COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 306

ON TRANSITIONS BETWEEN STEADY STATES

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January 18, 1971

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by

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1. The purpose of this paper is to prove the following theorem and discuss its implications:

Consider an economy of the usual activity analysis variety with a single primary factor, labor, and no joint production, which has available to itself two alternative technologies (i.e. specifications of the input-output matrix), denoted A and B . Assume the economy is in steady state using technology A. We shall prove that there exists an efficient[1] transition path of finite duration from the steady state using A to the steady state using B, along which all capital goods and labor are fully utilized at every point of time.[2]

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*The research described in this paper was supported by grants from the Guggenheim Foundation, the National Science Foundation, and the Ford Foundation. I should like to acknowledge the helpful comments of A.B. Atkinson, C.J. Bliss, F.H. Hahn, and A.K. Klevorick.

[1]Here as elsewhere, we use the word efficient in the conventional Phelps-Koopmans sense: it is impossible for the economy to increase its consumption of any commodity in any period without decreasing consumption of some commodity in some period.

[2]For a precise statement of the model and the assumptions underlying the analysis, the reader is referred to section 4 below.

2. The importance of this result is two-fold:

(a) It has often been suggested that the assumption in conventional neoclassical analysis of malleable capital is a crucial, but very unrealistic assumption. The economy inherits a particular vector of capital goods appropriate to one technology and except under very peculiar assumptions (Swan's[1] meccanno sets) these cannot be transformed into the vector of capital goods appropriate to another technology. Our theorem shows that even though capital goods cannot be instantaneously transformed one into another, i.e., capital is not malleable in the very short run, given enough time, the economy can transform its original vector of capital goods into the desired pattern; <u>the economy exhibits malleability in the long run</u>, even with a very limited degree of substitutibility in production (the availability of only two techniques). Moreover, this transformation can take place without any resources being wasted (unused) and in a <u>finite</u> number of periods. This, it seems to me, is exactly the meaning of malleability that most neoclassical growth theorists have in mind when they speak of "malleable capital."

(b) As a corollary of our theorem, we are able to establish the following result: Let r^* be a rate of interest at which the two technologies are competitive (r is often referred to as a switch point), i.e., the <u>steady state</u> relative prices at r^* corresponding to technology A are identical to those corresponding to B. Assume there is a single consumption good.

[1] T. Swan, "Economic Growth and Capital Accumulation," <u>Economic Record</u>, November 1956.

Assume the economy is in steady state using A . Let $\hat{c}$ denote the consumption vector (so $\hat{c}_t$ is consumption at time t) for any path beginning with the initial endowments appropriate to steady state A , going to steady state B in a finite number of periods, fully utilizing all capital goods and labor along the way, and remaining in steady state B thereafter. (Our theorem assures us of the existence of at least one such path.) Define δ as the solution(s) to

$$(1) \qquad \frac{c^*(A)}{\delta} \equiv \sum_{t=1}^{\infty} c^*(A)\left(\frac{1}{1+\delta}\right)^t = \sum_{t=1}^{\infty} \hat{c}_t\left(\frac{1}{1+\delta}\right)^t$$

where $c^*(A)$ is the steady state consumption of A . δ is conventionally called the rate of return to adopting technology B .[1] Note that it is a purely technological concept. No use of prices has been made. No behavioral assumption (such as perfect competition) has been employed.

There is an alternative way of writing the rate of return in those cases where the transition from one steady state to another may be made in one period:

$$(2) \qquad \delta = \frac{c^*(B) - c^*(A)}{c^*(A) - \hat{c}_1}$$

where $c^*(B)$ is the steady state consumption for technology B , and $\hat{c}_1$

[1] For an extensive discussion of the concept of rate of return, see R.M. Solow, Capital Theory and the Rate of Return (Amsterdam, 1963). See also I. Fisher, The Rate of Interest (New York, 1906).

is the consumption during the single transition period. Note that, in fact, (2) is a special case of (1). The interpretation of (2) is clear: δ equals the ratio of the steady state permanent increase (decrease) in consumption to the one period loss (gain) in going from A to B. In general, one period transitions which fully utilize resources are not feasible, and hence the necessity to use the more general formulation (1).

We can show then that if δ_i is the i^{th} solution to (1) either (a) $\delta_i = r^*$, the rate of return is equal to the rate of interest at a switch point or (b) at $r = \delta_i$, there is some other technology whose steady state prices are less than those of B or A (i.e., if A and B were the only technologies, then it would be a switch point, but there is some other technology which, at the given interest rate, dominates A and B).

3. Our analysis reveals several further properties of particular interest to optimal growth theory.

(a) The rate(s) of return in going from A to B are independent of the path chosen, provided that all resources are fully utilized.

