, \ldots, n$. Then $\alpha_i > 0$ for all i and $\sum_0^n \alpha_i = 1$; furthermore $\sum_{i=0}^{n} \alpha_i \bar{r}_i = \frac{\alpha}{\alpha+1} \bar{r} + \frac{\alpha}{\alpha+1} \sum_{i=1}^{n} \gamma_i \bar{r}_i = \frac{\alpha}{\alpha+1} (r-\bar{r}) = \bar{0}$, and for all i, $r_i \in G(t_i)$. Then $t_i \in G^{-1}(\bar{r}_i)$, and since $t_i \in U$, it follows that $\bar{r}_i \notin N$, hence for all i $|G^{-1}(\bar{r}_i)|/\omega \not\simeq 0$. Suppose $n = 2$, then we will consider two cases. First suppose $|G^{-1}(\bar{r}_2) - G^{-1}(\bar{r}_1)|/\omega \not\simeq 0$. Choose any δ such that $0 \lesssim \delta \lesssim \min\left\{\frac{|G^{-1}(\bar{r}_1)|}{\omega\alpha_1}, \frac{|G^{-1}(\bar{r}_2) - G^{-1}(\bar{r}_1)|}{\omega\alpha_2}\right\}_{i=1,2}$ and define $\rho_i = \omega\delta\alpha_i$, $\theta_i = [\rho_i]$ for $i = 1, 2$. Then $\frac{1}{\omega} \sum_{t=1}^{\theta_i} 1 = \frac{\theta_i}{\omega} \simeq \frac{\rho_i}{\omega} = \delta\alpha_i$ for $i = 1, 2$. If $|G^{-1}(\bar{r}_2) - G^{-1}(\bar{r}_1)|/\omega \simeq 0$, then let $\eta = |G^{-1}(\bar{r}_1) \cap G^{-1}(\bar{r}_2)|/2$ and choose any δ such that $0 \lesssim \delta \lesssim \min\{\eta/\omega\alpha_i\}_{i=1,2}$ and again define $\rho_i = \omega\delta\alpha_i$, $\theta_i = [\rho_i]$ for $i = 1, 2$. Then, as before, $\frac{1}{\omega} \sum_{t=1}^{\theta_i} 1 = \frac{\theta_i}{\omega} \simeq \frac{\rho_i}{\omega} = \delta\alpha_i$

for $i = 1, 2$. Assume that for all $k < n$ there exists $\{A_i\}_{i=1}^{k}$ such that $A_i \subseteq G^{-1}(\bar{r}_i)$ for $i = 1, 2, \ldots, k$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and there exists $\delta \gtrapprox 0$ such that $|A_i|/\omega \simeq \delta\alpha_i$. Now suppose we have n traders and let $B_i = G^{-1}(\bar{r}_i) - G^{-1}(\bar{r}_n)$, $i = 1, 2, \ldots, n-1$. If $|B_i|/\omega \not\simeq 0$ for $i = 1, 2, \ldots, n-1$, then by the induction hypothesis there exists $\{A_i\}_{i=1}^{n-1}$ such that $A_i \subseteq G^{-1}(\bar{r}_i)$, $A_i \cap A_j = \emptyset$ and there exists $\delta \gtrapprox 0$ such that $|A_i|/\omega \simeq \delta\alpha_i$, we need only consider $G^{-1}(\bar{r}_n)$. If $|G^{-1}(\bar{r}_n)|/\omega \geq \delta\alpha_n$ we are done. So suppose $|G^{-1}(\bar{r}_n)|/\omega < \delta\alpha_n$, then choose a $0 \lessapprox \delta' \lessapprox |G^{-1}(\bar{r}_n)|/\omega\alpha_n$. It is clear that there exists $\{A_i'\}_{i=1}^{n}$ such that $A_i' \subseteq G^{-1}(\bar{r}_i)$, $A_i' \cap A_j' = \emptyset$ and $|A_i'|/\omega \simeq \delta'\alpha_i$ for $i = 1, 2, \ldots, n$. For the other case assume there exists j such that $|B_j|/\omega \simeq 0$. Then by the induction hypothesis we can carry out the construction for all i expecting $i = j, n$. We have shown above that we can carry out the construction for the two remaining sets, which we do for $G^{-1}(\bar{r}_j)$ and $G^{-1}(\bar{r}_n)$. We now take the smaller of the two δ' in these constructions, call it δ'', and we see that there exists $\{A_i'\}_{i=1}^{n}$ such that $A_i' \subseteq G^{-1}(\bar{r}_i)$, $A_i' \cap A_j' = \emptyset$ and $|A_i'|/\omega \simeq \delta''\alpha_i$ for $i = 0, 2, \ldots, n$. Let $A' = \bigcup_{i=0}^{n} A_i'$ and define the following internal function

$$Y(t) = \begin{cases} \bar{r}_i + I(t) & \text{for } t \in A_i' \\ I(t) & \text{for } t \notin A' \end{cases}$$

It is obvious that $Y(t) \in \Omega$ for $t \notin A'$, and for $t \in A_i'$ it follows from $\bar{r}_i \in G(t)$, i.e. $A_i' \subset G^{-1}(\bar{r}_i)$, hence $\bar{r}_i + I(t) \in F(t) \subset {}^*\Omega_n$.

Next $\frac{1}{\omega} \sum_{t \in A'} Y(t) = \frac{1}{\omega} \sum_{i=0}^{n} \sum_{t \in A_i'} Y(t) = \frac{1}{\omega} \sum_{i=0}^{n} \sum_{t \in A_i'} \{\bar{r}_i + I(t)\} = \frac{1}{\omega} \sum_{i=0}^{n} \sum_{t \in A_i'} \bar{r}_i +$

$\frac{1}{\omega} \sum_{i=0}^{n} \sum_{t \in A_i'} I(t) = \frac{1}{\omega} \sum_{i=0}^{n} |A_i| \bar{r}_i + \frac{1}{\omega} \sum_{t \in A'} I(t) \simeq \delta'' \sum_{i=0}^{n} \alpha_i \bar{r}_i + \frac{1}{\omega} \sum_{t \in A'} I(t) \simeq \bar{0} +$

$\frac{1}{\omega} \sum_{t \in A'} I(t) = \frac{1}{\omega} \sum_{t \in A'} I(t)$ and therefore A' is effective for Y ; since $Y(t) = I(t)$ for $t \notin A'$, it follows that Y is an allocation. Finally from $A_i' \subset G^{-1}(\bar{r}_i)$ it follows that $\bar{r}_i + I(t) \succ_t X(t)$ for $t \in A_i'$, i.e. $Y(t) \succ_t X(t)$ for $t \in A'$.

Since A' is not negligible, we have shown that X is not in the core, contrary to assumption.

Theorem (1). If $\mathcal{E}$ is a nonstandard exchange economy satisfying the assumptions of Section II, then an allocation X is in the core of $\mathcal{E}$ if and only if there exists a price vector, $\bar{p}$, such that $\langle \bar{p}, X \rangle$ is a competitive equilibrium of $\mathcal{E}$.

Proof: Suppose $\langle \bar{p}, X \rangle$ is a competitive equilibrium of $\mathcal{E}$ and X is not in the core of $\mathcal{E}$. Then there exists an internal $S \subseteq T$ such that $|S|/\omega \not\simeq 0$ and an allocation Y where for some internal $R \subseteq S$ we have

(1) $|R|/\omega \simeq |S|/\omega$

(2) For all $t \in R$, $Y(t) \succ_t X(t)$ and $\mu(Y(t)) \cap \mu(X(t)) = \emptyset$

(3) For every $t \in R$, $Y(t) \not\lesssim \bar{p} \cdot I(t)$

(3) implies that for every $t \in R$, $\bar{p} \cdot (Y(t) - I(t)) \not\lesssim 0$. Since

$\bar{p}\cdot(Y(t) - I(t))$ is finite and R is an internal set, there exists $t_0 \in R$ such that for all $t \in R$, $\bar{p}\cdot(Y(t) - I(t)) \geq \bar{p}\cdot(Y(t_0) - I(t_0))$. But $\bar{p}\cdot(Y(t_0) - I(t_0)) \gtrsim 0$, hence there exists $\epsilon \gtrsim 0$ such that $\bar{p}\cdot(Y(t_0) - I(t_0)) \gtrsim \epsilon \gtrsim 0$. Consequently $\frac{1}{\omega} \sum_{t \in R} \bar{p}\cdot(Y(t) - I(t)) \geq \frac{1}{\omega} \sum_{t \in R} \bar{p}\cdot(Y(t_0) - I(t_0))$ $\geq \frac{|R|}{\omega} \epsilon \gtrsim 0$. Therefore $\bar{p}\cdot \sum_{t \in R} Y(t)/\omega = \frac{1}{\omega} \sum_{t \in R} \bar{p}\cdot Y(t) \gtrsim \frac{1}{\omega} \sum_{t \in R} \bar{p}\cdot \sum_{t \in R} I(t)/\omega$ which contradicts the effectiveness of S, since $\frac{1}{\omega} \sum_{t \in S} Y(t) = \frac{1}{\omega} \sum_{t \in R} Y(t)$ $+ \frac{1}{\omega} \sum_{t \in S-R} Y(t) \simeq \frac{1}{\omega} \sum_{t \in R} I(t) + \frac{1}{\omega} \sum_{t \in S-R} I(t) = \frac{1}{\omega} \sum_{t \in S} I(t)$ if S is effective. But $\sum_{t \in S-R} Y(t)/\omega \simeq \sum_{t \in S-R} I(t)/\omega \simeq 0$, hence $\bar{p}\cdot \sum_{t \in R} Y(t)/\omega \simeq \bar{p}\cdot \sum_{t \in R} I(t)/\omega$.

Suppose X is the core of $\mathcal{E}$ and let U be the full set of traders in the Principal lemma. Then by the S-separation lemma there exists a standard $\bar{p} \neq \bar{0}$ such that for all $\bar{x} \in$ S-Int(ΔU), $\bar{p}\cdot\bar{x} \gtrsim 0$. Hence for all $\bar{y} \in$ S-Int$(G(t))$, $\bar{p}\cdot\bar{y} \gtrsim 0$. This is equivalent to saying that $\bar{p}\cdot\bar{x} \gtrsim \bar{p}\cdot I(t)$ for all $\bar{x} \in$ S-Int$(F(t))$. Suppose $\bar{z}$ is a standard vector such that $\bar{z} \geq \bar{0}$. Then $\bar{z} + X(t) \in F(t)$ for all $t \in T$, by desirability. Moreover since $\mu(\bar{z} + X(t)) \cap \mu(X(t)) = \emptyset$, there exists a standard positive number δ such that for all $\bar{w} \in S(\bar{z} + X(t), \delta)$, $\bar{w} \succ_t X(t)$. Consequently $\bar{z} + X(t) \in$ S-Int$(F(t))$ for all standard $\bar{z} \geq \bar{0}$. Therefore $\bar{p}\cdot(\bar{z} + X(t)) \gtrsim \bar{p}\cdot I(t)$. By definition $X(t), I(t) \in {}^*\Omega_n$ for all t. Therefore for all $\bar{z} \in \Omega_n$, $\bar{p}\cdot(\bar{z} + X(t)) \gtrsim 0$. But if for some i, $p_i < 0$, we can choose a $\bar{z} \in \Omega_n$ such that $\bar{p}\cdot(\bar{z} + X(t)) \lesssim 0$. Thus $\bar{p} \geq 0$.