(b) Thus, the solution to the optimal growth problem

$$\max \sum_{1}^{\infty} \frac{c_t}{(1+\delta)^t}$$

may not be unique; for instance, starting in steady state A any path which maintains full employment of resources and uses only technologies A and B is optimal if $\delta = r^*$ (the interest rate at a switch point).

(c) On the other hand, the asymptotic solution to the optimal growth problem

$$\max \sum_{1}^{\infty} \frac{U(c)}{(1+\delta)^t}, \quad U'' < 0$$

depends on initial conditions. If $\delta = r^*$, if the economy starts in steady state in A, it remains there, regardless of the form of $U(c)$, and if it starts in steady state in B, it remains there. (In contrast, in the simpler growth models analyzed thus far, although the path of the economy in the short run does depend on initial conditions, the asymptotic trajectory does not.)[1] History is important.

(d) There may exist paths going from A to B which are efficient and which do not fully utilize resources at every point of time. The rate of return for such paths will not in general be equal to the rate of interest at the switch point. These paths may even be optimal.

(e) There exist efficient (and even optimal) paths along which one set of techniques is used over an interval of time, followed by a period in which another set of techniques is used, followed by still a third period in which the first set of techniques are used.

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[1] An exception to this generalization is the Koopmans-Beals model (Tjalling C. Koopmans and Richard Beals, "Maximizing Stationary Utility in a Constant Technology," SIAM J. Appl. Math., Vol. 17, No. 5, 1969) with a stationary but non-additive utility function). Koopmans has obtained similar results in unpublished work analyzing stationary states in linear models. In descriptive growth theory, see A.B. Atkinson and J.E. Stiglitz, "A New View of Technical Change" (Economic Journal, September 1969) for a theory of economic growth in which history matters even asymptotically.

4. Our central theorem is an immediate consequence of two lemmas which we establish in this and the next section. First, however, we set forth our model and its assumptions somewhat more explicitly that we have thus far.

We assume there are $n+1$ commodities. One or more of these commodities can be used for consumption. For simplicity, we consider in detail only the case where there is a single consumption good, the $n+1^{st}$ commodity.[1, 2] Commodities $1, \ldots, n$ can only be used as capital goods. The extension of our argument to the general case is easy. A technology is characterized then by its $n+2 \times n+1$ matrix of input requirements per unit of output. a_{ij} (b_{ij}) is the requirement of the i^{th} commodity per unit of output of the j^{th} commodity. a_{oj} (b_{oj}) is the requirement of labor per unit of output. We assume no joint production and a single primary (non-produced) factor labor, the supply of which is constant in every period and which we normalize at unity.[3] For simplicity, we assume that all capital is circulating. The results can easily be extended to the case of fixed capital goods under the not unreasonable assumption that all capital goods live only a finite number of periods. We make only two further (conventional) assumptions about our technology: (a) We assume that both A and B are viable, i.e., there exists a vector x_A such that

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[1]It makes no difference to the analysis whether this good can or cannot be used for production. For generality, we shall, in Section 5, assume that it can be.

[2]The other polar case, where every capital good can be used as a consumption good, has been investigated in detail by Solow in "The Interest Rate and Transition Between Techniques," in C.H. Feinstein, ed., Socialism, Capitalism and Economic Growth (Cambridge University Press, 1967). In that case Solow has actually been able to exhibit full employment transition paths although the transition does not occur in a finite number of periods.

[3]Both of these assumptions are made for expositional convenience and may be dropped.

$$Ax_A < x_A$$

Similarly, there exists a vector X_B such that

$$Ax_B < x_B$$

(b) We assume that every capital good used in B can be produced indirectly or directly by capital goods used in A ; i.e., a transition from B to A is feasible.[1] In our proof, we shall make a slightly stronger assumption--that in fact the commodities used and produced in technology A are identical to those used and produced in technology B . The analysis may again easily be extended, as we indicate in a footnote at the appropriate point.

We are now ready to state and prove

Lemma 1. There exists a path going in a finite number of periods from the steady state corresponding to technology A to that corresponding to technology B along which resources are fully utilized.

We define $[x^*(\lambda), c^*(\lambda)]$ as the steady state vector of outputs of capital goods and consumption corresponding to the technology[2,3]

$$D(\lambda) \equiv \lambda B + (1-\lambda)A \qquad 0 \leq \lambda \leq 1$$

$$d_0(\lambda) \equiv \lambda b_0 + (1-\lambda)a_0$$

where $a_0(b_0)$ is the vector of labor requirements per unit of output. Thus

(3a) $$[\lambda B + (1-\lambda)A]x^*(\lambda) = x^*(\lambda) - c^*(\lambda)$$

(3b) $$[\lambda b_0 + (1-\lambda)a_0]x^*(\lambda) = 1$$

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[1] This is analogous to the assumption in the turnpike literature of the "recoverability property."

[2] For the analysis, it makes no difference whether we consider λ to be a scalar or a vector. For simplicity we take it to be a scalar.

[3] (x^*, c^*) has n+2 elements, $x_1^* \ldots x_{n+1}^*$ and c^* .

Thus $(x^*(0),\ c^*(0))$ is the steady state vector of output of capital goods and the level of consumption good corresponding to technology A and $(x^*(1),\ c^*(1))$ is the steady state vector corresponding to technology B. For convenience, we shall also denote these two steady state vectors by $(x^*(A),\ c^*(A))$ and $(x^*(B),\ c^*(B))$, respectively.

By assumption $(x^*(0),\ c^*(0))$ and $(x^*(1),\ c^*(1))$ are strictly positive, and it is easy to establish that this implies that $(x^*(\lambda),\ c^*(\lambda))$ is strictly positive.

If x_{it} is the output of commodity i at time t and c_t consumption at time t, then, assuming full employment of all resources, the dynamics of the economy may be described by[1]

$$[\lambda_t B + (1-\lambda_t)A]x_t + c_{t-1} = x_{t-1} \tag{4}$$

$$[\lambda_t b_0 + (1-\lambda_t)a_0]x_t = 1 \tag{5}$$

The last equation can be solved for

$$x_{n+1,\,t} = \frac{1 - \sum_{i=1}^{n} x_{it}(\lambda_t b_{0i} + (1-\lambda_t)a_{0i})}{\lambda_t b_{0n+1} + (1-\lambda_t)a_{0n+1}} \tag{6}$$

and substituted into (4) to obtain for the n capital goods

$$\hat{D}(\lambda_t)x_t = x_{t-1} + e(\lambda_t) \tag{7a}$$

and for consumption

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[1] If λ is considered a vector, then (5) should read $\Sigma[\lambda_{it}b_{0i} + (1-\lambda_{it})a_{0i}]x_{it} = 1$. Similarly, we must modify the definition of $\hat{d}_{ij}(\lambda)$ below.

(7b) $$c_{t-1} = x_{n+1,t-1} - \Sigma\, \hat{d}_{n+1,i}(\lambda_t)x_{it} + e_{n+1}(\lambda_t)$$

where

$$\hat{d}_{ij}(\lambda_t) \equiv \lambda_t b_{ij} + (1-\lambda_t)a_{ij} - \frac{(\lambda_t b_{in+1} + (1-\lambda_t)a_{in+1})(\lambda_t b_{0j} + (1-\lambda_t)a_{0j})}{\lambda_t b_{0n+1} + (1-\lambda_t)a_{0n+1}}$$

$$e_i \equiv -\frac{\lambda_t b_{in+1} + (1-\lambda_t)a_{in+1}}{\lambda_t b_{0n+1} + (1-\lambda_t)a_{0n+1}}$$

Given x_t, we can solve for x_0 as a function of $(\lambda_1, \ldots, \lambda_t)$; in particular, let ψ_i be the function giving the value of x_{i0} defined by (7) when $\lambda_1, \ldots, \lambda_{n+2}$ and x_{n+2} are given:

(8) $$x_{i0} = \psi_i(\lambda_1, \ldots, \lambda_{n+2}; x_{n+2})$$

and similarly let (6) define ψ_{n+1}.

It is easy to establish that there exists a set of

$$\{\hat{\lambda}_1, \ldots, \hat{\lambda}_\nu\}, \quad 0 = \hat{\lambda}_1 < \hat{\lambda}_2 < \ldots < \hat{\lambda}_{\nu-1} < \hat{\lambda}_\nu = 1$$

such that if

$$x_{n+2} = x^*(\hat{\lambda}_{i+1})$$

and

$$\hat{\lambda}_i \leq \lambda_t \leq \hat{\lambda}_{i+1}$$

then, in equations (6) and (7),

$$x_t > 0, \quad c_t > 0$$

for $0 \leq t \leq n+2$.

This follows from observing that (6) and (7) are simply linear equations, and if $\lambda_t = \hat{\lambda}_i$, all t , $x_t > 0$, $c_t > 0$ [1] and then using straightforward continuity arguments.[2]

We now wish to establish that if $x_{n+2} = x(\hat{\lambda}_{i+1})$ then there exists a set of $\{\lambda_1, \ldots, \lambda_{n+2}\}$ such that $\hat{\lambda}_i \leq \lambda_t \leq \hat{\lambda}_{i+1}$ and

$$\psi_i(\lambda_1, \ldots \lambda_{n+2}; x^*(\hat{\lambda}_{i+1})) = x_i^*(\hat{\lambda}_i) .$$