We will now show that for all $t \in T$, $X(t) \in$ S-cl[S-Int$(F(t))$]. Every S-ball centered on $X(t)$ contains a $\bar{z}$ which by desirability trader

t prefers to $X(t)$, and $\mu(\bar{z}) \cap \mu(X(t)) = \emptyset$. Thus by continuity there exists a δ for all $\bar{w} \in S(\bar{z}, \delta)$, $\bar{w} \succ X(t)$, i.e. $S(\bar{z}, \delta) \subset F(t)$.

Since $\bar{p} \cdot \bar{x} \gtrsim \bar{p} \cdot I(t)$ for all $\bar{x} \in$ S-Int(F(t)) and $\bar{p} \cdot \bar{x}$ is a continuous function of x in the S-topology, we conclude that $\bar{p} \cdot X(t) \gtrsim \bar{p} \cdot I(t)$ for all $t \in T$.

We can now show that except for at most a negligible set of t , $\bar{p} \cdot X(t) \lesssim \bar{p} \cdot I(t)$. If for some non negligible internal set, S , we have $\bar{p} \cdot X(t) \gnsim \bar{p} \cdot I(t)$, then $\frac{1}{\omega} \sum_{t \in S} \bar{p} \cdot X(t) \gnsim \frac{1}{\omega} \sum_{t \in S} \bar{p} \cdot I(t)$, which contradicts the assumption that X is in the core. Thus $\bar{p} \cdot X(t) \lesssim \bar{p} \cdot I(t)$ except for at most a negligible internal set of traders.

To complete the proof we must show that $X(t)$ is maximal in t's budget set i.e. that $\bar{p} \cdot \bar{x} \gnsim \bar{p} \cdot I(t)$ for $\bar{x} \in F(t)$. We first show that $\bar{p} \gnsim \bar{0}$. Suppose not; let $p^1 = 0$, say. Since $\bar{p}$ is standard, some coordinate of $\bar{p}$ is not infinitesimal; say $p^2 \gnsim 0$. But $\frac{1}{\omega} \sum_{t \in T} I^2(t) \gnsim 0$. Since X is an allocation it follows that $\frac{1}{\omega} \sum_{t \in T} X^2(t) \gnsim 0$, so there must be a non negligible internal set of traders, S , for whom $X^2(t) \gnsim 0$. Now for any trader t , it follows from desirability that $X(t) + (1, 0, \ldots, 0) \succ_t X(t)$. Choosing $t \in S$ we see, by continuity, that for some sufficiently small $\epsilon \gnsim 0$ $X(t) + (1, -\epsilon, 0, \ldots, 0) \succ_t X(t)$, hence $X(t) + (1, -\epsilon, 0, \ldots, 0) \in F(t)$. Therefore $\bar{p} \cdot I(t) \lesssim \bar{p} \cdot [X(t) + (1, -\epsilon, 0, \ldots, 0)] = \bar{p} \cdot X(t) + p^1 - \epsilon p^2 \lnsim \bar{p} \cdot X(t)$, i.e. $\bar{p} \cdot I(t) \lnsim \bar{p} \cdot X(t)$ for all $t \in S$. Since $|S|/\omega \not\simeq 0$, this contradicts $\bar{p} \cdot X(t) \lesssim \bar{p} \cdot I(t)$ except for at most a negligible internal set of t . Therefore $\bar{p} \gnsim \bar{0}$.

Now suppose $\bar{x} \in F(t)$, $\mu(\bar{x}) \cap \mu(X(t)) = \emptyset$, and that $I(t) \gtrsim \bar{0}$; then $\bar{p} \cdot I(t) \gtrsim 0$ because $\bar{p} \gtrsim \bar{0}$. Since $\bar{p} \cdot \bar{x} \gtrsim \bar{p} \cdot I(t)$ it follows that $\bar{p} \cdot \bar{x} \gtrsim 0$, hence there is a j such that $x^j \gtrsim 0$; let $j = 1$, say. From continuity it then follows that $\bar{x} - (\epsilon, 0, \ldots, 0) \in F(t)$ for sufficiently small $\epsilon \gtrsim 0$. Then $\bar{p} \cdot I(t) \lesssim \bar{p} \cdot [\bar{x} - (\epsilon, 0, \ldots, 0)] = \bar{p} \cdot \bar{x} - \epsilon p' \lesssim \bar{p} \cdot \bar{x}$, i.e. $\bar{p} \cdot I(t) \lesssim \bar{p} \cdot \bar{x}$. If $I(t) \simeq \bar{0}$ and $\bar{x} \gtrsim \bar{0}$, then clearly $\bar{p} \cdot \bar{x} \gtrsim 0 \simeq \bar{p} \cdot I(t)$. Finally suppose $I(t) \simeq \bar{0}$ and $\bar{x} \simeq \bar{0}$, $\mu(\bar{x}) \cap \mu(X(t)) = \emptyset$. Since $\bar{x} \in F(t)$, this means that $I(t) \succ_t X(t)$, by continuity. If the set of traders t for whom this happens is negligible, then it can be ignored; if, on the other hand, it is non negligible, then $I(t)$ dominates X via S contradicting the membership of X in the core. This completes the proof of the theorem.

Let $\mathcal{A}$ be a countable family of large but finite economies, i.e. $\mathcal{A} = \{\mathcal{E}_n\}_{n=1}^{\infty}$, where for every n , $|\mathcal{E}_n| < |\mathcal{E}_{n+1}|$, $|\mathcal{E}_n| < \infty$, and $|\mathcal{E}_1| > 0$. $\mathcal{E}_n$ is completely specified by the initial endowments and preferences of the traders in $\mathcal{E}_n$. Hence let $\mathcal{E}_n = \{I_n(t), P_n(t)\}$ where for all n , $I_n(t) \in \Omega_m^{|\mathcal{E}_n|}$ and $P_n(t) \in \mathcal{P}(\Omega_m \times \Omega_m)^{|\mathcal{E}_n|}$. We shall assume that $\mathcal{E}$ has the following properties:

(1) There exists $\bar{r}_0 \in \Omega_m$ such that for all n and every $t \in \mathcal{E}_n$, $I_n(t) \leq \bar{r}_0$.

(2) $\liminf_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} I_n(t) > \bar{0}$.

(3) For all n and every $t \in \mathcal{E}_n$, we shall assume that

(α) $P_n(t)$ is irreflexive

(β) If $\bar{x} \geq \bar{y}$, then $\bar{x} P_n(t) \bar{y}$

(γ) For all $\{\bar{x}_n\}_1^\infty$, $\{\bar{y}_n\}_1^\infty$, $\{t_n\}_1^\infty$, if $\liminf_{n \to \infty} |\bar{x}_n - \bar{y}_n| > 0$ and there exists an n_0 such that for all $n \geq n_0$, $\bar{x}_n P_n(t_n)\bar{y}_n$ then there exists a $\delta > 0$ such that if $\bar{w}_n \in S(\bar{x}_n, \delta)$ then there exists an n_1 such that $n \geq n_1$ implies $\bar{w}_n P_n(t_n)\bar{y}_n$.

Let $\mathcal{H}$ be any subsequence of economies of $\mathcal{G}$, then the following notions are defined with respect to $\mathcal{H}$.

$\{Y_n(t)\}_{n=1}^\infty$ is an allocation for $\mathcal{H}$, if for all n, $Y_n(t) \in \Omega_m^{|\mathcal{E}_n|}$, and there exists an $\bar{r}_1 \in \Omega_m$ such that for all n and every $t \in \mathcal{E}_n$, $Y_n(t) \leq \bar{r}_1$. Also $\liminf_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} I_n(t) - \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} Y_n(t) = \bar{0}$.

$\{\mathcal{S}_n\}_1^\infty$ is a coalition of $\mathcal{H}$, if for all n, $\mathcal{S}_n \subseteq \mathcal{E}_n$. $\{\mathcal{S}_n\}_1^\infty$ is said to be negligible if $\liminf_{n \to \infty} \frac{|\mathcal{S}_n|}{|\mathcal{E}_n|} = 0$. Note that if $\{\mathcal{S}_n\}_1^\infty$ is negligible, then for all allocations $\{Y_n(t)\}_1^\infty$ $\liminf_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} Y_n(t) = 0$.

A coalition, $\{\mathcal{S}_n\}_1^\infty$, is effective for an allocation $\{Y_n(t)\}_1^\infty$ if $\liminf_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \ _n} I_n(t) - \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} Y_n(t) = \bar{0}$.

An allocation, $\{Y_n(t)\}_1^\infty$ dominates an allocation $\{X_n(t)\}_1^\infty$ via a coalition, $\{\mathcal{S}_n\}_1^\infty$, if for all $\{t_n\}_1^\infty$ such that for every n, $t_n \in \mathcal{S}_n$, implies $\liminf_{n \to \infty} |Y_n(t_n) - X_n(t_n)| > 0$ and there exists an n_0 such that for all $n \geq n_0$, $Y_n(t_n)P_n(t_n)X_n(t_n)$; and $\{\mathcal{S}_n\}_1^\infty$ is effective for $\{Y_n(t)\}_1^\infty$.

The <u>core of</u> $\mathcal{H}$ is the set of all allocations which are not dominated via any non negligible coalition.

An allocation for $\mathcal{G}$ is said to be in the <u>core of</u> $\mathcal{G}$ if for every subsequence of economies, $\mathcal{H}$, the allocation for $\mathcal{G}$ restricted to $\mathcal{H}$ is in the core of $\mathcal{H}$.

$\{\bar{p}_n\}_1^\infty$ is a price vector if for all n, $\bar{p}_n \in \Omega_m$, and $\liminf_{n \to \infty} \bar{p}_n > 0$.

$\langle\{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty\rangle$ is a <u>competitive equilibrium for</u> $\mathcal{H}$ if the following conditions hold:

(α) $\{X_n(t)\}_1^\infty$ is an allocation.

(β) $\{\bar{p}_n\}_1^\infty$ is a price vector.

(γ) There does not exist a coalition $\{\mathcal{S}_n\}_1^\infty$ such that $\liminf_{n \to \infty} \frac{|\mathcal{S}_n|}{|\mathcal{E}_n|} > 0$ and for all $\{t_n\}_{n=1}^\infty$ such that $t_n \in \mathcal{S}_n$, there exist $\{\bar{y}_n\}_1^\infty$, $\bar{y}_n \in \Omega_m$, for which $\liminf_{n \to \infty} \bar{p}_n \cdot I_n(t_n) - \bar{p}_n \cdot \bar{y}_n \geq 0$, $\liminf_{n \to \infty} |\bar{y}_n - X_n(t_n)| > 0$, and for some n_0, $\bar{y}_n P_n(t_n) X_n(t_n)$ for all $n \geq n_0$.