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[1] It is at this point that we are using our assumption that every commodity used and produced in technology A is used and produced in technology B . The analysis may, however, easily be extended. Assume the process for producing the first commodity differs in the two technologies; B uses a process which requires a commodity, x' not produced in A (although by assumption, producible by the commodities produced in A). If $c^*(0) > 0$, it is clear that it is possible to produce from $x^*(0)$ a strictly positive vector $\tilde{x}$ (i.e., including x' in positive amount) such that $d(\tilde{\lambda})\tilde{x} = 1$ for some $\tilde{\lambda}$, $0 < \tilde{\lambda} < 1$. There exists an $\epsilon > 0$ such that $x^{n+2} = x^*(\tilde{\lambda} + \epsilon)$ and $\tilde{\lambda} < \lambda_t < \tilde{\lambda} + \epsilon$, $0 < t \leq n+2$, $x^t \geq 0$, $c^t \geq 0$. By arguments analogous to those presented below, we can show that we can go in a finite number of periods from $\tilde{x}$ to $x^*(\tilde{\lambda} + \epsilon)$. From here the analysis procedes exactly as in the text.

[2] We can show that there exists an ϵ such that for any arbitrary $\hat{\lambda}_{i+1}$ if

$$\hat{\lambda}_{i+1} - \epsilon \leq \lambda_t \leq \hat{\lambda}_{i+1}$$

and $x_{n+2} = x^*(\hat{\lambda}_i)$

$$x_t > 0 , \quad c_t > 0$$

for $0 \leq t \leq n+2$.

To do this, we use Brouwer's Fixed Point Theorem. Let

$$\lambda_t \equiv \mu_t \hat{\lambda}_i + (1 - \mu_t)\hat{\lambda}_{i+1}$$

Define the continuous mapping from the set $\{0 \leq \mu_t \leq 1\}$ onto itself:

$$\hat{\mu}_t \equiv \min\left\{ \frac{\psi_t / x_t^*(\hat{\lambda}_i)}{\sum_{i=0}^{n} x_t / \psi_t^*(\hat{\lambda}_i)(n+1)} \mu_t,\ 1 \right\} \qquad t = 1, \ldots, n+1$$

$$\hat{\mu}_0 \equiv \min\{ (\sum_{i=0}^{n} \psi_t / x_t^*(\hat{\lambda}_{i+1})(n+1))\mu_0,\ 1\}$$

It has a fixed point μ_t^*. For $t = 1, \ldots, n+1$, either

(a) $\mu_t^* < 1$ implying $\dfrac{\psi_t / x_t^*}{\sum_{i=0}^{n} \psi_t / x_t^*(n+1)} = 1$

or

(b) $\mu_t^* = 1$ implying $\dfrac{\psi_t / x_t^*}{\Sigma \psi_t / x_t^*(n+1)} \geq 1$

Summing over all t, $t = 1, \ldots, n+1$, we immediately obtain the result that

$$\psi_i / x_i^* = \gamma \qquad i = 1, \ldots, n+1$$

Either

(a) $\mu_0 = 1$, in which case, using (3b), full employment of labor requires $\gamma = 1$

or

(b) $\mu_0 < 1$ in which case $\gamma = 1$.

Q.E.D.

5. In this section, we show that:

<u>Lemma 2</u>. <u>Any feasible path going from steady state A to steady state B which fully utilizes all resources along the way is efficient.</u>[1]

To establish this, we show that any such path is in fact a solution to a intertemporal utility maximization problem

$$\text{(9a)} \qquad \text{Max} \sum_{t=0}^{\infty} c_t \left(\frac{1}{1+\delta}\right)^t$$

(where δ is chosen to be equal to r^*, the rate of interest at the switch point) subject to the resource utilization constraints:

$$\text{(9b)} \qquad d_0(\lambda_t)x_t \leq 1$$

$$\text{(9c)} \qquad D(\lambda_t)x_t \leq x_{t-1} - c_{t-1}$$

where $0 \leq \lambda_t \leq 1$ and where $x_0 = x^*(A)$, the initial endowment is that corresponding to an economy in steady state using A.

It is well known that sufficient conditions for a path, which can be characterized by $\{\tilde{x}_t, \tilde{c}_t, \tilde{\lambda}_t\}$, to be optimal are that we can find a set of shadow prices of commodities, q_t (q_{it} is the price of commodity i at time t) and of labor, v_t (all measured in utility numeraire, or equivalently, in this problem in consumption numeraire) such that,

$$\text{(10)} \qquad q_{t+1} \leq (q_t D(\tilde{\lambda}_t) + v_t d_0(\tilde{\lambda}_t))(1+\delta)$$

with equality holding except when $x_{it+1} = 0$,

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[1]This lemma was originally proved in a somewhat different manner in R.M. Solow, "The Interest Rate and Transition between Techniques," <u>op.cit</u>., and in J. Stiglitz, "Accumulation in a Leontief-Sraffa Economy and the Reswitching of Techniques," M.I.T., 1966.

(10')
$$c_t = x_{n+1,\, t} \quad \text{if} \quad q_{0t} < 1$$
$$0 \leq c_t \leq x_{n+1,\, t} \quad \text{if} \quad q_{0t} = 1$$
$$c_t = 0 \quad \text{if} \quad q_{0t} > 1$$

(11)
$$q_t D(\tilde{\lambda}_t) + v_t d_0(\tilde{\lambda}_t) \leq q_t D(\lambda_t) + v_t d_0(\lambda_t) , \qquad 0 \leq \lambda_t \leq 1$$

and

(12)
$$\lim_{t \to \infty} q_t (x_t - c_t) \left(\frac{1}{1+\delta} \right)^t \to 0$$

Condition (10) says that the value of output must be equal to the value of input if the commodity is produced; (10') has the straightforward interpretation that, if the $n+1^{st}$ commodity can be used either as a capital good or for consumption, it is allocated to the use with the highest social value (shadow price); condition (11) says that the choice of technique must be cost minimizing (at the given shadow prices)[1] and condition (12) is the well known transversality condition.

But it is apparent that if we choose as our price system the competitive price system at the switch point, i.e., if

$$q_t = q^* \quad \text{for all} \quad t$$
$$v_t = v^*$$

[1]Although our formulation has considered so far only two possible alternative technologies, it is trivial to extend this to the more general case.

where

$$(13) \qquad q^* = (1+r^*)a_0v^*(I - (1+r^*)A)^{-1} = b_0v^*(1+r^*)(I - (1+r^*)B)^{-1} ,$$

$$q^*_{n+1} = 1$$

any feasible path going from steady state A to B along which resources are fully utilized, will satisfy (10)-(12), and hence will be optimal.

An immediate corollary of this result is that if $\hat{c}$ is the consumption vector along any such transition path,

$$(14) \qquad \Sigma c^*(A)\left(\frac{1}{1+\delta}\right)^t = \Sigma\hat{c}\left(\frac{1}{1+\delta}\right)^t$$

since the path which remains in steady state in A also satisfies (10)-(12), i.e., the rate of interest at the switch point is equal to the rate of return. The converse of this proposition, that every solution to (14) is either a rate of interest at a switch point or there is some technology which dominates (at that rate of interest) A and B also follows immediately.

Another interesting implication of this result is that the optimal trajectory is not unique. This should not be surprising, since our objective function is concave, but not strictly concave.

A geometrical interpretation of these results might be helpful. Consider any economy producing only three commodities, C_1, C_2, and C_3. If the production technology is linear, then the production possibilities schedule will consist of the intersection of a number of planes. Assume preferences are concave but not strictly concave, i.e., are of the form $\delta_1C_1 + \delta_2C_2 + \delta_3C_3$. Optimality requires the tangency between the plane of preferences and the opportunity set. This may either be a point, a line or a plane. In the latter two cases there is an indeterminancy of the optimal allocation. Moreover, the same point or line may be chosen for numerous different values of $\{\delta_1, \delta_2, \delta_3\}$.

Finally, we note that the first lemma implies that there exists paths which maintain full employment and go from A to B in finite time, and then return, in finite time back to A . The results just proved establish that such a path is efficient, and indeed, for one special utility function, (9a), is optimal.[1] Elsewhere, I have discussed at greater length[2] this phenomena of recurrence, where a technique (or set of techniques) is used over one interval of time, another technique over a subsequent period, to be followed by a third period in which the original set of techniques are used. The dynamic phenomena of recurrence has a certain superficial similarity to the phenomena of reswitching, where a set of techniques is used in an economy in steady state equilibrium at one interest rate, another at a higher interest rate, and the first is used at still another higher interest rate.

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[1]The fact that paths with recurrences could be efficient had been noted earlier in J.E. Stiglitz, "Accumulation in a Leontief-Sraffa Economy and the Reswitching of Techniques," op.cit. Subsequently, it was conjectured that although such paths were efficient, they would never be optimal for utility functions of the usual additive type with constant discount rates. Counterexamples to this conjecture were first presented in J.E. Stiglitz, "The Badly Behaved Economy with the Well-Behaved Production Function," in J. Mirrlees, ed., Models of Economic Growth, IEA Conference, Jerusalem, 1970.

[2]See J.E. Stiglitz, "The Badly Behaved Economy with the Well-Behaved Production Function," op.cit.

But it should be observed that reswitching is concerned with comparisons of different economies, while recurrences are concerned with growth paths of dynamic economies. Recurrences may occur for technologies in which reswitching is impossible, and need not occur in technologies in which reswitching is possible. In short, the reswitching phenomena is of no clear relevance for the more important and interesting phenomena of recurrence.

6. Although all transition paths which maintain full employment of all resources are efficient, it is important to emphasize that the converse is not true. Paths which do not maintain full employment may also be efficient. In order to have a very high level of consumption in one period, one may use up the stock of some commodity. Other factors, the utilization of which requires this complementary factor, remain unemployed until the stock can be replenished. Not only can this happen theoretically, but in war-time situations instances of this have actually occurred. Recall that efficiency requires only that consumption one period cannot be increased without decreasing consumption other periods. The fact that the undiscounted sum of consumption may be lower on one path than on another does not mean that the first path is inefficient.