$\langle\{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty\rangle$ is a <u>competitive equilibrium for</u> $\mathcal{G}$ if $\langle\{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty\rangle$ restricted to $\mathcal{H}$, where $\mathcal{H}$ is any subsequence of economies of $\mathcal{G}$, is a competitive equilibrium for $\mathcal{H}$.

Since $\mathcal{G}$ is a pair of sequences of functions, $\{I_n(t)\}_1^\infty$ and

$\{P_n(t)\}_{n=1}^{\infty}$, we may define $^*\mathcal{Y}$ as the nonstandard extensions of these two sequences. Hence $^*\mathcal{Y} = \langle\{^*I_n(t)\}_1^{\infty}, \{^*P_n(t)\}_1^{\infty}\rangle$ where n now ranges over the nonstandard integers, $^*N$. For each infinite integer $\omega \in {}^*N - N$ we define the nonstandard exchange economy $\mathcal{E}_\omega = \{^*I_\omega(t), {}^*P_\omega(t)\}$.

<u>Lemma (6)</u>. $\{X_n(t)\}_1^{\infty}$ is in the core of $\mathcal{Y}$, $\mathcal{C}(\mathcal{Y})$, if and only if for all infinite integers ω , $X_\omega(t)$ is in the core of $\mathcal{E}_\omega$, $\mathcal{C}(\mathcal{E}_\omega)$.

<u>Proof</u>: Suppose there exists an infinite integer ω such that $X_\omega(t) \notin \mathcal{C}(\mathcal{E}_\omega)$. Then there exists an allocation $Y_\omega(t)$ where $\frac{1}{\omega} \sum_{t\in\mathcal{E}_\omega} Y_\omega(t) \simeq \frac{1}{\omega} \sum_{t\in\mathcal{E}_\omega} I_\omega(t)$, a coalition $\mathcal{A}_\omega \subseteq \mathcal{E}_\omega$ where for all $t \in \mathcal{A}_\omega$, $Y_\omega(t) P_\omega(t) X_\omega(t)$, $\frac{1}{\omega} \sum_{t\in\mathcal{A}_\omega} Y_\omega(t) \simeq \frac{1}{\omega} \sum_{t\in\mathcal{A}_\omega} I_\omega(t)$, and there exists $\epsilon_1, \epsilon_2 \gtrsim 0$ such that for every $t \in \mathcal{A}_\omega$, $|Y_\omega(t) - X_\omega(t)| > \epsilon_1$, and $|\mathcal{A}_\omega|/|\mathcal{E}_\omega| \geq \epsilon_2$. Therefore for all $n \in N$ and every positive $\delta \in R$, the following sentence is true in our nonstandard universe, $^*U$: $(\exists \nu \in {}^*N)(\exists Y_\nu \in {}^*\Omega_m^{|\mathcal{E}_\nu|})(\exists \mathcal{A}_\nu \subseteq \mathcal{E}_\nu)[\nu > n \wedge |\frac{1}{\nu} \sum_{t\in\mathcal{E}_\nu} Y_\nu(t) - \frac{1}{\nu} \sum_{t\in\mathcal{E}_\nu} I_\nu(t)| < \delta \wedge |\frac{1}{\nu} \sum_{t\in\mathcal{A}_\nu} Y_\nu(t) - \frac{1}{\nu} \sum_{t\in\mathcal{A}_\nu} I_\nu(t)| < \delta \wedge (\forall t \in \mathcal{A}_\nu)\{Y_\nu(t) P_\nu(t) X_\nu(t) \wedge |Y_\nu(t) - X_\nu(t) \geq \epsilon_1\} \wedge |\mathcal{A}_\nu|/|\mathcal{E}_\nu| \geq \epsilon_2]$. Hence this sentence is true when translated in U , our standard universe. Therefore for every $n \in N$ and $\delta \in R$, $\delta > 0$, there exists $m \in N$, $Y_m \in \Omega_m^{|\mathcal{E}_m|}$ and $\mathcal{A}_m \subseteq |\mathcal{E}_m|$ such that $m > n$ $|\frac{1}{m} \sum_{t\in\mathcal{E}_m} Y_m(t) - \frac{1}{m} \sum_{t\in\mathcal{E}_m} I_m(t)| < \delta \wedge |\frac{1}{m} \sum_{t\in\mathcal{A}_m} Y_m(t) - \frac{1}{m} \sum_{t\in\mathcal{A}_m} I_m(t)| \wedge (\forall t \in \mathcal{A}_m)\{Y_m(t) P_m(t) X_m(t) \wedge$

$|Y_m(t) - X_m(t)| \geq \epsilon_1\} \wedge |\mathcal{A}_m|/|\mathcal{E}_m| \geq \epsilon_2$.

Consequently there exists a subsequence of economies, $\mathcal{H}$, which has the allocation $\{Y_m(t)\}_{m=1}^{\infty}$ and a non negligible coalition $\{\mathcal{A}_m\}_1^{\infty}$ such that $\{Y_m(t)\}_{m=1}^{\infty}$ dominates $\{X_n(t)\}_1^{\infty}$ via $\{\mathcal{A}_m\}_1^{\infty}$, hence $\{X_n(t)\}_1^{\infty} \notin \mathcal{C}(\mathcal{Y})$.