Consider the following simple example. There is a single final consumption good, C. There are two processes by which it is produced. One requires 1 unit of labor and 2 units of C to produce 3.4 units of C. The other requires .5 units of labor, one unit of C, and 1 unit of capital good X to produce 2 units of C. 1 unit of labor produces 2 units of X. The pricing equation for the first process is

$$p = \frac{1+r}{3.4} + \frac{2p(1+r)}{3.4}$$

$$= 1+r/(3.4 - 2(1+r))$$

and for the second is

$$p = \left(\frac{1}{4} + \frac{p}{2}\right)(1+r) + \frac{(1+r)^2}{4}$$

$$= \frac{\frac{1+r}{4} + \frac{(1+r)^2}{4}}{1 - \frac{(1+r)}{2}}$$

The two techniques are equally profitable at $1+r = 1.5$. In the interval between 1 and 1.5, the first process is cheaper; for $r > .5$, the second process is.

Consider the two steady states; in the first, there is a net consumption per worker each period of 1.4. In the second, 1/2 of the labor force works to produce X , 1/2 to produce C . The net consumption per worker each period is 1. Assume the economy is in steady state with the first process. A transition to the second technology maintaining full employment is possible: In the first period, instead of consuming 1.4, consume 2.4. 1/2 the labor force is used to produce X , the other half uses C to produce C with the first technology. The output of C per capita is then 1.7. Of this amount, .7 is consumed. The economy will then be in a position to maintain the steady state equilibrium associated with the second technology. The rate of return, i.e. that rate of discount, δ , for which

$$\sum_0^\infty 1.4 \frac{1}{(1+\delta)^t} = 2.4 + \frac{.7}{1+\delta} + \frac{1}{(1+\delta)^2} \sum_0^\infty \frac{1}{(1+\delta)^t}$$

is just .5 (as our theorem assures us it must be). We can increase consumption the first period even further and still maintain full employment. We let C the first period equal 2.66, .37 labor plus .74 C yield (using the first technology) 1.26 units of C , while the remaining .63 units of labor produce 1.26 units of X . The next period there is zero consumption. .63 units of labor with 1.26 units of X and 1.26 units of C produce 2.52 units of C , and the remaining .37 units of labor produce .74 units of X . If .26 units of C are combined with .13 units of labor using the first process, .74 units of C are combined with .37 units of labor and the .74 units of X in the second process, to produce a total of 1.92 units of C , and the remaining labor is used to produce X , in the following period the economy can be in long run equilibrium using only the second technology. Again we can calculate the rate of return, which turns out to be 1.5.

If we wish to increase our consumption the first period still further, we can only do so by creating unemployment in some subsequent period. Assume we wish to consume 2.82 units the first period. Then we either have to have unemployment of labor the first period or of X the second. (The two are really equivalent.) Thus, we take the remaining .58 units of C and, with .29 laborers produce 1 unit of C . With .5 units of labor, we produce 1 unit of X . .31 units of labor are unemployed. Next period, we consume nothing, and we are then prepared to remain in long run equilibrium using the second technology. This is an efficient path with unemployment. The rate of return to this transition is .53, not .5: the rate of return is not equal to the

the rate of interest. As compared to our first transition path, we have an increase in consumption of .42 the first period and a reduction of .7 the second.[1] Which of these paths we prefer depends on our relative "needs" (preferences) for consumption in different periods.[2] There is nothing sacred about full employment of resources.

7. L. Pasinetti[3] has recently discussed the concept of the rate of return at some length.

He distinguishes between two definitions of the rate of return, one the analogue to our (1) and the other that of our (2). We have noted that in the special case of a one period transition, these are equivalent, while in the more general case (2) is not defined: there is no unambiguous meaning to the reduction in consumption. The "cost" of going from A to B which Pasinetti uses--and which he incorrectly ascribes to Solow--is the change in the value of the capital stock (in consumption numeraire). This takes two alternative forms: in the first, no account is taken of the capital which might become "redundant." Let $p(r)$ be the price vector (steady state) corresponding to the interest rate r, and $\underline{x}(A)$ and $\underline{x}(B)$ be the vectors of steady state capital stocks for the two technologies; then if we define

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[1] Compared to the other transition path, C the first period is increased by .16, the second period is unchanged, and the third period, C is smaller by .52.

[2] There is no reason at this point to restrict ourselves to simple additive utility functions with constant rates of time preference.