Suppose $\{X_n(t)\}_1^{\infty} \notin \mathcal{C}(\mathcal{Y})$, then there exist subsequences $\{Y_m(t)\}_1^{\infty}$ and $\{\mathcal{A}_m\}_1^{\infty}$ such that $\liminf_{m \to \infty} \frac{1}{|\mathcal{E}_m|} \sum_{t \in \mathcal{A}_m} Y_m(t) - \sum_{t \in \mathcal{E}_m} I_m(t) = 0$, $\liminf_{m \to \infty} \frac{1}{|\mathcal{E}_m|} \sum_{t \in \mathcal{A}_m} Y_m(t) - \frac{1}{|\mathcal{E}_m|} \sum_{t \in \mathcal{A}_m} I_m(t) = 0$, $\liminf_{n \to \infty} |\mathcal{A}_m|/|\mathcal{E}_m| \neq 0$, and for all $\{y_m\}_1^{\infty}$ such that for every m , $t_m \in \mathcal{A}_m$, $\liminf_{n \to \infty} |Y_m(t_m) - X_m(t_m)| > 0$, and there exists m_0 , such that for all $m \geq m_0$, $Y_m(t_m) P_m(t_m) X_m(t_m)$. Hence there exists an infinite integer ω for which the following statements hold about the nonstandard extensions of $\{X_n(t)\}_1^{\infty}$, $\{Y_n(t)\}_1^{\infty}$, $\{P_n(t)\}_1^{\infty}$, $\{\mathcal{A}_m\}_1^{\infty}$ and $\{I_n(t)\}_1^{\infty}$: $\frac{1}{\omega} \sum_{t \in \mathcal{E}_\omega} X_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{E}_\omega} I_\omega(t)$, $\frac{1}{\omega} \sum_{t \in \mathcal{E}_\omega} Y_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{E}_\omega} I_\omega(t)$, $\frac{1}{\omega} \sum_{t \in \mathcal{A}_\omega} Y_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{A}_\omega} I_\omega(t)$, $|\mathcal{A}_\omega|/|\mathcal{E}_\omega| \not\simeq 0$, for all $t \in \mathcal{A}_\omega$, $\mu(Y_\omega(t)) \cap \mu(X_\omega(t)) \neq 0$ and $Y_\omega(t) P_\omega(t) X_\omega(t)$. Hence $X_\omega(t) \notin \mathcal{C}(\mathcal{E}_\omega)$.

<u>Lemma (7)</u>. $<\{X_n(t)\}_1^{\infty}, \{\bar{p}_n\}_1^{\infty}>$ is a competitive equilibrium for $\mathcal{Y}$ if and only if for all infinite integers ω , $<X_\omega(t), \bar{p}_\omega>$ is a competitive equilibrium for $\mathcal{E}_\omega$.

Proof: Suppose there exists an infinite integer $\omega \in N^* - N$ such that $\langle X_\omega(t), \bar{p}_\omega \rangle$ is not a competitive equilibrium for $\mathcal{E}_\omega$. Then there exists an internal set of traders, $\mathcal{A}_\omega$, and $\epsilon_1, \epsilon_2 \gtrsim 0$ such that $|\mathcal{A}_\omega|/|\mathcal{E}_\omega| \geq \epsilon_2$, and for all $t \in \mathcal{A}_\omega$, there exists a $\bar{y}_t \in B_{\bar{p}}(t)$ such that $\bar{y}_t P_\omega(t) X_\omega(t)$ and $|\bar{y}_t - X_\omega(t)| \geq \epsilon_1$. Hence for all $n \in N$ and for all $\delta \in R$, $\delta > 0$, the following sentence is true in $^*U$, our nonstandard universe:

$$(\exists \nu \in {}^*N)(\exists \bar{p}_\nu \in {}^*\Omega_m)(\exists \mathcal{A}_\nu \subseteq \mathcal{E}_\nu)[\nu > n \wedge (\forall t \in \mathcal{A}_\nu)(\exists \bar{y}_t \in {}^*\Omega_m)\{(|\bar{p}_\nu \cdot \bar{y}_t - \bar{p}_\nu \cdot I_\nu(t)| < \delta \vee \bar{p}_\nu \cdot \bar{y}_t \leq \bar{p}_\nu \cdot I_\nu(t)) \wedge |\bar{y}_t - X_\nu(t)| \geq \epsilon_1 \wedge \bar{y}_t P_\nu(t) X_\nu(t)\} \wedge |\mathcal{A}_\nu|/|\mathcal{E}_\nu| \geq \epsilon_2] .$$

Therefore this sentence is true when translated in U, our standard universe. Hence for every $n \in N$ and $\delta \in R$, $\delta > 0$, there exists $m > n$, $\bar{p}_m \in \Omega_m$, $\mathcal{A}_m \subseteq \mathcal{E}_m$ such that $m > n \wedge (\forall t \in \mathcal{A}_m)(\exists \bar{y}_t \in \Omega_m)\{(|\bar{p}_m \cdot \bar{y}_t - \bar{p}_m \cdot I_m(t)| < \delta \vee \bar{p}_m \cdot \bar{y}_t \leq \bar{p}_m \cdot I_m(t)) \wedge |\bar{y}_t - X_m(t)| \geq \epsilon_1 \wedge \bar{y}_t P_m(t) X_m(t)\} \wedge |\mathcal{A}_m|/|\mathcal{E}_m| \geq \epsilon_2$.