[3] L. Pasinetti, "Switches of Techniques and the 'Rate of Return' in Capital Theory," Economic Journal, September 1969.

$$\rho \equiv \frac{p(r)\cdot(\underline{c}(B) - \underline{c}(A))}{p(r)\cdot(\underline{x}(B) - \underline{x}(A))} \tag{15}$$

it is a fairly trivial theorem that the rate of profit is equal to the rate of return at a switch point. For at a switch point the wage, in consumption numeraire, must be the same for the two technologies, and wage payments plus payments to capital must be equal to the value of net output:[1]

$$w^* = p(r^*)[c(A) - x(A)r^*] = p(r^*)[c(B) - x(B)r^*]$$

Eliminating w from these two equations, the result is immediate. (Pasinetti in describing this result as a tautology, seems to have confused an easily proved theorem with a tautology.) In the second form, account is taken of the possibility of redundant factors, and the denominator of (15) becomes $p(r)(x(B) - x(A) - x_R(A))$ where $x_R(A)$ is the vector of redundant capital.[2] As we have noted, (15) may not be equal to the rate of profit, if there are redundant factors. But our theorem establishes the fact that there always exist paths of transition which fully utilize resources; the assumption of capital malleability is not required for this.

8. (15)[3] is not, however, what is usually meant by the rate of return; our definition (1), on the other hand is.

Our objection to (15) is two-fold. First, our definition (1) is purely

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[1] Wages are assumed to be paid at the end of the period of production.

[2] $x_R(A)$ is never precisely defined. At times, he seems to be suggesting that if B requires less of some capital good than A (in steady state) the difference is redundant capital; i.e., $x_R(A) = \max\{0, x(A) - x(B)\}$.

[3] With or without the correction for redundant capital.

technological, while (15) utilizes a price system. We are able to do this because we assume only a single consumption good. If there are many consumption goods, we have the usual index number problems of comparing economies with different consumption bundles. There are two approaches which we might take.

The simplest is to do our calculations in terms of a fixed consumption bundle, i.e., insist that the ratio in which the different consumption goods are supplied remains unchanged over time (and throughout the transition). It is easy to extend our theorem to show that no matter what the proportions we choose, there exists feasible full employment transitions from one steady state to another, and that the internal rate of return (defined in (1) where c is the number of units of our consumption bundle) is equal to the rate of interest.

The second approach is to use relative prices to weight the different consumption goods. Then (1) becomes

$$\Sigma p^*(\hat{c} - c^*(A))\left(\frac{1}{1+\delta}\right)^t = 0 \tag{16}$$

or in the special case where a transition is possible in one period, this becomes

$$\delta = \frac{p^*(c^*(B) - c^*(A))}{p^*(c^*(A) - \hat{c})} \tag{17}$$

The internal rate of return defined by (16) will not in general be equal to the rate of interest except if the price vector is identical to those at the switch point between A and B. But those are the natural prices to use in discussing a transition from technology A to technology B, for in competitive equilibrium, they give the marginal rates of substitution at which individuals would be willing to substitute one consumption good for another.

Note that neither of these is equivalent to any of Pasinetti's definitions: he always measures the cost of the transitions in terms of the change in the value of capital, not, as we have done, in terms of the change in consumption. But clearly since we wish to evaluate alternative paths (from the point of view of the economy as a whole) in terms of the consumption goods which they can deliver, it is far more natural to measure the "cost" in terms of foregone consumption.

This is our second objection to definition (15): it is misleading (if not simply incorrect) to measure the "cost" of a transition by the difference in the value of capital. The change in the value of capital (in comparing one steady state with another) has little if anything to do with a measurement of the true cost of going from one steady state to another, which is the cost of the consumption foregone along the way.[1]

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[1]Pasinetti's confusions in these respects have a long and noble tradition. Wicksell ("Real Capital and Interest," Appendix 2, Lectures on Political Economy, London, 1934) was very much concerned about the possibility of a higher value of capital (in steady state) being associated with a lower level of consumption (i.e. with the possibility that the pseudo-production function be negatively sloped). By assuming constant elasticity longevity functions in his model, he was able to show that this could not occur. When, however, this assumption is dropped, it is possible that consumption increase may as the value of capital decreases. This does not mean, however, that in going from the steady state with the lower level of consumption to that with the higher, there need not be "real savings," i.e., consumption still must be foregone along the transition. See J.E. Stiglitz, "The Badly Behaved Economy with the Well-Behaved Production Function," in J. Mirrlees, ed., IEA Conference Volume on Models of Economic Growth (forthcoming).