Consequently there exists a subsequence of economies, $\mathcal{H}$, such that $\langle \{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty \rangle$ restricted to $\mathcal{H}$ is not a competitive equilibrium for $\mathcal{H}$, hence $\langle \{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty \rangle$ is not a competitive equilibrium for $\mathcal{G}$.

Suppose $\langle \{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty \rangle$ is not a competitive equilibrium for $\mathcal{G}$, then there exists subsequences $\{X_m(t)\}_1^\infty$, $\{\bar{p}_m\}_1^\infty$, a sequence $\{\mathcal{A}_m\}_1^\infty$ and $\epsilon > 0$ such that $\liminf_{m \to \infty} \frac{|\mathcal{A}_m|}{|\mathcal{E}_m|} \geq \epsilon$ and for all $\{t_m\}_1^\infty$, where $t_m \in \mathcal{A}_m$, there exists $\{\bar{y}_m\}_1^\infty$, where $\bar{y}_m \in \Omega_m$, for which $\liminf_{n \to \infty} |\bar{y}_m - X_m(t_m)| > 0$, $\liminf_{n \to \infty} \bar{p}_m \cdot I_m(t_m) - \bar{p}_m \cdot \bar{y}_m \geq 0$, and for some

m_0, $\bar{y}_m P_m(t_m) X_m(t_m)$ for all $m \geq m_0$. Hence there exists an infinite integer ω for which the following statements hold about the nonstandard extensions of $\{X_n(t)\}_1^\infty$, $\{I_n(t)\}_1^\infty$, $\{\mathcal{A}_m\}_1^\infty$, $\{P_n(t)\}_1^\infty$: $\frac{1}{\omega} \sum_{t \in \mathcal{E}_\omega} I_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{E}_\omega} X_\omega(t)$, $|\mathcal{A}_\omega| / |\mathcal{E}_\omega| \not\simeq 0$, for all $t \in \mathcal{A}_\omega$, there exists $\bar{y}_t \in B_{\bar{p}_\omega}(t)$ such that $\mu(\bar{y}_t) \cap \mu(X_\omega(t)) = \emptyset$ and $\bar{y}_t P_\omega(t) X_\omega(t)$. Hence $\langle X_\omega(t), \bar{p}_\omega \rangle$ is not a competitive equilibrium for $\;_\omega$.

<u>Theorem (2)</u>. $\{X_n(t)\}_1^\infty$ is in the core of $\mathcal{H}$ if and only if there exists $\{\bar{p}_n\}_1^\infty$ such that $\langle \{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty \rangle$ is a competitive equilibrium for $\mathcal{H}$.

<u>Proof</u>: We need only show that every $\mathcal{E}_\omega$ is a nonstandard exchange economy satisfying the assumptions of Theorem (1). Then the proof is an immediate consequence of Lemmas (6) and (7). All the assumptions are obviously met except possibly (iv-γ), which we shall show holds also. Suppose (iv-γ) is false in some $\;_\omega$, then there exists a trader t, $\bar{x}, \bar{y}, \bar{x}' \in {}^*\Omega_m$ and an $\epsilon \underset{\not\simeq}{>} 0$ such that $|\bar{x} - \bar{y}| > \epsilon$, $\bar{x}' \in \mu(\bar{x})$, $\bar{x} \succ_t \bar{y}$, and $\bar{x}' \not\succ_t \bar{y}$. Therefore for all $n \in N$ and every positive $\delta \in R$ the following sentence is true in our nonstandard universe ${}^*U$: $(\exists \nu \in {}^*N)(\exists t \in \mathcal{E}_\nu)(\exists \bar{x}, \bar{y}, \bar{x}' \in {}^*\Omega_m)[\nu > n \wedge |\bar{x} - \bar{y}| > \epsilon \wedge |\bar{x} - \bar{x}'| < \delta \wedge \bar{x} \succ_t \bar{y} \quad \bar{x}' \not\succ_t \bar{y}]$. Hence this sentence is true when translated in U, our standard universe. Therefore for every $n \in N$, $\delta \in R$, $\delta > 0$, there exists $m \in N$, $t_m \in \mathcal{E}_m$, $\bar{x}_m, \bar{y}_m, \bar{x}_m' \in \Omega_m$ such that $|\bar{x}_m - \bar{y}_m| > \epsilon$, $|\bar{x}_m - \bar{x}_m'| < \delta$, $\bar{x}_m P_m(t_m) \bar{y}_m$ and $\bar{x}_m'$ not preferred to $\bar{y}_m$. But this contradicts our assumption (3-γ) about the preferences of $\mathcal{H}$. To complete the proof suppose $\langle \{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty \rangle$

is a competitive equilibrium for $\mathcal{H}$ then by Lemma (7) $\langle X_\omega(t), \bar{p}_\omega\rangle$ is a competitive equilibrium for $\mathcal{E}_\omega$ for every infinite integer ω. Hence by Theorem (1) for all infinite ω, X_ω is in the core of $\mathcal{E}_\omega$ and consequently by Lemma (6) $\{X_n(t)\}_1^\infty$ is in the core of $\mathcal{H}$. Suppose $\{X_n(t)\}_1^\infty$ is in the core of $\mathcal{H}$, then by Lemma (6) $X_\omega(t)$ is in the core of $\mathcal{E}_\omega$ for all infinite integers ω. Hence by Theorem (1), there exists $\bar{p}_\omega$ such that $\langle X_\omega(t), \bar{p}_\omega\rangle$ is a competitive equilibrium for $\mathcal{E}_\omega$ for every infinite integer ω. Consequently for every positive $\delta_1, \delta_2 \in R$ the following sentence is true in $^*U$: $(\exists\nu \in {}^*N)(\forall\theta \in {}^*N)[\theta \geq \nu \Longrightarrow (\exists\bar{p}_\theta \in {}^*\Omega_m)[(\forall\mathcal{S}_\theta \subseteq \mathcal{E}_\theta)\{(\forall t \in \mathcal{S}_\theta)(\exists\bar{y}_t \in {}^*\Omega_m)\{[\bar{p}_\theta\cdot\bar{y}_t \leq \bar{p}_\theta\cdot I_\theta(t) \vee |\bar{p}_\theta\cdot\bar{y}_t - \bar{p}_\theta\cdot I_\theta(t)| < \delta_2] \wedge \bar{y}_t P_\theta(t) X_\theta(t) \wedge |\bar{y}_t - X_\theta(t)| \geq \delta_1 \Longrightarrow |\mathcal{S}_\theta|/|\mathcal{E}_\theta| < \delta_2]]$. Translating this sentence in U for $\delta_1 = \delta_2 = \frac{1}{n}$, $n = 1, 2, \ldots,$ we generate a sequence of prices $\{\bar{p}_j\}_1^\infty$ and integers $\{n_j\}_{j=1}^\infty$ where n_j is the first integer such that the sentence is true for $\delta_1 = \delta_2 = \frac{1}{j}$, $\bar{p}_j$ is a corresponding price, and $n_j < n_{j+1}$. Let $n_0 = 0$ and assign economies $n_j + 1$ through n_{j+1} prices $\bar{p}_j$ where $j = 0, 1, 2, \ldots$. We claim that $\langle\{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty\rangle$ is a competitive equilibrium for $\mathcal{H}$. Suppose not then there exists $\{\mathcal{S}_n\}_1^\infty$, $\{t_n\}_1^\infty$, $\{\bar{y}_n\}_1^\infty$ such that $\liminf_{n\to\infty} \frac{|\mathcal{S}_n|}{|\mathcal{E}_n|} > 0$, for all n, $t_n \in \mathcal{S}_n$ and $\bar{y}_n \in \Omega_m$, and $\liminf_{n\to\infty} \bar{p}_n\cdot I_n(t_n) - \bar{p}_n\cdot\bar{y}_n \geq 0$, $\liminf_{n\to\infty} |\bar{y}_n - X_n(t_n)| > 0$, for some n_0, $\bar{y}_n P_n(t_n) X_n(t_n)$ for all $n \geq n_0$. But for all j such that $\frac{1}{j} < \min\{\liminf_{n\to\infty} \frac{|\mathcal{S}_n|}{|\mathcal{E}_n|}, \liminf_{n\to\infty} \bar{p}_n\cdot I_n(t_n) - \bar{p}_n\cdot\bar{y}_n,$ $\liminf_{n\to\infty} |\bar{y}_n - X_n(t_n)|\}$ and $j > n_0$ this contradicts the properties of

$\langle X_{n_j}(t), \bar{p}_{n_j} \rangle$ for economy $\mathcal{E}_{n_j}$.

Theorem (3). If all but a finite number of the economies of $\mathcal{U}$ have a competitive equilibrium and $\liminf_{n \to \infty} I_n(t) > \bar{0}$, then each nonstandard economy of $^*\mathcal{U}$ has a competitive equilibrium.

Proof: Let $\langle \{X_n(t)\}_1^\infty, \{\bar{p}_n\}_1^\infty \rangle$ be a sequence of competitive equilibria and suppose there exists an $\eta \in {}^*N$ such that $\langle X_\eta(t), \bar{p}_\eta \rangle$ is not a competitive equilibrium for $\mathcal{E}_\eta$. Hence there exists a trader t and a $\bar{y}_t \in {}^*\Omega_m$ such that $\bar{y}_t \succ_t X_\eta(t)$, $\mu(\bar{y}_t) \cap (X_\eta(t)) = \emptyset$ and $\bar{p}_\eta \cdot \bar{y}_t \lesssim \bar{p}_\eta \cdot I_\eta(t)$. Suppose $\bar{p}_\eta \cdot \bar{y}_t = \bar{p}_\eta \cdot I_\eta(t) + \epsilon$ where $\epsilon \approx 0$ and positive. $I_\eta(t) \not\approx 0$ and $\sum_{i=1}^n p_\eta^i = 1$, hence $\bar{p}_\eta \cdot I_\eta(t) \not\approx 0$. Therefore there exists $\bar{w}_t \in \mu(\bar{y}_t)$ such that $\bar{p}_\eta \cdot \bar{w}_t = \bar{p}_\eta \cdot I_\eta(t)$ and $\bar{w}_t \succ_t X_\eta(t)$. Hence the following sentence is true in $^*U$

$$: \quad (\exists \nu \in {}^*N)(\exists t \in \mathcal{E}_\nu)(\exists \bar{w}_t \in {}^*\Omega_m)(\bar{p}_\nu \cdot \bar{w}_t = \bar{p}_\nu \cdot I_\nu(t) \wedge \bar{w}_t \succ_t X_\eta(t))$$

. Therefore this sentence is true when translated into U our standard universe. That is there exists $n \in N$, $t \in T$, $\bar{y}_t \in \Omega_m$ such that $\bar{p}_n \cdot \bar{y} = \bar{p}_n \cdot I_n(t)$ and $\bar{y}_t \succ_t X_n(t)$ which contradicts the definition of $X_n(t)$ as a competitive allocation.

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