9. Although it is clear that the concept of the rate of return does not suffer from the kinds of difficulties which the "Cambridge economists" have ascribed to it, it is not clear to me that it is anywhere near as useful as Professor Solow has suggested. My objections to its use in this context very much parallel the conventional objections to using the internal rate of return for making the choice of technique at the micro-economic level. Most importantly, it implicitly (or explicitly) assumes the marginal rate of substitution for consumption between any two adjacent periods is constant. In a growing economy, out of steady state, this is not likely to be the case. Even if the pure rate of time preference were constant, the variations in consumption necessary to facilitate a transition from one steady state to another would, if the utility function is strictly concave, result in differing marginal rates of substitution. Indeed, it is possible to show that, if the representative man's intertemporal utility function (the planner's utility function) is of the form

$$\sum_{1}^{\infty} U(c)\left(\frac{1}{1+\delta}\right)^{t} \qquad U'' < 0$$

and if $\delta = r^*$, the rate of interest at the switch point between technologies A and B, then if the economy were initially in steady state using A, it would remain there: a transition to B would entail a loss of utility; and if it were initially in steady state using B, it would remain there. <u>The steady state configuration of the economy is not independent of the past history,</u>

as in the simpler conventional models.[1] Note that if the planner used the rate of return criterion, he would say that economy is indifferent between remaining in A and going to B (along a full employment path).

10. The advantage of the use of the rate of return as defined by (1) is that it provides a technologically determined number (when there is only a single consumption good) by which a proposed deviation from a given steady state path may be evaluated. It has, however, as we noted in the previous section, some important disadvantages. We propose the use of the following measure, which is not restricted to economies initially in steady state: We shall call it the Present Discounted Value Range (PDVR); it too has the advantage that it is completely technologically determined. The PDVR is the natural generalization of the present discounted value criterion to linear technologies (where there will in general be several "price systems" corresponding to any production point).

Assume the economy is on an efficient path. Any efficient path may be described as the solution to a problem of the form

(18) $$\text{Max} \quad c_0$$

(18a) $$\text{s.t.} \quad c_t = \bar{c}_t \qquad t > 1$$

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[1]The proof of this proposition is straightforward; Jensen's inequality says that, if $U'' < 0$

$$\Sigma U(c_t) f(t) < U(\Sigma c_t f(t))$$

where $\Sigma f(t) = 1$. By assumption,

$$\Sigma \hat{c} \left(\frac{1}{1+\delta} \right)^t = \Sigma c^*(A) \left(\frac{1}{1+\delta} \right)^t = c^*(A)/\delta$$

Let

$$f(t) = \delta \left(\frac{1}{1+\delta} \right)^t ; \quad \Sigma f(t) = 1 ; \quad c^*(A) = \Sigma \hat{c}_t f(t) = \Sigma \hat{c}_t \delta \left(\frac{1}{1+\delta} \right)^t$$

and the result is immediate.

given, of course, the initial endowments and the technologies available to the economy. We can then define the absolute value of the left and right handed derivatives of c_0 with respect to $\bar{c}_t$ (because of the linearity of the technology, these will clearly exist), which we will denote by R^{t+} and R^{t-} . It is easy to show that $R^{t-} \leq R^{t+}$. The PDVR is defined by the interval PDV^+ and PDV^- : if Δc_t is the proposed deviation from the given path,

$$PDV^+ \equiv \sum_t \max(R^{t+}\Delta c_t, R^{t-}\Delta c_t) = \Sigma R^{t+}\Delta c_t^+ + \Sigma R^{t-}\Delta c_t^-$$

$$PDV^- \equiv \sum_t \min(R^{t+}\Delta c_t, R^{t-}\Delta c_t) = \Sigma R^{t+}\Delta c_t^- + \Sigma R^{t-}\Delta c_t^+$$

where Δc_t^+ are the positive elements of the Δc_t vector, Δc_t^- are the negative elements (i.e., $\Delta c_t^+ = \max\{0, \Delta c_t\}$, $\Delta c_t^- = \min\{0, \Delta c_t\}$). If $PDV^- > 0$, then the proposed deviation (if it is sufficiently small) should clearly be adopted, if $PDV^+ < 0$ it should be rejected. But if $PDV^+ > 0$, while $PDV^- < 0$, the result is ambiguous. But this is just as it should be. In competitive equilibrium, the marginal rate of substitution between c_0 and c_t will lie in the interval $[R^{t+}, R^{t-}]$. And it is the marginal rate of substitution which we wish to use to evaluate consumption of one period relative to that of another. PDV^+ gives the highest possible value of the gain from adopting the proposed deviation, PDV^- gives the lowest. If the former is positive and the latter is negative, the true benefit may be positive or negative depending on preferences. As is conventional in such linear models, in general to make the decision of whether to adopt the deviation Δc a knowledge of the technology will not be sufficient: economic choices must be based on preferences.

This measure, although it has the advantage over the rate of return criterion of not being restricted to steady states, may still be misleading if it is used to evaluate any but infinitesimal changes in consumption. For if there is diminishing marginal utility of consumption, non-infinitesimal changes in consumption will change marginal rates of substitution of consumption in one period for that in another. Our criterion will no longer be valid: any purely technologically based criteria can at best be a halfway house toward a real evaluation based on preferences.